JENSEN'S INEQUALITY FOR SPECTRAL ORDER AND SUBMAJORIZATION

JORGE ANTEZANA, PEDRO MASSEY AND DEMETRIO STOJANOFF

Abstract. Let $\mathcal{A}$ be a C^*-algebra and $\phi : \mathcal{A} \to L(H)$ be a positive unital map. Then, for a convex function $f : I \to \mathbb{R}$ defined on some open interval and a self-adjoint element $a \in \mathcal{A}$ whose spectrum lies in I, we obtain a Jensen's-type inequality $f(\phi(a)) \preceq \phi(f(a))$ where $\preceq$ denotes an operator preorder (usual order, spectral preorder, majorization) and depends on the class of convex functions considered i.e., monotone convex or arbitrary convex functions. Some extensions of Jensen's-type inequalities to the multi-variable case are considered.

1. Introduction

Jensen's inequality is the continuous version of the usual definition of convex function and it can be stated in the following way: let I be an open interval and $f : I \to \mathbb{R}$ a convex map. Then, for every probability space (X, P) and every integrable map $g : X \to I$,

$$f\left(\int_X g\, dP\right) \le \int_X f \circ g\, dP.$$

In the context of C^*-algebras the simplest generalization of Jensen's inequality can be made by taking a state φ and a selfadjoint element a of a C^*-algebra $\mathcal{A}$ such that $\sigma(a) \subseteq I$. In this case,

$$f(\varphi(a)) \le \varphi(f(a)) \;, \tag{1}$$

because the state φ restricted to the C^*-algebra generated by a can be represented as an integral with respect to a probability measure. This type of inequality becomes false (for general convex functions) if one replaces φ by a unital positive map between two C^*-algebras. Indeed, given an infinite dimensional Hilbert space $\mathcal{H}$ and a function F defined on some open interval I, then F is **operator convex** if and only if

$$F(\phi(a)) \le \phi(F(a)) \;, \quad \text{for every} \quad a \in L(\mathcal{H})_{sa} \quad \text{with} \quad \sigma(a) \subseteq I \;, \tag{2}$$

and for every unital positive map $\phi : L(\mathcal{H}) \to L(\mathcal{H})$. This equivalence can be deduced from the well known characterizations of operator convexity (see for example

2000 *Mathematics Subject Classification.* Primary 47A63 ; Secondary 46L05, 46L10, 15A45.
Key words and phrases. Jensen's inequality, convex functions, positive maps, majorization.
Partially supported by Universidad de La Plata (UNLP 11 X350) and ANPCYT (PICT 03-09521) .

[6], [8] or [4]), using the Stinespring's theorems. Although in some particular cases (e.g. when $\phi(a)$ and $\phi(f(a))$ commute) we can obtain Jensen's type inequalities like (2) for arbitrary convex functions, the mentioned equivalence tells us that in general we can not consider the usual order for this kind of inequalities and convex funtions which are not operator convex.

However, previous works on the matter, such as Brown-Kosaki's and Hansen-Pedersen's ([5], [8]), suggest the idea of considering Jensen's type inequalities with respect to other preorders, such as the spectral order and submajorization (see section 2 for their definitions).

In this paper we study different Jensen's type inequalities in which, as in the case of Eq. (2), a positive (unital) map plays the roll of a non-commutative integral.

The paper is organized as follows. In section 2 we recall some definitions and we introduce the notations used throughout this paper. Section 3 is divided in two subsections, depending on the preorder relation and the assumptions on the convex functions. In the first subsection we consider monotone convex functions and, following Brown-Kosaki's ideas on the matter, we prove Jensen's type inequalities with respect to the spectral preorder. In the second subsection we obtain Jensen's type inequalities for arbitrary convex functions either by considering restrictions on the algebra $\mathcal{B}$ or by using the submajorization preorder. We list the main results of this section: Let $\mathcal{A}$, $\mathcal{B}$ be unital C^*-algebras, $\phi : \mathcal{A} \to \mathcal{B}$ a positive unital map, f a convex function defined on an open interval I and $a \in \mathcal{A}$, such that $a = a^*$ and $\sigma(a) \subseteq I$.

(1) If f is monotone and $\mathcal{B}$ is a von Neumann algebra, then

$$f(\phi(a)) \precsim \phi(f(a)) \quad \text{(spectral preorder)}.$$

(2) If $\mathcal{B}$ is abelian or, more generally, if $\phi(f(a))$ and $\phi(a)$ commute, then

$$f(\phi(a)) \leq \phi(f(a)).$$

(3) If $\mathcal{B}$ a finite factor, then $f(\phi(a))) \prec_w \phi(f(a))$ (submajorization).

We remark that all these inequalities still hold for *contractive* positive maps, under the assumption that $0 \in I$ and $f(0) \leq 0$. In section 4 we briefly describe the multi-variable functional calculus and obtain in this context similar results to those of section 3 by using essentially the same techniques. In section 5 we apply the results obtained to the finite dimensional case, where there exist fairly simple expressions for the spectral preorder and for the submajorization; we show that our results generalize those appeared in a recent work by J. S. Aujla and F. C. Silva [3]. Several results of this work are included in a Los Alamos preprint version (see [1]) in 2004. Since then, they have appeared some related results, see for example [2].

2. Preliminaries

Let $\mathcal{A}$ be a C^*-algebra, throughout this paper $\mathcal{A}_{sa}$ denotes the real vector space of self-adjoint elements of $\mathcal{A}$, $\mathcal{A}^+$ the cone of positive elements, $\mathrm{Gl}(A)$ the group of invertible elements of $\mathcal{A}$ and $\mathcal{U}(\mathcal{A})$ its unitary group. We also assume that all the C^*-algebras in consideration are unital. Given a Hilbert space $\mathcal{H}$, we denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For every $c \in L(\mathcal{H})$ its range will be denoted by $R(c)$, its null space by $\ker c$ and its spectrum by $\sigma(c)$. If p, $q \in L(\mathcal{H})$ are orthogonal projections, we denote by $p \wedge q$ (resp. $p \vee q$) the orthogonal projection onto the intersection of their ranges (resp. the closed subspace generated by their ranges).

Spectral Preorder and Majorization. In what follows $E_a[I]$ denotes the spectral projection of a self-adjoint operator a in a von Neumann algebra $\mathcal{A}$, corresponding to a (Borel) subset $I \subseteq \mathbb{R}$. Let us recall the notion of spectral preorder.

Definition 2.1. Let $\mathcal{A}$ be a von Neumann algebra. Given $a, b \in \mathcal{A}_{sa}$, we say that $a \lesssim b$ if $E_a[(\alpha, +\infty)]$ is equivalent, in the sense of Murray-von Neumann, to a subprojection of $E_b[(\alpha, +\infty)]$ for every real number α.

In finite factors the following result can be proved (see [5] and [12]).

Proposition 2.2. *Let $\mathcal{A}$ be a finite factor, with normalized trace tr. Given $a, b \in \mathcal{A}_{sa}$, the following conditions are equivalent:*

1. $a \lesssim b$.
2. $\mathrm{tr}(f(a)) \leq \mathrm{tr}(f(b))$ *for every continuous increasing function f defined on an interval containing both $\sigma(a)$ and $\sigma(b)$.*
3. *There exists a sequence $\{u_n\}$ in $\mathcal{U}(\mathcal{A})$ such that $u_n b u_n^* \xrightarrow[n\to\infty]{} c$ and $c \geq a$.*

Finally, Eizaburo Kamei defined the notion of majorization and submajorization in finite factors (see [11]):

Definition 2.3. Let $\mathcal{A}$ be a finite factor with normalized trace tr. Given $a, b \in \mathcal{A}_{sa}$, we say that a is submajorized by b, and denote $a \prec_w b$, if the inequality:

$$\int_0^\alpha e_a(t)\,dt \leq \int_0^\alpha e_b(t)\,dt$$

holds for every real number α, where $e_c(t) = \inf\{\gamma : \, tr(E_c[(\gamma, \infty)]) \leq t\}$ for $c \in \mathcal{A}_{sa}$. An equivalent condition (see [10]) is that

$$\sup\left\{ \mathrm{tr}(ap) : p \in \mathcal{P}_k \right\} \leq \sup\left\{ \mathrm{tr}(bp) : p \in \mathcal{P}_k \right\} \ , \quad 0 \leq k \leq 1 \ ,$$

where $\mathcal{P}_k = \{p \in \mathcal{A}_{sa} : p^2 = p \text{ and } \mathrm{tr}\, p = k\}$. We say that a is majorized by b (and denote $a \prec b$) if $a \prec_w b$ and $\mathrm{tr}(a) = \mathrm{tr}(b)$. ▲

The following characterizarion of submajorization also appears in [11].

Proposition 2.4. *Let $\mathcal{A}$ be a finite factor with normalized trace tr. Given $a, b \in \mathcal{A}_{sa}$, the following conditions are equivalent:*

(1) $a \prec_w b$.
(2) $\operatorname{tr}(g(a)) \leq \operatorname{tr}(g(b))$ *for every non-decreasing convex function g defined on an interval containing both $\sigma(a)$ and $\sigma(b)$.*

3. Jensen's type inequalities.

Throughout this section ϕ is a positive unital map from a C^*-algebra $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$, $f : I \to \mathbb{R}$ is a convex function defined on an open interval I and $a \in \mathcal{A}_{sa}$ whose spectrum lies in I. Note that the spectrum of $\phi(a)$ is also contained in I. As we mentioned in the introduction, we can not expect a Jensen's type inequality of the form

$$f(\phi(a)) \leq \phi(f(a)) \tag{3}$$

for an arbitrary convex function f without other assumptions. This is the reason why, in order to study inequalities similar to (3) for different subsets of convex functions, we shall use the spectral and submajorization (pre)orders, or we shall change the hypothesis made over $\mathcal{B}$. Although most of the inequalities considered in this section involve unital positive maps, similar results can be obtained for contractive positive maps by adding some extra hypothesis on f.

Monotone convex and concave functions. Spectral Preorder. In this section we shall consider monotone convex and concave functions. The following result, due to Brown and Kosaki [5], indicates that the appropriate order relation for this class of functions is the spectral preorder. Let $\mathcal{A}$ be a semi-finite von Neumann algebra, and let $v \in \mathcal{A}$ be a contraction; then, for every positive operator $a \in \mathcal{A}$ and every continuous monotone convex function f defined in $[0, +\infty)$ such that $f(0) = 0$, it holds that

$$v^* f(a) v \lesssim f(v^* a v).$$

The following statement is an analogue of Brown and Kosaki's result, in terms of positive unital maps and monotone convex functions. The proof we give below follows essentially the same lines as that in [5].

Theorem 3.1. *If $\mathcal{B}$ is a von Neumann algebra, and f is monotone convex, then*

$$f(\phi(a)) \lesssim \phi(f(a)) \ . \tag{4}$$

Proof. We shall prove that, given $\alpha \in \mathbb{R}$, there exists a projection $q \in \mathcal{A}$ such that

$$E_{f(\phi(a))}\left[(\alpha, +\infty)\right] = E_{\phi(a)}\left[\{f > \alpha\}\right] \sim q \leq E_{\phi(f(a))}\left[(\alpha, +\infty)\right].$$

We claim that $E_{\phi(a)}\left[\{f > \alpha\}\right] \wedge E_{\phi(f(a))}\left[(-\infty, \alpha]\right] = 0$. Consider a unit vector $\overline{\eta} \in R\left(E_{\phi(a)}\left[\{f > \alpha\}\right]\right)$. Since f is monotone we have that $\alpha < f(\langle \phi(a)\overline{\eta}, \overline{\eta}\rangle)$ and, using Jensen's inequality for states, Eq.(1), we get $\alpha < \langle \phi(f(a))\overline{\eta}, \overline{\eta}\rangle$. On the other hand, if $\overline{\xi} \in R\left(E_{\phi(f(a))}\left[(-\infty, \alpha]\right]\right)$ is a unitary vector, then $\alpha \geq \langle \phi(f(a))\overline{\xi},\ \overline{\xi}\rangle$. So, using Kaplansky's formula, we have

$$\begin{aligned} E_{f(\phi(a))}\left[(\alpha, +\infty)\right] &= E_{f(\phi(a))}\left[(\alpha, +\infty)\right] - \left(E_{f(\phi(a))}\left[(\alpha, +\infty)\right] \wedge E_{\phi(f(a))}\left[(-\infty, \alpha]\right]\right) \\ &\sim \left(E_{f(\phi(a))}\left[(\alpha, +\infty)\right] \vee E_{\phi(f(a))}\left[(-\infty, \alpha]\right]\right) - E_{\phi(f(a))}\left[(-\infty, \alpha]\right] \\ &\leq I - E_{\phi(f(a))}\left[(-\infty, \alpha]\right] = E_{\phi(f(a))}\left[(\alpha, +\infty\right]. \end{aligned}$$

■

Remark 3.2. Note that we can not infer from the above result a similar one for monotone concave functions, because it is not true that $a \lesssim b \Rightarrow -b \lesssim -a$. However, using the same arguments a similar result can be proved for monotone concave function. ▲

If $\phi(a)$ and $\phi(f(a))$ are compact operators and $f(0) = 0$, Theorem 3.1 can be rephrased in terms of the following interpretation of the spectral order:

Proposition 3.3. *Let $\mathcal{H}$ be a Hilbert space. Let $a, b \in L(\mathcal{H})^+$ be compact operators, and $\{\lambda_n\}_{n\leq N}$ (resp. $\{\mu_n\}_{n\leq M}$) the decreasing sequence of positive eigenvalues of a (resp. b), counted with multiplicity ($N, M \in \mathbb{N} \cup \{\infty\}$). Suppose that $a \lesssim b$. Then*

1. *$\lambda_m \leq \mu_m$ for every $m \leq \min\{N, M\}$.*
2. *There exists a partial isometry $u \in L(\mathcal{H})$ with initial space $\overline{R(a)}$ such that, if $c = uau^*$, $c \leq b$ and $cb = bc$.*

Moreover, if $\dim(\mathcal{H}) < \infty$, then (2) holds for $a, b \in L(\mathcal{H})_{sa}$, for some $u \in \mathcal{U}(\mathcal{H})$.

Proof. Given $\alpha > 0$, let $n_\alpha = \max\{m \leq N : \lambda_m > \alpha\}$ and $m_\alpha = \max\{m \leq M : \mu_m > \alpha\}$ (we take $n_\alpha = 0$ if $\alpha \geq \|a\|$ and similarly for m_α). By hypothesis

$$n_\alpha = \operatorname{tr} E_a\left[(\alpha, +\infty)\right] \leq \operatorname{tr} E_b\left[(\alpha, +\infty)\right] = m_\alpha\ .$$

Taking $\alpha = \lambda_m - \varepsilon$ (for every $0 < \varepsilon < \lambda_m$), one deduces that $\lambda_m \leq \mu_m$. Let $\{\xi_n\}_{n\leq N}$ be an orthonormal basis of $\overline{R(a)}$ of eigenvectors associated to the sequence $\{\lambda_n\}_{n\leq N}$ of a. Define in a similar way, $\{\eta_m\}_{m\leq M}$ associated to $\{\mu_m\}_{m\leq M}$ for b. Consider the isometry $u : \overline{R(a)} \to \overline{R(b)}$, given by $u(\xi_n) = \eta_n$, $n \leq N$, and extend u to a partial isometry with $\ker u = \ker a$. Let $c = uau^*$. Then $cb = bc$ and $c \leq b$, since $\{\eta_m\}_{m\leq N}$ is an orthonormal basis of $\overline{R(c)}$ of eigenvectors associated to $\{\lambda_n\}_{n\leq N}$. If $\dim \mathcal{H} = n < \infty$ and $a, b \in L(\mathcal{H})_{sa}$, we can include the eventual nonpositive eigenvalues to the previous argument, getting orthonormal basis of $\mathcal{H}$, so that u becomes unitary. ■

Remark 3.4. It can be shown, by a slight modification of the proof of the previous Lemma, that Item 2 still holds in case that b is diagonalizable but not compact. ▲

Remark 3.5. In [7] Farenick and Manjegani stated a question that, in the particular case of $L(\mathcal{H})$, can be formulated in the following way: given $a, b \in L(\mathcal{H})_{sa}$ such that $a \lesssim b$, is there an isometry $v \in L(\mathcal{H})$ such that $v^*bv \geq a$? Proposition 3.3

gives a partial answer to this question, and the next example shows that, in $L(\mathcal{H})$, this answer is the best one, even if we restric ourselves to compact operators.

Example 3.6. Let $B = \{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of the separable Hilbert space $\mathcal{H}$ and let $b \in L(\mathcal{H})^+$ be the diagonal operator (w.r.t. B) with diagonal $\{2^{-n}\}_{n\in\mathbb{N}}$. If s is the backward shift (w.r.t. B), let $a = s^*bs$, i.e. $a\xi_1 = 0$ and $a\xi_n = 2^{n-1}\xi_n$ for $n \geq 2$. Then, $a \precsim b$ because $\operatorname{tr}[E_a(\lambda,\infty)] = \operatorname{tr}[E_b(\lambda,\infty)]$ for every $\lambda \geq 0$. Suppose that there exists an isometry $v \in L(\mathcal{H})$ such that $v^*bv \geq a$. Then

$$\langle bv(\xi_2),\, v(\xi_2)\rangle \geq \langle a(\xi_2),\, a(\xi_2)\rangle = \frac{1}{2} \implies v(\xi_2) = \xi_1 \ .$$

Similarly, using that v is an isometry, it can proved that $v(\xi_n) = \xi_{n-1}$ for every $n \geq 2$. Therefore, $v = s$ which is not an isometry. ▲

Arbitrary convex functions. The following example, due to J. S. Aujla and F. C. Silva, shows that Theorem 3.1 may be false if the function f is not monotone.

Example 3.7. Consider the positive map $\phi : \mathcal{M}_4 \to \mathcal{M}_2$ given by

$$\phi\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{A_{11}+A_{22}}{2}$$

Take $f(t) = |t|$ and let A be the following matrix $A = \left(\begin{array}{cc|cc} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$. Then $\phi(f(A)) = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and $f(\phi(A)) = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$. Therefore

$$\operatorname{rank}(E_{\phi(f(A))}\,[(0.5,+\infty)]) = 1 < 2 = \operatorname{rank}(E_{f(\phi(A))}\,[(0.5,+\infty)]).$$

Nevertheless, a Jensen's type inequality holds with respect to the usual order for every convex function, if the map ϕ takes values in a commutative algebra $\mathcal{B}$. ▲

Proposition 3.8. *If $\mathcal{B}$ is abelian, then $f(\phi(a)) \leq \phi(f(a))$.*

Proof. For every character Γ of the algebra $\mathcal{B}$, $\Gamma\circ\phi$ is an state over the C^*-algebra $\mathcal{A}$. Thus, using Eq. (1) (Jensen's inequality for states), $\Gamma(f(\phi(a))) = f(\Gamma(\phi(a))) \leq \Gamma(\phi(f(a)))$. ■

Now, we shall prove a Jensen inequality for arbitrary convex functions, with respect to the submajorization (pre)-order.

Theorem 3.9. *Suppose that $\mathcal{B}$ is a finite factor. Then*

$$f(\phi(a))) \prec_w \phi(f(a)). \tag{5}$$

In order to prove this theorem, we need the following lemma.

Lemma 3.10. *Let* tr *be a trace defined on $\mathcal{B}$, and let $b \in \mathcal{B}$. Then, there exist a Borel measure μ defined on $\sigma(b)$ and a positive unital linear map $\Psi : \mathcal{B} \to L^\infty(\sigma(b),\mu)$ such that:*

(1) $\Psi(f(b)) = f$ *for every* $f \in C(\sigma(b))$.
(2) $\operatorname{tr}(x) = \displaystyle\int_{\sigma(b)} \Psi(x)(t)\, d\mu(t)$ *for every* $x \in \mathcal{B}$.

Proof. First of all, note that for every continuous function g defined on the spectrum of b, the map

$$g \to \operatorname{tr}(g(b))\,,$$

is a bounded linear functional on $C(\sigma(b))$. Therefore, by the Riesz's representation theorem, there exists a Borel measure μ defined on the Borel subsets of $\sigma(b)$, such that for every continuous function g on $\sigma(b)$,

$$\operatorname{tr}(g(b)) = \int_{\sigma(b)} g(t)\, d\mu(t).$$

Now, given $x \in \mathcal{B}^+$, define the following functional on $C(\sigma(B))$

$$g \to \operatorname{tr}(xg(b)).$$

Since for every $y \in \mathcal{B}^+$, $\operatorname{tr}(xy) = \operatorname{tr}(y^{1/2}xy^{1/2}) \leq \|x\| \operatorname{tr}(y)$, this functional is not only bounded but also dominated by the functional defined before. So, there exists an element h_x of $L^\infty(\sigma(b), \mu)$ such that, for every $g \in C(\sigma(b))$,

$$\operatorname{tr}(xg(b)) = \int_{\sigma(b)} g(t)\, h_x(t)\, d\mu(t).$$

The map $x \mapsto h_x$, extended by linearization, defines a positive unital linear map $\Psi : \mathcal{B} \to L^\infty(\sigma(b), \mu)$ which satisfies conditions (1) and (2) because

$$\operatorname{tr}(f(b)g(b)) = \operatorname{tr}(fg(b)) = \int_{\sigma(b)} g(t)f(t)\, d\mu(t)\ , \quad \text{and}$$

$$\operatorname{tr}(x) = \operatorname{tr}(x\ 1(b)) = \int_{\sigma(b)} 1\ \Psi(x)(t)\, d\mu(t) = \int_{\sigma(b)} \Psi(x)(t)\, d\mu(t)\ .$$

∎

Proof of Theorem 3.9. By Proposition 2.4, it is enough to prove that

$$\operatorname{tr}[g(f(\phi(a)))] \leq \operatorname{tr}[g(\phi(f(a)))]$$

for every non-decreasing convex function g such that $g(f(\phi(a)))$ and $g(\phi(f(a)))$ are well defined. Fix such a function g. Let $\Psi : \mathcal{B} \to L^\infty(\sigma(b), \mu)$ be the positive unital linear map given by Lemma 3.10 for $b = \phi(a)$. Define the map

$$\Phi : C(\sigma(a)) \to L^\infty(\sigma(b), \mu) \quad \text{by} \quad \Phi(h) = \Psi(\phi(h(a)))\ .$$

Then, Φ is bounded, unital and positive. Moreover, given $h \in C(\sigma(a))$,

$$\operatorname{tr}(\phi(h(a))) = \int_{\sigma(b)} \Psi(\phi(h(a)))(t)\, d\mu(t) = \int_{\sigma(b)} \Phi(h)(t)\, d\mu(t).$$

Then, using Proposition 3.8 and the fact that g is non-decreasing,

$$\begin{aligned}\operatorname{tr}(g[f(\phi(a))]) = \operatorname{tr}(g\circ f(b)) &= \int_{\sigma(b)} g\circ f(t)\,d\mu(t) = \int_{\sigma(b)} g[f(\Phi(Id))]\,d\mu(t)\\ &\le \int_{\sigma(b)} g[\Phi(f)(t)]\,d\mu(t) = \int_{\sigma(b)} g[\Psi(\phi(f(a)))(t)]\,d\mu(t)\\ &\le \int_{\sigma(b)} \Psi(g[\phi(f(a))])(t)\,d\mu(t) = \operatorname{tr}(g[\phi(f(a))]),\end{aligned}$$

which completes the proof. ■

Remark 3.11. Let $\mathcal{C}\subseteq\mathcal{B}$ be C^*-algebras. A conditional expectation $\mathcal{E}:\mathcal{B}\to\mathcal{C}$ is a positive $\mathcal{C}$-linear projection from $\mathcal{B}$ onto $\mathcal{C}$ of norm 1. The centralizer of $\mathcal{E}$ is the C^*-subalgebra of $\mathcal{B}$ defined by $\mathcal{B}^{\mathcal{E}}=\{b\in\mathcal{B}:\ \mathcal{E}(ba)=\mathcal{E}(ab),\ \forall a\in\mathcal{B}\}$. Following similar ideas as those in Lemma 3.10 and the proof of Theorem 3.9 it can be proved that, if $\phi(a)$ belongs to $\mathcal{B}^{\mathcal{E}}$, then

$$\mathcal{E}(g[f(\phi(a))]) \le \mathcal{E}(g[\phi(f(a))]) \tag{6}$$

for every non-decreasing convex function $g:J\to\mathbb{R}$ defined on some open interval J such that $f(I)\subseteq J$. This fact is similar to Hansen and Pedersen's results obtained in [8]. ▲

Remark 3.12. With slight modifications in their proofs (and using Theorem 4.2 below), Proposition 3.8, and Theorems 3.1 and 3.9 can be restated for a *contractive* map ϕ (instead of unital), if the function f satisfies that $f(0)\le 0$. ▲

4. The multi-variable case.

In this section we shall be concerned with the restatement, in the multi-variable context, of several results obtained in section 3. For related results, see [9] and [13].

Multivariated functional calculus. Let $\mathcal{A}$ be a unital C*-algebra and let $a_1,\dots,a_n$ be mutually commuting elements of $\mathcal{A}_{sa}$. If $\mathcal{B}=C^*(a_1,\dots,a_n)$ denotes the unital C*-subalgebra of $\mathcal{A}$ generated by these elements, then $\mathcal{B}$ is abelian. So there exists a compact Hausdorff space X such that $\mathcal{B}$ is *-isomorphic to $C(X)$. Actually X is (up to homeomorphism) the space of characters of $\mathcal{B}$.

Recall that in the case of one operator $a\in\mathcal{A}_{sa}$, X is homeomorphic to $\sigma(a)$. In general, characters of the algebra $\mathcal{B}$ are associated in a continuous and injective way to n-tuples $(\lambda_1,\dots,\lambda_n)\in\prod_{i=1}^n\sigma(a_i)$ by the correspondence $\gamma\mapsto(\gamma(a_1),\dots,\gamma(a_n))$. Thus X is homeomorphic to its image under this map, which we call joint spectrum and denote $\sigma(a_1,\dots,a_n)$.

Let $f\in C(\sigma(a_1,\dots,a_n))$. Denote by $f(a_1,\dots,a_n)\in C^*(a_1,\dots,a_n)$, the element that corresponds to f by the above *-isomorphism. Note that by Tietze's extension theorem we can consider functions defined on $\prod_{i=1}^n\sigma(a_i)\subseteq\mathbb{R}^n$ without loss of generality. Therefore the association $f\mapsto f(a_1,\dots,a_n)$ is a *-homomorphism from $C(\prod_{i=1}^{n}\sigma(a_i))$ onto $\mathcal{B}$, which generealizes the functional calculus of one variable.

Jensen's type inequality in several variables. Submajorization. Throughout this section $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a positive unital map. We fix U, an open convex subset of $\mathbb{R}^n$, a convex function $f : U \to \mathbb{R}$ and a mutually commuting n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}_{sa}^n$ such that $\prod_{i=1}^{n}\sigma(a_i) \subseteq U$. We denote $\vec{\phi}(\mathbf{a}) = (\phi(a_1), \ldots, \phi(a_n)\,)$.

Remark 4.1. Let $K \subseteq U$ be compact. Then, there is a countable family of linear functions $\{f_i\}_{i\geq 1}$ such that for each $x \in K$ it holds that $f(x) = \sup_{i\geq 1} f_i(x)$. ▲

Theorem 4.2. *Suppose that $\phi(f(\mathbf{a}))$, $\phi(a_1), \ldots, \phi(a_n)$ are also mutually commuting. Then*

$$f(\phi(a_1), \ldots, \phi(a_n)) = f(\vec{\phi}(\mathbf{a})\,) \leq \phi(f(\mathbf{a})\,) = \phi(f(a_1, \ldots, a_n)). \tag{7}$$

Moreover, if $\vec{0} = (0, \ldots, 0) \in U$ and $f(\vec{0}) \leq 0$ then equation (7) holds even if ϕ is positive contractive.

Proof. Denote by $\widehat{\mathcal{B}}$ the abelian C^*-subalgebra of $\mathcal{B}$ generated by $\phi(a_1), \ldots, \phi(a_n)$ and $\phi(f(\mathbf{a}))$. On the other hand, let $\{f_i\}_{i\geq 1}$ be the linear functions given by Remark 4.1. Since $f \geq f_i$ $(i \geq 1)$ we have that $f(\mathbf{a}) \geq f_i(\mathbf{a})$ and therefore

$$\phi(f(\mathbf{a})) \geq \phi(f_i(\mathbf{a})) = f_i(\vec{\phi}(\mathbf{a})\,)\,, \tag{8}$$

where the last equality holds because f_i is linear. As $f_i(\vec{\phi}(\mathbf{a})\,) \in \widehat{\mathcal{B}}$ for every $i \geq 1$ and also $\phi(f(\mathbf{a})) \in \widehat{\mathcal{B}}$, which is abelian,

$$\phi(f(\mathbf{a})) \geq \max_{1\leq i\leq n} f_i(\vec{\phi}(\mathbf{a})\,) = \left(\max_{1\leq i\leq n} f_i\right)(\vec{\phi}(\mathbf{a})\,).$$

Now, since $\max_{1\leq i\leq n} f_i \xrightarrow[n\to\infty]{} f$ uniformly on compact sets, we can deduce from Dini's theorem that $\phi(f(\mathbf{a})) \geq f(\vec{\phi}(\mathbf{a})\,)$. If ϕ is contractive and $f(0) \leq 0$, the functions f_i also satisfy that $f_i(0) \leq 0$ and we can replace Eq. (8) by: $\phi(f(\mathbf{a})) \geq \phi(f_i(a)) \geq f_i(\phi(a))$. Then we can repeat the same argument to get the desired inequality. ∎

The following results are the multi-variable versions of Lemma 3.10, Remark 3.11 and Theorem 3.9. The proofs of those results were chosen in such a way that they still hold in the multivariable case without substantial differences.

Lemma 4.3. Let φ be a state defined on $\mathcal{B}$, and let $\mathbf{b} = (b_1, \ldots, b_n) \in (\mathcal{B}^{\varphi})^n$ be a n-tuple with mutually commuting entries. Then, there exist a Borel measure μ defined on $K := \sigma(\mathbf{b})$ and a positive unital linear map $\Psi : \mathcal{B} \to L^\infty(K, \mu)$ such that:

- **i.:** $\Psi(f(\mathbf{b})) = f$ for every $f \in C(K)$.
- **ii.:** $\varphi(x) = \displaystyle\int_K \Psi(x)(t)\; d\mu(t)$ for every $x \in \mathcal{B}$.

Theorem 4.4. Let $\mathcal{E} : \mathcal{B} \to \mathcal{C}$ be a conditional expectation onto the C*-subalgebra $\mathcal{C}$. If $\phi(a_1), \ldots, \phi(a_n) \in \mathcal{B}^{\mathcal{E}}$ and are mutually commuting, then

$$\mathcal{E}(g[f(\vec{\phi}(\mathbf{a})]) \leq \mathcal{E}(g[\phi(f(\mathbf{a}))]). \tag{9}$$

for every convex increasing map $g : J \to \mathbb{R}$ such that $f(I) \subseteq J$.

Theorem 4.5. If $\mathcal{B}$ is a finite factor and $\phi(a_1), \ldots, \phi(a_n)$ are mutually commuting, then $f(\vec{\phi}(\mathbf{a}))) \prec_w \phi(f(\mathbf{a}))$.

5. The finite dimensional case.

In this section we rewrite the already obtained Jensen's inequalities in the finite dimensional case. We use the notations $\mathcal{M}_n = L(\mathbb{C}^n)$, $\mathcal{M}_n^{sa}$ for the space of self-adjoint matrices, $\mathcal{M}_n^+$ for the cone of positive matrices and $\mathcal{U}(n)$ the unitary group of $\mathcal{M}_n$. Given $A \in \mathcal{M}_n^{sa}$, by means of $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A)\,)$ we denote the eigenvalues of A counted with multiplicity and arranged in non-increasing order. Now, we recall the aspect of the spectral preorder and majorization in $\mathcal{M}_n^{sa}$:

5.1. Let $A, B \in \mathcal{M}_n^{sa}$.

1. By Proposition 3.3, the following conditions are equivalent
   (a) $A \lesssim B$.
   (b) There is $U \in \mathcal{U}(n)$ such that $(UAU^*)B = B(UAU^*)$ and $UAU^* \leq B$.
   (c) $\lambda_i\,(A) \leq \lambda_i\,(B)$ $(1 \leq i \leq n)$.
2. Straightforward calculations show that, given a selfadjoint matrix C, the functions $e_C(t)$ considered in the definition of majorization satisfy that $e_C(t) = \lambda_k\,(C)$ for $\frac{k-1}{n} \leq t < \frac{k}{n}$, $1 \leq k \leq n$. Therefore, $A \prec_w B$ if and only if $\lambda(A) \prec_w \lambda(B)$ (as vectors).

In the next Proposition, we summarize the different versions, in this setting, of Jensen's inequality obtained in section 3.

Proposition 5.2. *Let $\mathcal{A}$ be an unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{M}_n$ a positive unital map. Suppose that $a \in \mathcal{A}_{sa}$ and $f : I \to \mathbb{R}$ is a function with $\sigma(a) \subseteq I$.*

1. *If f is monotone convex, $\lambda_i\,(f(\phi(a))) \leq \lambda_i\,(\phi(f(a)))$, for $1 \leq i \leq n$.*
2. *If f is convex, $\displaystyle\sum_{i=1}^{k} \lambda_i\,(f(\phi(a))) \leq \sum_{i=1}^{k} \lambda_i\,(\phi(f(a)))$, $1 \leq k \leq n$.*

If $0 \in I$ and $f(0) \leq 0$ the above inequalities also hold for contractive positive maps.

Example 5.3. Given two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we denote $A \circ B = (a_{ij}b_{ij})$ their Schur's product. It is a well known fact that the map $A \mapsto A \circ B$ is completely positive for each positive matrix B, and if we further assume that $I \circ B = I$ then it is also unital. So the above inequalities can be rewritten taking $\phi : \mathcal{M}_n \to \mathcal{M}_n$ given by $\phi(A) = A \circ B$ where B satisfies the mentioned properties. ▲

Example 5.4. Let $\mathcal{A} \subseteq L(\mathcal{H})$ be a C^*-algebra. Take $\phi : \mathcal{A}^r \to \mathcal{M}_n$ given by $\phi(A_1, \cdots, A_r) = \sum_{i=1}^r W_i^* A_i W_i$, where $W_1, \ldots, W_r \in L(\mathbb{C}^n, \mathcal{H})$ are bounded operators such that $\sum_{i=1}^r W_i^* W_i = I$. Since this map is positive, one can apply Proposition 5.2 to get new versions of the inequalities appearing in [6] and [8]. ▲

Example 5.5. Given $\alpha \in (0,1)$, consider the positive unital map $\phi_\alpha : \mathcal{M}_{2n} \to \mathcal{M}_n$ defined by $\phi\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \alpha A + (1-\alpha)D$. Using these maps in Proposition 5.2 and taking diagonal block matrices we get that for every monotone convex function f

$$\lambda_i \left(f(\alpha A + (1-\alpha)D)\right) \leq \lambda_i \left(\alpha f(A) + (1-\alpha)f(D)\right) \qquad 1 \leq i \leq n$$

and for general convex functions f

$$\sum_{i=1}^r \lambda_i \left(f(\alpha A + (1-\alpha)D)\right) \leq \sum_{i=1}^r \lambda_i \left(\alpha f(A) + (1-\alpha)f(D)\right) \qquad 1 \leq i \leq n$$

where $A, D \in \mathcal{M}_n^{sa}$ with $\sigma(A)$ and $\sigma(D)$ contained in the domain of f. This inequalities were proved by J. S. Aujla and F. C. Silva in [3]. ▲

References

[1] J. Antezana, P. Massey and D. Stojanoff, Jensen's Inequality and majorization, preprint arXiv.org - math/0411442 (2004).

[2] J. S. Aujla J. C. Bourin, Eigenvalues inequalities for convex and logconvex functions, preprint arXiv.org math/0601757 (2006)

[3] J. S. Aujla and F. C. Silva, Weak majorization inequalities and convex functions, Linear Algebra Appl. 369 (2003), 217-233.

[4] Rajendra Bhatia, Matrix Analysis, Berlin-Heildelberg-New York, Springer 1997.

[5] Lawrence G. Brown and Hideki Kosaki, Jensen's inequality in semi-finite von Neumann algebras, Journal of Operator Theory 23 (1990), 3-19.

[6] C. Davis, Notions generalizing convexity for functions defined on spaces of matrices, Proc. Sympos. Pure Math., Vol. VII (1963), 187–201.

[7] D.R. Farenick and S.M. Manjegani, Young's Inequality in Operator Algebras, J. of the Ramanujan Math. Soc., to appear (see http://xxx.lanl.gov/abs/math.OA/0303318).

[8] Frank Hansen and Gert K. Pedersen, Jensen's operator inequality, Bulletin of London Mathematical Society 35 (2003), 553-564.

[9] Frank Hansen and Gert K. Pedersen, Jensen's trace inequality in several variables, Internat. J. Math. 14 (2003), 667-681.

[10] R. V. Kadison, Non-commutative conditional expectations and their applicactions, volume 365 of Contemporary Mathematics, pages 143-179. Amer. Math Soc. 2004. Operator algebras, quantization and non-commutative geometry.

[11] Eizaburo Kamei, Majorization in finite factors, Math. Japonica 28, No. 4 (1983), 495-499.

[12] P. Massey, Refinements of spectral resolutions and modelling of operators in II_1 factors, preprint arXiv.org - math/0605070 (2006).

[13] G. K. Pedersen, Convex trace functions of several variables on C^*-algebras. J. Operator Theory 50 (2003), 157-167.

Jorge Antezana, Pedro Massey and Demetrio Stojanoff, Depto. de Matemática, FCE-UNLP and IAM-CONICET, Calles 50 y 115 (1900), La Plata, Argentina

E-mail address: antezana@mate.unlp.edu.ar , massey@mate.unlp.edu.ar

E-mail address: demetrio@mate.unlp.edu.ar