DIFFERENTIAL STRUCTURE OF THE THOMPSON COMPONENTS OF SELFADJOINT OPERATORS.

GUILLERMINA FONGI AND ALEJANDRA MAESTRIPIERI

Abstract. Different equivalence relations are defined in the set $L(\mathcal{H})^s$ of selfadjoint operators of a Hilbert space $\mathcal{H}$ in order to extend a very well known relation in the cone of positive operators. As in the positive case, for $a \in L(\mathcal{H})^s$ the equivalence class C_a admits a differential structure, which is compatible with a complete metric defined on C_a. This metric coincides with the Thompson metric when a is positive.

1. Introduction

Let $L(\mathcal{H})$ be the algebra of bounded linear operators of a Hilbert space $\mathcal{H}$ and $L(\mathcal{H})^+$ the cone of positive operators. Consider the partial order $\leq$ induced in $L(\mathcal{H})$ by $L(\mathcal{H})^+$. There is a natural equivalence relation in $L(\mathcal{H})^+$ which arises from this order (namely, given $a, b \in L(\mathcal{H})^+$, $a \sim b$ if and only if there exist positive numbers α, β such that $a \leq \alpha b$ and $b \leq \beta a$). Each class or *Thompson component* admits a complete metric, known as the *Thompson part metric*. In fact, this construction can be done in every closed convex cone of a Banach space. The reader is referred to [10] and [11] to find definitions, properties and applications of this metric.

On the other hand, the set $GL(\mathcal{H})^s$ of invertible selfadjoint operators admits a differential structure as a homogeneous space of the group $GL(\mathcal{H})$ of invertible operators, see [5]. The orbit of the identity is the set $GL(\mathcal{H})^+$ of invertible positive operators and it is also the Thompson component of every $a \in GL(\mathcal{H})^+$. More generally, the components of positive operators are parameterized by the ranges of their square roots, see [2] and [3]. Also, in [2] and [3] the differential and metrical properties of the component of a positive operator were studied in detail. The aim of this paper is to extend this equivalence relation to the set $L(\mathcal{H})^s$ of selfadjoint operators and to define a metric on each component, that coincides with the Thompson metric in the positive case; also to study the geometrical structure of the components and finally, to compare both approaches.

The contents of the paper are the following: In Section 2 we recall some classical results about decompositions of selfadjoint operators which are used along the paper. Also, the definition and some properties of the Thompson part metric for positive operators are stated. Section 3 contains a brief description of the homogeneous structure of $GL(\mathcal{H})^s$. We give a new characterization of the orbit of an invertible selfadjoint operator by means of its polar decomposition. In Section 4 we define three equivalence relations on $L(\mathcal{H})^s$ that extend the one defined in the cone $L(\mathcal{H})^+$ and characterize the corresponding equivalence classes, C_a, C_a^1, C_a^2. In

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Section 5, we show that C_a is homeomorphic to the product of two Thompson components of positive operators. This result allows to define the Thompson metric d_T on the component of a selfadjoint operator. It is proved that, for $a \in L(\mathcal{H})^s$, C_a admits a natural structure of homogeneous space of an appropriated group and given $b, c \in C_a$ there exists a unique geodesic $\gamma_{b,c}$ joining them. A Finsler structure is given in C_a which allows to define the length of a curve $\gamma \subseteq C_a$, $L(\gamma)$. It holds that the geodesic $\gamma_{b,c}$ has minimal length among all curves with endpoints b and c. Finally if $d(b,c)$ is the geodesic distance between b and c, then d is a complete distance in C_a and $d(b,c) = d_T(b,c) = L(\gamma_{b,c})$.

2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space, $L(\mathcal{H})$ the algebra of linear bounded operators in $\mathcal{H}$, $L(\mathcal{H})^s$ the subset of selfadjoint operators and $L(\mathcal{H})^+$ the cone of positive operators of $L(\mathcal{H})$. Denote by $GL(\mathcal{H})$ the group of invertible operators of $L(\mathcal{H})$ and $GL(\mathcal{H})^s = L(\mathcal{H})^s \cap GL(\mathcal{H})$, the set of invertible selfadjoint operators. Let $\mathcal{U}$ be the subgroup of $GL(\mathcal{H})$ of unitary operators and $\mathcal{P}$ the subset of *reflections* of $\mathcal{U}$, i.e., $\mathcal{P} = \{u \in L(\mathcal{H}) : u = u^{-1} = u^*\}$. The range and the nullspace of $a \in L(\mathcal{H})$ are $R(a)$ and $N(a)$ respectively. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, $p_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$. Observe that $p_{\mathcal{S}}$ induces a matrix decomposition as follows: if $p = p_{\mathcal{S}}$ each $a \in L(\mathcal{H})$ can be written as $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where $a_{11} = pap_{|_{\mathcal{S}}} \in L(\mathcal{S})$, $a_{12} = pa(1-p)_{|_{\mathcal{S}^\perp}} \in L(\mathcal{S}^\perp, \mathcal{S})$, $a_{21} = (1-p)ap_{|_{\mathcal{S}}} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and $a_{22} = (1-p)a(1-p)_{|_{\mathcal{S}^\perp}} \in L(\mathcal{S}^\perp)$.

Every operator $a \in L(\mathcal{H})$ admits a *polar decomposition* $a = v|a|$ where $|a| = (a^*a)^{1/2}$ is positive and v is a partial isometry from $N(a)^\perp$ onto $\overline{R(a)}$. Observe that, in general, this decomposition is not unique. To fix the isometric part, define v_a as the partial isometry verifying that $v_a : N(a)^\perp \to \overline{R(a)}$ is an isometry and $N(v_a) = N(a)$. If a is selfadjoint, the isometric part of the polar decomposition can be defined to obtain a reflection: in this case $R(a)^\perp = N(a)$ so that if $u_a = v_a + p_{N(a)}$, it is easy to see that $u_a \in \mathcal{P}$ and $a = u_a|a| = |a|u_a$. Notice that $u_{a|_{N(a)}} = id_{|_{N(a)}}$ and if $p_a = p_{\overline{R(a)}}$, $u_a p_a = v_a$. If $q_a = \frac{u_a+1}{2}$ then q_a is an orthogonal projection and $u_a q_a = q_a$.

From now on, given $a \in L(\mathcal{H})^s$ denote by $a = u_a|a|$ the polar decomposition of a with $u_a \in \mathcal{P}$ and $a = v_a|a|$ the polar decomposition of a with v_a the partial isometry defined before.

Lemma 2.1. *Consider $a \in L(\mathcal{H})^s$ with polar decomposition $a = u_a|a|$. Then a admits a unique decomposition $a = a_1 - a_2$ such that a_1, a_2 are (semidefinite) positive and $a_1 a_2 = 0$. Moreover $a_1 = \frac{|a|+a}{2} = aq_a$ and $a_2 = \frac{|a|-a}{2} = -a(1-q_a)$, where $q_a = \frac{u_a+1}{2}$.*

Proof. Straightforward. □

Given $a \in L(\mathcal{H})^s$ we will refer to the decomposition of Lemma 2.1 as the *positive decomposition* of a.

Consider $a \in L(\mathcal{H})^s$ with positive decomposition $a = a_1 - a_2$ then $|a| = a_1 + a_2$; if $p_c = p_{\overline{R(c)}}$ then $v_a = p_{a_1} - p_{a_2}$, $u_a = p_{a_1} - p_{a_2} + p_{N(a)}$ and $p_a = p_{a_1} + p_{a_2}$. Notice

also that $R(a_1)$ and $R(a_2)$ are closed if and only if $R(a)$ is closed. Finally observe that $R(q_a) = \overline{R(a_1)} \oplus N(a)$ and $N(q_a) = \overline{R(a_2)}$.

Thompson metric for positive operators. Every closed convex cone K of a real Banach space E defines an order relation: if $x, y \in E$, $x \leq y$ if and only if $y - x \in K$. Suppose that K is normal, i.e. there exists a constant $r > 0$ such that $\|x\| \leq r\|y\|$ if $0 \leq x \leq y$. Consider the following equivalence relation in K: $x \sim y$ if there exist constants $r, s > 0$ such that $x \leq ry$ and $y \leq sx$. The *Thompson component* of $x \in E$, C_x, is the equivalence class of x. Thompson [13] proved that

$$d_T(x, y) = \log \max\{\inf\{r > 0 : x \leq ry\}, \inf\{s > 0 : y \leq sx\}\}$$

defines a complete distance on each component of K. The distance d_T is known as the *part metric* or *Thompson metric*. In the papers by R. Nussbaum [10] and [11] some applications and examples of this metric can be found.

Observe that if $\mathcal{H}$ is a Hilbert space then $L(\mathcal{H})^+$ is a closed normal convex cone and the above relation can be considered in $L(\mathcal{H})^+$.

The following proposition shows that the components C_a, $a \in L(\mathcal{H})^+$, are parametrized by the ranges of $a^{1/2}$.

Proposition 2.2. *Consider $a, b \in L(\mathcal{H})^+$ then $b \in C_a$ if and only if $R(a^{1/2}) = R(b^{1/2})$.*

Proof. See Corollary 3.3 of [2]. □

Each C_a admits a structure of homogeneous space of an appropiated group. In particular, if $a \in L(\mathcal{H})^+$ has closed range then C_a identifies with $GL(R(a))^+$. There is a natural connection on the associated tangent bundle, TC_a, and given $b, c \in C_a$ there exists a unique geodesic in C_a joining b to c, namely, $\gamma_{b,c}(t) = b^{1/2}(b^{-1/2}cb^{-1/2})^t b^{1/2}$, $t \in [0, 1]$. Also the geodesic distance, $d(b, c) = \inf\{L(\gamma)\}$ (where the infimun is taken over all smooth curves in C_a, joining b to c and $L(\gamma)$ is the lenght of γ) coincides with the Thompson metric, d_T, on each component. Moreover, if $b, c \in C_a$ it holds that

$$d_T(b, c) = L(\gamma_{b,c}) = \| \log(b^{-1/2}cb^{-1/2})\|, \tag{2.1}$$

where $b^{-1/2} = (b^{1/2})^\dagger{}_{|_{R(a)}}$ and $(b^{1/2})^\dagger$ denotes the *Moore-Penrose* pseudoinverse of $b^{1/2}$. See [2] and [3] for a complete exposition of these facts.

3. Homogeneous structure of $GL(\mathcal{H})^s$.

In what follows we recall some results about the geometry of the set of invertible selfadjoint operators, $GL(\mathcal{H})^s$, that can be found in [5]. The main result of this section is a characterization of the orbit of an invertible selfadjoint operator given by the action of $GL(\mathcal{H})$ in terms of the polar decomposition of its elements.

Given $a \in GL(\mathcal{H})^s$ define in $\mathcal{H}$ the following sesquilinear indefinite form

$$\langle x, y\rangle_a = \langle ax, y\rangle, \quad x, y \in \mathcal{H}.$$

The adjoint of $u \in L(\mathcal{H})$ with respect to $\langle\, ,\rangle_a$ or the *a-adjoint* of u is $u^{*_a} = a^{-1}u^*a$. It is easy to see that the group $\mathcal{U}_a$ of a-unitary elements consists on the operators $u \in GL(\mathcal{H})$ such that $u^{-1} = a^{-1}u^*a$.

Consider the following action of $GL(\mathcal{H})$ on $L(\mathcal{H})^s$:

$L : GL(\mathcal{H}) \times L(\mathcal{H})^s \to L(\mathcal{H})^s,\ L_g a = gag^*, \quad \text{for } a \in L(\mathcal{H})^s \text{ and } g \in GL(\mathcal{H}).$

Given $a \in L(\mathcal{H})^s$, the *orbit* of a corresponding to the action L is the set

$$\mathcal{O}_a = \{gag^* : g \in GL(\mathcal{H})\}.$$

Observe that if a is invertible then $\mathcal{O}_a \subseteq GL(\mathcal{H})^s$, so that $L : GL(\mathcal{H}) \times GL(\mathcal{H})^s \to GL(\mathcal{H})^s$. The action L is locally transitive in $GL(\mathcal{H})^s$, see [5].

The *isotropy group* of $a \in GL(\mathcal{H})^s$, I_a, is the group of operators $g \in GL(\mathcal{H})$ such that $L_g a = a$, i.e $I_a = \{g \in GL(\mathcal{H}) : gag^* = a\}$. Then $I_a = \mathcal{U}_{a^{-1}}$.

Proposition 3.1. *Given $a \in GL(\mathcal{H})^s$, consider the map $f_a : GL(\mathcal{H}) \to \mathcal{O}_a$, $f_a(g) = L_g a = gag^*$. Then $(GL(\mathcal{H}), \mathcal{O}_a, f_a)$ is a principal fibre bundle with structural group I_a.*

Proof. See Proposition 1.1 of [5]. □

The next result shows that the orbit of $a \in GL(\mathcal{H})^s$ is the orbit of its unitary part.

Lemma 3.2. *Consider $a \in GL(\mathcal{H})^s$ with polar decomposition $a = u_a|a|,\ u_a \in \mathcal{P}$, then $\mathcal{O}_a = \mathcal{O}_{u_a}$.*

Proof. Since a is selfadjoint, $a = u_a|a| = |a|^{1/2}u_a|a|^{1/2}$. Then $a \in \mathcal{O}_{u_a}$ so that $\mathcal{O}_a = \mathcal{O}_{u_a}$. □

Given $w \in \mathcal{U}$ the *unitary orbit* of w is the set $\mathcal{U}\mathcal{O}_w = \{uwu^* : u \in \mathcal{U}\}$. The following theorem relates the orbits $\mathcal{O}_a$ and $\mathcal{U}\mathcal{O}_{u_a}$.

Theorem 3.3. *Consider $a, b \in GL(\mathcal{H})^s$ with polar decompositions $a = u_a|a|$, $b = u_b|b|$ and positive decompositions $a = a_1 - a_2$, $b = b_1 - b_2$. Then the following conditions are equivalent:*

1. $b \in \mathcal{O}_a$,
2. $u_b \in \mathcal{U}\mathcal{O}_{u_a}$,
3. $\dim R(a_i) = \dim R(b_i)$, *for* $i = 1, 2$.

Proof. (1) → (2): Consider $b \in \mathcal{O}_a$ then, by Lemma 3.2, $u_b \in \mathcal{O}_{u_a}$, so that there exists $g \in GL(\mathcal{H})$ such that $u_b = gu_a g^*$. Therefore $u_b = gu_a g^* = (g^*)^{-1}u_a g^{-1}$, or $g^* g u_a g^* g = u_a$. Consider $\lambda = g^* g$; then $\lambda u_a \lambda = u_a$ or $\lambda^{-1} = u_a \lambda u_a$. Since $\lambda > 0$ and $u_a \in \mathcal{P}$, it follows that $\lambda^{-1/2} = u_a \lambda^{1/2} u_a$, or $\lambda^{-1/2} u_a = u_a \lambda^{1/2}$. If $g = w\lambda^{1/2}$ is the polar decomposition of g with $w \in \mathcal{U}$, we get that $u_b = w\lambda^{1/2}u_a\lambda^{1/2}w^* = wu_a w^*$, so that $u_b \in \mathcal{U}\mathcal{O}_{u_a}$.

(2) → (3): Consider $u_b \in \mathcal{U}\mathcal{O}_{u_a}$, then there exists $u \in \mathcal{U}$ such that $u_b = uu_a u^*$. Therefore $p_{b_1} - p_{b_2} = u_b = uu_a u^* = up_{a_1}u^* - up_{a_2}u^*$. It follows from Lemma 2.1 that $p_{b_i} = up_{a_i}u^*$ so that $\dim R(a_i) = \dim R(b_i)$, for $i = 1, 2$.

(3) → (1): Since $\dim R(b_i) = \dim R(a_i)$ for $i = 1, 2$, there exists a partial isometry u_i from $\mathcal{H}$ onto $R(b_i)$ with $N(u_i) = R(a_i)^\perp$. Consider $u = u_1 + u_2$ then $u \in \mathcal{U}$ because $R(a_1) \oplus R(a_2) = \mathcal{H}$ and $R(a_1)^\perp = R(a_2)$. Also $uu_a u^* = u(p_{a_1} - p_{a_2})u^* = p_{b_1} - p_{b_2} = u_b$, so that $u_b \in \mathcal{O}_{u_a}$, or equivalently $b \in \mathcal{O}_a$. □

Consider the map

$$\pi : GL(\mathcal{H})^s \to \mathcal{P}, \quad \pi(a) = u_a, \ \text{ where } a \in GL(\mathcal{H})^s, a = u_a|a|.$$

It follows from the definition of π that the *fibre* of $u \in \mathcal{P}$, $\pi^{-1}(\{u\})$, is the subset of $GL(\mathcal{H})^s$ of elements $a = u_a|a|$ such that $u_a = u$, i.e. $\pi^{-1}(\{u\}) = \{a \in GL(\mathcal{H})^s : a = u\alpha, \alpha > 0\}$. In particular, the fibre of 1 is $GL(\mathcal{H})^+$.

The following lemma is similar to Proposition 4.1 of [5].

Lemma 3.4. *Consider $u \in \mathcal{P}$, then the following conditions are equivalent:*

(1) $a \in \pi^{-1}(\{u\})$,
(2) $a = a^*$, $au > 0$,
(3) $au = ua$, $au > 0$.

Proof. Straightforward. □

4. Equivalence relations in $L(\mathcal{H})^s$

From now on, given $c \in L(\mathcal{H})^s$, $c = c_1 - c_2$ denotes the positive decomposition of c defined in the Preliminaries and $p_c = p_{\overline{R(c)}}$. Recall that $c = u_c|c| = v_c|c|$ denotes the polar decompositions of c with $u_c \in \mathcal{P}$ and v_c a partial isometry.

Given v_a and v_b partial isometries, v_a is unitarily equivalent to v_b if there exits $u \in \mathcal{U}$ such that $v_b = uv_au^*$. The *unitary orbit* of v_a is the set $\mathcal{UO}_{v_a} = \{uv_au^* : u \in \mathcal{U}\}$.

Given a reflection $v \in \mathcal{P}$, consider the indefinite sesquilinear form given by

$$\langle x, y\rangle_v = \langle vx, y\rangle, \qquad x, y \in \mathcal{H},$$

and define in $L(\mathcal{H})$ the following order: given $a, b \in L(\mathcal{H})$, then $a \leq_v b$ if and only if $\langle (b-a)x, x\rangle_v \geq 0 \; \forall\, x \in \mathcal{H}$, or equivalently, $v(b-a) \geq 0$.

In what follows we define three equivalence relations in $L(\mathcal{H})^s$ that extend the relation in $L(\mathcal{H})^+$ discussed in the Preliminaries. Consider $a, b \in L(\mathcal{H})^s$ then

(1) $a \sim b$, if there exist $\alpha, \beta > 0$ such that $a \leq_{u_a} \alpha b$ and $b \leq_{u_b} \beta b$.
(2) $a \sim_1 b$, if there exist $\alpha, \beta > 0$ such that $|a| \leq \alpha|b|$, $|b| \leq \beta|a|$ and v_a is unitarily equivalent to v_b.
(3) $a \sim_2 b$, if there exist $\alpha, \beta > 0$ such that $|a| \leq \alpha|b|$ and $|b| \leq \beta|a|$.

It is not difficult to see that $\sim_1$ and $\sim_2$ are equivalence relations.

To see that $\sim$ is an equivalence relation we need the following lemma.

Lemma 4.1. *Consider $a, b \in L(\mathcal{H})^s$ with polar decompositions $a = u_a|a|, b = u_b|b|$. If $a \sim b$, then $u_a = u_b$.*

Proof. If $a \sim b$, then there exist $\alpha, \beta > 0$ such that $u_a(\alpha b - a) \geq 0$ and $u_b(\beta a - b) \geq 0$. Then $c = \alpha u_a b \geq |a| \geq 0$, and $b = \alpha^{-1}u_a c$. Since the polar decomposition is unique in $\overline{R(b)}$, it follows that $u_a = u_b$ in $\overline{R(b)}$. It remains to prove that $u_a = u_b$ in $N(b)$. In fact $N(a) = N(b)$: if $x \in N(b)$, then $0 \leq \langle (c - |a|)x, x\rangle = -\langle |a|x, x\rangle$ so that $|a|x = 0$. Hence, $x \in N(|a|) = N(a)$ and $N(b) \subseteq N(a)$. In the same way, using that $u_b(\beta a - b) \geq 0$ it follows that $N(a) \subseteq N(b)$. Then $u_a = u_b$ in $N(a)$ by definition (see the Preliminaries). □

Corollary 4.2. *The relation $\sim$ is an equivalence relation.*

Proof. Apply Lemma 4.1. □

Denote by C_a, C_a^1 and C_a^2 the equivalence classes or *components* of a corresponding to $\sim$, $\sim_1$ and $\sim_2$ respectively.

As a corollary of Lemma 4.1 we obtain a characterization of the component C_a of a in terms of its polar decomposition.

Corollary 4.3. *Consider $a \in L(\mathcal{H})^s$, with polar decomposition $a = u_a|a|$. Then*

$$C_a = \{b \in L(\mathcal{H})^s : R(|b|^{1/2}) = R(|a|^{1/2}) \text{ and } u_b = u_a\}.$$

Proof. Consider $b \in C_a$ with polar decomposition $b = |b|u_b$, then by Lemma 4.1, it follows that $u_a = u_b$. Also there exist $\alpha, \beta > 0$ such that $u_a(\alpha b - a) \geq 0$ and $u_b(\beta a - b) \geq 0$ so that $|a| \leq \alpha|b|$ and $|b| \leq \beta|a|$. Therefore $|a| \sim |b|$ and by Proposition 2.2, this is equivalent to $R(|b|^{1/2}) = R(|a|^{1/2})$. The converse is similar. □

Corollary 4.4. Consider $a = v_a|a| \in L(\mathcal{H})^s$ with closed range, then $C_a = C_{v_a}$.

Proof. Since a has closed range, $b \in C_a$ if and only if $R(b) = R(a)$ and $u_b = u_a$ or equivalently, $R(b) = R(a)$ and $v_b = v_a$. But observe that, in this case, $R(v_a) = R(a)$. Therefore $v_a \in C_a$. □

It follows from Proposition 2.2 that $C_a^1 = \{b \in L(\mathcal{H})^s : R(|b|^{1/2}) = R(|a|^{1/2})$ and $v_b \in \mathcal{U}\mathcal{O}_{v_a}\}$ and $C_a^2 = \{b \in L(\mathcal{H})^s : R(|b|^{1/2}) = R(|a|^{1/2})\}$.

Given $a \in L(\mathcal{H})^s$, denote by $\tilde{a} = a|_{\overline{R(a)}}$. If a has closed range then $\tilde{a} \in GL(R(a))^s$. Consider the orbit of $\tilde{a}$ in $R(a)$, i.e., $\mathcal{O}_{\tilde{a}} = \{g\tilde{a}g^* : g \in GL(R(a))\}$. Observe that it follows from Theorem 3.3 that if $a \in GL(\mathcal{H})^s$ then $\mathcal{O}_a = C_a^1$. More generally,

Proposition 4.5. *Let $a \in L(\mathcal{H})^s$ be a closed range operator and $b \in L(\mathcal{H})^s$. Then the following conditions are equivalent:*

(1) $b \in C_a^1$,
(2) $R(b) = R(a)$ *and* $\dim R(b_i) = \dim R(a_i)$, *for* $i = 1, 2$,
(3) $\tilde{b} \in \mathcal{O}_{\tilde{a}}$.

Proof. (1) $\to$ (2): If $b \in C_a^1$, then $R(|b|^{1/2}) = R(|a|^{1/2})$. Since a has closed range $R(a) = R(|a|^{1/2})$ so that $R(|b|^{1/2})$ is closed. Then $R(|b|^{1/2}) = |b|^{1/2}(N(b)^\perp) = |b|^{1/2}(R(|b|^{1/2})) = R(b)$. Therefore $R(b) = R(a)$. Also $v_b \in \mathcal{U}\mathcal{O}_{v_a}$, so that $v_b = uv_au^*$ for $u \in \mathcal{U}$. In the same way as in the proof of (2) $\to$ (3) of Theorem 3.3 it can be proved that $\dim R(b_i) = \dim R(a_i)$ for $i = 1, 2$.

(2) $\to$ (3): Notice that $\tilde{b} \in GL(R(a))$ because $R(b) = R(a)$. Then apply Theorem 3.3 to $GL(R(a))^s$.

(3)$\to$ (1): Consider $b \in L(\mathcal{H})^s$ such that $\tilde{b} = g\tilde{a}g^*$ with $g \in GL(R(a))$. Then $R(b) = R(\tilde{b}) = R(a)$. By Theorem 3.3, $u_{\tilde{b}} \in \mathcal{U}\mathcal{O}_{u_{\tilde{a}}}$. If $c \in L(\mathcal{H})^s$ it holds that $u_{\tilde{c}} = \tilde{v_c}$, therefore $\tilde{v_b} \in \mathcal{U}\mathcal{O}_{\tilde{v_a}}$, i.e. there exists $u \in \mathcal{U}(R(a))$ such that $\tilde{v_b} = u\tilde{v_a}u^*$. If $w = u + p_{N(a)}$ then $w \in \mathcal{U}$ and $v_b = wv_aw^*$. Therefore $b \in C_a^1$. □

Remark 4.6. In Section 3 we defined the map

$$\pi : GL(\mathcal{H})^s \to \mathcal{P},\ \pi(b) = u_b, \quad \text{for } b \in GL(\mathcal{H})^s \text{ with } b = u_b|b|.$$

From the definitions and properties studied in Section 3, if $u \in \mathcal{P}$, it holds that

$$\pi^{-1}(\{u\}) \subseteq \mathcal{O}_u \subseteq GL(\mathcal{H})^s.$$

Notice that $\pi^{-1}(\{u\}) = C_u$, $\mathcal{O}_u = C_u^1$ and $GL(\mathcal{H})^s = C_u^2$. Analogously, if a is a closed range operator, consider the map

$$\pi_a : GL(R(a))^s \to \mathcal{P}(R(a)), \quad \pi(b) = u_b,$$

where $GL(R(a))^s$ is the set of invertible selfadjoint operators in $R(a)$, $\mathcal{P}(R(a))$ is the subset of reflections in $R(a)$ and $b = u_b|b|$ is the polar decomposition of $b \in GL(R(a))^s$. Then, by Corollary 4.3 and Proposition 4.5, we can identify

$$\pi_a^{-1}(\{\tilde{u}_a\}) \approx C_a, \quad \mathcal{O}_{\tilde{a}} \approx C_a^1, \quad GL(R(a))^s \approx C_a^2.$$

More generally, given $a \in L(\mathcal{H})^s$, it holds that $C_a \subset C_a^1 \subset C_a^2$. Therefore $\sim, \sim_1$ and $\sim_2$ extend to $L(\mathcal{H})^s$ the relation in $L(\mathcal{H})^+$ described in the Preliminaries. In fact, if $a \in L(\mathcal{H})^+$ and $b \in L(\mathcal{H})^s$ is such that $b \in C_a$, then $1 = u_a = u_b$ so that $b \in L(\mathcal{H})^+$ and there exist $\alpha, \beta > 0$ such that $a \leq \alpha b$ and $b \leq \beta a$. Therefore, when a is positive, C_a coincides with the Thompson component of a.

5. The Thompson components of $L(\mathcal{H})^s$

The next theorem proves that given $a \in L(\mathcal{H})^s$, C_a is homeomorphic to the product of two Thompson components of positive operators. This characterization naturally leads to the definition of a metric on C_a that extends the Thompson metric. From now on we will refer to C_a as the *Thompson component* of a.

Theorem 5.1. *Consider $a \in L(\mathcal{H})^s$ with positive decomposition $a = a_1 - a_2$. Then C_a is homeomorphic to $C_{a_1} \times C_{a_2}$.*

Proof. Given $b \in L(\mathcal{H})^s$ with positive decomposition $b = b_1 - b_2$, define $\psi : L(\mathcal{H})^s \to L(\mathcal{H})^+ \times L(\mathcal{H})^+$, $\psi(b) = (b_1, b_2)$. The map ψ is well defined because, by Lemma 2.1, the decomposition is unique; also ψ is injective. If $b \in C_a$, $u_a = u_b$ and there exist $\alpha, \beta > 0$ such that $(\alpha b - a)u_a \geq 0$ and $(\beta a - b)u_a \geq 0$, then $q_a(\alpha b - a)u_a q_a = q_a(\alpha b - a)q_a \geq 0$, with $q_a = \frac{u_a+1}{2}$. Since $bq_a = b_1$ and $aq_a = a_1$, it follows that $\alpha b_1 - a_1 \geq 0$. Similarly, $\beta a_1 \geq b_1$. Hence $b_1 \in C_{a_1}$. In the same way, $b_2 \in C_{a_2}$.

To see that ψ is surjective consider $(b_1, b_2) \in C_{a_1} \times C_{a_2}$, then $b = b_1 - b_2 \in L(\mathcal{H})^s$, $b_1, b_2 \geq 0$ and $b_1 b_2 = 0$. Hence, by Lemma 2.1, if $b = b_1 - b_2$, then $b \in L(\mathcal{H})^s$. Moreover $b \in C_a$: since $b_i \in C_{a_i}$ there exist $\alpha_i > 0$ such that $a_i \leq \alpha_i b_i$ for $i = 1, 2$. Hence $|a| \leq \alpha_1 b_1 + \alpha_2 b_2 \leq \alpha'(b_1 + b_2) \leq \alpha'|b|$, where $\alpha' = \max\{\alpha_1, \alpha_2\}$. In a similar way, there exists $\beta' > 0$ such that $|b| \leq \beta'|a|$. Therefore $|b| \in C_{|a|}$. It remains to prove that $u_a = u_b$, or equivalently that $q_a = q_b$. Since $|b| \in C_{|a|}$, then $N(a) = N(b)$. Also $\overline{R(a_1)} = \overline{R(b_1)}$ because $b_1 \in C_{a_1}$. Then $R(q_a) = \overline{R(a_1)} \oplus N(a)$ (see the Preliminaries). Therefore $R(q_a) = R(q_b)$ so that $q_a = q_b$, because q_a and q_b are orthogonal projectors. □

Remark 5.2. Consider $a \in L(\mathcal{H})^s$ with positive decomposition $a = a_1 - a_2$. Then, from the proof of Theorem 5.1, the component of a can be written as

$$C_a = \{b_1 - b_2 \in L(\mathcal{H})^s : b_i \in C_{a_i},\ i = 1, 2\}.$$

Consider $a, b, c \in L(\mathcal{H})^s$ with positive decompositions $a = a_1 - a_2, b = b_1 - b_2$ and $c = c_1 - c_2$. If $b, c \in C_a$, define

$$d(b, c) = \max\{d_T(b_i, c_i), i = 1, 2)\},$$

where $d_T(b_i, c_i) = \log \max\{\inf\{r > 0 :\ b_i \leq r c_i\}, \inf\{s > 0 :\ c_i \leq s b_i\}\}$ for $i = 1, 2$, is the Thompson metric for positive operators considered in the Preliminaries.

Observe that d is well defined because of the uniqueness of the positive decomposition of selfadjoint operators and the fact that $b_i, c_i \in C_{a_i}$, $i = 1, 2$, where d_T is defined. It is easy to see that d defines a complete metric on C_a. Furthermore observe that the metric d coincides with d_T on each positive component. From now on, denote by $d = d_T$, the *Thompson metric* on C_a.

If $c \in L(\mathcal{H})^s$ has closed range, denote by $c^{-1} = (c_{|_{R(c)}})^{-1} = c^\dagger{}_{|_{R(c)}}$, where $c^\dagger$ is the Moore-Penrose pseudoinverse of c. Observe that $c^{-1} \in GL(R(c))$.

Proposition 5.3. *Given $a \in L(\mathcal{H})^s$ with closed range, consider $b, c \in C_a$ with positive decompositions $b = b_1 - b_2$ and $c = c_1 - c_2$ then*

$$d_T(b, c) = \| \log[((b_1^{1/2})^\dagger c_1 (b_1^{1/2})^\dagger + (b_2^{1/2})^\dagger c_2 (b_2^{1/2})^\dagger)_{|_{R(b)}}] \|.$$

Proof. Since $b, c \in C_a$ and $a = a_1 - a_2$ is the positive decomposition of a then for $i = 1, 2$, $b_i, c_i \in C_{a_i}$. Since $d_i = b_i^{-1/2} c_i b_i^{-1/2} \in GL(R(b_i))^+$, then, by (2.1), $d_T(b, c) = \max\{\| \log d_i \|,\, i = 1, 2\} = \left\| \begin{pmatrix} \log d_1 & 0 \\ 0 & \log d_2 \end{pmatrix} \right\| = \left\| \log \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \right\|$
$= \| \log[((b_1^{1/2})^\dagger c_1 (b_1^{1/2})^\dagger + (b_2^{1/2})^\dagger c_2 (b_2^{1/2})^\dagger)_{|_{R(b)}}] \|$. □

Corollary 5.4. *Consider $a \in L(\mathcal{H})^s$ with closed range. If $b, c \in C_a$ then*

$$d_T(b, c) = d_T(|b|, |c|).$$

Proof. If $b, c \in C_a$ then $|b|, |c| \in C_{|a|}$ and, by Theorem 3.11 of [2], it holds that $d_T(|b|, |c|) = \| \log(|b|^{-1/2} |c| |b|^{-1/2}) \|$. On the other hand, it is easy to see that $(|b|^{1/2})^\dagger |c| (|b|^{1/2})^\dagger = (b_1^{1/2})^\dagger c_1 (b_1^{1/2})^\dagger + (b_2^{1/2})^\dagger c_2 (b_2^{1/2})^\dagger$ where $b = b_1 - b_2$ and $c = c_1 - c_2$ are the positive decompositions of b and c respectively. Then, applying Proposition 5.3, it follows that $d_T(b, c) = d_T(|b|, |c|)$. □

As in the positive case, the metrical structure of (C_a, d_T) is related to a differential structure. Consider $a \in L(\mathcal{H})^s$ with closed range and positive decomposition $a = a_1 - a_2$, then C_{a_i} identifies with $GL(R(a_i))^+$ for $i = 1, 2$ (see [2], [3]) and, as it was pointed out in the Preliminaries, C_{a_i} is a homogeneous space of $GL(R(a_i))$, $i = 1, 2$. The reader is referred to [3] and [9] in order to have a full description of this structure. Then, since C_a is homeomorphic to $C_{a_1} \times C_{a_2}$ (see Theorem 5.1), C_a admits a natural structure of homogeneous space of $GL(R(a_1)) \times GL(R(a_2))$.

A natural connection can be defined in C_a, which induces the concept of parallel field along a curve. A curve $\gamma \subseteq C_a$ is a *geodesic* if $\dot\gamma$ is parallel along γ and, in this case γ satisfies $\ddot\gamma = \dot\gamma \gamma^\dagger \dot\gamma$ (see [2], [3]).

For $b \in C_a$, denote by $(TC_a)_b$ the tangent space of C_a at b. If $X \in (TC_a)_b$ it follows from the product structure considered on C_a, that $X = X_1 p_{a_1} + X_2 p_{a_2}$, where $X_i \in L(R(a_i))^s$, $i = 1, 2$.

The unique geodesic γ such that $\gamma(0) = b$ and $\dot\gamma = X \in (TC_a)_b$ is

$$\gamma(t) = e^{\frac{t}{2} X b^\dagger} b e^{\frac{t}{2} b^\dagger X}.$$

It follows easily that $\gamma(t) = \gamma_1(t) - \gamma_2(t)$ where $\gamma_i(t) = e^{\frac{t}{2} X b_i{}^\dagger} b_i e^{\frac{t}{2} b_i{}^\dagger X}$ is the geodesic such that $\gamma_i(0) = b_i$ and $\dot\gamma_i(0) = X_i p_{a_i} \in (TC_{a_i})_{b_i}$, $i = 1, 2$.

Given $b, c \in C_a$ there is only one geodesic $\gamma_{b,c}$ such that $\gamma_{b,c}(0) = b$ and $\gamma_{b,c}(1) = c$, namely

$$\gamma_{b,c}(t) = |b|^{1/2}(|b|^{-1/2}|c||b|^{-1/2})^t|b|^{1/2}v_a = \gamma_{|b|,|c|}(t)v_a. \tag{5.1}$$

It is easy to see that

$$\gamma_{b,c}(t) = \gamma_{b_1,c_1}(t) - \gamma_{b_2,c_2}(t), \tag{5.2}$$

where $\gamma_{b_i,c_i}(t) = b_i^{1/2}(b_i^{-1/2}c_ib_i^{-1/2})^t b_i^{1/2}$ is the geodesic in C_{a_i} joining b_i and c_i, $i = 1, 2$.

The norm

$$\|X\|_b = \|(|b|^{1/2})^\dagger X(|b|^{1/2})^\dagger\|, \qquad X \in (TC_a)_b,$$

defines a *Finsler structure* on the tangent bundle TC_a, i.e. a smooth assignation of a complete norm on each tangent space $(TC_a)_b$, $b \in C_a$. If $X = X_1p_{a_1} + X_2p_{a_2}$, with $X_i \in L(R(a_i))^s$, $i = 1, 2$ it follows that $\|X\|_b = \max\{\|X_i\|_{b_i}, i = 1, 2\}$.

For a C^∞ curve $\gamma : [0, 1] \to C_a$ define the *length of* γ by

$$L(\gamma) = \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}dt.$$

Given $\gamma \subseteq C_a$ then $L(\gamma) = \max\{L(\gamma_1), L(\gamma_2)\}$, where $\gamma = \gamma_1 - \gamma_2$ is the positive decomposition of γ. In fact, $\|\dot\gamma(t)\|_{\gamma(t)} = \max\{\|\dot\gamma_i(t)\|_{\gamma_i(t)}, i = 1, 2\}$.

Lemma 5.5. *Given $a \in L(\mathcal{H})^s$ with closed range, consider $b, c \in C_a$. If $\gamma_{b,c}$ is the unique geodesic joining b and c then*

$$L(\gamma_{b,c}) = L(\gamma_{|b|,|c|}) = \max\{\|\log(b_i^{-1/2}c_ib_i^{-1/2})\|,\ i = 1, 2\}.$$

Proof. The first equality follows from (5.1). From (5.2) we get that $L(\gamma_{b,c}) = \max\{L(\gamma_{b_i,c_i}) : i = 1, 2\}$ but $L(\gamma_{b_i,c_i}) = \|\log(b_i^{-1/2}c_ib_i^{-1/2})\|$ (see the Preliminaries or Corollary 2.9 of [2]). □

Now we show that among all curves in C_a joining b and c, the geodesic $\gamma_{b,c}$ has minimal length.

Proposition 5.6. *Given a closed range operator $a \in L(\mathcal{H})^s$, consider $b, c \in C_a$. If $\delta : [0, 1] \to C_a$ is a C^∞ curve such that $\delta(0) = b$ and $\delta(1) = c$, then $L(\gamma_{b,c}) \leq L(\delta)$.*

Proof. Consider a curve $\delta : [0, 1] \to C_a$, $\delta(t) = |\delta(t)|v_a$ such that $\delta(0) = b$ and $\delta(1) = c$. If $b = |b|v_a, c = |c|v_a$ then it holds that $|\delta| : [0, 1] \to C_{|a|}$ is a curve joining $|b|$ with $|c|$. Then, by Theorem 2.10 of [2], it follows that $L(\gamma_{|b|,|c|}) \leq L(|\delta|)$. But $L(\gamma_{|b|,|c|}) = L(\gamma_{b,c})$ and $L(\delta) = L(|\delta|)$.

□

Consider $b, c \in C_a$, the *geodesic distance* in C_a is defined by

$$d(b, c) = \inf\{L(\gamma) :\ \gamma : [0, 1] \to C_a,\ C^\infty,\ \gamma(0) = b, \gamma(1) = c\}.$$

Corollary 5.7. *Given $a \in L(\mathcal{H})^s$ with closed range and $b, c \in C_a$, then*

$$d(b, c) = L(\gamma_{b,c}).$$

As in the positive case, the Thompson metric coincides with the geodesic distance on each component.

Corollary 5.8. *Consider a closed range operator $a \in L(\mathcal{H})^s$. If $b, c \in C_a$, then*

$$d_T(b,c) = d(b,c) = L(\gamma_{b,c})$$

where $\gamma_{b,c}$ is the geodesic in C_a joining b and c.

Proof. Apply (2.1), Corollary 5.4 and Lemma 5.5. □

Final remarks. The geometrical structure of C_a as a homogeneous space, discussed in Section 5 when a has closed range, is still valid in the general case. However the computations are more complicated. The reader can find in [3] these computations in detail when a is a general positive operator. In Section 3 the orbit of an invertible selfadjoint operator was described by means of the polar decomposition of its elements. More generally, it is interesting to study the orbit of a closed range operator $a \in L(\mathcal{H})^s$, i.e. the set $\mathcal{O}_a = \{gag^*, g \in GL(\mathcal{H})\}$. The geometrical and metrical structures of $\mathcal{O}_a$ were studied in the positive case, in [4], where it was proved that $\mathcal{O}_a$ admits a structure of differential manifold, if an appropriated metric is considered.

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Instituto Argentino de Matemática, CONICET, Saavedra 15, 1083 Buenos Aires, Argentina

Current address: Instituto Argentino de Matemática, CONICET, Saavedra 15, 1083 Buenos Aires, Argentina

E-mail address: fongig@fceia.unr.edu.ar

Instituto de Ciencias, Universidad Nacional General Sarmiento, 1613 Los Polvorines, Argentina

E-mail address: amaestri@ungs.edu.ar