

Localization of semi-Heyting algebras

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ABSTRACT. In this note, we introduce the notion of ideal on semi-Heyting algebras which allows us to consider a topology on them. Besides, we define the concept of \mathfrak{F} -multiplier, where \mathfrak{F} is a topology on a semi-Heyting algebra L , which is used to construct the localization semi-Heyting algebra $L_{\mathfrak{F}}$. Furthermore, we prove that the semi-Heyting algebra of fractions L_S associated with an \wedge -closed system S of L is a semi-Heyting of localization. Finally, in the finite case we prove that L_S is isomorphic to a special subalgebra of L . Since Heyting algebras are a particular case of semi-Heyting algebras, all these results generalize those obtained in [11].

Key words and phrases. localization, \mathfrak{F} -multipliers, semi-Heyting algebras, \wedge -closed system.

1. Introduction

Starting from the example of the ring, J. Schmid introduces in [20], [21] the notion of maximal lattice of quotients for a distributive lattice. The central role in this construction is played by the concept of multipliers, defined by W. H. Cornish in [13], [14]. Using the model of localization ring, in [15] is defined for a bounded distributive lattice L the localization lattice $L_{\mathfrak{F}}$ of L with respect to a topology \mathfrak{F} on L and is proved that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of regular ideals). The same theory is also valid for the lattice of fractions of a distributive lattice with 0 and 1 relative to an \wedge -closed system. A theory of localization for Hilbert and Hertz algebras was developed in [5]; for the case of commutative bounded BCK-algebras see [17]; for the case of n -valued Łukasiewicz-Moisil algebras see [12]; for the case of Heyting algebras see [11], for the case of MV and pseudo MV-algebras see [6, 7], for the case of BL and pseudo BL-algebras see [8, 9, 10] and for the case of hoop-algebras see [16].

On the other hand, semi-Heyting algebras were introduced as a new equational class by H. P. Sankappanavar in [19] (see also [1, 2, 3]). These algebras represent a generalization of Heyting algebras. Nevertheless the behavior of semi-Heyting algebras is much more complicated than that of Heyting algebras.

An algebra $\mathcal{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a semi-Heyting algebra if the following conditions hold:

- (sH1) $\mathcal{L} = \langle L, \vee, \wedge, 0, 1 \rangle$ is a lattice with 0 and 1.
- (sH2) $x \wedge (x \rightarrow y) = x \wedge y$.
- (sH3) $x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$.

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(sH4) $x \rightarrow x = 1$.

We will denote by \mathcal{SH} the variety of semi-Heyting algebras. The variety \mathcal{H} of Heyting algebras is the subvariety of \mathcal{SH} characterized by the equation $(x \wedge y) \rightarrow x = 1$.

These new algebras share with Heyting algebras some important properties (see [19]). For instance, they are pseudocomplemented, with the pseudocomplement given by $x^* = x \rightarrow 0$, congruences on them are determined by filters and the variety of semi-Heyting algebras is arithmetic. But, at the same time, semi-Heyting algebras present strong differences with Heyting algebras. For example, the operation of implication on a semi-Heyting algebra \mathcal{L} is not determined by the order of the underlying structure of lattice of \mathcal{L} : there are two non-isomorphic structures of semi-Heyting definable on the two-element lattice and ten on the three-element lattice. That is, we can have many operations of semi-Heyting implications on a given lattice. Among all these implications, the Heyting implication is the biggest one.

The aim of this paper is to generalize some of the results established in [11], using the model of bounded distributive lattices from [15] to semi-Heyting algebras. To this end, we introduce the notion of ideal on semi-Heyting algebras, dual to that of filter, which allows us to consider a topology on them. Besides, we define the concept of \mathfrak{F} -multiplier, where \mathfrak{F} is a topology on an semi-Heyting algebra L , which is used to construct the localization semi-Heyting algebra $L_{\mathfrak{F}}$. Furthermore, we prove that the semi-Heyting algebra of fractions L_S associated with an \wedge -closed system S of L is an semi-Heyting algebra of localization. In the last part of this paper we give an explicit description of the semi-Heyting algebras $L_{\mathfrak{F}}$ and L_S in the finite case.

2. \mathfrak{F} -multipliers and localization of semi-Heyting algebras

Taking into account the notion of topology for bounded distributive lattices introduced in [15], we will consider this concept in the particular case of semi-Heyting algebras.

Definition 2.1. Let $\mathcal{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$ be a semi-Heyting algebra. A non-empty subset I of L is an ideal of L , if I is an ideal of the bounded lattice $\langle L, \vee, \wedge, 0, 1 \rangle$.

It is worth noting that $\{0\}$ and L are ideals of L . We will denote by $I(L)$ the set of all ideals of L .

If X is a non-empty subset of L , we will denote by $\langle X \rangle$ the ideal generated by X . In particular, if $X = \{a\}$ we will write $\langle a \rangle$ instead of $\langle \{a\} \rangle$. We have that $\langle X \rangle = \{y \in L : \text{there are } x_1, \dots, x_n \in X \text{ such that } y \leq \bigvee_{i=1}^k x_i\}$. Moreover, if $a \in L$ then $\langle a \rangle = \{x \in L : x \leq a\}$.

Definition 2.2. Let L be a semi-Heyting algebra. A nonempty set \mathfrak{F} of elements $I \in I(L)$ will be called a topology on L if the following properties hold:

- (1) If $I_1 \in \mathfrak{F}$, $I_2 \in I(L)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathfrak{F}$ (hence $L \in \mathfrak{F}$),
- (2) If $I_1, I_2 \in \mathfrak{F}$, then $I_1 \cap I_2 \in \mathfrak{F}$.

Clearly, if \mathfrak{F} is a topology on L , then $(L, \mathfrak{F} \cup \{\emptyset\})$ is a topological space. Any intersection of topologies on L is a topology, hence the set $\mathcal{T}(L)$ of all topologies of L is a complete lattice with respecto to inclusion.

\mathfrak{F} is a topology on L iff \mathfrak{F} is a filter of the lattice of power set of L , for this reason a topology on L is usually called a Gabriel filter on $I(L)$.

Definition 2.3. Let \mathfrak{F} be a topology on a semi-Heyting algebra L . Define the relation $\theta_{\mathfrak{F}}$ on L as follows: $(x, y) \in \theta_{\mathfrak{F}}$ iff there exists $I \in \mathfrak{F}$ such that $e \wedge x = e \wedge y$ for any $e \in I$ for all $x, y \in L$.

Theorem 2.1. $\theta_{\mathfrak{F}}$ is a congruence on L .

Proof. The proofs of reflexivity and the symmetry of $\theta_{\mathfrak{F}}$ are straightforward. We show that $\theta_{\mathfrak{F}}$ is transitive. Suppose that $(x, y), (y, z) \in \theta_{\mathfrak{F}}$. Then there exist $I_1, I_2 \in \mathfrak{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1$, and $f \wedge y = f \wedge z$ for every $f \in I_2$. Put $I = I_1 \cap I_2 \in \mathfrak{F}$. Then for every $g \in I$, we have $g \wedge x = g \wedge z$. Hence $(x, z) \in \theta_{\mathfrak{F}}$.

Now suppose that $(x, y), (z, w) \in \theta_{\mathfrak{F}}$. Then there exist $I_1, I_2 \in \mathfrak{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1$, and $f \wedge z = f \wedge w$ for every $f \in I_2$. Put $I = I_1 \cap I_2 \in \mathfrak{F}$. Then $g \wedge x = g \wedge y$ and $g \wedge z = g \wedge w$, for every $g \in I$. Hence, we get that $g \wedge (x \wedge z) = (g \wedge y) \wedge z = (g \wedge w) \wedge y = g \wedge (y \wedge w)$. Then $(x \wedge z, y \wedge w) \in \theta_{\mathfrak{F}}$. On the other hand, $g \wedge (x \vee z) = (g \wedge x) \vee (g \wedge z) = (g \wedge y) \vee (g \wedge w) = g \wedge (y \vee w)$. Then $(x \vee z, y \vee w) \in \theta_{\mathfrak{F}}$. By (sH3), we have $g \wedge (x \rightarrow z) = g \wedge [(g \wedge x) \rightarrow (g \wedge z)] = g \wedge [(g \wedge y) \rightarrow (g \wedge w)] = g \wedge (y \rightarrow w)$. Thus $(x \rightarrow z, y \rightarrow w) \in \theta_{\mathfrak{F}}$. Hence $\theta_{\mathfrak{F}}$ is a congruence on L . \square

Remark 2.1. Theorem 2.1 proves that $L/\theta_{\mathfrak{F}}$ is a semi-Heyting algebra. We will denote by $[x]_{\theta_{\mathfrak{F}}}$ the congruence class of an element $x \in L$ and by $p_{\mathfrak{F}} : L \rightarrow L/\theta_{\mathfrak{F}}$ the canonical morphism of semi-Heyting algebras.

Definition 2.4. Let \mathfrak{F} be a topology on a semi-Heyting algebra L . An \mathfrak{F} -multiplier is a mapping $f : I \rightarrow L/\theta_{\mathfrak{F}}$ where $I \in \mathfrak{F}$ and for every $x \in I$ and $e \in L : f(e \wedge x) = [e]_{\theta_{\mathfrak{F}}} \wedge f(x)$.

Example 2.1. (a) The map $1 : L \rightarrow L/\theta_{\mathfrak{F}}$ defined by $1(x) = [x]_{\theta_{\mathfrak{F}}}$ for every $x \in L$ is an \mathfrak{F} -multiplier.

(b) The map $0 : L \rightarrow L/\theta_{\mathfrak{F}}$ defined by $0(x) = [0]_{\theta_{\mathfrak{F}}}$ for every $x \in L$ is an \mathfrak{F} -multiplier.

(c) For $a \in L$ and $I \in \mathfrak{F}$, $f_a : I \rightarrow L/\theta_{\mathfrak{F}}$ defined by $f_a(x) = [a]_{\theta_{\mathfrak{F}}} \wedge [x]_{\theta_{\mathfrak{F}}}$ for every $x \in I$, is an \mathfrak{F} -multiplier. If $\text{dom}(f_a) = L$, we denote f_a by \underline{f}_a .

Lemma 2.2. For each \mathfrak{F} -multiplier $f : I \rightarrow L/\theta_{\mathfrak{F}}$ the following properties hold:

- (a) $f(x) \leq [x]_{\theta_{\mathfrak{F}}}$, for all $x \in I$,
- (b) $f(x \wedge y) = f(x) \wedge f(y)$,
- (c) $f(x \vee y) = f(x) \vee f(y)$,
- (d) $[x]_{\theta_{\mathfrak{F}}} = [y]_{\theta_{\mathfrak{F}}} \wedge f(x)$.

Proof. It is straightforward. \square

We denote the set of all the \mathfrak{F} -multipliers having the domain $I \in \mathfrak{F}$ by $M(I, L/\theta_{\mathfrak{F}})$ and $M(L/\theta_{\mathfrak{F}}) = \bigcup_{I \in \mathfrak{F}} M(I, L/\theta_{\mathfrak{F}})$.

If $I_1, I_2 \in \mathfrak{F}$ and $I_1 \subseteq I_2$, we have a canonical mapping $\varphi_{I_1, I_2} : M(I_2, L/\theta_{\mathfrak{F}}) \rightarrow M(I_1, L/\theta_{\mathfrak{F}})$, defined by $\varphi_{I_1, I_2}(f) = f|_{I_1}$ for $f \in M(I_2, L/\theta_{\mathfrak{F}})$.

Consider the directed system of sets $\langle \{M(I, L/\theta_{\mathfrak{F}})\}_{I \in \mathfrak{F}}, \{\varphi_{I_1, I_2}\}_{I_1, I_2 \in \mathfrak{F}, I_1 \subseteq I_2} \rangle$ and denote the inductive limit (in the category of sets) by $L_{\mathfrak{F}}$:

$$L_{\mathfrak{F}} = \varinjlim_{I \in \mathfrak{F}} M(I, L/\theta_{\mathfrak{F}}).$$

For any \mathfrak{F} -multiplier $f : I \rightarrow L/\theta_{\mathfrak{F}}$, we denote by $\widehat{(I, f)}$ the equivalence class of f in $L_{\mathfrak{F}}$.

Remark 2.2. If $f_i : I_i \rightarrow L/\theta_{\mathfrak{F}}$, $i = 1, 2$ are \mathfrak{F} -multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $L_{\mathfrak{F}}$) iff there exists $I \in \mathfrak{F}$ and $I \subseteq I_1 \cap I_2$ such that $f_1|_I = f_2|_I$.

Let $f_1 \in M(I_1, L/\theta_{\mathfrak{F}})$, $f_2 \in M(I_2, L/\theta_{\mathfrak{F}})$ two \mathfrak{F} -multipliers and we consider the mappings $f_1 \vee f_2, f_1 \wedge f_2, f_1 \rightarrow f_2 : I_1 \cap I_2 \rightarrow L/\theta_{\mathfrak{F}}$ defined by:

- $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$
- $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$
- $(f_1 \rightarrow f_2)(x) = [f_1(x) \rightarrow f_2(x)] \wedge p_{\mathfrak{F}}(x)$ for any $x \in I_1 \cap I_2$.

The fact that $f_1 \vee f_2$ and $f_1 \wedge f_2$ are \mathfrak{F} -multipliers is proved in [15].

Proposition 2.3. $f_1 \rightarrow f_2$ defined as before is a \mathfrak{F} -multiplier.

Proof. Let $x \in I_1 \cap I_2$ and $e \in L$. Then by (sH3) we have

$$\begin{aligned} (f_1 \rightarrow f_2)(x \wedge y) &= p_{\mathfrak{F}}(x \wedge y) \wedge [f_1(x \wedge y) \rightarrow f_2(x \wedge y)] \\ &= p_{\mathfrak{F}}(x) \wedge p_{\mathfrak{F}}(y) \wedge [(f_1(x) \wedge p_{\mathfrak{F}}(y)) \rightarrow (f_2(x) \wedge p_{\mathfrak{F}}(y))] \\ &= p_{\mathfrak{F}}(x) \wedge p_{\mathfrak{F}}(y) \wedge [f_1(x) \rightarrow f_2(x)] \\ &= (f_1 \rightarrow f_2)(x) \wedge p_{\mathfrak{F}}(y). \end{aligned}$$

Hence $f_1 \rightarrow f_2 \in M(I_1 \cap I_2, L/\theta_{\mathfrak{F}})$. □

We will define on $L_{\mathfrak{F}}$ the following operations:

- $\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)} = \widehat{(I_1 \cap I_2, f_1 \wedge f_2)}$
- $\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)} = \widehat{(I_1 \cap I_2, f_1 \vee f_2)}$

and for $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)}$ we define the element

- $\widehat{(I_1, f_1)} \rightarrow \widehat{(I_2, f_2)} = \widehat{(I_1 \cap I_2, f_1 \rightarrow f_2)}$.

We denote by $\mathbf{0}$ the equivalence class of $(L, 0)$ and by $\mathbf{1}$ the equivalence class of $(L, 1)$.

Lemma 2.4. For each $I \in \mathfrak{F}$, $\langle M(I, L/\theta_{\mathfrak{F}}), \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a semi-Heyting algebra.

Proof. It is easy to verify that $\langle M(I, L/\theta_{\mathfrak{F}}), \wedge, \vee, 0, 1 \rangle$ is a bounded lattice. To prove that it is a semi-Heyting algebra, we have for all $f_1, f_2, f_3 \in M(I, L/\theta_{\mathfrak{F}})$ and $x \in I$,

$$\begin{aligned} \text{(sH1)} \quad (f_1 \wedge (f_1 \rightarrow f_2))(x) &= f_1(x) \wedge (f_1 \rightarrow f_2)(x) \\ &= f_1(x) \wedge (f_1(x) \rightarrow f_2(x)) \wedge [x]_{\theta_{\mathfrak{F}}} \\ &= f_1(x) \wedge (f_1(x) \rightarrow f_2(x)) \\ &= f_1(x) \wedge f_2(x) \\ &= (f_1 \wedge f_2)(x) \end{aligned}$$

$$\begin{aligned} \text{(sH2)} \quad (f_1 \wedge [(f_1 \wedge f_2) \rightarrow (f_1 \wedge f_3)])(x) &= f_1(x) \wedge [(f_1 \wedge f_2) \rightarrow (f_1 \wedge f_3)](x) \\ &= f_1(x) \wedge [(f_1 \wedge f_2)(x) \rightarrow (f_1 \wedge f_3)(x)] \wedge [x]_{\theta_{\mathfrak{F}}} \\ &= f_1(x) \wedge (f_2(x) \rightarrow f_3(x)) \wedge [x]_{\theta_{\mathfrak{F}}} \\ &= f_1(x) \wedge (f_2 \rightarrow f_3)(x) \\ &= (f_1 \wedge (f_2 \rightarrow f_3))(x). \end{aligned}$$

$$\begin{aligned} \text{(sH3)} \quad (f_1 \rightarrow f_1)(x) &= (f_1(x) \rightarrow f_1(x)) \wedge [x]_{\theta_{\mathfrak{F}}} \\ &= [1]_{\theta_{\mathfrak{F}}} \wedge [x]_{\theta_{\mathfrak{F}}} \\ &= [1 \wedge x]_{\theta_{\mathfrak{F}}} \end{aligned}$$

$$\begin{aligned} &= [x]_{\theta_{\mathcal{F}}} \\ &= 1(x). \end{aligned}$$

□

Theorem 2.5. $\langle L_{\mathfrak{F}}, \vee, \wedge, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$ is a semi-Heyting algebra.

Proof. It follows as a special case of Corollary 2.1 in [18]. Indeed, condition (ii) in Definition 2.2 is stronger than the property of being down directed, the operations $\vee, \wedge, \rightarrow, 0$ and 1 of $M(I, L/\theta_{\mathfrak{F}})$ obviously satisfy conditions (2.1) and (2.2) in [18, Section 2.1] and $M(I, L/\theta_{\mathfrak{F}})$ is a semi-Heyting algebra by Lemma 2.4. □

The semi-Heyting algebra $L_{\mathfrak{F}}$ will be called the localization semi-Heyting algebra of L with respect to the topology \mathfrak{F} .

3. Semi-Heyting algebra of fractions relative to an \wedge -closed system

Definition 3.1. A subset $S \subseteq A$ is called \wedge -closed system if $1 \in S$ and if $x, y \in S$ implies $x \wedge y \in S$.

We will denote by $S(L)$ the set of all \wedge -closed system of L .

Lemma 3.1. Let $S \in S(L)$. Then, the binary relation θ_S defined by $(x, y) \in \theta_S \Leftrightarrow$ there is $s \in S$ such that $x \wedge s = y \wedge s$ is a congruence on L .

Proof. We will only prove that θ_S is compatible with \rightarrow . Suppose that $(x, y), (z, w) \in \theta_S$. Then, there exists $s, e \in S$ such that $x \wedge s = y \wedge s$ and $z \wedge e = w \wedge e$. From these statements and (sH2), we have that

$$\begin{aligned} s \wedge e \wedge (x \rightarrow z) &= s \wedge e \wedge [(s \wedge e \wedge x) \rightarrow (s \wedge e \wedge z)] \\ &= s \wedge e \wedge [(s \wedge e \wedge y) \rightarrow (s \wedge e \wedge w)] \\ &= s \wedge e \wedge (y \rightarrow w). \end{aligned}$$

Thus, $(x \rightarrow z, y \rightarrow w) \in \theta_S$. Hence θ_S is a congruence on L . □

For $x \in L$ we denote the equivalence class of x relative to θ_S by $[x]_S$ and by $L[S] = L/\theta_S$. We denote the canonical map $p_S : L \rightarrow L[S]$ defined by $p_S(x) = [x]_S$ for every $x \in L$. Clearly, $L[S]$ become a semi-Heyting algebra.

Remark 3.1. Since for every $s \in S$, $s \wedge s = s \wedge 1$, we deduce that $[s]_S = [1]_S$, hence $p_S(S) = \{[1]_S\}$.

Theorem 3.2. If L is a semi-Heyting algebra and $f : L \rightarrow L'$ is an morphism of semi-Heyting algebras such that $f(S) = \{1\}$, then there is a unique morphism of semi-Heyting algebras $f' : L[S] \rightarrow L'$ such that $f' \circ p_S = f$.

Proof. It follows from [18, Theorem 4.1] and Remark 3.1. □

Theorem 3.2 allows us to call $L[S]$ the semi-Heyting algebra of fractions relative to the \wedge -closed system S .

Remark 3.2. From Theorem 3.2 we have that

- (i) If $S = \{1\}$, then θ_S coincides with the identity congruence on L and so, $L[S] \simeq L$.
- (ii) If S is an \wedge -closed system of L such that $0 \in S$ (for example $S = L$), then $\theta_S = L \times L$. Hence, $L[S] = \{[0]_S\}$.

Remark 3.3. Let L a semi-Heyting algebra and $S \in S(L)$. Then $\mathfrak{F}_S = \{I \in I(L) : I \cap S \neq \emptyset\}$ is a topology on L .

The topology \mathfrak{F}_S will be called the topology associated with the \wedge -closed system S of L .

Lemma 3.3. Let \mathcal{F}_S be the topology associated with the \wedge -closed system S . Then $\theta_{\mathcal{F}_S} = \theta_S$.

Proof. Let $(x, y) \in \theta_{\mathcal{F}_S}$. Then there is $I \in \mathfrak{F}_S$ such that $s \wedge x = s \wedge y$, for all $s \in I$. Since there exists $s_o \in I \cap S$ verifying $s_o \wedge x = s_o \wedge y$, we infer that $(x, y) \in \theta_S$. Conversely, let $(x, y) \in \theta_S$. Then there is $s_o \in S$ such that $x \wedge s_o = y \wedge s_o$. By considering $I = \langle s_o \rangle$ we conclude that $(x, y) \in \theta_{\mathfrak{F}_S}$. \square

Remark 3.4. From Lemma 3.3, we have that $L/\theta_{\mathfrak{F}_S} = L[S]$. Then an \mathfrak{F}_S -multiplier can be consider as a map $f : I \rightarrow L[S]$ where $I \in \mathfrak{F}_S$ and $f(e \wedge x) = [e]_S \wedge f(x)$ for all $x \in I$ and $e \in I$.

Lemma 3.4. Let $(\widehat{I_1, f_1}), (\widehat{I_2, f_2}) \in L_{\mathfrak{F}_S}$ be such that $(\widehat{I_1, f_1}) = (\widehat{I_2, f_2})$. Then there exists $I \subseteq I_1 \cap I_2$ such that $f_1(s_o) = f_2(s_o)$ for all $s_o \in I \cap S$.

Proof. From the hypothesis and Remark 2.2, we have that there exists $I \in \mathfrak{F}_S, I \subseteq I_1 \cap I_2$ such that $f_1|_I = f_2|_I$ and so, $f_1(s_o) = f_2(s_o)$ for each $s_o \in I \cap S$. \square

Theorem 3.5. Let L be a semi-Heyting algebra. If \mathfrak{F}_S is the topology associated with the \wedge -closed system S , then $L_{\mathfrak{F}_S}$ is isomorphic to $L[S]$.

Proof. Let $\alpha : L_{\mathfrak{F}_S} \rightarrow L[S]$ be defined by $\alpha(\widehat{I, f}) = f(s)$ for all $s \in I \cap S$. From Lemma 3.4, we have that α is well-defined. Besides, α is one-to-one. Indeed, suppose that $\alpha(\widehat{I_1, f_1}) = \alpha(\widehat{I_2, f_2})$. Then there exist $s_1 \in I_1 \cap S$ and $s_2 \in I_2 \cap S$ such that $f_1(s_1) = f_2(s_2)$. Hence, by considering $f_1(s_1) = [x]_S$ and $f_2(s_2) = [y]_S$, we have that there is $s \in S$ verifying $x \wedge s = y \wedge s$. If $s' = s \wedge s_1 \wedge s_2$, then we infer that $f_1(s') = f_1(s' \wedge s_1) = [s']_S \wedge f_1(s_1) = [s']_S \wedge f_2(s_2) = f_2(s')$. Let $I = \langle s' \rangle$. So, $I \in \mathfrak{F}_S, I \subseteq I_1 \cap I_2$ and $f_1|_I = f_2|_I$. Remark 2.2 allows us to infer that $(\widehat{I_1, f_1}) = (\widehat{I_2, f_2})$. In order to prove that α is surjective, let $[a]_S \in L[S]$ and $f_a : L \rightarrow L[S]$ defined by $f_a(x) = [a \wedge x]_S$ for all $x \in L$. It is simple to verify that f_a is an \mathfrak{F}_S -multiplier. Moreover, from Remark 3.1, $\alpha(\widehat{L, f_a}) = f_a(s) = [a \wedge s]_S = [a]_S$, being $s \in S$. It is simple to verify that this map is an morphism of semi-Heyting algebras. \square

4. Localization and fractions in finite semi-Heyting algebras

In this section, our attention is focus on considering the above results in the particular case of finite semi-Heyting algebras. More precisely, we will prove that for each finite semi-Heyting algebra L and $S \in S(L)$ the algebra $L[S]$ is isomorphic to a special subalgebra of L . In order to do this, the following propositions will be fundamental.

Proposition 4.1. Let L be a finite semi-Heyting algebra and $I \subseteq L$. Then, the following conditions are equivalent:

- (i) $I \in I(A)$,
- (ii) $I = \langle a \rangle$ for some $a \in L$.

Proof. It is straightforward. \square

Proposition 4.2. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then $\mathfrak{F}_S = \{\langle a \rangle : \bigwedge_{x \in S} x \leq a\}$.*

Proof. Let us consider $\mathfrak{T} = \{\langle a \rangle : \bigwedge_{x \in S} x \leq a\}$. Assume that $I \in \mathfrak{F}_S$. Then, by Proposition 4.1 we have that $I = \langle a \rangle$ for some $a \in L$. On the other hand, from Remark 3.3 there is $s \in S \cap \langle a \rangle$ which implies that $\bigwedge_{x \in S} x \leq s \leq a$. Therefore, $I \in \mathfrak{T}$. Conversely, suppose that $I \in \mathfrak{T}$. Hence, $\bigwedge_{x \in S} x \in I \cap S$. Furthermore by Proposition 4.1 we have that $I \in I(L)$. From these last assertions and Remark 3.3 we conclude that $I \in \mathfrak{F}_S$. \square

Proposition 4.3. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then, the following conditions are equivalent:*

- (i) $(x, y) \in \theta_{\mathfrak{F}_S}$
- (ii) $x \wedge b = y \wedge b$ where $b = \bigwedge_{x \in S} x$.

Proof. It is routine. \square

Lemma 4.4. ([19, Lemma 6.2]) *Let L be a finite semi-Heyting algebra and $\langle a \rangle \in I(L)$. Then, $L_a = \langle \langle a \rangle, \vee, \wedge, \rightarrow^a, 0, a \rangle$ is a semi-Heyting algebra.*

Finally, we obtain our desired goal.

Theorem 4.5. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then $L[S]$ is isomorphic to L_b where $b = \bigwedge_{x \in S} x$.*

Proof. Let $\beta : L \rightarrow L_b$ be the function defined by the prescription $\beta(x) = x \wedge b$. By [19, Lemma 6.2] β is a semi-Heyting morphism. It is easy to check that β is surjective. Therefore, β is an semi-Heyting epimorphism. Moreover, $x \in [1]_{\theta_S} \Leftrightarrow (x, 1) \in \theta_S \Leftrightarrow$ there is $s \in S$ such that $x \wedge s = s \Leftrightarrow x \wedge b = b \Leftrightarrow \beta(x) = b \Leftrightarrow x \in \ker(\beta)$. Therefore, taking into account a well-known result of universal algebra (see [4, p. 59]) we conclude that $L[S]$ is isomorphic to L_b . \square

Corollary 4.6. *Let L be a finite semi-Heyting algebra and $S \in S(L)$. Then, $L_{\mathfrak{F}_S}$ is isomorphic to L_b where $b = \bigwedge_{x \in S} x$. More precisely, $L_{\mathfrak{F}_S} = \{\widehat{(\langle b \rangle, f_x)} : x \in \langle b \rangle\}$.*

Proof. It follows as a consequence of Theorem 3.5 and Theorem 4.5. \square

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