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On the influence of external stochastic excitation on linear oscillators with subcritical self-excitation applied to brake squeal

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Abstract: A characteristic of linear systems with self-excitation is the occurrence of non-normal modes. Because of this non-normality, there may be a significant growth in the vibration amplitude at the beginning of the transient process even in the case of solely negative real parts of the eigenvalues, i.e. asymptotic stability of the trivial solution. If such a system is excited additionally with white noise, this process is continually restarted and a stationary vibration with dominating frequencies and comparably large amplitudes can be observed. Similar observations can be made during brake squeal, a high-frequency noise resulting from self-excitation due to the frictional disk-pad contact. Although commonly brake squeal is considered as a stable limit cycle with the necessity of corresponding nonlinearities, comparable noise phenomena can in the described model even observed in a pure linear case when the trivial solution is asymptotically stable.

1. Introduction

In a lot of applications, including but not limited to cutting machines, bridges under wind, and disk brakes, self-excited vibrations may appear as an unwanted phenomenon that reduces the effectiveness of the machines, causes inconvenience or even leads to destruction. In linear analysis, a self-excited system is usually modeled as a system of homogeneous linear ordinary differential equations (ODE) which may be obtained for a general continuous system by discretization and linearization. As a consequence, a trivial solution exists. The most popular criterion to determine, whether harmful vibrations happen or not is based in that type of mathematical models on the stability of the trivial solution. If the trivial solution is asymptotically stable, any difference between the initial state and the trivial solution is reduced to a negligible amount after a period of time called the transient process. The common disinterest in the transient process is supported by the fact that it is usually so short that it is far less representative for the behavior of the system than the steady state. However, the importance of the transient process is remarkable when there is an appearance of transient growth: vibration amplitudes may increase at the beginning of the transient process even when the largest Lyapunov exponent is negative. Transient growth is of more interest in fluid dynamics to study turbulence [1, 2]. In the field of mechanics, some studies led by Hoffmann show that transient growth may cause beating

[3] or initiate stick-slip [4] in friction-induced vibration problems. So far, this phenomenon is known from literature although probably not aware to many engineers in this field.

In reality, there may be sources for additional external forces in the self-excited systems so that their governed ODE are not homogeneous. Instead, white noise excitation, for example, can be added to the mathematical models. In this case, stability analysis of the trivial solution alone is not enough to characterize the behavior of the system. The asymptotic stability of the respective homogeneous system only means that the stochastic process in case of Gaussian white noise excitation is not drifting away. The result is a Gaussian probability density distribution around zero. But asymptotic stability together with the maximum Lyapunov exponent does not say anything about how likely large deviations from zero are.

These deviations may be important, noting the fact that harmful phenomena may occur even with small vibration amplitudes: the amplitudes of mechanical parts during brake squeal [5, 6] – a type of uncomfortable noise with kHz-frequency that may happen when an automotive mechanical brake system is activated – lie in the micrometer range. Therefore, the effect of stochastic excitation on linear systems, especially systems with the above-mentioned transient growth, should be studied.

By comparing a normal and a non-normal system with same maximum real part of the eigenvalues, the reason for transient growth is introduced, and then the effect of stochastic excitation on such equations examined.

2. Non-normality and transient growth in linear systems with self-excitation

2.1. Properties of an EDKN system

Consider a system of two linear ordinary differential equations for self-excited vibrations, which is in its basic structure similar to those, which are obtained from minimum models for brake squeal [4, 7]. These equations are written in the form

$$\ddot{\mathbf{x}} + (\mathbf{K} + \mathbf{N})\mathbf{x} = \mathbf{0}\,, \tag{1}$$

where $\mathbf{x}$ is a 2-by-1 vector representing in mechanical systems displacements or angles, $\mathbf{K}$ is a 2-by-2 symmetric positive definite matrix (stiffness matrix) and $\mathbf{N}$ is a 2-by-2 skew-symmetric matrix representing the self-excitation (circulatory matrix). If linear damping is added to the model, its equations read [4]

$$\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{N})\mathbf{x} = \mathbf{0}\,, \tag{2}$$

where $\mathbf{D}$ is a 2-by-2 symmetric positive definite matrix (damping matrix). In general, minimal brake squeal models may have equations of motions, where the mass matrix is not an identity matrix (e.g. in [8]), i.e. they read as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{N})\mathbf{x} = \mathbf{0} , \tag{3}$$

where **M** is the 2-by-2 symmetric positive definite mass matrix.

A system governed by Eq. (3) is called an MDKN system [9], implying that its equations include four matrices denoted by these four letters. The special case Eq. (2) of this with mass matrix being identity matrix **E** can be called an EDKN system. Any MDKN system whose matrix **M** is a diagonal matrix with all diagonal elements equal to each other can be easily written in the form of an EDKN system by multiplying to the left of its equations the inverse matrix of **M**. The positive definite assumptions for **M**, **D** and **K** can be reduced to positive semi-definite for generalization, but it is not the case considered in this paper. The appearance of **N** originates from non-conservative circulatory forces and may result in instability of the trivial solution, which is in linear brake squeal models considered to be the mechanism of squeal. Even in the case of asymptotically stable trivial solution, a system with non-vanishing **N** is a system with self-excitation in which it can be called more specifically a system with subcritical self-excitation. Without **N**, we have the well-known MDK system which has always an asymptotically stable trivial solution provided that **M**, **D** and **K** are all symmetric positive definite.

2.2. Non-normality and transient growth

The basic effect of transient growth for non-normal systems is known from literature also with application to friction induced vibrations [3, 4]. Nevertheless it shall be repeated here as an introduction of the considered systems and the effects resulting from additional stochastic excitation to be described in section 3.

To visualize the concept of non-normality and transient growth, consider an EDKN system whose matrices are chosen as

$$\mathbf{D} = \begin{bmatrix} 0.1 & -0.01 \\ -0.01 & 0.1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 & -n \\ n & 0 \end{bmatrix}, \tag{4}$$

where n is a real parameter, and a corresponding EDK system

$$\ddot{\mathbf{x}} + \alpha\mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} , \tag{5}$$

where α is chosen so that both systems have the same maximum real part of the eigenvalues. For simplicity purposes, all the parameters including the time are considered as dimensionless in the following.

As long as the trivial solution is asymptotically stable (subcritical self-excitation) and its characteristic polynomial has no repeated roots, the general real solution of it has the form

$$\mathbf{x}(t) = C_1\mathbf{u}_1 e^{-\lambda_1 t}\cos(\omega_1 t) + C_2\mathbf{u}_2 e^{-\lambda_2 t}\cos(\omega_2 t) + C_3\mathbf{u}_3 e^{-\lambda_1 t}\sin(\omega_1 t) + C_4\mathbf{u}_4 e^{-\lambda_2 t}\sin(\omega_2 t)\,, \tag{6}$$

where λ_1, λ_2, ω_1 and ω_2 are positive values. The Euclidean norms of modal vectors $\mathbf{u}_i$ (i = 1, 2, 3, 4) are chosen as 1. C_1, C_2, C_3 and C_4 are coefficients to be determined from the initial condition.

If we consider the initial condition

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} x_{10} & x_{20} \end{bmatrix}^T, \quad \dot{\mathbf{x}}(0) = \mathbf{0}\,, \tag{7}$$

with

$$\|\mathbf{x}\|_2 = x_{10}^2 + x_{20}^2 = 1\,, \tag{8}$$

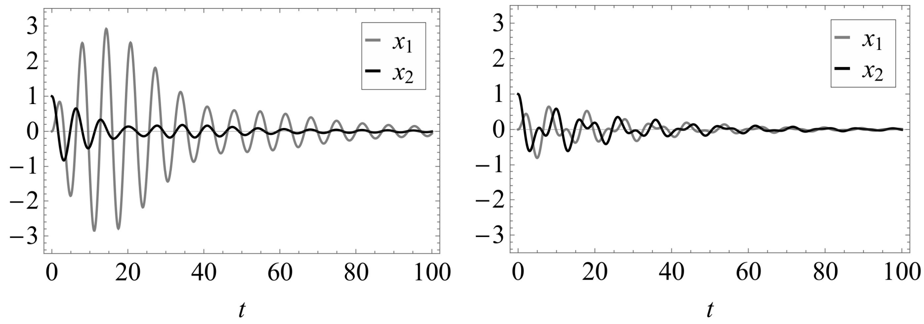
both C_3 and C_4 are equal to zero while C_1 and C_2 can be found by solving the linear algebraic equations

$$\mathbf{Uc} = \mathbf{x}_0 \tag{9}$$

where

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}^T. \tag{10}$$

Varying n, one gets different pairs of $\mathbf{u}_1$, $\mathbf{u}_2$ and different angles between them. When the angle is close to π/2, the Euclidean norm of **c** always stay near 1. In contrast, when the angle close to 0 or π and with appropriate initial conditions, either C_1 or C_2 or both can take a value much higher than 1, i.e. the initial modal vectors can be much larger than the vector of initial conditions. We can say that the latter case shows a strong *non-normality* and a system with this characteristic is called a *non-normal system*. The explanation of the concept of non-normality can also be found in [10]. As a result of the large initial vectors, a non-normal system may have a transient growth: even when the system is exponentially stable, its vibration amplitude initially increases before decaying to zero (Fig. 1a). This behavior cannot be seen in a typical MDK system (Fig. 1b). The maximum real part of the eigenvalues for both systems is approximately –0.0324 and the initial conditions are as described in Eq. (7), with $x_{10} = 0$ and $x_{20} = 1$. It should be noted that whether transient growth occurs or not also depends on the choice of the initial conditions as described in [3].



a) b)

Figure 1. a) Transient growth and subsequent decaying vibration of an EDKN system (2) with $n = 0.48$.

b) Decaying vibration without transient growth of an EDK system (5) when $n = 0$, $\alpha = 0.72$.

3. Comparison of vibrational behavior EDK and EDKN systems subjected to external stochastic excitation

In the operation of real systems, may they contain self-excitation or not, it can be expected, that small external disturbances are present, which are in the following modeled by Gaussian white noise. As we consider linear systems with Gaussian white noise most of the following steps can be performed analytically with well-known relations.

The governing ODEs in this case become inhomogeneous by adding white noise to the right-hand side of equation (2) and (5)

$$\ddot{\mathbf{X}}_t + \mathbf{D}\dot{\mathbf{X}}_t + (\mathbf{K} + \mathbf{N})\mathbf{X}_t = \boldsymbol{\sigma}\xi_t \,, \tag{11}$$

$$\ddot{\mathbf{X}}_t + \alpha\mathbf{D}\dot{\mathbf{X}}_t + \mathbf{K}\mathbf{X}_t = \boldsymbol{\sigma}\xi_t \,. \tag{12}$$

Herein ξ_t is a scalar Gaussian white noise with zero mean and the 2-by-1 vector $\boldsymbol{\sigma}$ contains their intensity coefficients. The considered equations now form a system of linear stochastic differential equations (SDE). In the following we use stationary probability density functions p (PDF) for comparing the two systems under consideration. The probability density function can either be calculated using numerical integration (Monte-Carlo simulation) or by solving the corresponding Fokker-Planck equation. In both cases, (11) and (12) are rewritten as a first-order system

$$\mathrm{d}\mathbf{Q}_t = \mathbf{A}\mathbf{Q}_t + \mathbf{g}\,\mathrm{d}W_t \,, \tag{13}$$

$$\mathrm{d}\mathbf{Q}_t = \mathbf{A}'\mathbf{Q}_t + \mathbf{g}\,\mathrm{d}W_t \tag{14}$$

respectively, where $\mathbf{Q}_t$ is the vector of the random state processes

$$\mathbf{Q}_t = \begin{bmatrix} \mathbf{X}_t \\ \dot{\mathbf{X}}_t \end{bmatrix}. \tag{15}$$

W_t is the Wiener process corresponding to ξ_t , and the other matrices are determined as follows

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ -(\mathbf{K}+\mathbf{N}) & -\mathbf{D} \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ -\mathbf{K} & -\alpha\mathbf{D} \end{bmatrix}, \tag{16}$$

$$\mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{\sigma} \end{bmatrix}. \tag{17}$$

The diffusion matrix is defined as

$$\mathbf{B} = \mathbf{g}\mathbf{g}^T . \tag{18}$$

To save space, the equations from now on are only written for **A**, but they also hold for **A′**. In order to find a PDF $p(\mathbf{q})$ of the stationary process $\mathbf{Q}_t$, the stationary Fokker-Planck equation associated with Eq. (13) and (14)

$$\sum_{i=1}^{4} \frac{\partial}{\partial q_i}\left[p(\mathbf{q}) \sum_{j=1}^{4} a_{ij} q_j \right] - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial^2}{\partial q_i \partial q_j}\left[p(\mathbf{q}) b_{ij} \right] = 0 \tag{19}$$

has to be solved, where a_{ij} and b_{ij} are the elements in row i and column j of matrix **A** and matrix **B**, respectively.

Since we have a linear system with Gaussian excitation, the corresponding solution is also Gaussian. Following [11], the solution has the form

$$p(\mathbf{q}) = \mathbf{N}\ (\boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\Lambda}) = \frac{1}{(2\pi)^2 |\boldsymbol{\Lambda}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \mathbf{q}^T (\boldsymbol{\Lambda})^{-1} \mathbf{q} \right) \tag{20}$$

with mean value vector $\boldsymbol{\mu} = \mathbf{0}$ due to missing asymmetry and covariance matrix $\boldsymbol{\Lambda}$. Hence, Eq. (19) yields to the following algebraic equation [12]

$$[(\mathbf{E} \otimes \mathbf{A}) + (\mathbf{A} \otimes \mathbf{E})]\, \mathrm{vec}(\boldsymbol{\Lambda}) + \mathrm{vec}(\mathbf{B}) = 0 , \tag{21}$$

where $\otimes$ denotes the Kronecker product and $\mathrm{vec}(\cdot)$ denotes vectorization operator. For the results discussed later on, it is chosen that

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{22}$$

Solving (21) for $\mathbf{\Lambda}$ and computing (20), one obtains $p(\mathbf{q})$. Marginal PDF p_{X_1} and p_{X_2} can then calculated by

$$p_{X_1}(x_1) = p_{Q_1}(q_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\mathbf{q})\ \mathrm{d}q_2\mathrm{d}q_3\mathrm{d}q_4\,, \tag{23}$$

$$p_{X_2}(x_2) = p_{Q_2}(q_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\mathbf{q})\ \mathrm{d}q_1\mathrm{d}q_3\mathrm{d}q_4\,. \tag{24}$$

In the following, corresponding results are discussed. Figure 2 shows time responses X_{1t} and X_{2t} obtained by Monte-Carlo simulation performed by using the Euler-Maruyama method. It can be observed, that the vibration amplitudes of X_{2t} of both cases are in a similar range, while for X_{1t} the EDKN system produces much higher amplitudes.

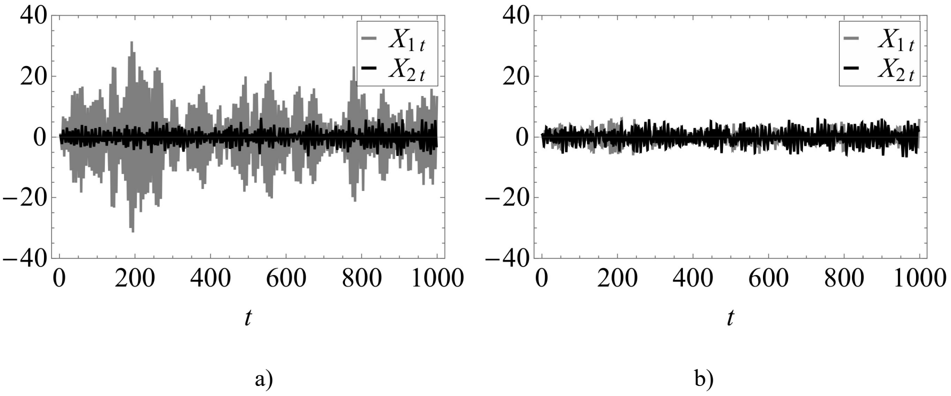


Figure 2. Time responses X_{1t} and X_{2t} for a) EDKN system and b) EDK system with same white noise excitation according to Eq. (22).

A similar behavior can be observed when considering the PDF. Figures 3 and 4 show the marginal PDF p_{X_1} and p_{X_2} for both EDKN and EDK system as analytical solution of the Fokker-Planck equation according to Eqs. (20) – (24) compared with Monte-Carlo simulation results. Again p_{X_1} is spreading much more for the EDKN system compared to the EDK system while p_{X_2} is comparable in both cases.

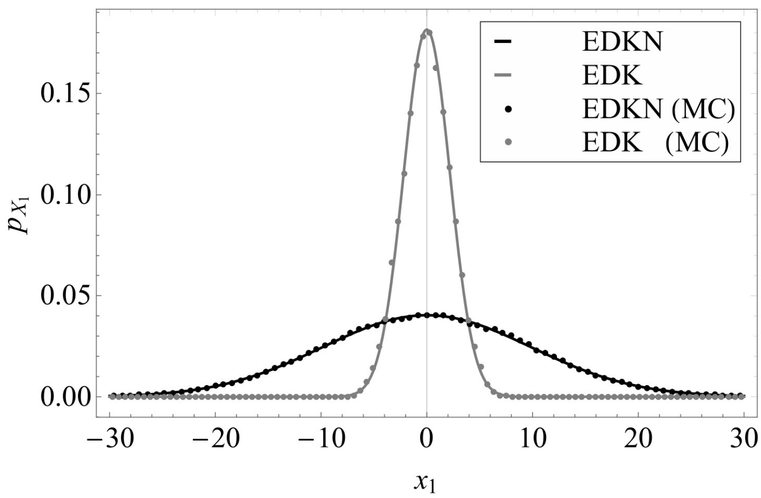


Figure 3. Marginal PDFs p_{X_1} of stochastically excited EDKN and EDK system respectively obtained from solving the Fokker-Planck equation (lines) and from Monte-Carlo simulation (dots).

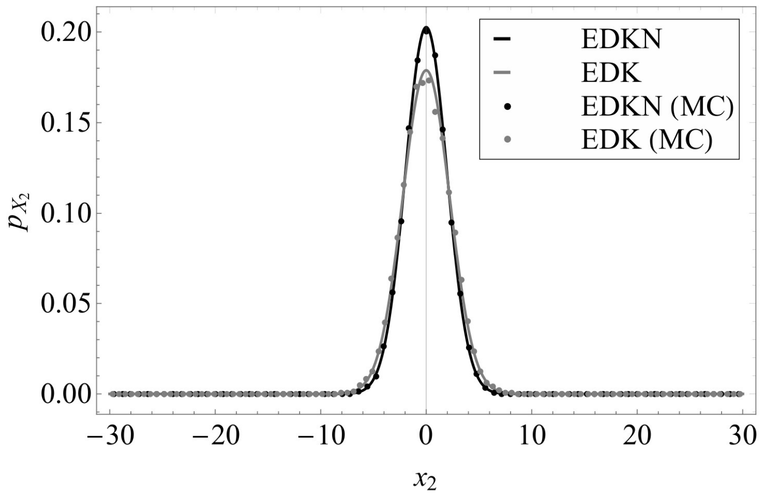


Figure 4. Marginal PDFs p_{X_2} of stochastically excited EDKN and EDK system respectively obtained from solving the Fokker-Planck equation (lines) and from Monte-Carlo simulation (dots).

Finally the frequency characteristic of the responses X_{1t} and X_{2t} for the EDKN system shall be considered by the absolute values of corresponding transfer functions $H_1(\Omega)$ and $H_2(\Omega)$ in the case of single excitation of the second equation in Eq. (2) which is in accordance with excitation Eq. (22). It can be seen, that for X_{1t} distinct vibrations can be expected with (as the two eigenfrequencies are close together) almost one single dominating frequency which is very close to the behavior observed in brake squeal (Fig. 5).

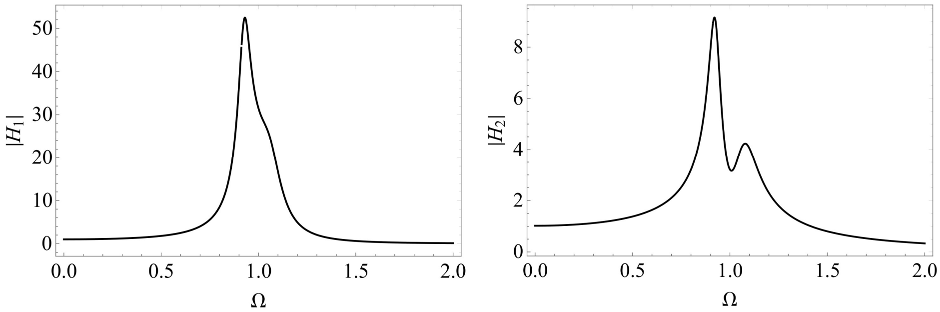


Figure 5. Transfer functions $|H_1(\Omega)|$ and $|H_2(\Omega)|$ for excitation according to (21).

4. Conclusions and outlook

In this paper a non-normal EDKN system with corresponding suitable system and initial condition parameters producing a negative maximum real part of the eigenvalues, i.e. a system with transient growth and sub-critical self-excitation, has been considered in comparison with an EDK system showing similar stability behavior. Stochastic excitation has been added to both systems to compare its effect in both cases. In a system without self-excitation and stable trivial solution this will result in vibrations according to the excitation level around the zero solution. In contrast to this, in a system with similar maximum real part of the eigenvalues but self-excitation, much larger vibrations may result as the transient growth behavior is continually restarted by the stochastic excitation. The resulting vibrations remember to what can be observed during brake squeal. This surprisingly happens for a linear system, with stable trivial solution only needing some external noise excitation to get comparably large responses, while general explanation of brake squeal is that of a stable limit cycle in a nonlinear system.

In future work, we intend to consider full MDGKN systems resulting from minimal models of brakes.

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