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# On some aspects of the dynamic behavior of the softening Duffing oscillator under harmonic excitation

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**Abstract** The Duffing oscillator is probably the most popular example of a nonlinear oscillator in dynamics. Considering the case of softening Duffing oscillator with weak damping and harmonic excitation and performing standard methods like harmonic balance or perturbation analysis zero mean solutions with large amplitudes are found for small excitation frequencies. These solutions produce a "nose-like" curve in the amplitude-frequency diagram and merge with the inclining resonance curve for decreasing (but non-vanishing) damping. These results are presented without any additional discussion in several textbooks. The present paper discusses the accurateness of these solutions by introducing an error estimation in the harmonic balance method showing large errors. Performing a modified perturbation analysis leads to solutions with non-vanishing mean value, showing very small errors in the harmonic balance error analysis.

**Keywords** Softening Duffing oscillator, harmonic balance, perturbation analysis

## 1 Introduction

The German engineer Georg Duffing (1861-1944) investigated in his original work 1918 an oscillator with quadratic and cubic stiffness and linear viscous damping performing free or forced harmonic vibrations [1]. Nevertheless the term "Duffing equation" is nowadays in general used for any nonlinear equation of motion including a cubic stiffness term. The Duffing equation is capable to show many phenomena of nonlinear vibrations, as the dependency of natural period on the vibration amplitude in case of undamped free vibrations, the possibility of multiple solutions in the case of forced vibrations or the occurrence of superharmonics in the response in case of harmonic (monofrequent) excitation, as it is discussed in many textbooks e.g. [2]. The Duffing equation is also capable to show chaotic responses e.g. in the case of negative linear and positive cubic stiffness under harmonic excitation or phenomena like period multiplication. In contrast to the second classical popular nonlinear oscillator, the van der Pol oscillator (being capable to show self excited vibrations and limit cycle oscillations), a Duffing oscillator is not too serious to realize in a pure mechanical experiment. E.g. a pendulum shows, if the sine restoring term is expanded around zero up to the third order

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term, a negative ("softening") cubic restoring. Using special kinematics with springs, also a positive ("stiffening") restoring term can easily be realized, see e.g. [1]. Therefore the Duffing equation is probably the most famous example of a nonlinear oscillator to be found in vibration classes. Since decades also textbooks deal with the example of the Duffing oscillator as an introduction into nonlinear vibrations. The present paper considers a classic problem of this type: the softening Duffing oscillator under harmonic excitation in the presence of weak linear damping. The authors also use classic methods like harmonic balance, Lindstedt-Poincaré perturbation analysis or numerical integration for getting the results for stationary vibrations. The essential point in this paper is, that a class of solutions with high amplitudes obtained by harmonic balance or perturbation analysis in the case of low excitation frequencies is considered. The paper discusses in detail these solutions and argues about their non-accurateness. For this purpose an error estimation based on the harmonic balance method is used. An alternative perturbation analysis is performed in order to identify the character of qualitatively different solutions, which are then also checked by the error estimation of the harmonic balance. In many textbooks the original harmonic balance solutions are simply plotted without any further discussion on the validity of these results. As this example is widely taught in nonlinear vibration classes the authors see the urgent necessity of additional argumentation when this problem is presented.

Extensive scientific work has been done on the Duffing equation. An overview is given in the book "The Duffing Equation" edited by Kovacic and Brennan [1]. The softening Duffing oscillator with weak damping and harmonic excitation has been studied especially in the late 1980s and 1990s special focus given to chaotic behavior and paths to chaos. Compared to the example considered in this paper, most authors investigate the resonance behavior of even weaker damped or stronger excited systems, showing rich varieties of bifurcation and chaotic behavior. Nayfeh and Sanchez [3] considered the bifurcation behavior of the softening Duffing oscillator and calculated attractors. Szemplińska-Stupnicka [4] studied as well chaos in such systems as Dowell and coauthors [5]. The work by Tsuda and coauthors [6] present broad varieties of regular solutions and investigates their stability but also focuses mainly on resonance. Like Tsuda we will restrict to regular solutions but (after a broader introduction) will focus on low excitation frequencies.

## 2 Classic analysis of the softening Duffing oscillator

Let's consider the Duffing oscillator

$$m\ddot{x} + d\dot{x} + cx + \alpha x^3 = F_0 \cos \Omega t \tag{1}$$

with $m$ being the oscillator mass, $d$ the damping coefficient, $c$ the linear stiffness, $\alpha$ the coefficient of the nonlinear stiffness, $F_0$ the excitation force amplitude and $\Omega$ the circular excitation frequency. For the softening Duffing oscillator $\alpha$ is negative, while $m$, $d$, $c$ and $F_0$ are positive. This equation can be rewritten by introducing the circular frequency of the undamped free linear vibrations $\omega_0^2 = c/m$, the damping ratio $D = d/(2\sqrt{cm})$ and the dimensionless time $\tau = \omega_0 t$ as

$$x''(\tau) + 2Dx'(\tau) + x(\tau) + \varepsilon x^3(\tau) = f \cos(\eta\tau) \tag{2}$$

with $()' = \mathrm{d}()/\mathrm{d}\tau$, $\varepsilon = \alpha/(m\omega_0^2)$, $f = F_0/(m\omega_0^2)$ and $\eta = \Omega/\omega_0$. The restoring characteristic is plotted in Fig. 1. In most cases $\varepsilon$ is considered to be a small parameter, i.e.

$$1 \gg |\varepsilon| x^2. \tag{3}$$

This means that we are in the region around $x$ being close to zero, where almost no influence of the cubic term is visible. Please notice, that in most argumentations afterwards, this relation (3) will not hold! As a standard method to obtain the solution of (2) harmonic balance is used. In the simplest case the ansatz

$$x(\tau) = C \cos(\eta\tau - \varphi) \tag{4}$$

is performed. Inserting this into (2) and neglecting the higher order frequency term proportional to $\cos 3\eta\tau$, the amplitude $C$ and phase shift $\varphi$ can be obtained by solving the nonlinear algebraic equations

$$\left(1 - \eta^2 + \frac{3}{4}\varepsilon C^2\right)^2 C^2 + 4D^2\eta^2 C^2 = f^2, \qquad \tan\varphi = -\frac{2D\eta}{1 - \eta^2 + \frac{3}{4}\varepsilon C^2}. \tag{5}$$

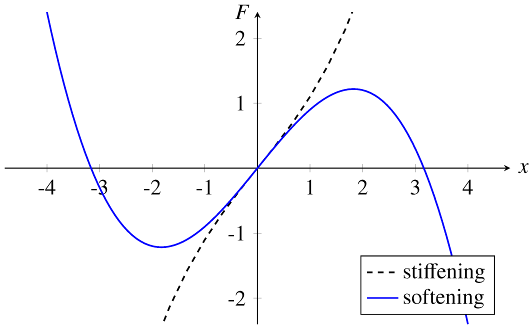


**Fig. 1** Restoring characteristic $F(x)$ of the Duffing oscillator, $\varepsilon = 0.1$ (stiffening case, dashed line) and $\varepsilon = -0.1$ (softening case, solid line).

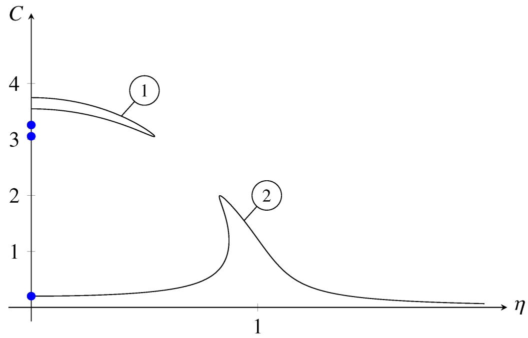


**Fig. 2** Amplitude frequency dependence of the softening Duffing oscillator according to perturbation analysis or harmonic balance (equations (4, 5)), exact solution in case of $\eta = 0$ (blue dots) for parameters $D = 0.06$, $\varepsilon = -0.1$ and $f = 0.2$.

These results for the amplitude $C$ are plotted in Figure 2. Beside the well known behavior in resonance with the curve having an inclination to the left and the occurrence of multiple solutions (denoted as solutions ②) additional "nose-like" solutions for small $\eta$ can be found with large amplitudes $C$ (denoted as solutions ①). For weaker (but non-vanishing) damping and/or larger excitation amplitude the "nose-like" solutions and the inclined resonance peak converge and are finally merging. Now, the oncoming considerations will focus on the accurateness of these "nose-like" solutions and - after having found several arguments for the non-accurateness - on alternative solutions for the low frequency range. Beside the already mentioned books [1] and [2] some textbooks on nonlinear dynamics have been examined, starting from the 1950s to actual ones [7] - [11] with respect to the point, whether these "nose-like" solutions are plotted or not, when the softening Duffing oscillator is examined. In these books several semi-analytical solution methods have been used in order to get the amplitude characteristic. As a result, in approximately one half of the books the "nose-like" solution is plotted and in most of the others the plotting is restricted to $\eta \approx 1$. The only book of these plotting the "nose-like" solution but formulating some doubts on the correctness is by Klotter [10], but some doubts can also be found e.g. in the paper [12]. Let's first consider some initial doubts on these "nose-like" solutions. As an alternative standard method for the solution of (2) Lindstedt-Poincaré perturbation analysis can be used, where for getting results with multiple solutions also small damping and excitation has to be considered, i.e. equation (2) is rewritten in the form

$$x''(\tau) + 2\varepsilon\delta x'(\tau) + x(\tau) + \varepsilon x^3(\tau) = \varepsilon \hat{f} \cos(\eta\tau) \tag{6}$$

with $\delta = D/\varepsilon$ and $\hat{f} = f/\varepsilon$. This assumption is satisfied in the case of weak damping and excitation frequencies with $\eta$ close to one (resonance case), where small excitation amplitudes result in large response amplitudes. Due to Lindstedt-Poincaré the following expansions are performed:

$$x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + ..., \qquad \eta = 1 + \varepsilon\eta_1 + ... \quad . \tag{7}$$

Solving the problem up to order $x_0$ including a vanishing secular term in the first order equation similar results as for the harmonic balance can be obtained. Nevertheless the perturbation analysis allows a much better proof, whether the obtained solutions are valid, due to the assumptions made. In harmonic balance, higher order frequency terms are neglected without any additional proof of dimensions or relations between the occurring terms. Looking on the assumptions made in the beginning, it is stated in equation(3) that $1 \gg |\varepsilon|x^2$ should hold. For the "nose-like" solution this assumption is obviously violated, as we are in the range of $x$, where the restoring characteristic in Figure 1 is crossing zero, i.e. $1 \approx |\varepsilon|x^2$. Another assumption due to equation (7) was, that $\eta \approx 1$. Here we have $\eta$ close to zero. Additionally, for $\eta = 0$ (static case), an exact solution of (2) can be calculated by solving the equation

$$C + \varepsilon C^3 = f. \tag{8}$$

This result is marked by blue dots in Figure 2. Obviously the two upper solutions are unstable. There are some deviations between the amplitudes calculated from this and from the perturbation analysis.

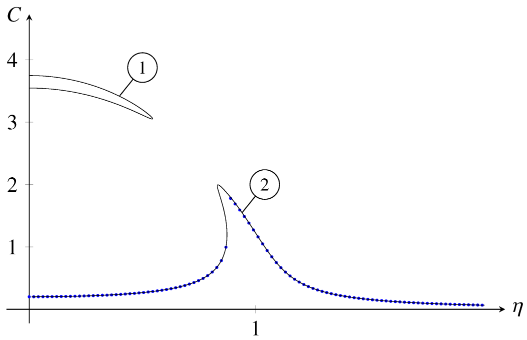


**Fig. 3** Comparison of the result of harmonic balance analysis (equations (4, 5)) with numerical integration (dotted) for parameters $D = 0.06$, $\varepsilon = -0.1$ and $f = 0.2$.

Using numerical integration the "nose-like" solutions couldn't be found as can be seen in Figure 3. Of course this would also happen, if both "nose-like" solutions are unstable. Now, these "nose-like" solutions will be investigated more in detail by adding an error estimation to the harmonic balance method.

## 3 Investigation using refined harmonic balance

We are now again considering the harmonic balance method for the solution of (2). The ansatz (4) is replaced by the truncated Fourier series

$$x(\tau) = \sum_{k=1}^{n} \left(a_k \cos(k\eta\tau) + b_k \sin(k\eta\tau)\right) \ . \tag{9}$$

Introducing this in the Duffing equation (2) and expansion as Fourier series yields

$$\sum_{k=1}^{n} \left(\tilde{a}_k \cos(k\eta\tau) + \tilde{b}_k \sin(k\eta\tau)\right) = f\cos(\eta\tau) - \sum_{k=n+1}^{3n} \left(\tilde{a}_k \cos(k\eta\tau) + \tilde{b}_k \sin(k\eta\tau)\right) \tag{10}$$

where the coefficients $\tilde{a}_k,\ \tilde{b}_k$ are nonlinear functions of the original ansatz coefficients $a_k,\ b_k$. Following harmonic balance, the higher order frequency terms

$$\sum_{k=n+1}^{3n} \left(\tilde{a}_k \cos(k\eta\tau) + \tilde{b}_k \sin(k\eta\tau)\right) \tag{11}$$

in (10) are neglected. and the coefficients $a_k,\ b_k$ are calculated from the thereby modified equation (10). In the following, we will refine harmonic balance in that way, that we use the neglected terms (11) as a measure for the approximation error. These terms can be calculated from the found solution $a_k,\ b_k$. As an error we choose

$$\hat{e} = \max_{0 \le \tau \le \frac{2\pi}{\eta(n+1)}} \left\{ \sum_{k=n+1}^{3n} \left(\tilde{a}_k \cos(k\eta\tau) + \tilde{b}_k \sin(k\eta\tau)\right) \right\} \tag{12}$$

which gives the maximum of the neglected term (11) over one period in time. Restricting the order to $n = 1$ as in the section before, the resulting error $\hat{e}$ is plotted in Figure 4(a). Herein the large errors for small $\eta$ belong to the "nose-like" solution (denoted as solution ① in Figure 2 and afterwards). Compared to this, the error of solution ② in Figure 2 is very small. Enlarging the order to $n = 3$ in the harmonic balance additional solutions occur with large amplitudes for small $\eta$ while the behavior for $\eta \approx 1$ remains qualitatively unchanged. Those new solutions contain also large errors, that are not plotted here. The solutions ① and ② in Figure 2 change slightly for $n = 3$ and the corresponding

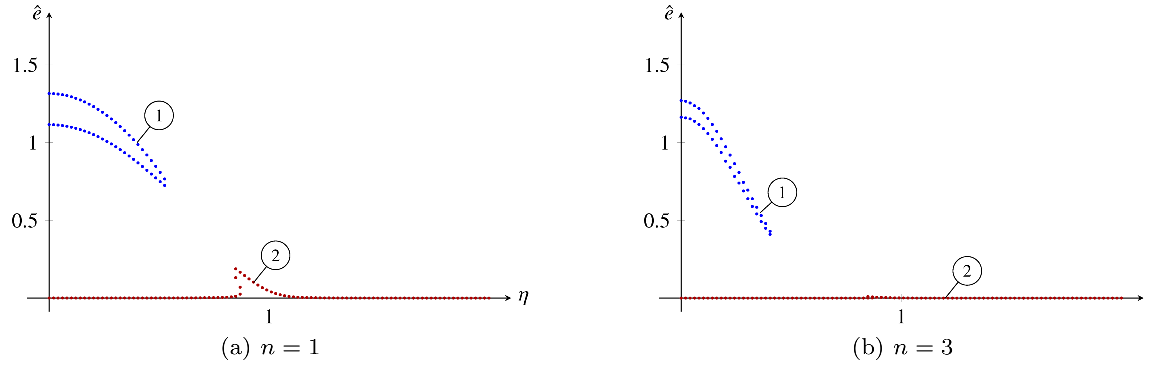


**Fig. 4** Error $\hat{e}$ for different approximation orders $n$, see equation (12), and parameters $D = 0.06$, $\varepsilon = -0.1$ and $f = 0.2$.

errors are plotted in Figure 4(b). It is clearly visible, that the solutions with large amplitudes at small $\eta$ ① still have large errors while the errors at $\eta \approx 1$ for solutions ② have decreased. From these results it can be concluded, that the "nose-like" solutions ① seem to be very inaccurate. But how do potential additional solutions in this frequency range look like?

## 4 Perturbation analysis for small excitation frequencies

We consider again equation (1). In contrast to the transformation used to get equation (2) we now use the circular excitation frequency $\Omega$ for performing the transformation

$$\tau = \Omega t. \tag{13}$$

Considering small excitation frequencies, the frequency ratio $\eta$ can now be considered to be small and is therefore replaced by $\varepsilon = \eta$. With this we get the equation

$$\varepsilon^2 x''(\tau) + \varepsilon\, 2Dx'(\tau) + x(\tau) + \beta x^3(\tau) = f \cos(\tau) \tag{14}$$

with $\beta = \alpha/(m\omega_0^2)$. Please notice, that the inertia term is now of order $\varepsilon^2$ and the damping term of order $\varepsilon$. It is known from the solution of boundary value problems that in such cases singular perturbation analysis has to be applied, see e.g. [13], i.e. the order of all terms has to be reconsidered carefully. In order to check again the dimensions of all terms within our problem in time, it should be mentioned, that the assumption of the inertia term being of order $\varepsilon^2$ and of damping term being of order $\varepsilon$ is only valid, if not only the excitation frequency is small, but also the characteristic frequency of the response. If solutions with displacement amplitudes in the range of the "nose-like" solutions are considered, it can be clearly stated, that the linear and the nonlinear restoring term are not small and of same order. In fact these terms are equal in the equilibrium position. If solutions in the displacement amplitude range of the "nose-like" solutions are considered, the excitation can be considered to be small. Therefore $\hat{f} = f/\varepsilon$ is introduced and equation (14) is rewritten as

$$\varepsilon^2 x''(\tau) + \varepsilon\, 2Dx'(\tau) + x(\tau) + \beta x^3(\tau) = \varepsilon \hat{f} \cos(\tau). \tag{15}$$

As we are outside resonance simple perturbation analysis is sufficient for solving equation (15) by using the ansatz

$$x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \ldots \quad . \tag{16}$$

For the terms of order $\varepsilon^0$ this results in

$$x_0(\tau) + \beta x_0^3(\tau) = 0 \tag{17}$$

and for order $\varepsilon^1$ in

$$x_1(\tau) + 3\beta x_0^2(\tau) x_1(\tau) = -2Dx_0'(\tau) + \hat{f} \cos \eta\tau \quad . \tag{18}$$

The solutions of equation (17) are given by the three equilibrium positions namely

$$\bar{x}_{01} = -\sqrt{-1/\beta},\ \bar{x}_{02} = 0 \text{ and } \bar{x}_{03} = \sqrt{-1/\beta}. \tag{19}$$

Please remember that for the softening Duffing oscillator $\beta$ is negative. Taking into account that $x_0'$ vanishes at any of these three cases, inserting (19) into (18), solving this equation and summarizing the solutions (16), the approximate solutions

$$x_{01}(\tau) = -\sqrt{-\frac{1}{\beta}} - \frac{1}{2} f \cos(\tau),\ x_{02}(\tau) = f \cos(\tau),\ x_{03}(\tau) = \sqrt{-\frac{1}{\beta}} - \frac{1}{2} f \cos(\tau). \tag{20}$$

can be calculated. The main difference compared to the "nose-like" solutions calculated by harmonic balance is, that the solutions $x_{01}$ and $x_{03}$ are vibrations about nonzero (unstable) equilibrium positions, while the "nose-like" solutions are vibrations with zero mean value. Such solutions with non-vanishing mean values are usually expected for quadratic nonlinearities. The third solution is in both cases a small vibration amplitude solution about the (stable) zero equilibrium position. It should be mentioned, that if as a starting point for the perturbation analysis (14) instead of (15) is chosen, the same results (20) are obtained if linearized about equilibrium positions.

With this a-priori knowledge an improved ansatz can be used for the solution of equation (2) with harmonic balance

$$x(\tau) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(k\eta\tau) + b_k \sin(k\eta\tau)\right) \tag{21}$$

by adding a constant $a_0$. Restricting to the case $n = 1$ produces the following solutions to be seen in Figure 5 and the corresponding error

$$\hat{e} = \max_{0 \le \tau \le \frac{2\pi}{3\eta}} \left\{ \tilde{a}_3 \cos(3\eta\tau) + \tilde{b}_3 \sin(3\eta\tau) \right\} \tag{22}$$

also plotted in Figure 5. All results are presented in this Figure with their absolut value, i.e. each branch of the solutions ①, ③ and ④ represents two solutions. The parameters are the same as in the results before. Denoted with ① there are again the "nose-like" solutions with large amplitude and the low amplitude solutions denoted by ②. Both of them have vanishing mean value, i.e. $a_0 = 0$. Then, there are two additional solutions denoted with ③ and ④, both of them with non-vanishing mean value $a_0$. Solutions ③ are the solutions from equation (20) with nonzero mean while solution ④ produces again "nose-like" solutions but now with non-vanishing $a_0$. Whether these solutions are artifacts or not can be seen in the error in Figure 5. From this it can be concluded that the new solutions with nonzero mean from (20) ③ and the zero mean solutions ② have errors tending to zero while the two "nose-like" solutions ① and ④ again have large errors and can therefore be considered as artifacts. This does also not change if these four types of solution are calculated for $n = 3$ as can be seen in Figure 6. The solutions change again only slightly while the error of the non-artifact solutions ② and ③ decreases significantly compared to $n = 1$ while the errors of the artifact solutions ① and ④ remain on a high level.

In fact some authors already investigated solutions of the softening Duffing oscillator including constant terms and additional even instead of only odd superharmonics. As there occur bifurcations of asymmetric solutions such extended solutions can be found in [3] and [4]. Dowell and coauthors [5] also formulated an ansatz containing a constant term which is later intensively used by Tsuda and coauthors [6] in considering regular solutions mainly in resonance conditions. The last paper from 1998 seems to be widely unknown as their are no citations of it in Web of Science within end of May 2015. Therefore also this special type of solutions discussed in this paper seems to be widely unknown, as shows the corresponding presentation of the softening Duffing oscillator in textbooks.

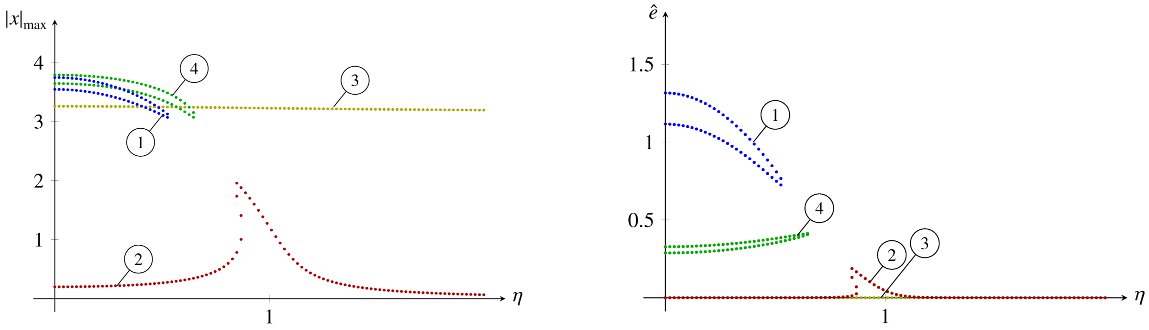


**Fig. 5** Maximum displacements $|x|_{max}$ and errors $\hat{e}$ for different types of solutions with refined harmonic balance according to (21) for $n = 1$

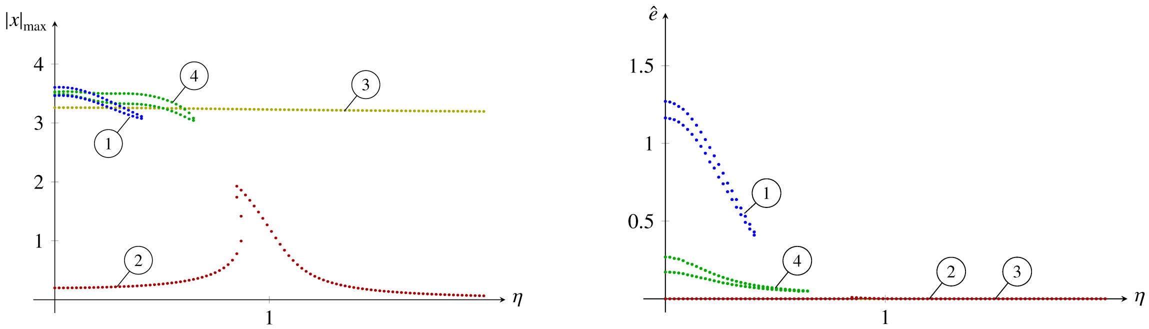


**Fig. 6** Maximum displacements $|x|_{max}$ and errors $\hat{e}$ for different types of solutions with refined harmonic balance according to (21) for $n = 3$

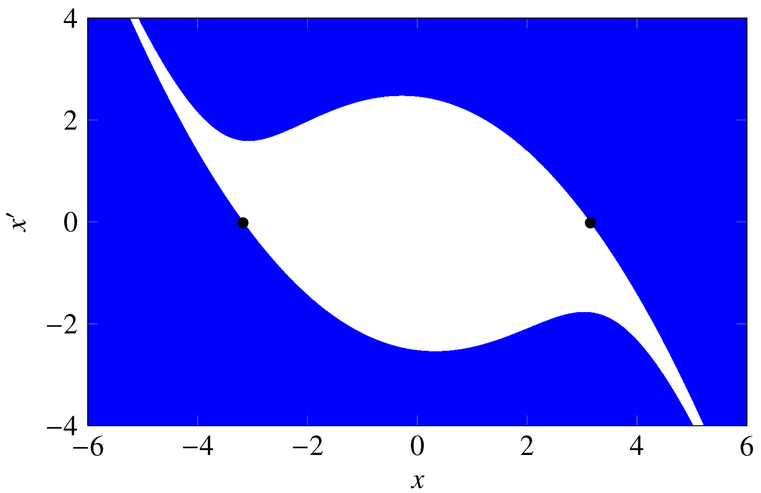


**Fig. 7** Basins of attraction for $\eta = 0.02$. White points correspond to initial conditions at $\tau = 0$ leading to the solution ② in Figures 5 and 6 ($x_{02}$ from equation (20)), blue points denote initial conditions for unbounded solutions drifting to plus or minus infinity. The two black points denote displacement and velocity for $\tau = 0$ of solutions ③ in Figures 5 and 6 ($x_{01}$ and $x_{03}$ from equation (20)).

Finally the stability and basins of attraction shall be discussed for the found solutions. It is well known, that for the solution(s) of type ② in Figures 5 and 6 there exist close to $\eta \approx 1$ three solutions and it is also well known, that the solution with medium amplitude is unstable, while the solutions with small and large amplitude respectively are stable solutions. It can be expected, that the found new solutions ③ in Figures 5 and 6 are unstable, as they represent small forced vibrations around unstable equilibrium positions. The behavior for small $\eta$ shall be discussed shortly in the following with the basins of attraction as plotted in Figure 7 for $\eta = 0.02$ and other parameters chosen as above. Each point in this figure represents an initial condition (amplitude $x$ and velocity $x'$) at $\tau = 0$ and the corresponding color (white or blue) of the point denotes, of which type is the solution resulting from this initial condition for $\tau$ to infinity. These results were calculated by simply applying numerical integration in time for equation (2). White points correspond to initial conditions leading to the solution ② in Figures 5 and 6 ($x_{02}$ from equation (20)) while blue points denote initial solutions for unbounded solutions drifting to plus or minus infinity. There can be found two "tails" of white points, i.e. initial conditions leading to the solution of type ② for large negative displacements and large positive velocities and vice versa. This coincides with intuitive understanding of system's behavior. Solutions of types ① and ④ in Figures 5 and 6 were found to be artifacts in the calculations above and can therefore not be found in Figure 7. Also no initial conditions could be found resulting in the solution of type ③ in Figures 5 and 6 ($x_{01}$ and $x_{03}$ from equation (20)). The two black points in Figure 7 in fact denote the solutions $x_{01}$ and $x_{03}$ for $\tau = 0$. If these solutions would be asymptotically stable these initial conditions would result without any transient behavior in the stationary oscillation and there would be a region of initial conditions around it leading to the same solution. In fact, nothing like this can be observed but the points are located on the borderline between white and blue region, i.e. a small disturbance in initial conditions either result in the solution of type ② (solution $x_{02}$ in equation (20)) or in drifting away. This clearly shows, that the solutions of type ③ (solution $x_{01}$ and $x_{03}$ in equation (20)) are unstable. Corresponding results were also found for other values of $\eta$.

## 5 Conclusions

The Duffing oscillator is probably the most popular example of a nonlinear oscillator in dynamics. Considering the case of softening oscillator with weak damping and harmonic excitation and performing standard methods like harmonic balance or perturbation analysis zero mean solutions with large amplitudes are found for small excitation frequencies. These "nose-like" solutions in the amplitude-frequency diagram merge with the inclining resonance curve for decreasing (but non-vanishing) damping. These "nose-like" solutions are investigated with a refined harmonic balance using the neglected terms as a criterion for the error. From this it can be found, that the "nose-like" solutions have large errors. In order to find the character of non-artificial solutions in this large amplitude range, a perturbation analysis is performed for small excitation frequencies finding solutions with non-vanishing mean value. Based on this the harmonic balance is extended with a constant term and corresponding solutions are calculated. It can again be found that the "nose-like" solutions contain large errors while the newly found solution with non-vanishing mean value has a small error. As those "nose-like" solutions are presented in many textbooks on nonlinear oscillations without any additional comments there seems to be the necessity for broader knowledge about these results.

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