REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 48, Número 1, 2007, Páginas 7–15

NON POSITIVELY CURVED METRIC IN THE SPACE OF POSITIVE DEFINITE INFINITE MATRICES

ESTEBAN ANDRUCHOW AND ALEJANDRO VARELA

Abstract. We introduce a Riemannian metric with non positive curvature in the (infinite dimensional) manifold Σ_∞ of positive invertible operators of a Hilbert space H, which are scalar perturbations of Hilbert-Schmidt operators. The (minimal) geodesics and the geodesic distance are computed. It is shown that this metric, which is complete, generalizes the well known non positive metric for positive definite complex matrices. Moreover, these spaces of finite matrices are naturally imbedded in Σ_∞.

1. Introduction

The space $M_n^+(\mathbb{C})$ of positive definite (invertible) matrices is a differentiable manifold, in fact an open subset of the real euclidean space of hermitian matrices. Let X, Y be hermitian matrices and A positive definite, the formula

$$< X, Y >_A = tr(XA^{-1}YA^{-1})$$

endows $M_n^+(\mathbb{C})$ with a Riemannian metric, which makes it a non positively curved, complete symmetric space. This metric is natural: it is the Riemannian metric obtained by pushing the usual trace norm on matrices to $M_n^+(\mathbb{C})$ by means of the identification

$$\mathcal{G}l(n)/\mathcal{U}(n) \simeq M_n^+(\mathbb{C}),$$

$$G + \mathcal{U}(n) \mapsto G^*G,$$

where $\mathcal{G}l(n)$ and $\mathcal{U}(n)$ are, respectively, the linear and unitary groups of $\mathbb{C}^n$. This example has a universal property: every symmetric space of noncompact type can be realized isometrically as a complete totally geodesic submanifold of $M_n^+(\mathbb{C})$ [7]. These facts are well known, have been used in a variety of contexts, and have motivated several extensions. For example, in the interpolation theory of Banach and Hilbert spaces [6], [16], in partial differential equations [15], and in mathematical physics [12], [17], [9]. They have also been generalized to infinite dimensions, i.e. Hilbert spaces and operator algebras: [17], [3], [2] [5], [1].

2000 *Mathematics Subject Classification.* 58B20, 47B10, 47B15.
Key words and phrases. positive operator, Hilbert-Schmidt class.

The purpose of this note is to introduce a Riemannian metric in the space Σ_∞ of *infinite* positive definite matrices, that is the set of positive and invertible operators on an infinite dimensional Hilbert space H, which are of the form

$$A = \lambda I + A_0 = \lambda + A_0,$$

where $\lambda \in \mathbb{C}$ and $A_0 \in \mathcal{B}_2(H)$, the class of Hilbert-Schmidt operators, i.e. elements $B \in \mathcal{B}(H)$ such that $tr(B^*B) < \infty$. We shall regard Σ_∞ as an infinite dimensional manifold, in fact as an open subset of an apropriate infinite dimensional euclidean space, and introduce a Riemannian metric in Σ_∞, whichs looks formally identical to the metric for finite matrices. It will be shown that with this metric Σ_∞ becomes a non positively curved, complete Riemannian manifold, which contains in a natural (isometric, flat) manner all spaces $M_n^+(\mathbb{C})$.
Therefore Σ_∞ can be regarded as a universal model, containing isometric and totally geodesic copies of all finite dimensional symmetric spaces of non compact type.
Let us fix some notation. We shall denote by $\| \; \|$ the usual norm of $\mathcal{B}(H)$ and by $\| \; \|_2$ the Hilbert-Schmidt norm: $\|B\|_2 = tr(B^*B)^{1/2}$. Denote by

$$\mathcal{H} = \{\lambda + X \in \mathcal{B}(H) : \lambda \in \mathbb{C}, \; X \in \mathcal{B}_2(H)\},$$

and

$$\mathcal{H}_\mathbb{R} = \{\lambda + B \in \mathcal{H} : (\lambda + B)^* = \lambda + B\}.$$

Note that since H is infinite dimensional, the scalars λ and the operators X, B in $\mathcal{B}_2(H)$ are linearly independent. In particular, one has that $\lambda + B \in \mathcal{H}_\mathbb{R}$ if and only if $\lambda \in \mathbb{R}$ and $B^* = B$. Formally, $\mathcal{H} = \mathbb{C} \oplus \mathcal{B}_2(H)$ and $\mathcal{H}_\mathbb{R} = \mathbb{R} \oplus \mathcal{B}_2(H)_h$, where $\mathcal{B}_2(H)_h$ denotes the real Hilbert space of selfadjoint Hilbert-Schmidt operators. Let us define

$$< \lambda + X, \mu + Y >= \lambda\overline{\mu} + tr(Y^*X).$$

Clearly this inner product makes $\mathcal{H}$, $\mathcal{H}_\mathbb{R}$, respectively, complex and real Hilbert spaces, where the scalars λ and the operators (in $\mathcal{B}_2(H)$) are orthogonal. The space $\Sigma_\infty \subset \mathcal{H}_\mathbb{R}$ will be considered with the relative topology induced by this inner product norm. It follows that the maps $\mathcal{H} \to \mathbb{C}$, $\lambda + X \mapsto \lambda$ and $\mathcal{H} \to \mathcal{B}_2(H)$, $\lambda + X \mapsto X$ are orthogonal projections and their adjoints are the inclusions, which are therefore isometric.
Note that $\lambda + A \geq 0$ means that $\lambda \geq 0$ and the spectrum of A is a subset of $[-\lambda, +\infty)$. Indeed, the first assertion follows from the fact that $0 \in \sigma(A)$, and the second is obvious.
In what follows, we denote by $\| \; \|_2$ the norm of $\mathcal{H}$. No confusion should arise with the norm of $\mathcal{B}_2(H)$, because the former extends the latter.

2. BASIC PROPERTIES OF Σ_∞

Let us prove some elementary facts concerning the topology of Σ_∞.

Proposition 2.1. *Σ_∞ is open and convex in $\mathcal{H}_\mathbb{R}$.*

Proof. The fact that Σ_∞ is convex is apparent. Let $X_0 = \lambda_0 + A_0 \in \Sigma_\infty$. Since X_0 is positive and invertible, it follows that the eigenvalues of A_0 are bounded from below by $-\lambda_0$, and do not aproach $-\lambda_0$. Then there exists $r > 0$ such that $-\lambda_0 + r < A_0$, or in other words, $\lambda_0 - r + A_0$ is positive and invertible. Consider the ball

$$\mathcal{D}_{r/2}(X_0) = \{X = \mu + B \in \mathcal{H}_{\mathbb{R}} : \|X - X_0\|_2 < r/2\}.$$

We claim that if $X \in \mathcal{D}_{r/2}(X_0)$, then $X \in \Sigma_\infty$. Indeed, $\|X - X_0\|_2^2 = (\lambda - \lambda_0)^2 + \|B - A_0\|_2^2$. Then $|\lambda - \lambda_0| < r/2$ and the operator norm $\|B - A_0\| \le \|B - A_0\|_2 < r/2$. Then $\lambda > \lambda_0 - r/2$ and $B - A_0 \ge -r/2$, and therefore

$$\lambda + B > \lambda_0 - r/2 + B = \lambda_0 - r/2 + (B - A_0) + A_0 \ge \lambda + A_0 - r,$$

which is positive and invertible. It follows that $X \in \Sigma_\infty$. □

The following elementary estimations will be useful.

Lemma 2.2. *Let $X = \lambda + B, Y = \mu + C \in \mathcal{H}$, then*

(1) $\|X\|_2 \ge \frac{\sqrt{2}}{2}\|X\|$.
(2) $\|XY\|_2 \le 2\|X\|_2\|Y\|_2$

Proof. Let β_n be the singular values of B. Then $\|X\|_2^2 = \lambda^2 + \sum_{n\ge 1}\beta_n^2$. On the other hand, $\|X\| = sup\{|\lambda + \beta_n| : n \ge 1\}$. Since the singular values accumulate eventually only at 0, clearly one has $\|X\| = |\lambda|$ or $\|X\| = |\lambda + \beta_k|$ for some k. In either case

$$2(\lambda^2 + \sum_{n\ge 1}\beta_n^2) \ge 2(\lambda^2 + \beta_k^2) \ge (|\lambda| + |\beta_k|)^2 \ge (\lambda + \beta_k)^2,$$

which proves the first assertion. For the second, $\|(\lambda + B)(\mu + C)\|_2 \le |\lambda||\mu| + |\mu|\|B\|_2 + |\lambda|\|C\|_2 + \|BC\|_2$. Since $B, C \in \mathcal{B}_2(H)$, $\|BC\|_2 \le \|B\|\|C\|_2 \le \|B\|_2\|C\|_2$. Then

$$\|(\lambda + B)(\mu + C)\|_2 \le (|\lambda| + \|B\|_2)(|\mu| + \|C\|_2).$$

By an argument similar to the one given above, $|\lambda| + \|B\|_2 \le \sqrt{2}\|\lambda + B\|_2$, and the second assertion follows. □

If $X, Y \in \mathcal{B}_2(H)$, one has the usual inequalities $\|X\| \le \|X\|_2$ and $\|XY\|_2 \le \|X\|_2\|Y\|_2$. As a consequence of 2.2, one has that the product $(X, Y) \to XY$ is continuous, and therefore smooth, as a map from $\mathcal{H} \times \mathcal{H}$ to $\mathcal{H}$.
Next we show that the inversion map $\sigma : \Sigma_\infty \to \Sigma_\infty$, $\sigma(A) = A^{-1}$ is smooth. The second inequality in 2.2, shows that $\mathcal{H}$ can be renormed in order to become a Banach algebra. Indeed, putting $\|X\|_0 = 2\|X\|_2$, one obtains

$$\|XY\|_0 \le \|X\|_0\|Y\|_0, \quad X, Y \in \mathcal{H}.$$

Corollary 2.3. *The map*

$$\sigma : \Sigma_\infty \to \Sigma_\infty, \quad \sigma(A) = A^{-1}$$

is C^∞.

Proof. The map Σ_∞ is the restriction of the inversion map of the regular group of the Banach algebra $(\mathcal{H}, \| \ \|_0)$, which is an analytic map [14], to the smooth submanifold Σ_∞. □

Note that in fact $A \mapsto A^{-1}$ is real analytic in Σ_∞ (Σ_∞, being open in $\mathcal{H}_\mathbb{R}$, has in fact real analytic structure).
We finish this section establishing certain identities which are satisfied by the inner product of $\mathcal{H}$. Because it is defined in terms of the trace, this inner product inherits certain symmetries. But not others: for example, it is easy to see that if $A \in \mathcal{H}_\mathbb{R}$ (selfadjoint) and $X, Y \in \mathcal{H}$, then $< AX, Y >$ may not be equal to $< X, AY >$.

Lemma 2.4. *Let $X, Y \in \mathcal{H}$ and $A, B \in \mathcal{H}_\mathbb{R}$, then the following hold:*

1. $< AX, YA >=< XA, AY >$.
2. $< AX, YB > + < BX, YA >=< XA, BY > + < XB, AY >$.

Proof. The proof is a simple verification, and is left to the reader. The only issues here are the properties of the trace and the fact that scalars are orthogonal to operators in $\mathcal{B}_2(H)$. □

3. NON POSITIVELY CURVED METRIC ON Σ_∞

For $A \in \Sigma_\infty$, consider the following inner product on $\mathcal{H}_\mathbb{R}$ (regarded as the tangent space $(T\Sigma_\infty)_A$):

$$< X, Y >_A =< A^{-1}X, YA^{-1} > . \tag{3.1}$$

First note that in fact it is a positive definite form, which varies smoothly with A, because the inversion map is smooth. Also note that it looks formally similar to the nonpositively curved metric for the space $M_n^+(\mathbb{C})$ of positive definite *finite* matrices. However there are significant differences. For instance, if H is finite dimensional, clearly Σ_∞ is $M_n^+(\mathbb{C})$ (n = dimension of H), but the inner product defined on $\mathcal{H}$ Is not the same as the trace inner product. An evidence of this is that in general $< AX, Y > \neq < X, AY >$ for $A \in \Sigma_\infty$, $X, Y \in \mathcal{H}$.
Nevertheless, the known formulas for the geodesics and curvature from the finite dimensional case, can be extended in this context. The reason for this is that the covariant derivative has the same formula as in the matrix case.

Proposition 3.1. *The Riemannian connection of the metric defined in (3.1) is given by*

$$\nabla_X Y = X\{Y\} - \frac{1}{2}(XA^{-1}Y + YA^{-1}X), \tag{3.2}$$

where X is a tangent vector at $A \in \Sigma_\infty$, and Y is a vector field. Here $X\{Y\}$ denotes derivation of the field Y in the X direction, performed in the ambient space $\mathcal{H}$.

Proof. The formula (3.2) defines a connection in Σ_∞. It clearly takes values in $\mathcal{H}_\mathbb{R}$, which is the tangent space of Σ_∞ at any point, and also verifies the formal identities of a connection. Also it is apparent that it is a symmetric connection. Therefore, in order to prove that it is the Riemannian connection of the metric from (3.1), it suffices to show that the connection and the metric are compatible.

This amounts to proving that if γ is a smooth curve in Σ_∞ and X, Y are tangent vector fields along γ, then

$$\frac{d}{dt} < X, Y >_\gamma = < \frac{DX}{dt}, Y >_\gamma + < X, \frac{DY}{dt} >_\gamma,$$

where as is usual notation, $\frac{DX}{dt} = \nabla_{\dot\gamma} X$. On one hand, using that $(X\dot{\gamma^{-1}}) = \dot X \gamma^{-1} + X(\dot{\gamma^{-1}})$, and that $(\dot{\gamma^{-1}}) = -\gamma^{-1}\dot\gamma\gamma^{-1}$, one has

$$\frac{d}{dt} < X, Y >_\gamma = < \gamma^{-1}\dot X, Y\gamma^{-1} > - < \gamma^{-1}\dot\gamma\gamma^{-1}X, Y\gamma^{-1} > +$$
$$+ < \gamma^{-1}X, \dot Y\gamma^{-1} > - < X\gamma^{-1}, Y\gamma^{-1}\dot\gamma\gamma^{-1} > . \quad (3.3)$$

On the other hand

$$< \frac{DX}{dt}, Y >_\gamma + < X, \frac{DY}{dt} >_\gamma = < \gamma^{-1}\dot X, Y\gamma^{-1} > - \frac{1}{2} < \gamma^{-1}\dot\gamma\gamma^{-1}X, Y\gamma^{-1} > -$$
$$-\frac{1}{2} < \gamma^{-1}X\gamma^{-1}\dot\gamma, Y\gamma^{-1} > + < \gamma^{-1}X, \dot Y\gamma^{-1} > - \frac{1}{2} < \gamma^{-1}X, \dot\gamma\gamma^{-1}Y\gamma^{-1} > -$$
$$-\frac{1}{2} < \gamma^{-1}X, Y\gamma^{-1}\dot\gamma\gamma^{-1} > . \quad (3.4)$$

In the expression above, one may use the first identity in Lemma 2.4 to replace $< \gamma^{-1}X\gamma^{-1}\dot\gamma, Y\gamma^{-1} >$ by $< X\gamma^{-1}\dot\gamma\gamma^{-1}, \gamma^{-1}Y >$ and $< \gamma^{-1}X, \dot\gamma\gamma^{-1}Y\gamma^{-1} >$ by $< X\gamma^{-1}, \gamma^{-1}\dot\gamma\gamma^{-1}Y >$. Then proving the equality of (3.3) and (3.4) is equivalent to prove that

$$\begin{aligned} < \gamma^{-1}\dot\gamma\gamma^{-1}X, Y\gamma^{-1} > + \quad & < \gamma^{-1}X, Y\gamma^{-1}\dot\gamma\gamma^{-1} > \quad = \\ = \quad & < X\gamma^{-1}\dot\gamma\gamma^{-1}, \gamma^{-1}Y > \quad + < X\gamma^{-1}, \gamma^{-1}\dot\gamma\gamma^{-1}Y > . \end{aligned}$$

This is the same as the second identity in Lemma 2.4, with $A = \gamma^{-1}\dot\gamma\gamma^{-1}$ and $B = \gamma^{-1}$. □

As a consequence, one obtains that the curvature tensor has the same expression as in the matrix case [3] [5]:

$$\mathcal{R}_A(X, Y)Z = -\frac{1}{4} A\big[[A^{-1}X, A^{-1}Y], A^{-1}Z\big], \quad (3.5)$$

for $A \in \Sigma_\infty$, $X, Y, Z \in \mathcal{H}_\mathbb{R}$. Here [,] denotes the usual commutator for operators.

Theorem 3.2. *Σ_∞ has non positive sectional curvature.*

Proof. Compute

$$\begin{aligned} < \mathcal{R}_A(X, Y)Y, X >_A \quad &= \quad -\tfrac{1}{4} < \big[[A^{-1}X, A^{-1}Y], A^{-1}Y\big], XA^{-1} > \\ &= \quad -\tfrac{1}{4}\{< A^{-1}X(A^{-1}Y)^2, XA^{-1} > - \\ & \quad -2 < A^{-1}YA^{-1}XA^{-1}Y, XA^{-1} > + \\ & \quad + < (A^{-1}Y)^2A^{-1}X, XA^{-1} >\}. \end{aligned}$$

Again we may use the same identity from Lemma 2.4 as follows:

$$< A^{-1}X(A^{-1}Y)^2, A^{-1}X > = < A^{-1/2}XA^{-1/2}(A^{-1/2}YA^{-1/2})^2, A^{-1/2}XA^{-1/2} > .$$

The other terms above can be modified likewise. Let us denote $\bar{X} = A^{-1/2}XA^{-1/2}$ and $\bar{Y} = A^{-1/2}YA^{-1/2}$. Then $< \mathcal{R}(X,Y)Y, X >_A$ equals

$$-\frac{1}{4}\{< \bar{X}(\bar{Y})^2, \bar{X} > -2 < \bar{Y}\bar{X}\bar{Y}, \bar{X} > + < (\bar{Y})^2\bar{X}, \bar{X} >\}.$$

Let us compare $< \bar{X}(\bar{Y})^2, \bar{X} >$ and $< \bar{Y}\bar{X}\bar{Y}, \bar{X} >$. Note that $\bar{X}, \bar{Y} \in \mathcal{H}_{\mathbb{R}}$, let $\bar{X} = \lambda + X'$, $\bar{Y} = \mu + Y'$. Then

$$\begin{aligned} < \bar{X}(\bar{Y})^2, \bar{X} > \;&=\; (\lambda^2\mu^2 + 2\lambda\mu\ tr(Y'X') + \lambda\ tr((Y')^2X') + \mu^2\ tr((X'')^2) + \\ &\quad +2\mu\ tr(X''X') + tr((X')^2(Y')^2). \end{aligned}$$

Analogously

$$\begin{aligned} < \bar{Y}\bar{X}\bar{Y}, \bar{X} > \;&=\; \lambda^2\mu^2 + 2\lambda\mu\ tr(Y'X') + \mu^2\ tr((X')^2) + \mu\ tr(X'Y'X') + \\ &\quad +\lambda\ tr((Y')^2X') + \mu\ tr(Y'(X')^2) + tr(Y'X'Y'X'). \end{aligned}$$

After cancellations, in order to compare $< \bar{X}(\bar{Y})^2, \bar{X} >$ and $< \bar{Y}\bar{X}\bar{Y}, \bar{X} >$ it suffices to compare $tr(X'X'Y'Y')$ and $tr(Y'X'Y'X')$. By the Cauchy-Schwarz inequality for the trace, one has

$$\begin{aligned} tr(Y'X'Y'X') \;&= tr\ \ ((X'Y')^*Y'X') \\ &\leq\ \ tr((X'Y')^*X'Y')^{1/2}\ tr((Y'X')^*Y'X')^{1/2} \\ &=\ \ tr(X'X'Y'Y'). \end{aligned}$$

Therefore

$$< \bar{Y}\bar{X}\bar{Y}, \bar{X} > \leq < \bar{X}(\bar{Y})^2, \bar{X} > .$$

Analogously one proves that

$$< \bar{Y}\bar{X}\bar{Y}, \bar{X} > \leq < (\bar{Y})^2\bar{X}, \bar{X} > .$$

It follows that $< \mathcal{R}(X,Y)Y, X >_A \leq 0$. □

Remark 3.3. *As was stated above, the fact that the formula to compute the Riemannian connection looks formally equal for Σ_∞ and for the space of positive definite finite matrices (in fact, also for positive invertible operators of an abstract C^*-algebra* [3], [5]*) implies that one knows the explicit form of the geodesic curves. Let $A, B \in \Sigma_\infty$. Then the curve*

$$\gamma_{A,B}(t) = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2} \tag{3.6}$$

is a geodesic, which is defined for all $t \in \mathbb{R}$, and joins A and B.

Then Σ_∞ is a simply connected (in fact convex) manifiold on non positive sectional curvature. It follows [10], [11] that the geodesic (3.6) is the unique geodesic joining A and B. In fact one has the following:

Corollary 3.4. *The curve given in (3.6) is the unique geodesic joining A and B, and it realizes the geodesic distance. The manifold Σ_∞ is complete with the geodesic distance.*

The geodesic distance of a non positively curved simply connected manifold has also the following property [10]: if γ_1 and γ_2 are two geodesics of Σ_∞, then the map

$$f : \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad f(t) = d(\gamma_1(t), \gamma_2(t))$$

is convex. As in [5], we obtain the following consequence of this fact:

Corollary 3.5. *Let $X, Y \in \mathcal{H}_\mathbb{R}$ and $A = e^X, B = e^Y \in \Sigma_\infty$. Then*

$$\|X - Y\|_2 \leq d(A,B) = \|log(A^{-1/2}BA^{-1/2})\|_2.$$

Proof. The proof follows as in Thm. 3 of [5]. We outline the argument. Let $\gamma_1(t) = e^{tX}$ and $\gamma_2(t) = e^{tY}$. These are geodesics of Σ_∞ which start at I and verify $\gamma_1(1) = A$ and $\gamma_2(1) = B$. The function $f(t) = d(\gamma_1(t), \gamma_2(t))$ is convex, with $f(0) = 0$. Then $f(t)/t \leq f(1)$ for $t \in [0,1]$. Note that $d(e^{tX}, e^{tY}) = \|log(e^{-tX/2}e^{tY}e^{-tX/2})\|_2$. Then

$$\frac{f(t)}{t} = \|\frac{1}{t}\ log(e^{-tX/2}e^{tY}e^{-tX/2})\|_2.$$

If $t > 0$ is small, then $e^{-tX/2}e^{tY}e^{-tX/2}$ is close to I, and therefore one has the usual power series for the logarithm, $log(1+T) = I - T + \frac{1}{2}T^2 - \frac{1}{3}T^3 + \dots$. Using also the power series of the involved exponentials and taking limit $t \to 0$, one obtains

$$d(A,B) = f(1) \geq \lim_{t\to 0} \|\frac{1}{t}\ log(e^{-tX/2}e^{tY}e^{-tX/2})\|_2 = \|Y - X\|_2.$$

□

The map $\sigma : \Sigma_\infty \to \Sigma_\infty$ provides a symmetry for Σ_∞. It is clearly a diffeomorphism with $\sigma^2 = id$. Note that it is isometric. Indeed, if γ is a curve in Σ_∞ with $\gamma(0) = A$ and $\dot{\gamma}(0) = X$, then $(\dot{\sigma(\gamma)}) = -\gamma^{-1}\dot{\gamma}\gamma^{-1}$. Then

$$d\sigma_A(X) = -A^{-1}XA^{-1}.$$

Then

$$< d\sigma_A(X), d\sigma_A(X) >_{\sigma(A)} = < A^{-1}XA^{-1}, A^{-1}XA^{-1} >_{A^{-1}} = < XA^{-1}, A^{-1}X >$$

which equals $< X, X >_A$ by the first identity in Lemma 2.4.

4. THE INMERSION OF $M_n^+(\mathbb{C})$ IN Σ_∞.

Fix a positive integer n and let H_n be an n-dimensional subspace of H. Let P be the orthogonal projection onto H_n, and $\bar{P} = 1 - P$. The space $M_n^+(\mathbb{C})$ of $n \times n$ positive definite (invertible) matrices identifies naturally with the space Σ_{H_n} of positive invertible operators of H_n. We shall consider the manifold Σ_{H_n} with the Riemannian metric

$$< X, Y >_A = tr(XA^{-1}YA^{-1}), \quad A \in \Sigma_{H_n},\ X, Y \in \mathcal{B}(H_n)_h.$$

This metric is well known in differential geometry ([15],[16]), and has been thoroughly studied and generalized to various infinite dimensional contexts ([3], [5]). There is also a natural map from Σ_{H_n} into Σ_∞,

$$i_{H_n} : \Sigma_{H_n} \to \Sigma_\infty, \quad i_{H_n}(A) = A + \bar{P}.$$

Note that i_{H_n} is well defined: A has finite rank and therefore $A+\bar{P}$ is a finite rank perturbation of the indentity.

Proposition 4.1. *The map i_{H_n} is an isometric imbedding.*

Proof. Clearly, it is injective. Let $A \in \Sigma_{H_n}$ and X, Y be hermitian elements of $\mathcal{B}(H_n)$, regarded as tangent vectors of Σ_{H_n} at A. Apparently, $d(i_{H_n})_A(X) = X + 0\bar{P} = X$. In particular, the range of $d(i_{H_n})_A(X)$ is $\mathcal{B}(H_n)_h \oplus 0$ which is complemented in $\mathcal{B}(H)_h$. One has

$$< d(i_{H_n})_A(X), d(i_{H_n})_A(Y) >_{A+\bar{P}} = < (A+\bar{P})^{-1}X, Y(A+\bar{P})^{-1} > .$$

Note that $(A+\bar{P})^{-1} = A^{-1} + \bar{P}$, where A^{-1} denotes the inverse of A in $\mathcal{B}(H_N)$. Also $X\bar{P} = 0$, and then $X(A+\bar{P})^{-1} = XA^{-1}$ is a finite rank operator, in particular, $X(A+\bar{P})^{-1} \in \mathcal{B}_2(H)$. Then

$$< (A+\bar{P})^{-1}X, Y(A+\bar{P})^{-1} > = tr(XA^{-1}YA^{-1}).$$

□

Remark 4.2. *Another implication of the fact that the connections of Σ_{H_n} and Σ_∞ look formally identical, is that the maps i_{H_n} are flat inclusions. The spaces Σ_{H_n} regarded as submanifolds of Σ_∞, are not curved in Σ_∞. In particular these submanifolds are geodesically complete, or in other words, geodesics of Σ_{H_n} are also geodesics of the ambient space Σ_∞. One may fix $\{e_k : k \geq 1\}$ an orthonormal basis for H, and consider H_n the span of $\{e_1, \dots, e_n\}$. Let $i_n = i_{H_n}$. Then via this family of imbeddings, one may think of Σ_∞ as an ambient for all spaces $M_n^+(\mathbb{C})$ of positive definite matrices, of all possible sizes.*

References

[1] Corach, G., Maestripieri, A.L.; *Differential and metrical structure of positive operators.* Positivity 3 (1999), 297–315.

[2] Corach, G., Porta, H., Recht, L.A.; *Splitting of the positive set of a C^*-algebra*, Indag. Mathem., N.S. 2 (4), 461-468.

[3] Corach, G., Porta, H., Recht, L.A.; *The geometry of the space of selfadjoint invertible elements in a C^*-algebra.* Integral Equations Operator Theory 16 (1993), no. 3, 333–359.

[4] Corach, G., Porta, H., Recht, L.A.; *Geodesic and operator means in the space of positive operators*, International J. Math. 4 (1993), 193-202.

[5] Corach, G., Porta, H., Recht, L.A.; *Convexity of the geodesic distance on spaces of positive operators.* Illinois J. Math. 38 (1994), no. 1, 87–94.

[6] Donoghue, W.; *The interpolation of quadratic norms*, Acta Math. 118 (1967), 251-270.

[7] Eberlein, P.; *Structure of manifolds of nonpositive curvature.* Global differential geometry and global analysis 1984, Lecture Notes in Mathematics 1156, Springer, Berlin, 1985.

[8] Kobayashi, M., Nomizu, K.; *Foundations of Differential Geometry*, Interscience, New York, 1969.

[9] Kosaki, H.; *Interpolation theory and the Wigner-Yanase-Dyson-Lieb concavity*, Comm. Math. Phys. 87 (1982/83), 315-329.

[10] Lang, S.; *Differential and Riemannian Manifolds*, Springer, New York, 1995.

[11] McAlpin, J.; *Infinite dimensional manifolds and Morse theory*, Ph. D. thesis, Columbia University, 1965.

[12] Pusz, W., Woronowicz, S.L.; *Functional calculus for sesquilinear forms and the purification map*, Rep. Math. Phys. 8 (1975), 159-170.

[13] Raeburn, I.; *The relationship between a commutative Banach algebra and its maximal ideal space* J. Functional Analysis 25 (1977), no. 4, 366–390.

[14] Rickart, C.E.; *General theory of Banach algebras* , van Nostrand, New York, 1960.

[15] Rochberg, R.; *Interpolation of Banach spaces and negatively curved vector bundles.* Pacific J. Math. 110 (1984), no. 2, 355–376.

[16] Semmes, S.; *Interpolation of Banach spaces, differential geometry and differential equations.* Rev. Mat. Iberoamericana 4 (1988), no. 1, 155–176.

[17] Uhlmann, A.; *Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory*, Comm. Math. Phys. 54 (1977), 21-32.

Esteban Andruchow
Instituto de Ciencias,
Universidad Nacional de Gral. Sarmiento,
J. M. Gutierrez 1150,
(1613),Los Polvorines, Argentina
eandruch@ungs.edu.ar

Alejandro Varela
Instituto de Ciencias,
Universidad Nacional de Gral. Sarmiento,
J. M. Gutierrez 1150,
(1613),Los Polvorines, Argentina
avarela@ungs.edu.ar

Recibido: 19 de mayo de 2005
Aceptado: 30 de noviembre de 2006