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MINIMAL HERMITIAN MATRICES WITH FIXED ENTRIES OUTSIDE THE DIAGONAL

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Dedicated to the memory of Mischa Cotlar. Teacher and friend.

ABSTRACT. We survey some results concerning the problem of finding the complex hermitian matrix or matrices of least supremum norm with variable diagonal. Some qualitative general results are given and more specific descriptions are shown for the 3×3 case. We also comment some results and examples concerning this approximation problem.

CONTENTS

1. NOTATION AND PRELIMINARIES

The problem of finding a complex hermitian matrix of least supremum norm with variable diagonal originally aroused when describing short curves in certain flag manifolds. A finite dimensional flag manifold is one such that its elements are chains of vector spaces included strictly. That is

$$\{0\} \subsetneq V_r \subsetneq V_s ... \subsetneq V_k \subset \mathbb{C}^n$$

where $\dim V_i = i$. The metric considered here is the invariant Finsler metric that will be described further on.

We will be interested in the special case of complete flags, that is when $k = n$.

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The flags could be seen from different points of view. For example if we choose bases for the respective subspaces (that can be taken orthonormal), that is if

$$V_i = [w_1, \cdots, w_i]$$

we can describe the previous chain listing the elements of the base in order (loosing the unicity of the representation).

They can also be described choosing sets of orthonormal projections of rank one which sum gives the identity, that is:

$$\{P_1, P_2, \cdots, P_n\} \text{ such that } \sum_{i=1}^{n} P_i = \mathbb{I}_{\mathbb{C}^n}$$

Yet another way of representing flag manifolds is by the quotient between the group of $n \times n$ unitary complex matrices $\mathcal{U}$ over the subgroup of the unitary diagonals $\mathcal{U}_{\text{diag}}$:

$$\mathcal{U}\,/\mathcal{U}_{\text{diag}} = \left\{ \begin{array}{ll} [v]: & x \in [v] \Leftrightarrow \\ & x = u.v, \text{ with } u \in \mathcal{U}_{\text{diag}} \end{array} \right\}.$$

Therefore the elements of $\mathcal{U}\,/\mathcal{U}_{\text{diag}}$ are the classes $[v]$ with elements of the form

$$d.v = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \cdot \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,n} \end{pmatrix} = \begin{pmatrix} \lambda_1\, v_{1,1} & \lambda_1\, v_{1,2} & \cdots & \lambda_1\, v_{1,n} \\ \lambda_2\, v_{2,1} & \lambda_2\, v_{2,2} & \cdots & \lambda_2\, v_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n\, v_{n,1} & \lambda_n\, v_{n,2} & \cdots & \lambda_n\, v_{n,n} \end{pmatrix}$$

We will use the following notation

$$M_{n\times n}(\mathbb{C}), \quad M^h_{n\times n}(\mathbb{C}), \quad M^{ah}_{n\times n}(\mathbb{C})$$

for the $n \times n$ algebra of matrices of complex coefficients, the hermitian and the anti-hermitian respectively.

We will also denote with

$$\mathcal{D}_{n\times n}, \quad \mathcal{D}^h_{n\times n}, \quad \mathcal{D}^{ah}_{n\times n},$$

the subagebra of the diagonal matrices of $M_{n\times n}(\mathbb{C})$, the hermitian (that is real) and anti-hermitian respectively. Observe that $\mathcal{D}^{ah}_{n\times n} = i\mathcal{D}^h_{n\times n}$.

We will consider the quotient spaces

$$M^h_{n\times n}(\mathbb{C})\,/\mathcal{D}^h_{n\times n}\,, \qquad M^{ah}_{n\times n}(\mathbb{C})\,/\mathcal{D}^{ah}_{n\times n}$$

with the usual quotient norm for $M \in M^h_{n\times n}(\mathbb{C})$:

$$|\,[M]\,|_q = \inf_{D\in\mathcal{D}^h_{n\times n}} ||\,M + D\,||, \tag{1}$$

where $||\cdot||$ is the usual operator (supremum) norm (respectively $M \in M^{ah}_{n\times n}(\mathbb{C})$ and $D \in \mathcal{D}^{ah}_{n\times n}$ for the anti-hermitian case).

The quotient space $M^{ah}_{n\times n}(\mathbb{C})\,/\mathcal{D}^{ah}_{n\times n}$ can be identified with the tangent of the space $\mathcal{U}\,/\mathcal{U}_{\text{diag}}$ in the base point 1, since $\mathcal{U}\,/\mathcal{U}_{\text{diag}}$ is an homogeneous space under the natural action of left multiplication of elements of $\mathcal{U}$:

$$T\,(\mathcal{U}\,/\mathcal{U}_{\text{diag}})_{[1]} = T\,(\mathcal{U})_{[1]}\,/T\,(\mathcal{U}_{\text{diag}})_{[1]} \cong M^{ah}_{n\times n}(\mathbb{C})/\mathcal{D}^{ah}_{n\times n}.$$

Moreover, the Finsler metric considered in $T(\mathcal{U}/\mathcal{U}_{\text{diag}})_{[1]}$ coincides with the quotient norm (1) of $M^{ah}_{n\times n}(\mathbb{C})/\mathcal{D}^{ah}_{n\times n}$.

This will let us link geometric results to the problem we are interested to describe in this article. For instance, in theorem 1 of the next section, the existence of matrices D reaching the minimum of the norm quotient (1) is related with the existence of curves in $\mathcal{U}/\mathcal{U}_{\text{diag}}$ of the shortest posible length. In section 7, an example in 4×4 of infinite different curves of the shortest length joining arbitrarily close points is shown.

Operator approximation problems consist of finding, for a given operator, the element in some special class nearest to it, when distance is measured with a norm. These problems have been treated in the case of hermitian, positive and unitary approximants using different norms in [5], [6], [7], and others. The survey article [8] is related to matrix nearness. There, explicit formulas of operator approximation solutions are presented. Uniqueness results and algorithms for computing or estimating the minimal norm attained are also described, as well as the matrix or matrices sought in different contexts. Nevertheless, in that paper, the operator or supremum norm is not considered.

The problem of finding the minimum of $\|M + D\|$ for a given matrix $M \in M^h_{n\times n}(\mathbb{C})$ among all the diagonal matrices $D \in \mathcal{D}^h_{n\times n}$, and finding the matrix or matrices D that realize the minimum, is indeed an operator approximation problem. It has a trivial translation to the problem of finding a real diagonal matrix D' that satisfies that $M + D' \geq 0$ and that $\|M + D'\|$ is minimum.

In the $n \times n$ case, some bounds of this minimum were obtained in [3]. In that work the calculation of this minimum is related to the estimation of bounds of the norm of the operator $\mathcal{O} : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ that for any $n \times n$ matrix replaces all its diagonal entries by zeroes.

Definition 1. *We will call a matrix $Z \in M^h_{n\times n}(\mathbb{C})$* **minimal hermitian** *if*

$$\|Z\| \leq \|Z + D\|, \text{ for all } D \in \mathcal{D}^h_{n\times n}.$$

or equivalently if

$$\|Z\| = |[Z]|_q.$$

for $[Z] \in M^h_{n\times n}(\mathbb{C})/\mathcal{D}^h_{n\times n}$. In a similar way it can be defined a **minimal antihermitian** *matrix.*

Remark 1. *If $Z \in M^h_{n\times n}(\mathbb{C})$ is a minimal hermitian matrix and*

$$||Z|| = \lambda \ (> 0)$$

then

(1) *$i.Z$ is a minimal antihermitian matrix.*
(2) *$\pm\lambda$ are eigenvalues of Z.*
(3) *The diameter of the spectrum of Z is 2λ.*
(4) *If Z is of 3×3 then $\sigma(Z) = \{\lambda, -\lambda, \mu\}$ with $\mu = Tr(Z)$.*

2. Previous results

In Theorem I of [4], Durán, Mata-Lorenzo and Recht proved a result that, in the context of matrices, can be stated as follows:

Theorem 1. *Let* $[M] \in \mathcal{U}\,/\mathcal{U}_{diag}$ *and* $[X] \in M^{ah}_{n\times n}(\mathbb{C})\,/\mathcal{D}^{ah}_{n\times n} \cong T\left(\mathcal{U}\,/\mathcal{U}_{diag}\right)_{[M]}$. *If there exists a antihermitian minimal matrix* $Z \in M^{ah}_{n\times n}(\mathbb{C})$ *that projects in* $[X]$, *that is,* $[Z] = [X]$ *and* $||Z|| = |[X]|_q$, *then the curve*

$$\gamma(t) = L_{e^{tZ}} \cdot [M] = [e^{tZ}M]$$

has minimum length in the class of all the curves in $\mathcal{U}\,/\mathcal{U}_{diag}$ *that join* $\gamma(0)$ *with* $\gamma(t)$ *for each* t *with* $|t| \le \frac{\pi}{2||Z||}$.

Remark 2. *Note that this theorem implies that, in order to find the curve of minimum length such that* $\dot\gamma(0) = [X]$, *we have to find the matrix* Z *with the same off-diagonal entries as* X *and with a diagonal that makes* Z *a minimal hermitian matrix.*

This result shows the importance of the set of minimal matrices in the study of the shortest curves (in the geometric sense) in these homogeneus spaces.

With the ideas of Section 5 of [4], resctricted to the context of matrices, the following characterization of minimal hermitian matrices can be obtained:

Theorem 2. *A hermitian matrix* $Z \in M^h_{n\times n}$ *is minimal, if and only if, there exists a positive matrix* $P \in M_{n\times n}(\mathbb{C})$ *such that,*

- $P.Z^2 = \lambda^2\, P$, *where* $||Z|| = \lambda$.
- *The diagonal elements of the product* $P.Z$ *are all zero.*

3. Description of minimal hermitian matrices of 3×3

The previous theorem about minimal matrices together with auxiliary results in $\mathbb{C}^3$ (see [1]), allow the following characterization:

Theorem 3. *Let* $Z \in M^h_{3\times 3}(\mathbb{C})$ *with* $||Z|| = \lambda > 0$. *Then* Z *is minimal, if and only if, there exist two unitary eigenvectors,*

(1) v_+ *for the eigenvalue* λ, *and*
(2) v_- *for the eigenvalue* $-\lambda$,

such that their respective coordinates have the same absolute value

$$|(v_+)_i| = |(v_-)_i|, \text{ with } 1 \le i \le 3.$$

Remark 3. *Under these hypotheses, it can be proved that the eigenvectors* v_+ *and* v_- *are triangular, that is,* $|v_1|^2$, $|v_2|^2$ *and* $|v_3|^2$ *can represent the sides of a triangle (the sum of any two of them is always greater than the other one, see figure 1).*

Parametrization of the set of minimal matrices of 3×3

Definition 2. *Let $M \in M^h_{3\times 3}(\mathbb{C})$. We will say that M is of* **extremal** *type if there exist*

(1) $\eta \in [0, 2\pi)$,
(2) $\lambda > 0$,
(3) $\mu \in \mathbb{R}$ *with* $|\mu| \leq \lambda$,

such that M is some of the following three matrices:

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & \lambda\, e^{i\,\eta} \\ 0 & \lambda\, e^{-i\,\eta} & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 0 & \lambda\, e^{-i\,\eta} \\ 0 & \mu & 0 \\ \lambda\, e^{i\,\eta} & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & \lambda\, e^{-i\,\eta} & 0 \\ \lambda\, e^{i\,\eta} & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

Definition 3. *Let $M \in M^h_{3\times 3}(\mathbb{C})$. We will say that M is of* **non extremal** *type if there exist:*

(1) $\eta, \xi \in [0, 2\pi)$,
(2) $\lambda, \mu \in \mathbb{R}$ *with* $\lambda > 0$, $|\mu| \leq \lambda$,
(3) α, β, $\chi \in \mathbb{R}_{\geq 0}$, *with:* $\begin{cases} \alpha + \beta + \chi = \frac{1}{2}, \\ \alpha + \beta > 0, \ \beta + \chi > 0, \\ \alpha + \chi > 0. \end{cases}$

such that

$$M = \mu \begin{pmatrix} 2\alpha & n_{12} & \overline{n_{31}} \\ \overline{n_{12}} & 2\beta & n_{23} \\ n_{31} & \overline{n_{23}} & 2\chi \end{pmatrix} + \lambda \begin{pmatrix} 0 & m_{12} & \overline{m_{31}} \\ \overline{m_{12}} & 0 & m_{23} \\ m_{31} & \overline{m_{23}} & 0 \end{pmatrix}$$

where:

$$\begin{cases} n_{12} = \dfrac{-2\,\alpha\,\beta \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\chi)(\beta+\chi)}}\, e^{-i\eta} \\ n_{31} = \dfrac{-2\,\alpha\,\chi \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\beta+\chi)}}\, e^{-i\xi} \\ n_{23} = \dfrac{-2\,\beta\,\chi \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\alpha+\chi)}}\, e^{-i(\xi-\eta)} \end{cases} \qquad \begin{cases} m_{12} = \dfrac{\chi \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\chi)(\beta+\chi)}}\, e^{-i\eta} \\ m_{31} = \dfrac{\beta \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\beta+\chi)}}\, e^{-i\xi} \\ m_{23} = \dfrac{\alpha \pm i\,\sqrt{2\,\alpha\,\beta\,\chi}}{\sqrt{(\alpha+\beta)(\alpha+\chi)}}\, e^{-i(\xi-\eta)} \end{cases}$$

for one of the two corresponding choices of the signs.

Remark 4. *For matrices of both types (extremals and non extremals), the parameters $\lambda > 0$ and μ give the norm of M, $||M|| = \lambda$, and the trace of M, $Tr(M) = \mu$.*

In the previous Theorem 3 we have seen that $M \in M^h_{3\times 3}(\mathbb{C})$, with $\|M\| = \lambda$ is minimal, if and only if, there exist unitary eigenvalues v_+ and v_- of λ and $-\lambda$ such that their coordinates have equal absolute value.

The following theorem gives a description of all the minimal hermitian matrices of 3×3 in terms of the parameters α, β, χ, η, ξ, λ and μ.

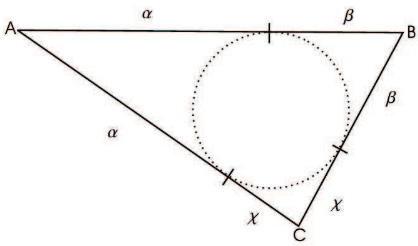


FIGURE 1. Construction of α, β and χ appearing in the parametrization.

Theorem 4. [Parametrization theorem] *Let $Z \in M^h_{3\times 3}(\mathbb{C})$, then, Z is minimal, if and only if, verifies any of the following mutually exclusive cases:*

(1) *The eigenvector v_+ of λ has a zero coordinate and Z is of extremal type.*

(2) *The eigenvector v_+ of λ has no zero coordinates and Z is of non extremal type.*

4. Minimal matrices in a class

Every matrix $M \in M^h_{3\times 3}(\mathbb{C})$ can be writen in the form,

$$M = \begin{pmatrix} a & x & \bar{y} \\ \bar{x} & b & z \\ y & \bar{z} & c \end{pmatrix},$$

where $a\,, b\,, c \in \mathbb{R}$ and $x\,, y\,, z \in \mathbb{C}$.

We already observed that, if M is minimal hermitian (not null), the eigenvalues of M are $\lambda = \|M\|$, $-\lambda$ and $\mu = Tr(M)$ (with $|\mu| \leq \lambda$).

If we consider the characteristic polynomial of M:

$$\det(M - \Lambda \mathbb{I}) = -\Lambda^3 + u\,\Lambda^2 + v\,\Lambda + w$$

these properties impose necessary conditions on the coefficients u, v and w:

$$\begin{cases} u\,v + w &= & 0 \\ v &= & \|M\|^2 \\ u &= & Tr(M) \\ u^2 &\leq & v \end{cases}$$

Then for specific class of a matrix $[M_0]$ (that is for x, y and z fixed)

$$M_0 = \begin{pmatrix} 0 & x & \bar{y} \\ \bar{x} & 0 & z \\ y & \bar{z} & 0 \end{pmatrix},$$

we consider the real manifold of $\mathbb{R}^3$ given by the equation:

$$\Delta := u\,v + w = 0$$

or, in terms of the matrix coefficients a, b and c:

$$\Delta := (a+b)(a+c)(b+c) - (a+b)|x|^2 - (a+c)|y|^2 - (b+c)|z|^2 - 2\,\mathrm{Re}(x\,y\,z) = 0.$$

Let us call with Δ this manifold.

Every minimal matrix M of $[M_0]$ must belong to this manifold and must minimize

$$\lambda^2 := \|M\|^2 = |x|^2 + |y|^2 + |z|^2 - ab - ac - bc$$

with the restriction $u^2 \leq v$.

To simplify the expression of the map Δ, the following linear change of variables can be introduced,

$$a = (r+s-t)/2\,, \qquad b = (t+r-s)/2\,, \qquad c = (s+t-r)/2.$$

The equations above change to give a new description of Δ

$$\Delta := r\,s\,t - r\,|x|^2 - s\,|y|^2 - t\,|z|^2 + 2\,\mathrm{Re}(x\,y\,z) = 0,$$

and a new expression for the function to minimize λ^2,

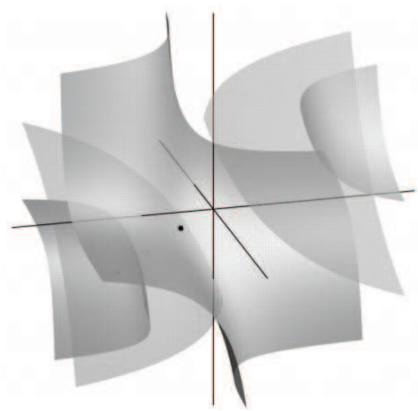
$$\lambda^2(r,s,t) = \frac{1}{4}(r^2+s^2+t^2) - \frac{1}{2}(r\,s + r\,t + s\,t) + |x|^2 + |y|^2 + |z|^2.$$

5. Different cases

To find the minimal matrix (or matrices) in the class $[M_0]$ we have to minimize λ^2 on Δ. We shall consider four cases depending on the triple (x, y, z). Figures representing Δ using the r, s and t variables are shown in each case. It can be proved that only in the fourth case there might be multiple minima in the given class $[M_0]$. Two rounded surfaces, shown in the first three figures, do not belong to Δ, they represent the bounding surfaces $\mu = \pm\lambda$ in between which the (unique) minimum is located.

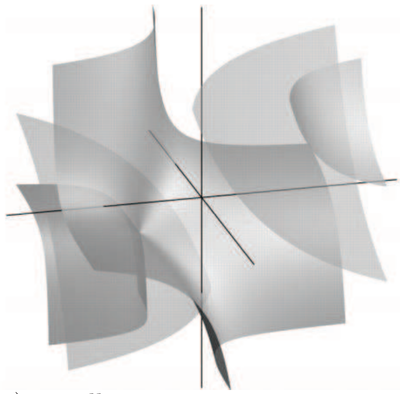
(1) When $\mathrm{Im}(x\,y\,z) \neq 0$.

In this case the surface Δ is regular (a smooth manifold) and the method of Lagrange multipliers can be used to find the unique minimum in the class. In the figure to the right, the middle portion represents the component satisfying $u^2 \leq v$, and the dark point indicates the minimum.



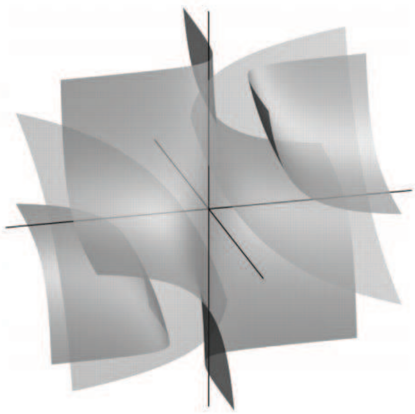
(2) When $\mathrm{Im}(x\,y\,z) = 0$ and $\mathrm{Re}(x\,y\,z) \neq 0$.

In this case the surface Δ is not regular, has one singular point which is the unique minimum in the class. In the figure to the right, two components of Δ touch at the singular point which is the minimum.



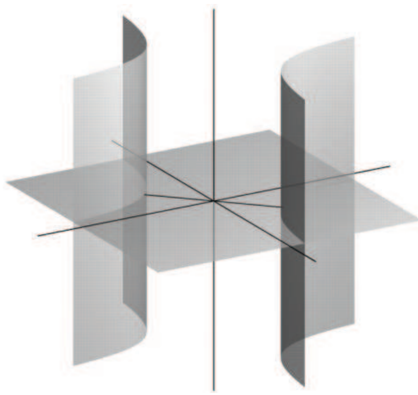
(3) When exactly one coordinate of (x, y, z) is null.

In this case the surface Δ is regular; the class has a unique minimum at the origin, $(r, s, t) = (0, 0, 0) = (a, b, c)$. Observe that in the figure the vertical t-axis lies in Δ.



(4) When exactly two coordinates of (x, y, z) are null.

In this case the surface Δ is not regular along two curves, the two branches of the hyperbola shown in the figure, and the class has multiple minima, represented by the segment shown in the figure joining the two branches of the hyperbola.



Remark 1. *Despite this last case, there is no multiplicity of minimal curves of matrices in* 3×3 *and all multiple minimal matrices* Z *produce the same curve* $\gamma(t) = [e^{tZ} M]$ *for* $M \in \mathcal{U} / \mathcal{U}_{diag}$.

If the matrix M is real with zero diagonal and we suppose that there is a diagonal D that reaches the minimum of $||M+D||$ and such that $M+D$ has all of its eigenvalues of equal absolute value (that is, the spectrum of $M+D$ is $\{\pm\lambda\}$ and λ or $-\lambda$ has double multiplicity), then a precise formula of D can be found in terms of the entries of M (this proposition is proved in [10] motivated by results of [9]). We state it here as the following remark:

Remark 2. *Let* $M = \begin{pmatrix} 0 & c & a \\ c & 0 & b \\ a & b & 0 \end{pmatrix}$, *with* $a, b, c \neq 0$, *be a matrix in* $\mathbb{R}^{3\times 3}$ *and* $D \in \mathbb{R}^{3\times 3}$ *be a diagonal such that* $||M+D||$ *is minimum and the eigenvalues of* $M+D$ *have equal absolute value. Then* D *must be of the form*

$$D = \begin{pmatrix} -\frac{1}{2}(\frac{bc}{a} + \frac{ab}{c} - \frac{ac}{b}) & 0 & 0 \\ 0 & -\frac{1}{2}(-\frac{bc}{a} + \frac{ab}{c} + \frac{ac}{b}) & 0 \\ 0 & 0 & -\frac{1}{2}(\frac{bc}{a} - \frac{ab}{c} + \frac{ac}{b}) \end{pmatrix}.$$

6. The topology of the set of minimal matrices of 3×3

Using the theorem of parametrization and considering the sets

$$\Sigma = \{\ (\alpha, \beta, \chi) \in \mathbb{R}^3 : \alpha + \beta + \chi = \frac{1}{2}, \alpha \geq 0,\ \beta \geq 0,\ \chi \geq 0\ \},$$

and

$$C = \left\{(\mu\,, \lambda) \in \mathbb{R}^2 : \lambda > 0,\ |\mu| \leq \lambda\right\},$$

we define

$$W = \Sigma \times S^1 \times S^1 \times C,$$

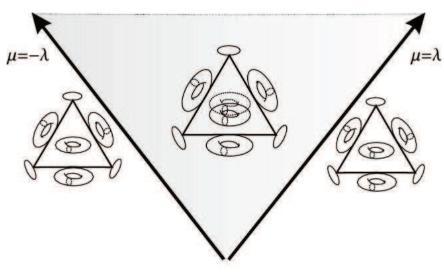
where S^1 is the unitary circle in the complex plane and

$$\nabla = W_+ \sqcup W_-,$$

is the disjoint union of two copies of W.

Let us consider in ∇ the smallest equivalent relation '$\sim$' that identifies the elements of ∇ that give the same matrix in the theorem of parametrization. Then, it can be proved that the set $\nabla/\sim$ is homemophic to the set of minimal hermitian matrices.

The following is a representation of the set of minimal hermitian matrices through $\nabla/\sim$.



7. An example in 4×4 using minimal matrices

We have remarked that in 3×3, despite of the existence of matrices that allow infinite minimizing diagonals, there are no multiplicity of short curves. This is not the case in 4×4.

Let us consider the manifold

$$S \subset \mathcal{P}(4) = \mathcal{U}_{4\times 4} \Big/ (\mathcal{U}_{\text{diag}})_{4\times 4}$$

defined by

$$\begin{aligned} S &= (SU(2) \times SU(2)) \Big/ (SU(2) \times SU(2))_{\text{diag}} \\ &\cong \left(SU(2) \Big/ SU(2)_{\text{diag}} \right) \times \left(SU(2) \Big/ SU(2)_{\text{diag}} \right) \end{aligned}$$

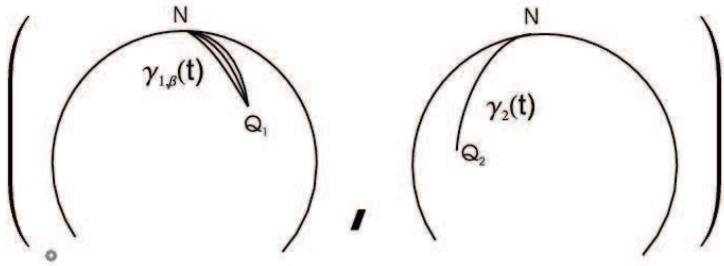
where $SU(2)$ is the special unitary group (unitary complex matrices with determinant equal to 1) and with the Finsler metric in $\mathcal{P}(4)$ as in the 3×3 case.

Using a suitable description (see [2] and [11] for details) we can consider $S \cong S^2 \times S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$. This gives a better geometrical view of the example. Let us outline which are the properties of the curves mentioned at the beginning of the section.

Let $\mathcal{N} = (N, N) \in S^2 \times S^2$ be the point whose coordinates are both the North Pole, $N \in S^2$. Let $\mathcal{Q} = (Q_1, Q_2) \in S^2 \times S^2$ be any point such that Q_1 has higher latitude than Q_2 in S^2 (Q_1 is closer to N than Q_2).

> *$\mathcal{Q}$ is going to be fixed so that Q_2 is above the equator line (and Q_1 is even higher). In [2], using the characterizations of minimal matrices seen previously, a family of minimal curves $\Gamma_\beta(t) = (\gamma_{1,\beta}(t), \gamma_2(t))$, for $t \in [0, 1]$, was constructed, all joining $\mathcal{N}$ to $\mathcal{Q}$, with the following properties.*
>
> - *The curve $\gamma_2(t)$ in S^2 will trace the smaller arc of the great circle that contains N and Q_2.*
> - *The family of curves $\gamma_{1,\beta}(t)$ will vary continuously with the parameter β.*
> - *Each of the curves of the family $\gamma_{1,\beta}(t)$ will parameterize the smaller arc of some circle in S^2 that joins N to Q_1; the arcs will not be great circles but for $\beta = 0$.*
> - *The curve $\gamma_2(t)$ runs over a great circle in S^2.*
> - *The curve $\gamma_{1,\beta}(t)$ varies continuously with the parameter β.*
> - *The curve $\gamma_{1,\beta}(t)$ has constant speed in S^2.*
> - *The curve $\gamma_2(t)$ has constant speed in S^2.*

The following is a representation of these curves in $S^2 \times S^2$:



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