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FREE Q-DISTRIBUTIVE LATTICE OVER AN n-ELEMENT CHAIN

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Abstract: In this note we provide an explicit construction of $FQ(\mathbf{n})$, the free Q-distributive lattice over an n-element chain, different from those given by Cignoli [4] and Abad–Díaz Varela [1], and prove that $FQ(\mathbf{n})$ can be endowed with a structure of a De Morgan algebra.

1 – Preliminaries

Quantifiers on distributive lattices were considered for the first time by Servi in [14], but it was Cignoli [3] who studied them as algebras, which he named Q-distributive lattices. In [4], Cignoli gave a construction of the free Q-distributive lattice over a set X. This construction generalizes that given by Halmos [6] for the free monadic Boolean algebra. The present paper is motivated by a result of Abad and Díaz Varela [1] on the free Q-distributive lattice, $FQ(I)$, over a finite ordered set I. They proved that $FQ(\mathbf{n}) \cong \mathbf{2}^{[\mathbf{2}^{[\mathbf{2}\times\mathbf{n}]}]}$, where $\mathbf{n}$ is an n-element chain. We provide an easy construction for the diagram of $\mathbf{2}^{[\mathbf{2}\times\mathbf{n}]}$, and we prove some properties of $FQ(\mathbf{n})$.

We include in this section some notation, definitions and results on distributive lattices which are used in the paper.

An element p of a lattice R is said to be join-irreducible if $p \neq 0$ and $p = x \vee y$ implies $p = x$ or $p = y$, for all $x, y \in R$. We denote the set of join-irreducible elements of R by $\mathcal{J}(R)$.

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The following result was proved by Antonio Monteiro (see [8], [12]) in his lectures held at the Universidad Nacional del Sur.

Lemma 1.1. *If R is a finite distributive lattice and $p \in R\backslash\{0\}$, then $p \in \mathcal{J}(R)$ if and only if there is a greatest element in the set $I(p) = \{x \in R:\ x < p\}$.* ∎

For $n \in \mathbb{N}$, we write $\mathbf{n}$ to denote the chain obtained by giving the n-element set $\{1, 2, ..., n\}$ the order in which $1 < 2 < \cdots < n$.

Given an ordered set X, we denote the dual of X by X^* and the bounded distributive lattice of all order-preserving functions from X into $\mathbf{2}$ by $\mathbf{2}^{[X]}$ (see [7]).

A subset Y of X is said to be a *down-set*, or *order ideal* if, whenever $b \in Y$, $a \in X$ and $a \leq b$, we have $a \in Y$. Given $x \in X$, it is clear that the set $(x] = \{y \in X:\ y \leq x\}$ is an order ideal; it is known as the principal order ideal generated by x. The notions of an *upper-set*, or *order filter* and of $[x)$ are dually defined.

If X is a finite ordered set, the lattice $\mathbf{2}^{[X]}$ is anti-isomorphic to the distributive lattice $\mathcal{O}(X)$ of all down-sets of X, and $\mathcal{J}(\mathcal{O}(X)) = \{(x]:\ x \in X\}$.

Lemma 1.2 (G. Birkhoff [2])**.** *For every finite ordered set X, $\mathcal{J}(\mathbf{2}^{[X]}) \cong X^*$, and conversely, for every finite distributive lattice R, $R \cong \mathbf{2}^{[\mathcal{J}(R)^*]}$.* ∎

We use the following notations: the cardinality of a finite set X is denoted by $N[X]$, and the $(0,1)$-sublattice generated by a subset X of a lattice R is denoted by $SL(X)$. $FD(G)$ denotes the free bounded distributive lattice generated by G. An ordered set X is called *self-dual* if X and X^* are isomorphic ordered sets.

The following result can be found in B. Jónsson [7].

Lemma 1.3. $FD(\mathbf{2} \times \mathbf{n}) \cong \mathbf{2}^{\left[\mathbf{2}^{[\mathbf{2} \times \mathbf{n}]}\right]}$. ∎

2 – Free Q-distributive lattice over an n-element chain

The aim of this section is to give a construction of the free Q-distributive lattice $FQ(\mathbf{n})$ over an n-element chain. A Q-distributive lattice is an algebra $\langle A, \wedge, \vee, 0, 1, \nabla\rangle$ of type $(2, 2, 0, 0, 1)$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and ∇ is an existential quantifier, that is, $\nabla 0 = 0$, $x \leq \nabla x$, $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$, $\nabla(x \vee y) = \nabla x \vee \nabla y$.

We introduce an ordered set $V(n)$ isomorphic to $\mathbf{2}^{[\mathbf{2}\times\mathbf{n}]}$, so that

$$\mathcal{O}(V(n)) \cong \mathbf{2}^{\left[\mathbf{2}^{[\mathbf{2}\times\mathbf{n}]}\right]} ,$$

and define an operator ∇ on $\mathcal{O}(V(n))$, in such a way that $\langle \mathcal{O}(V(n)), \nabla \rangle$ is isomorphic to the free Q-distributive lattice over an n-element chain.

Let $V(n) = \{(x, y) \in (\mathbf{n}+\mathbf{1}) \times (\mathbf{n}+\mathbf{1})\colon x \le y\}$. It is clear that $V(n)$ is a $(0,1)$-sublattice of $(\mathbf{n}+\mathbf{1}) \times (\mathbf{n}+\mathbf{1})$. Observe that $V(n)$ is self-dual and

$$N[V(n)] = \frac{(n+1)\,(n+2)}{2} .$$

Now we want to characterize the set $\mathcal{J}(V(n))$.

For every j, $2 \le j \le n+1$, $I((1,j)) = ((1, j-1)]$. So, from Lemma 1.1, we have that $(1,j) \in \mathcal{J}(V(n))$. If $i \ge 2$, then $I((i,i)) = ((i-1, i)]$. So, again from Lemma 1.1, $(i,i) \in \mathcal{J}(V(n))$. Since $(i,j) = (i,i) \vee (1,j)$, it follows that

$$\mathcal{J}(V(n)) \;=\; \bigcup_{j=2}^{n+1} \{(1,j)\} \,\cup\, \bigcup_{i=2}^{n+1} \{(i,i)\} .$$

It is clear that the ordered sets $\mathcal{J}(V(n))$ and $\mathbf{2}\times\mathbf{n}$ are isomorphic, and consequently the distributive lattices $V(n)$ and $\mathbf{2}^{[\mathbf{2}\times\mathbf{n}]}$ are isomorphic. From Lemma 1.3,

$$FD(\mathbf{2}\times\mathbf{n}) \;\cong\; \mathbf{2}^{[V(n)]} .$$

Let us see now that $N\left[\mathbf{2}^{[V(n)]}\right] = 2^{n+1}$, for every $n \in \mathbb{N}$. In the proof we make use of the following result of L. Monteiro [13]:

Lemma 2.1. *If X is a finite ordered set and f is an element of X which is neither first nor last element of X, then*

$$N\left[\mathbf{2}^{[X]}\right] = N\left[\mathbf{2}^{[X\setminus(f]]}\right] + N\left[\mathbf{2}^{[X\setminus[f)]}\right] . \blacksquare$$

To prove that $N\left[\mathbf{2}^{[V(n)]}\right] = 2^{n+1}$ we proceed by induction. For $n=1$, $N\left[\mathbf{2}^{[V(1)]}\right] = N\left[\mathbf{2}^{[V(1)\setminus((1,2)]]}\right] + N\left[\mathbf{2}^{[V(1)\setminus[(1,2))]}\right] = 2+2 = 2^{1+1}$. Suppose that $N\left[\mathbf{2}^{[V(n-1)]}\right] = 2^n$, for $n \ge 2$. Then

$$N\left[\mathbf{2}^{[V(n)]}\right] = N\left[\mathbf{2}^{[V(n)\setminus((1,n+1)]]}\right] + N\left[\mathbf{2}^{[V(n)\setminus[(1,n+1))]}\right] .$$

But $V(n) \setminus ((1, n+1)]$ and $V(n) \setminus [(1, n+1))$ are ordered sets isomorphic to $V(n-1)$. So $N\left[\mathbf{2}^{[V(n)]}\right] = 2 \cdot N\left[\mathbf{2}^{[V(n-1)]}\right] = 2 \cdot 2^n = 2^{n+1}$.

Let $m\colon V(n) \to V(n)$ be the map defined by $m(i,j) = (i, n+1)$. m is a lattice homomorphism that satisfies $x \le m(x)$ and $m(m(x)) = m(x)$.

Now we define the operator ∇ on $\mathcal{O}(V(n))$. For $X \in \mathcal{O}(V(n))$, let

$$\nabla X = \begin{cases} \emptyset, & \text{if } X = \emptyset \\ \bigcup\limits_{x\in X} (m(x)], & \text{if } X \in \mathcal{O}(V(n)) \setminus \{\emptyset\} \ . \end{cases}$$

Let us prove that $\langle \mathcal{O}(V(n)), \nabla\rangle$ is a Q-distributive lattice.

By the definition, $\nabla\emptyset = \emptyset$.

Since $x \le m(x)$, it follows that $\nabla X = \bigcup\limits_{x\in X} (m(x)] \supseteq \bigcup\limits_{x\in X} (x] = X$.

$$\nabla(X \cup Y) = \bigcup_{z\in X\cup Y} (m(z)] = \bigcup_{z\in X} (m(z)] \cup \bigcup_{z\in Y} (m(z)] = \nabla X \cup \nabla Y \ .$$

Finally, let us see that $\nabla(X \cap \nabla Y) = \nabla X \cap \nabla Y$. If $t \in \nabla(X \cap \nabla Y)$ then $t \le m(w)$, $w \in X \cap \nabla Y$. As $w \in X$, then $t \in \nabla X$. Since $w \in \nabla Y$, we have that $w \le m(y)$, for some $y \in Y$, and consequently, $t \le m(w) \le m(m(y)) = m(y)$. Thus $t \in \nabla Y$. Conversely, if $t \in \nabla X \cap \nabla Y$, then $t \le m(x)$, $x \in X$ and $t \le m(y)$, $y \in Y$. Since m is a homomorphism, $t \le m(x\wedge y)$, where $x\wedge y \in X\cap Y \subseteq X\cap\nabla Y$, that is, $t \in \nabla(X \cap \nabla Y)$.

Theorem 2.1. *$\langle \mathcal{O}(V(n)), \nabla\rangle$ is the free Q-distributive lattice over* $\mathbf{n}$.

Proof: First we prove that $\langle \mathcal{O}(V(n)), \nabla\rangle$ is generated by the chain

$$G = \Big\{g_i = ((i,i)]\colon\ 1 \le i \le n\Big\} \ .$$

Observe that

$$\nabla g_j = \nabla((j,j)] = \bigcup_{x\in((j,j)]} (m(x)] = (m(j,j)] = ((j, n+1)]\,, \qquad 1 \le j \le n \ .$$

So

$$G \cup \nabla G = \Big\{((i,i)]\colon\ 1 \le i \le n\Big\} \cup \Big\{((i,n+1)]\colon\ 1 \le i \le n\Big\} \ .$$

If $(i,j) \in V(n)$, then $(i,j) = (j,j) \wedge (i,n+1)$. Hence, if $j \ne n+1$,

$$((i,j)] = ((j,j)] \cap ((i,n+1)] = g_j \cap \nabla g_i \ .$$

That is, every principal down-set of $V(n)$ different from $V(n)$ is a meet of elements in $G \cup \nabla G$. Since every non-empty element of $\mathcal{O}(V(n))$ is a union of principal down-sets of $V(n)$, we have that every element of $\mathcal{O}(V(n))$ different from $\emptyset$ and $V(n)$ is a union of meets of elements of $G \cup \nabla G$, that is,

$$SL(G \cup \nabla G) = \mathcal{O}(V(n)) \ .$$

So it is clear that $\mathcal{O}(V(n))$ is generated by G as a Q-distributive lattice.

In order to prove that $\mathcal{O}(V(n))$ is the free Q-distributive lattice over $\mathbf{n}$, let $f\colon \mathbf{n} \to \mathcal{O}(V(n))$ be the mapping defined by $f(i) = g_i$, $1 \le i \le n$. f can be extended to a Q-homomorphism $\overline{f}\colon FQ(\mathbf{n}) \to \mathcal{O}(V(n))$. Now, $\overline{f}(FQ(\mathbf{n}))$ is a Q-sublattice of $\mathcal{O}(V(n))$ such that $G \subseteq \overline{f}(FQ(n))$ and since $\mathcal{O}(V(n))$ is the Q-distributive lattice generated by G, we have $\overline{f}(FQ(\mathbf{n})) = \mathcal{O}(V(n))$. So $\overline{f}$ is a Q-epimorphism. Since the sets $FQ(\mathbf{n})$ and $\mathcal{O}(V(n))$ are finite and they have the same cardinality, $\overline{f}$ is also injective. Consequently, $\overline{f}$ is an isomorphism. ∎

3 – A structure of De Morgan algebra on $FQ(\mathbf{n})$

The aim of this section is to define a De Morgan negation on $\mathcal{O}(V(n))$ in order to establish a relationship between $\mathcal{O}(V(n))$ and $\mathcal{O}(V(n-1))$.

Since $V(n)$ is self-dual, it is possible to define a De Morgan operation on $\mathcal{O}(V(n))$. Indeed, let $\varphi\colon V(n) \to V(n)$ be defined by

$$\varphi(x,y) \;=\; (n+2-y,\, n+2-x) \ .$$

It is clear that φ is an anti-isomorphism of period 2. So we can define on $\mathcal{O}(V(n))$ a De Morgan operation $\sim$ associated to φ in the usual way [9, 10, 11]: if $Y \in \mathcal{O}(V(n))$ then

$$\sim Y \;=\; \bigcup \Big\{ (\varphi(p)]\colon\, p \in V(n),\ p \notin Y \Big\} \ .$$

Let us see that

$$\sim g_j = \nabla g_{n+1-j}, \qquad \text{for } \ 1 \le j \le n \ . \tag{3.1}$$

First observe that $\varphi((p]) = [\varphi(p))$ and $\varphi([p)) = (\varphi(p)]$ for every $p \in V(n)$. Thus

$$\begin{aligned}
\sim g_j \;&= \sim ((j,j)] \\
&= \bigcup \Big\{ (\varphi(p)]\colon\, p \in V(n),\ p \notin ((j,j)] \Big\} \\
&= \bigcup \Big\{ \varphi([p))\colon\, p \in V(n),\ p \notin ((j,j)] \Big\} \\
&= \varphi\Big(\bigcup \Big\{ [p)\colon\, p \in V(n),\ p \notin ((j,j)] \Big\} \Big) \\
&= \varphi([(1,j+1))) \;=\; (\varphi(1,j+1)] \;=\; ((n+1-j, n+1)] \;=\; \nabla g_{n+1-j} \ .
\end{aligned}$$

Theorem 3.1. *$\langle \mathcal{O}(V(n)), \sim \rangle$ is a Kleene algebra.*

Proof: If A is a De Morgan algebra and $X \subseteq A$, let $SM(X)$ denote the De Morgan subalgebra of A generated by X. It is known that $SM(X) = SL(X \cup \sim X)$. So, as $G = \{g_1, g_2, ..., g_n\} \subseteq \mathcal{O}(V(n))$, by (3.1) we have that $SM(G) = SL(G \cup \nabla G) = \mathcal{O}(V(n))$. Hence the De Morgan algebra $\mathcal{O}(V(n))$ is generated by an n-element chain, and consequently it is a homomorphic image of the free De Morgan algebra $\mathcal{M}$ over an n-element chain. $\mathcal{M}$ is a Kleene algebra [5], so $\langle \mathcal{O}(V(n)), \sim \rangle$ is a Kleene algebra. ∎

Now we are going to prove that:

Theorem 3.2. *For $n \geq 2$, the down-set $(g_n]$ and the upper-set $[\nabla g_1)$ of $\mathcal{O}(V(n))$ are ordered sets isomorphic to $\mathcal{O}(V(n-1))$.*

Proof: Since $\sim$ is an order anti-isomorphism from $\mathcal{O}(V(n))$ on $\mathcal{O}(V(n))$, and

$$X \in (g_n] \quad \text{iff} \quad X \subseteq g_n \quad \text{iff} \quad \nabla g_1 = \sim g_n \subseteq \sim X \ ,$$

then $\sim$ induces an anti-isomorphism from $(g_n]$ on $[\nabla g_1)$, that is, $(g_n] \cong [\nabla g_1)^*$.

Now observe that $\mathcal{J}((g_n]) = \mathcal{J}(\mathcal{O}(V(n))) \cap (g_n]$, and this is clearly isomorphic to $V(n-1)$. Then $(g_n] \cong \mathcal{O}(V(n-1))$. In particular, $(g_n] \cong (g_n]^*$, thus

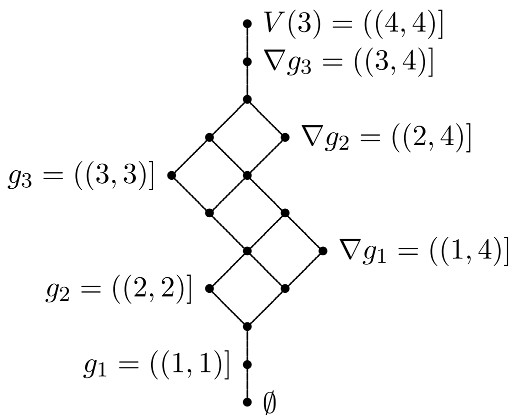
$$[\nabla g_1) \cong \mathcal{O}(V(n-1)) \cong (g_n] \ .$$

Observe that $\mathcal{O}(V(n))$ is the disjoint union of $[\nabla g_1)$ and $(g_n]$. Indeed, if $X \not\subseteq g_n = ((n,n)]$, there exists $(x_1, x_2) \in X$ such that $(x_1, x_2) \notin ((n,n)]$, that is, $(x_1, x_2) \not\leq (n,n)$. Then $(1, n+1) \leq (x_1, x_2)$, and consequently, $\nabla g_1 = ((1, n+1)] \subseteq X$. So $X \in [\nabla g_1)$. If $X \in [\nabla g_1) \cap (g_n]$ we have that $\nabla g_1 = ((1, n+1)] \subseteq X$. In particular, $(1, n+1) \in X$ and $X \subseteq g_n = ((n,n)]$. Thus $(1, n+1) \in ((n,n)]$, a contradiction. ∎

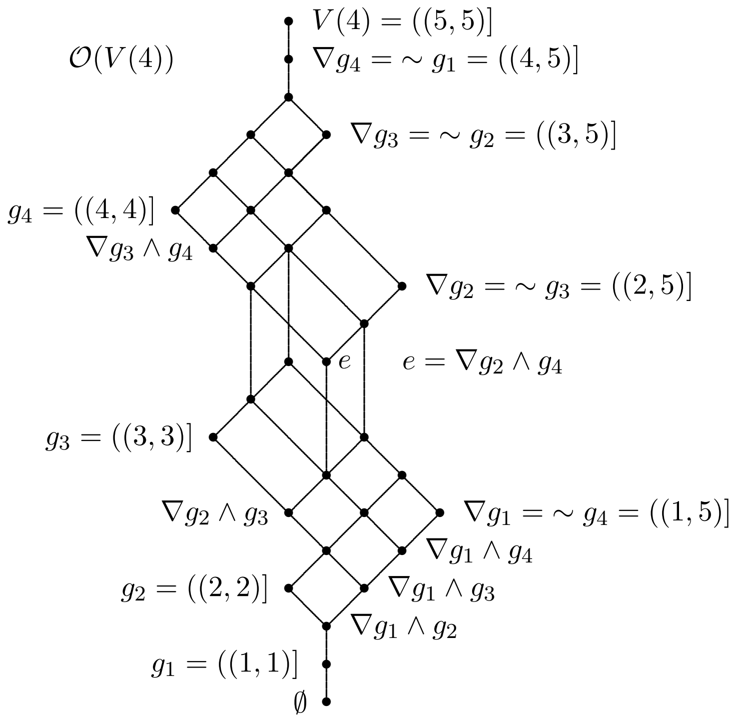
In particular, $N[\mathcal{O}(V(n))] = 2 \cdot N[\mathcal{O}(V(n-1))]$. This fact can be used to get another proof of the result $N[\mathcal{O}(V(n))] = 2^{n+1}$.

This situation is illustrated in the following figure.

$\mathcal{O}(V(3))$



$\mathcal{O}(V(4))$



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