Duality in reconstruction systems *†

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Abstract

We consider the notion of finite dimensional reconstructions systems (RS's), which includes the fusion frames as projective RS's. We study erasures, some geometrical properties of these spaces, the spectral picture of the set of all dual systems of a fixed RS, the spectral picture of the set of RS operators for the projective systems with fixed weights and the structure of the minimizers of the joint potential in this setting. We give several examples.

Contents

1 Introduction

In this paper we study the notion of finite dimensional reconstruction systems, which gives a new framework for fusion and vector frames. Fusion frames (briefly FF's) were introduced under the name of "frame of subspaces" in [12]. They arise naturally as a generalization of

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the usual frames of vectors for a Hilbert space $\mathcal{H}$. Several applications of FF's have been studied, for example, sensor networks [15], neurology [26], coding theory [6], [7] , [22], among others. We refer the reader to [14] and the references therein for a detailed treatment of the FF theory. Further developments can be found in [9], [13] and [27].

Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \dots, m\} \subseteq \mathbb{N}$. In the finite dimensional setting, a FF is a sequence $\mathcal{N}_w = (w_i \, , \mathcal{N}_i)_{i \in \mathbb{I}_m}$ where each $w_i \in \mathbb{R}_{>0}$ and the $\mathcal{N}_i \subseteq \mathbb{C}^d$ are subspaces that generate $\mathbb{C}^d$. The synthesis operator of $\mathcal{N}_w$ is usually defined as

$$T_{\mathcal{N}_w} : \mathcal{K}_{\mathcal{N}_w} \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{I}_m} \mathcal{N}_i \to \mathbb{C}^d \qquad \text{given by} \quad T_{\mathcal{N}_w} (x_i)_{i \in \mathbb{I}_m} = \sum_{i \in \mathbb{I}_m} w_i \, x_i \ .$$

Its adjoint, the so-called analysis operator of $\mathcal{N}_w$, is given by $T^*_{\mathcal{N}_w} y = (w_i \, P_{\mathcal{N}_i} \, y)_{i \in \mathbb{I}_m}$ for $y \in \mathbb{C}^d$, where $P_{\mathcal{N}_i}$ denotes the orthogonal projection onto $\mathcal{N}_i$. The frame $\mathcal{N}_w$ induces a linear encoding-decoding scheme that can be described in terms of these operators.

The previous setting for the theory of FF's presents some technical difficulties. For example the domain of $T_{\mathcal{N}_w}$ relies strongly on the subspaces of the fusion frame. In particular, any change on the subspaces modifies the domain of the operators preventing smooth perturbations of these objects. Moreover, this kind of rigidity on the definitions implies that the notion of a dual FF is not clear.

An alternative approach to the fusion frame (FF) theory comes from the theory of protocols introduced in [6] and the theory of reconstruction systems considered in [25] and [23]. In this context, we fix the dimensions $\dim \mathcal{N}_i = k_i$ and consider a universal space

$$\mathcal{K} = \mathcal{K}_{m \, , \mathbf{k}} \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{I}_m} \mathbb{C}^{k_i} \ , \qquad \text{where} \quad \mathbf{k} = (k_1 \, , \, \dots \, , \, k_m) \in \mathbb{N}^m \ .$$

A reconstruction system (RS) is a sequence $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m}$ such that $V_i \in L(\mathbb{C}^d \, , \, \mathbb{C}^{k_i})$ for every $i \in \mathbb{I}_m$, which allows the construction of an encoding-decoding algorithm (see Definition 2.1 for details). We denote by $\mathcal{RS} = \mathcal{RS}(m, \mathbf{k}, d)$ the set of all RS's with these fixed parameters. Observe that, if $\mathcal{N}_w = (w_i \, , \mathcal{N}_i)_{i \in \mathbb{I}_m}$ is a FF, it can be modeled as a system $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m} \in \mathcal{RS}$ such that $V_i^* V_i = w_i^2 \, P_{\mathcal{N}_i}$ for every $i \in \mathbb{I}_m$. These systems are called projective RS's.

On the other hand, a general RS arise from a usual vector frame by grouping together the elements of the frame. Thus, the coefficients involved in the encoding-decoding scheme of RS are vector valued, and they lie in the space $\mathcal{K}$.

The main advantage of the RS framework with respect to the fusion frame formalism is that each (projective) RS has many RS's that are *dual* systems. In particular, the canonical dual RS remains being a RS (for details and definitions see Section 2). In contrast, it is easy to give examples of a FF such that its canonical dual is not a fusion frame. There exists a notion of duality among fusion frames defined by Gavruta (see [19]), where the reconstruction formula of a fixed $\mathcal{V}$ involves the FF operator $S_{\mathcal{V}}$ of $\mathcal{V}$. Nevertheless, in the context of RS's, we show that the notion of dual systems can be described and characterized in a quite natural way. On the other hand, the RS framework (see Section 2 for a detailed description) allows to make not only a metric but also a differential geometric study of the set of RS's, which will be developed in Section 4 of this paper.

Let us fix the parameters $(m, \mathbf{k}, d)$ and the sequence $\mathbf{v} = (v_i)_{i \in \mathbb{I}_m} \in \mathbb{R}^m_{>0}$ of weights. In this work we study some properties of the sets $\mathcal{RS} = \mathcal{RS}(m, \mathbf{k}, d)$ of RS's and $\mathcal{PRS}_{\mathbf{v}} = \mathcal{PRS}_{\mathbf{v}} \, (m, \mathbf{k}, d)$ of projective systems with fixed weights $\mathbf{v}$. In section 3 we study erasures in this context. We show conditions which guarantee that, after erasing some of its components, the system keeps being a RS, and we exhibit adequate bounds for it. In section 4 we present a geometrical description of both sets $\mathcal{RS}$ and $\mathcal{PRS}_{\mathbf{v}}$, and give a sufficient condition (the notion of irreducible systems) in order that the operation of taking RS operators $\mathcal{PRS}_{\mathbf{v}} \ni \mathcal{V} \mapsto S_{\mathcal{V}}$ (see Definition 2.1) has smooth local cross sections. In section 5 we study the spectral picture of the set $\mathcal{D}(\mathcal{V})$ of all dual systems for a fixed $\mathcal{V} \in \mathcal{RS}$, and the set $\mathcal{OP}_{\mathbf{v}}$ of the RS operators of all systems in $\mathcal{PRS}_{\mathbf{v}}$.

Finally, in section 6 we focus on the main problem of the paper, which needs the results of the previous sections: Let $\mathcal{DP}_{\mathbf{v}} \stackrel{\text{def}}{=} \big\{ (\mathcal{V}, \mathcal{W}) \in \mathcal{PRS}_{\mathbf{v}} \times \mathcal{RS} \ : \ \mathcal{W} \in \mathcal{D}(\mathcal{V}) \big\}$. We look for pairs $(\mathcal{V}\,, \mathcal{W})$ which have the best minimality properties. If there exist tight systems in $\mathcal{PRS}_{\mathbf{v}}$ (systems whose RS operator is a multiple of the identity) then the pair $(\mathcal{V}\,, \mathcal{V}^{\#})$ is minimal, where $\mathcal{V}^{\#}$ is the canonical dual of $\mathcal{V}$ (see Defintion 2.3). Nevertheless, this is not always the case (see [10] or [25]). Therefore we define a joint RS potential given by $\mathcal{DP}_{\mathbf{v}} \ni (\mathcal{V}\,, \mathcal{W}) \mapsto \text{RSP}\,(\mathcal{V}\,, \mathcal{W}) = \text{tr } S_{\mathcal{V}}^2 + \text{tr } S_{\mathcal{W}}^2 \in \mathbb{R}_{>0}$, which is similar to the potential used in [11] for vector frames. The minimizers of RSP are those pairs which are the best analogue of a tight pair. The main results in this direction are that:

- There exist $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \mathbf{k}, d) \in \mathbb{R}^d_{>0}$ such that a pair $(\mathcal{V}\,, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is a minimizer for the RSP if and only if $\mathcal{W} = \mathcal{V}^{\#}$ and the vector of eigenvalues $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$.
- Every such $\mathcal{V}$ can be decomposed as a orthogonal sum of tight projective RS's, where the quantity of components and their tight constants are the same for every minimizer.

In section 7 we give some examples of these problems, showing sets of parameters for which the vector $\lambda_{\mathbf{v}}$ and all minimizers $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ can be explicitly computed. We also present a conjecture which suggest an easy way to compute the vector $\lambda_{\mathbf{v}}$, as the minimal element in the spectral picture of $\mathcal{OP}_{\mathbf{v}}$ with respect to the majorization (see Conjecture 7.4).

General notations.

Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \dots, m\} \subseteq \mathbb{N}$ and $\mathbb{1} = \mathbb{1}_m \in \mathbb{R}^m$ denotes the vector with all its entries equal to 1. For a vector $x \in \mathbb{R}^m$ we denote by $x^{\downarrow}$ the rearrangement of x in a decreasing order, and $(\mathbb{R}^m)^{\downarrow} = \{x \in \mathbb{R}^m : x = x^{\downarrow}\}$ the set of ordered vectors.

Given $\mathcal{H} \cong \mathbb{C}^d$ and $\mathcal{K} \cong \mathbb{C}^n$, we denote by $L(\mathcal{H}, \mathcal{K})$ the space of linear operators $T : \mathcal{H} \to \mathcal{K}$. Given an operator $T \in L(\mathcal{H}, \mathcal{K})$, $R(T) \subseteq \mathcal{K}$ denotes the image of T, $\ker T \subseteq \mathcal{H}$ the null space of T and $T^* \in L(\mathcal{K}, \mathcal{H})$ the adjoint of T. If $d \le n$ we say that $U \in L(\mathcal{H}, \mathcal{K})$ is an isometry if $U^*U = I_{\mathcal{H}}$. In this case, U^* is called a coisometry. If $\mathcal{K} = \mathcal{H}$ we denote by $L(\mathcal{H}) = L(\mathcal{H}\,, \mathcal{H})$, by $\mathcal{G}l(\mathcal{H})$ the group of all invertible operators in $L(\mathcal{H})$, by $L(\mathcal{H})^+$ the cone of positive operators and by $\mathcal{G}l(\mathcal{H})^+ = \mathcal{G}l(\mathcal{H}) \cap L(\mathcal{H})^+$. If $T \in L(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, by $\text{rk}\, T$ the rank of T, and by $\text{tr}\, T$ the trace of T. By fixing orthonormal basis (onb) of the Hilbert spaces involved, we shall identify operators with matrices, using the following notations:

By $\mathcal{M}_{n,d}(\mathbb{C}) \cong L(\mathbb{C}^d\,, \mathbb{C}^n)$ we denote the space of complex $n \times d$ matrices. If $n = d$ we write $\mathcal{M}_n(\mathbb{C}) = \mathcal{M}_{n,n}(\mathbb{C})$. $\mathcal{H}(n)$ is the $\mathbb{R}$-subspace of selfadjoint matrices, $\mathcal{G}l(n)$ the group of all invertible elements of $\mathcal{M}_n(\mathbb{C})$, $\mathcal{U}(n)$ the group of unitary matrices, $\mathcal{M}_n(\mathbb{C})^+$ the set of positive semidefinite matrices, and $\mathcal{G}l(n)^+ = \mathcal{M}_n(\mathbb{C})^+ \cap \mathcal{G}l(n)$. If $d \le n$, we denote by $\mathcal{I}(d\,, n) \subseteq \mathcal{M}_{n\,,\,d}(\mathbb{C})$ the set of isometries, i.e. those $U \in \mathcal{M}_{n\,,\,d}(\mathbb{C})$ such that $U^*U = I_d$.

If $W \subseteq \mathcal{H}$ is a subspace we denote by $P_W \in L(\mathcal{H})^+$ the orthogonal projection onto W, i.e. $R(P_W) = W$ and $\ker\, P_W = W^{\perp}$. For vectors on $\mathbb{C}^n$ we shall use the euclidean norm. On the other hand, for matrices $T \in \mathcal{M}_n(\mathbb{C})$ we shall use both

1. The spectral norm $\|T\| = \|T\|_{sp} = \max\limits_{\|x\|=1} \|Tx\|$.

2. The Frobenius norm $\|T\|_2 = (\text{tr } T^*T)^{1/2} = \Big(\sum\limits_{i,j\in \mathbb{I}_n} |T_{ij}|^2 \Big)^{1/2}$. This norm is induced by the inner product $\langle A,\ B\rangle = \text{tr } B^*A$, for $A, B \in \mathcal{M}_n(\mathbb{C})$.

2 Basic framework of reconstruction systems

In what follows we consider $(m, \mathbf{k}, d)$-reconstruction systems, which are more general linear systems than those considered in [4], [6], [7], [8], [20] and [24], that also have an associated reconstruction algorithm.

Definition 2.1. Let $m, d \in \mathbb{N}$ and $\mathbf{k} = (k_1\,,\,\dots\,,\,k_m) \in \mathbb{N}^m$.

1. We shall abbreviate the above description by saying that $(m, \mathbf{k}, d)$ is a set of parameters. We denote by $n = \text{tr } \mathbf{k} \stackrel{\text{def}}{=} \sum_{i\in \mathbb{I}_m} k_i$ and assume that $n \geq d$.

2. We denote by $\mathcal{K} = \mathcal{K}_{m\,,\,\mathbf{k}} \stackrel{\text{def}}{=} \bigoplus_{i\in \mathbb{I}_m} \mathbb{C}^{k_i} \cong \mathbb{C}^n$. We shall often write each direct summand by $\mathcal{K}_i = \mathbb{C}^{k_i}$.

3. Given a space $\mathcal{H} \cong \mathbb{C}^d$ we denote by
$$L(m, \mathbf{k}, d) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} L(\mathcal{H}\,,\,\mathcal{K}_i) \cong L(\mathcal{H}, \mathcal{K}) \cong \bigoplus_{i\in\,\mathbb{I}_m} \mathcal{M}_{k_i\,,\,d}(\mathbb{C}) \cong \mathcal{M}_{n,d}(\mathbb{C})\ .$$
A typical element of $L(m, \mathbf{k}, d)$ is a system $\mathcal{V} = \{V_i\}_{i\in\,\mathbb{I}_m}$ such that each $V_i \in L(\mathcal{H}\,,\,\mathcal{K}_i)$.

4. A family $\mathcal{V} = \{V_i\}_{i\in\,\mathbb{I}_m} \in L(m, \mathbf{k}, d)$ is an $(m, \mathbf{k}, d)$-reconstruction system (RS) for $\mathcal{H}$ if
$$S_{\mathcal{V}} \stackrel{\text{def}}{=} \sum_{i\in\,\mathbb{I}_m} V_i^* V_i \in \mathcal{G}l(\mathcal{H})^+\,, \tag{1}$$
i.e., if $S_{\mathcal{V}}$ is invertible. $S_{\mathcal{V}}$ is called the **RS operator** of $\mathcal{V}$. In this case, the m-tuple $\mathbf{k} = (k_1\,,\,\dots\,,\,k_m) \in \mathbb{N}^m$ satisfies that $n = \text{tr}\,\mathbf{k} \geq d$.
We shall denote by $\mathcal{RS} = \mathcal{RS}(m, \mathbf{k}, d)$ the set of all $(m, \mathbf{k}, d)$-RS's for $\mathcal{H} \cong \mathbb{C}^d$.

5. The system $\mathcal{V}$ is said to be **projective** if there exists a sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}_+^m$ of positive numbers, the weights of $\mathcal{V}$, such that
$$V_i\,V_i^* = v_i^2\,I_{\mathcal{K}_i}\ , \qquad \text{for every} \qquad i \in \mathbb{I}_m\ .$$
In this case, the following properties hold:
   (a) The weights can be computed directly, since each $v_i = \|V_i\|_{sp}$.
   (b) Each $V_i = v_i U_i$ for a coisometry $U_i \in L(\mathcal{H}\,,\,\mathcal{K}_i)$. Thus $V_i^*V_i = v_i^2\,P_{R(V_i^*)} \in L(\mathcal{H})^+$ for every $i \in \mathbb{I}_m$.
   (c) $S_{\mathcal{V}} = \sum_{i\in\,\mathbb{I}_m} v_i^2\,P_{R(V_i^*)}$ as in fusion frame theory.

   We shall denote by $\mathcal{PRS} = \mathcal{PRS}(m, \mathbf{k}, d)$ the set of all projective elements of $\mathcal{RS}$.

6. The **analysis** operator of the system $\mathcal{V}$ is defined by

$$T_{\mathcal{V}} \ : \ \mathcal{H} \to \mathcal{K} = \bigoplus_{i\in \mathbb{I}_m} \mathcal{K}_i \ \text{ given by } \ T_{\mathcal{V}}\, x = (V_1\, x \, , \, \dots \, , \, V_m\, x) \ , \quad \text{for} \quad x \in \mathcal{H} \ .$$

7. Its adjoint $T_{\mathcal{V}}^*$ is called the **synthesis** operator of the system $\mathcal{V}$, and it satisfies that

$$T_{\mathcal{V}}^* : \mathcal{K} = \bigoplus_{i\in \mathbb{I}_m} \mathcal{K}_i \to \mathcal{H} \quad \text{is given by} \quad T_{\mathcal{V}}^* \big((y_i)_{i\in \mathbb{I}_m} \big) = \sum_{i\in \mathbb{I}_m} V_i^*\, y_i \ .$$

   Using the previous notations and definitions we have that $S_{\mathcal{V}} = T_{\mathcal{V}}^*\, T_{\mathcal{V}}$.

8. The frame constants in this context are the following: $\mathcal{V}$ is a RS if and only if

$$A_{\mathcal{V}}\, \|x\|^2 \ \le \ \langle S_{\mathcal{V}}\, x \, , \, x\rangle \ = \sum_{i\in\mathbb{I}_m} \ \|V_i\, x\|^2 \ \le \ B_{\mathcal{V}}\, \|x\|^2 \tag{2}$$

   for every $x \in \mathcal{H}$, where $0 < A_{\mathcal{V}} = \lambda_{\min}(S_{\mathcal{V}}) = \|S_{\mathcal{V}}^{-1}\|^{-1} \le \lambda_{\max}(S_{\mathcal{V}}) = \|S_{\mathcal{V}}\| = B_{\mathcal{V}}$.

9. As usual, we say that $\mathcal{V}$ is **tight** if $A_{\mathcal{V}} = B_{\mathcal{V}}$. In other words, the system $\mathcal{V} \in \mathcal{RS}(m, \mathbf{k}, d)$ is tight if and only if $S_{\mathcal{V}} = \frac{\tau}{d}\, I_{\mathcal{H}}$, where $\tau = \sum_{i\in\mathbb{I}_m} v_i^2\, k_i$.

10. The Gram matrix of $\mathcal{V}$ is $G_{\mathcal{V}} = T_{\mathcal{V}}\, T_{\mathcal{V}}^* \in L(\mathcal{K})^+ \cong \mathcal{M}_n(\mathbb{C})^+$, where the size of $G_{\mathcal{V}}$ viewed as a matrix is $n = \operatorname{tr}\, \mathbf{k} = \sum_{i\in\mathbb{I}_m} \ k_i = \dim \mathcal{K}$.

11. Given $U \in \mathcal{G}l(d)$, we define $\mathcal{V} \cdot U \ \stackrel{\text{def}}{=} \ \{V_i\, U\}_{i\in \mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$. △

Remark 2.2. Let $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}$ such that every $V_i \neq 0$. In case that $\mathbf{k} = \mathbb{1}_m$, then $\mathcal{V}$ can be identified with a vector frame, since each $V_i : \mathbb{C}^d \to \mathbb{C}$ is in fact a vector $0 \neq f_i \in \mathbb{C}^d$. In the same manner, the projective RS's can be seen as fusion frames. Here the identification is given by $V_i \simeq \big(\|V_i\| \, , \, R(V_i^*) \big)$ for every $i \in \mathbb{I}_m$. △

Definition 2.3. For every $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$, we define the system

$$\mathcal{V}^{\#} \ \stackrel{\text{def}}{=} \ \mathcal{V} \cdot S_{\mathcal{V}}^{-1} = \{V_i\, S_{\mathcal{V}}^{-1}\}_{i\in \mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d) \ ,$$

called the **canonical dual** RS associated to $\mathcal{V}$. △

Remark 2.4. Given $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}$ with $S_{\mathcal{V}} = \sum_{i\in \mathbb{I}_m} V_i^* V_i$, then

$$\sum_{i\in \mathbb{I}_m} S_{\mathcal{V}}{}^{-1}\, V_i^* V_i = I_{\mathcal{H}} \ , \quad \text{and} \quad \sum_{i\in \mathbb{I}_m} V_i^* V_i \ S_{\mathcal{V}}{}^{-1} = I_{\mathcal{H}} \ . \tag{3}$$

Therefore, we obtain the reconstruction formulas

$$x = \sum_{i\in \mathbb{I}_m} S_{\mathcal{V}}^{-1}\, V_i^*\, (V_i\, x) = \sum_{i\in \mathbb{I}_m} V_i^*\, V_i (S_{\mathcal{V}}^{-1}\, x) \quad \text{for every} \quad x \in \mathcal{H} \ . \tag{4}$$

Observe that, by Eq. (3), we see that the canonical dual $\mathcal{V}^{\#}$ satisfies that

$$T_{\mathcal{V}^{\#}}^*\, T_{\mathcal{V}} = \sum_{i\in \mathbb{I}_m} S_{\mathcal{V}}{}^{-1}\, V_i^* V_i = I_{\mathcal{H}} \quad \text{and} \quad S_{\mathcal{V}^{\#}} = \sum_{i\in \mathbb{I}_m} S_{\mathcal{V}}^{-1}\, V_i^*\, V_i\, S_{\mathcal{V}}^{-1} = S_{\mathcal{V}}^{-1} \ . \tag{5}$$

Next we generalize the notion of dual RS's: △

Definition 2.5. Let $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m}$ and $\mathcal{W} = \{W_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}$. We say that $\mathcal{W}$ is a **dual** RS for $\mathcal{V}$ if $T_{\mathcal{W}}^* T_{\mathcal{V}} = I_{\mathcal{H}}$, or equivalently if $x = \sum_{i\in \mathbb{I}_m} W_i^* V_i\, x$ for every $x \in \mathcal{H}$.

We denote the set of all dual RS's for a fixed $\mathcal{V} \in \mathcal{RS}$ by $\mathcal{D}(\mathcal{V}) \stackrel{\text{def}}{=} \{\mathcal{W} \in \mathcal{RS} : T_{\mathcal{W}}^* T_{\mathcal{V}} = I_{\mathcal{H}}\}$. Observe that $\mathcal{D}(\mathcal{V}) \neq \emptyset$ since $\mathcal{V}^\# \in \mathcal{D}(\mathcal{V})$. △

Remark 2.6. Let $\mathcal{V} \in L(m, \mathbf{k}, d)$. Then $\mathcal{V} \in \mathcal{RS} \iff T_{\mathcal{V}}^*$ is surjective. In this case, a system $\mathcal{W} \in \mathcal{D}(\mathcal{V})$ if and only if its synthesis operator $T_{\mathcal{W}}^*$ is a pseudo-inverse of $T_{\mathcal{V}}$. Indeed, $\mathcal{W} \in \mathcal{D}(\mathcal{V}) \iff T_{\mathcal{W}}^* T_{\mathcal{V}} = I_{\mathcal{H}}$. Observe that the map $\mathcal{RS} \ni \mathcal{W} \mapsto T_{\mathcal{W}}^*$ is one to one. Thus, in the context of RS's each $(m, \mathbf{k}, d)$-RS has many duals that are $(m, \mathbf{k}, d)$-RS's. This is one of the advantages of the RS's setting.

Moreover, the synthesis operator $T_{\mathcal{V}^\#}^*$ of the canonical dual $\mathcal{V}^\#$ corresponds to the Moore-Penrose pseudo-inverse of $T_{\mathcal{V}}$. Indeed, notice that $T_{\mathcal{V}}\, T_{\mathcal{V}^\#}^* = T_{\mathcal{V}}\, S_{\mathcal{V}}^{-1} T_{\mathcal{V}}^* \in L(\mathcal{K})^+$, so that it is an orthogonal projection. From this point of view, the canonical dual $\mathcal{V}^\#$ has some optimal properties that come from the theory of pseudo-inverses.

On the other hand the map $L(m, \mathbf{k}, d) \ni \mathcal{W} \mapsto T_{\mathcal{W}}^* \in L(\mathcal{K}, \mathcal{H})$ is $\mathbb{R}$-linear. Then, for every $\mathcal{V} \in \mathcal{RS}$, the set $\mathcal{D}(\mathcal{V})$ of dual systems is convex in $L(m, \mathbf{k}, d)$, because the set of pseudoinverses of $T_{\mathcal{V}}$ is convex in $L(\mathcal{K}, \mathcal{H})$. △

3 Erasures and lower bounds.

It is a known result in frame theory that, for a given frame $\mathcal{F} = \{f_i\}_{i\in I}$, the set $\mathcal{F}' = \{f_i\}_{i\in I,\, i\neq j}$ is either a frame or a incomplete set for $\mathcal{H}$. In [13] P. Casazza and G. Kutyniok give examples where this situation does not occur in the fusion frame setting. Considering fusion frames as a particular case of reconstruction systems we can rephrase their result in the following way:

Theorem 3.1 (Casazza and Kutyniok)**.** Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}$ with bounds $A_{\mathcal{V}}$, $B_{\mathcal{V}}$. If $\sum_{i\in J} \|V_i\|^2 < A_{\mathcal{V}}$ then the sequence $\mathcal{V}_J \stackrel{\text{def}}{=} \{V_i\}_{i\in\mathbb{I}_m\setminus J}$ is a projective RS for $\mathcal{H} \cong \mathbb{C}^d$ with bounds $A_{\mathcal{V}_J} \geq A_{\mathcal{V}} - \sum_{i\in J} \|V_i\|^2$ and $B_{\mathcal{V}_J} \leq B_{\mathcal{V}}$.

As they notice in [13] with an example, this is not a necessary condition. On the other side, in [3], M. G. Asgari proves that, under certain conditions, a single element can be erased from the original fusion frame (in our setting, a projective RS), and he obtains different lower bounds for the resulting reconstruction system:

Theorem 3.2 (Asgari)**.** Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}$ with bounds $A_{\mathcal{V}}$, $B_{\mathcal{V}}$. Suppose that there exists $j \in \mathbb{I}_m$ such that $M_j \stackrel{\text{def}}{=} I_d - V_j^* V_j S_{\mathcal{V}}^{-1} \in \mathcal{G}l(d)$, then $\mathcal{V}^j = \{V_i\}_{i\neq j}$ is a projective RS for $\mathcal{H} \cong \mathbb{C}^d$ with bounds $A_{\mathcal{V}_J} \geq \frac{A_{\mathcal{V}}^2}{A_{\mathcal{V}} + \|V_j\|^2 \|M_j^{-1}\|^2}$ and $B_{\mathcal{V}_J} \leq B_{\mathcal{V}}$.

Actually, Asgari's result can be generalized to any subset J of $\mathbb{I}_m$ and general RS's. In the following statement we shall give necessary and sufficient conditions which guarantee that the erasure of $\{V_i\}_{i\in J}$ of a non necessary projective $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}$ provides another RS. Recall that the sharp bounds for $\mathcal{V}$ are given by $A_{\mathcal{V}} = \|S_{\mathcal{V}}^{-1}\|^{-1}$ and $B_{\mathcal{V}} = \|S_{\mathcal{V}}\|$.

Theorem 3.3. *Let $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$ with bounds $A_{\mathcal{V}}$, $B_{\mathcal{V}}$. Fix a subset $J \subset \mathbb{I}_m$ and consider the matrix $M_J \stackrel{\text{def}}{=} I_d - \sum_{i\in J} V_i^* V_i\, S_{\mathcal{V}}^{-1} \in \mathcal{M}_d(\mathbb{C})$. Then,*

$$\mathcal{V}_J = (V_i)_{i\in\mathbb{I}_m\setminus J} \quad \text{is a RS for} \quad \mathcal{H} \cong \mathbb{C}^d \iff M_J \in \mathcal{G}l(d) \ . \tag{6}$$

In this case $S_{\mathcal{V}_J} = M_J\, S_{\mathcal{V}}$ *and* $\mathcal{V}_J$ *has bounds* $A_{\mathcal{V}_J} \geq \frac{A_{\mathcal{V}}}{\|M_J^{-1}\|}$ *and* $B_{\mathcal{V}_J} \leq B_{\mathcal{V}}$ *.*

Proof. The equality $S_{\mathcal{V}_J} = M_J\, S_{\mathcal{V}}$ follows from the following fact:

$$M_J = I_d - \sum_{i\in J} V_i^* V_i\, S_{\mathcal{V}}^{-1} = S_{\mathcal{V}}\, S_{\mathcal{V}}^{-1} - \sum_{i\in J} V_i^* V_i\, S_{\mathcal{V}}^{-1} = \sum_{i\notin J} V_i^* V_i\; S_{\mathcal{V}}^{-1} = S_{\mathcal{V}_J}\, S_{\mathcal{V}}^{-1}\ .$$

This implies the equivalence of Eq. (6). On the other hand,

$$\frac{A_{\mathcal{V}}}{\|M_J^{-1}\|} = \|S_{\mathcal{V}}^{-1}\|^{-1}\, \|M_J^{-1}\|^{-1} \leq \|(M_J\, S_{\mathcal{V}})^{-1}\|^{-1} = \|S_{\mathcal{V}_J}^{-1}\|^{-1} = A_{\mathcal{V}_J}\ .$$

The fact that $0 < S_{\mathcal{V}_J} \leq S_{\mathcal{V}}$ assures that $B_{\mathcal{V}_J} \leq B_{\mathcal{V}}$. □

In the case $J = \{j\}$, the lower bound in Theorem 3.3 is greater than that obtained in [3]:

Proposition 3.4. *Let* $\mathcal{V}$*, and* M_J *be as in Theorem 3.3, with* $J = \{j\}$*. Then*

$$\frac{A_{\mathcal{V}}^2}{A_{\mathcal{V}} + \|V_j\|^2 \|M_J^{-1}\|^2} \;\leq\; \frac{A_{\mathcal{V}}}{\|M_J^{-1}\|}\ . \tag{7}$$

Proof. We can suppose $\|M_J^{-1}\| \geq 1$, since otherwise (7) is evident. Note that

$$\|V_j\|^2 \geq A_{\mathcal{V}} \quad\Longrightarrow\quad \tfrac{A_{\mathcal{V}}^2}{A_{\mathcal{V}}+\|V_j\|^2\|M_J^{-1}\|^2} \;\leq\; \tfrac{A_{\mathcal{V}}^2}{\|V_j\|^2\|M_J^{-1}\|^2} \;\leq\; \tfrac{A_{\mathcal{V}}}{\|M_J^{-1}\|^2} \;\leq\; \tfrac{A_{\mathcal{V}}}{\|M_J^{-1}\|}\ .$$

But if $\|V_j\|^2 < A_{\mathcal{V}}$, then $\|I_d - M_J\| = \|V_j^* V_j\, S_{\mathcal{V}}^{-1}\| \leq \frac{\|V_j\|^2}{A_{\mathcal{V}}} < 1$. Therefore

$$\|M_J^{-1}\| \leq \tfrac{A_{\mathcal{V}}}{A_{\mathcal{V}}-\|V_j\|^2} \quad\Longrightarrow\quad A_{\mathcal{V}}\, \|M_J^{-1}\| \leq A_{\mathcal{V}} + \|V_j\|^2\|M_J^{-1}\| \leq A_{\mathcal{V}} + \|V_j\|^2\|M_J^{-1}\|^2\ ,$$

which clearly implies the inequality of Eq. (7). □

Remark 3.5. Let $J \subseteq \mathbb{I}_m$, $\mathcal{V} \in \mathcal{RS}$, and M_J be as in Theorem 3.3. Assume that $\|\sum_{i\in J} V_i^*\, V_i\| < A_{\mathcal{V}}$ (compare with the hypothesis $\sum_{i\in J}\|V_i\|^2 < A_{\mathcal{V}}$ of Theorem 3.1). Then, as in the proof of Proposition 3.4, it can be shown that under this assumption it holds that $\|I_d - M_J\| < 1 \implies M_J \in \mathcal{G}l(d)$ and that the lower bounds satisfy

$$A_{\mathcal{V}} - \textstyle\sum_{i\in J}\|V_i\|^2 \;\;\leq\;\; A_{\mathcal{V}} - \|\sum_{i\in J} V_i^*\, V_i\| \leq \tfrac{A_{\mathcal{V}}}{\|M_J^{-1}\|} \leq A_{\mathcal{V}_J}\ .$$

Hence Theorem 3.3 generalizes Theorem 3.1 to general RS's with better bounds. The matrix M_J also serves to compute the canonical dual system $(\mathcal{V}_J)^\#$: If we denote $\mathcal{V}^\# = \{W_i\}_{i\in\mathbb{I}_m}$ and $\mathcal{V}_J^\# = \{W_i\}_{i\notin J}$, then the formula $S_{\mathcal{V}_J} = M_J\, S_{\mathcal{V}}$ of Theorem 3.3 gives the equality

$$\mathcal{V}_J^\# \cdot M_J^{-1} \;\stackrel{\text{def}}{=}\; \{W_i\, M_J^{-1}\}_{i\notin J} = \{V_i\, S_{\mathcal{V}}^{-1}\, M_J^{-1}\}_{i\notin J} = \{V_i\, S_{\mathcal{V}_J}^{-1}\}_{i\notin J} = (\mathcal{V}_J)^\#\ .$$

That is, $(\mathcal{V}_J)^\#$ is the truncation of the canonical dual $\mathcal{V}^\#$ modified with M_J^{-1} . △

4 Geometric presentation of $\mathcal{RS}$.

In this section we shall study several objects related with the sets $\mathcal{RS}$ from a geometrical point of view. On one hand, this study is of independent interest. On the other hand, some geometrical results of this section will be necessary in order to characterize the minimizers of the joint potential, a problem that we shall consider in Section 6.

4.1 General Reconstruction systems

4.1. Observe that we can use on $L(m,\mathbf{k},d)$ the natural metric $\|\mathcal{V}\|_2 = \big(\sum_{i\in\mathbb{I}_m} \|V_i\|_2\big)^{1/2}$ for $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in L(m,\mathbf{k},d)$. Note that

$$\|\mathcal{V}\|_2^2 = \sum_{i\in\mathbb{I}_m} \|V_i\|_2^2 = \|T_\mathcal{V}\|_2^2 \qquad \big(\text{ in the space } L(\mathcal{H},\mathcal{K}) \ \big) \ .$$

With this metric it is easy to see that in $\mathcal{RS} \subseteq L(m,\mathbf{k},d)$ the following conditions hold:

1. The space $\mathcal{RS}$ is open in $L(m,\mathbf{k},d)$, since the map RSO : $L(m,\mathbf{k},d) \to L(\mathcal{H})$ given by $\text{RSO}(\mathcal{V}) = S_\mathcal{V} = T_\mathcal{V}^* T_\mathcal{V}$ (for $\mathcal{V} \in L(m,\mathbf{k},d)$) is continuous.
2. On the other hand, if we fix $\mathcal{V} \in \mathcal{RS}$, then the set $\mathcal{D}(\mathcal{V})$ is **closed** in $L(m,\mathbf{k},d)$, because the map $L(m,\mathbf{k},d) \ni \mathcal{W} \mapsto T_\mathcal{W}^* T_\mathcal{V} \in L(\mathcal{H})$ is continuous. Observe that the equality $T_\mathcal{W}^* T_\mathcal{V} = I_\mathcal{H} \implies T_\mathcal{W}^*$ is surjective, so that $\mathcal{W} \in \mathcal{RS}$. $\triangle$

4.2. Given a **surjective** $A \in L(\mathcal{K}\,,\,\mathcal{H}\,)$, let us consider $P_{\mathcal{K}_i} A^* \in L(\mathcal{H}\,,\,\mathcal{K}_i)$ for every $i \in \mathbb{I}_m$. Then A produces a system $\mathcal{W}_A = (P_{\mathcal{K}_i} A^*)_{i\in\mathbb{I}_m} \in \mathcal{RS}$ such that

$$T_{\mathcal{W}_A}^* = A \ \text{ and } \ S_{\mathcal{W}_A} = AA^* \in \mathcal{G}l(\mathcal{H})^+ \ .$$

Therefore, given a fixed $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$, we can parametrize

$$\mathcal{RS} = \{U\cdot\mathcal{V} \stackrel{\text{def}}{=} (P_{\mathcal{K}_i}\, U\, T_\mathcal{V})_{i\in\mathbb{I}_m} : U \in \mathcal{G}l(\mathcal{K})\} \ .$$

In other words, the Lie group $\mathcal{G}l(\mathcal{K})$ acts transitively on $\mathcal{RS}$, where the action is given by the formula $U\cdot\mathcal{V} = (P_{\mathcal{K}_i}\, U\, T_\mathcal{V})_{i\in\mathbb{I}_m}$. Indeed, for every $x = (x_i)_{i\in\mathbb{I}_m} \in \mathcal{K}$,

$$T_{U\cdot\mathcal{V}}^*\, x \ = \sum_{i\in\mathbb{I}_m} T_\mathcal{V}^*\, U^*\, P_{\mathcal{K}_i}\, x = T_\mathcal{V}^*\, U^* \sum_{i\in\mathbb{I}_m} P_{\mathcal{K}_i}\, x = T_\mathcal{V}^*\, U^*\, x \ . \tag{8}$$

Therefore $T_{U\cdot\mathcal{V}}^* = T_\mathcal{V}^*\, U^* \in L(\mathcal{K},\mathcal{H})$, which is **surjective** for every $U \in \mathcal{G}l(\mathcal{K})$, so that $U\cdot\mathcal{V} \in \mathcal{RS}$. Hence $T_{U\cdot\mathcal{V}} = U\, T_\mathcal{V}$, which shows that this is indeed an action. On the other hand, for every $\mathcal{W} \in \mathcal{RS}$, since both $T_\mathcal{W}^*$ and $T_\mathcal{V}^*$ are surjective, then there exists $U \in \mathcal{G}l(\mathcal{K})$ such that $T_\mathcal{W}^* = T_\mathcal{V}^*\, U^*$. Therefore we have that $\mathcal{W} = U\cdot\mathcal{V}$.

Fix $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$. Then we can define a continuous surjective map

$$\pi_\mathcal{V} : \mathcal{G}l(\mathcal{K}) \to \mathcal{RS} \quad \text{ given by } \quad \pi_\mathcal{V}(U) = U\cdot\mathcal{V} \quad \text{ for } \quad U \in \mathcal{G}l(\mathcal{K}) \ .$$

The isotropy subgroup of this action is $\mathcal{I}_\mathcal{V} = \pi_\mathcal{V}^{-1}(\mathcal{V}) = \{U \in \mathcal{G}l(\mathcal{K}) : U\big|_{R(T_\mathcal{V})} = \text{ Id }\}$. Indeed, looking at Eq. (8) we see that $U\cdot\mathcal{V} = \mathcal{V} \iff T_\mathcal{V}^*\, U^* = T_\mathcal{V}^* \iff U\, T_\mathcal{V} = T_\mathcal{V}$. In [16] it is proved that these facts are sufficient to assure that $\mathcal{RS}$ is a smooth submanifold of $L(m,\mathbf{k},d)$ (actually it is an open subset) such that the map $\pi_\mathcal{V} : \mathcal{G}l(\mathcal{K}) \to \mathcal{RS}$ becomes a smooth submersion. On the other hand, we can parametrize $\mathcal{D}(\mathcal{V})$ in two different ways:

$$\begin{aligned}\mathcal{D}(\mathcal{V}) \ &= \Big\{\mathcal{W} \in L(m,\mathbf{k},d) \ : \ T_\mathcal{W}^* = T_{\mathcal{V}^\#}^* + G \ , \ \ G \in L(\mathcal{K},\mathcal{H}) \ \text{ and } \ G\big|_{R(T_\mathcal{V})} \equiv 0\Big\} \\ &= \Big\{U\cdot\mathcal{V}^\# \ : \ U \in \mathcal{G}l(\mathcal{K}) \ \text{ and } \ P\,U^*\,P = P\Big\} \ , \quad \text{where } P = P_{R(T_\mathcal{V})} \ .\end{aligned} \tag{9}$$

Indeed, just observe that $\ker T_{\mathcal{V}^\#}^* = \ker S_V^{-1} T_\mathcal{V}^* = R(T_\mathcal{V})^\perp = \ker P$. Therefore

$$T_{U\cdot\mathcal{V}^\#}^*\, T_\mathcal{V} = T_{\mathcal{V}^\#}^*\, U^*\, T_\mathcal{V} = I_\mathcal{H} \iff U^*\, x \in x + \ker T_{\mathcal{V}^\#}^* \ \text{ for every } x \in R(P) \ ,$$

which means exactly that $PU^*P = P$. $\triangle$

Remark 4.3. This geometric presentation is similar to the presentation of vector frames done in [16]. The relationship is based on the following fact:

The space $\mathcal{RS}$ can be seen as an agrupation in packets of vector frames. Recall that $n = \operatorname{tr} \mathbf{k} = \sum_{i\in\mathbb{I}_m} k_i$. Let us denote by $\mathcal{E}_i = \{e_1^{(i)}, \ldots, e_{k_i}^{(i)}\}$ the canonical ONB of each $\mathcal{K}_i = \mathbb{C}^{k_i}$, and the set $\mathcal{E} = \bigcup_{i\in\mathbb{I}_m} \mathcal{E}_i$, which is a reenumeration of the canonical ONB of the space $\mathcal{K} = \bigoplus_{i\in \mathbb{I}_m} \mathcal{K}_i \cong \mathbb{C}^n$. Then, there is a natural one to one correspondence

$$\mathcal{RS} \ni \mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \longleftrightarrow \Big((\, V_i^* \, e_j^{(i)}\,)_{j\in\mathbb{I}_{k_i}} \Big)_{i\in\mathbb{I}_m} = \big(T_{\mathcal{V}}^* \, e \,\big)_{e\in \mathcal{E}} \in \mathcal{H}^n , \tag{10}$$

where the right term is a general n-vector frame for $\mathcal{H}$. On the other hand, fixed the ONB $\mathcal{E}$ of $\mathcal{K}$, the set of n-vector frames for $\mathcal{H}$ can be also identified with the space $E(\mathcal{K}\,, \mathcal{H}) \stackrel{\text{def}}{=} \{A \in L(\mathcal{K}, \mathcal{H}) : A \text{ is surjective } \}$, via the map $A \longleftrightarrow \big(\, A\, e\,\big)_{e\in \mathcal{E}}$.

The geometrical representation of $\mathcal{RS}$ given before is the natural geometry of the space of epimorphisms $E(\mathcal{K}\,, \mathcal{H})$ under the (right) action of $\mathcal{G}l\,(\mathcal{K})$. Through all these identifications we get the correspondence $\mathcal{RS} \ni \mathcal{V} \longleftrightarrow T_{\mathcal{V}}^* \in E(\mathcal{K}\,, \mathcal{H})$.

Observe that, in terms of Eq. (10), a system $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}$ satisfy that $\mathcal{V} \in \mathcal{PRS} \iff$ each subsystem $(\, V_i^* \, e_j^{(i)}\,)_{j\in\mathbb{I}_{k_i}}$ is a multiple of an orthonormal system in $\mathcal{H}$. $\triangle$

4.2 Projective RS's with fixed weights

Given a fixed sequence of weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}_{>0}^m$, we define the set of projective RS's with fixed set of weights $\mathbf{v}$:

$$\mathcal{PRS}_{\mathbf{v}} \stackrel{\text{def}}{=} \Big\{ \mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{PRS} : \ \|V_i\|_{sp} = v_i \ \text{ for every } i \in \mathbb{I}_m \Big\} . \tag{11}$$

Denote by $\tau = \sum\limits_{i\in\mathbb{I}_m} v_i^2 \, k_i$. Observe that $\operatorname{tr} S_{\mathcal{V}} = \sum\limits_{i\in\mathbb{I}_m} \operatorname{tr} V_i^* \, V_i = \tau$ for every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$. In what follows we shall denote by

$$\mathcal{M}_d(\mathbb{C})^+_\tau \stackrel{\text{def}}{=} \{A \in \mathcal{M}_d(\mathbb{C})^+ \, : \, \operatorname{tr} A = \tau\} \qquad \text{and} \qquad \mathcal{G}l\,(d)^+_\tau \stackrel{\text{def}}{=} \mathcal{M}_d(\mathbb{C})^+_\tau \cap \mathcal{G}l\,(d)\,.$$

the set of $d \times d$ positive and positive invertible operators with fixed trace τ, endowed with the metric and geometric structure induced by those of $\mathcal{G}l\,(d)$.

In this section we look for conditions which assure that the smooth map

$$\operatorname{RSO} : \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l\,(d)^+_\tau \quad \text{ given by } \quad \operatorname{RSO}(\mathcal{V}) = S_{\mathcal{V}} = \sum_{i\in\mathbb{I}_m} V_i^* \, V_i \ , \tag{12}$$

for every $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}$, has smooth local cross sections. Before giving these conditions and the proof of their sufficiency, we need some notations and two geometrical lemmas: Fix $d \in \mathbb{N}$. For every $k \in \mathbb{I}_d$, we denote by $\mathcal{I}(k\,, d) = \{U \in L(\mathbb{C}^k\,, \mathbb{C}^d) : U^*U = I_k\}$ the set of isometries. Given an m-tuple $\mathbf{k} = (k_i)_{i\in\mathbb{I}_m} \in \mathbb{I}_d^m \subseteq \mathbb{N}^m$, we denote by

$$\mathcal{I}(\mathbf{k}\,, d) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \mathcal{I}(k_i\,, d) \subseteq \bigoplus_{i\in\mathbb{I}_m} L(\mathcal{K}_i\,, \mathcal{H}) \cong L(\mathcal{K}\,, \mathcal{H}) \ ,$$

endowed with the product (differential, metric) structure (see [1] for a description of the geometrical structure). Similarly, let $\mathrm{Gr}(k,d)$ denote the Grassmann manifold of orthogonal projections of rank k in $\mathbb{C}^d$ and let

$$\mathrm{Gr}\,(\mathbf{k}\,,\,d) \stackrel{\mathrm{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \mathrm{Gr}\,(k_i\,,\,d) \subseteq L(\mathcal{H})^m\ ,$$

with the product smooth structure (see [17]).

Lemma 4.4. *Consider the smooth map* $\Phi : \mathcal{I}(\mathbf{k}\,,\,d) \to \mathrm{Gr}\,(\mathbf{k}\,,\,d)$ *given by*

$$\Phi(\mathcal{W}) = (W_1\,W_1^*\,,\,\ldots\,,\,W_m\,W_m^*) \quad \textit{for every} \quad \mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{I}(\mathbf{k}\,,\,d)\ .$$

Then Φ *has smooth local cross sections around any point* $\mathcal{P} = (P_i)_{i\in\mathbb{I}_m} \in \mathrm{Gr}\,(\mathbf{k}\,,\,d)$ *toward every* $\mathcal{W} \in \mathcal{I}(\mathbf{k}\,,\,d)$ *such that* $\Phi(\mathcal{W}) = \mathcal{P}$*. In particular,* Φ *is open and surjective.*

Proof. Since both spaces have a product structure, it suffices to consider the case $m = 1$. It is clear that the map Φ is surjective.

For every $P \in \mathrm{Gr}\,(k\,,\,d)$, the C^∞ map $\pi_P : \mathcal{U}(d) \to \mathrm{Gr}\,(k\,,\,d)$ given by $\pi_P(U) = UPU^*$ for $U \in \mathcal{U}(d)$ is a submersion with a smooth local cross section (see [17])

$$h_P : U_P \stackrel{\mathrm{def}}{=} \{Q \in \mathrm{Gr}\,(k\,,\,d) : \|Q - P\| < 1\} \to \mathcal{U}(d) \quad \text{such that} \quad h_P(P) = I_d\ .$$

For completeness we recall that, for every $Q \in U_P$, the matrix $h_P(Q)$ is the unitary part in the polar decomposition of the invertible matrix $QP + (I_d - Q)(I_d - P)$. Then, fixed $W \in \mathcal{I}(k\,,\,d)$ such that $\Phi(W) = P$, we can define the following smooth local cross section for Φ:

$$s_{P\,,W} : U_P \to \mathcal{I}(k\,,\,d) \quad \text{given by} \quad s_{P\,,W}(Q) = h_P(Q)\,W\ , \quad \text{for every} \quad Q \in U_P\ . \qquad \square$$

We shall need the following result from [25]. In order to state it we recall the following notions and introduce some notations:

1. Fix $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$. We shall consider the smooth map

$$\Psi_{\mathbf{v}} : \mathcal{I}(\mathbf{k}\,,\,d) \to \mathcal{M}_d(\mathbb{C})^+ \quad \text{given by} \quad \Psi_{\mathbf{v}}(\mathcal{U}) = \sum_{i\in\mathbb{I}_m} v_i^2\ U_i\,U_i^* \tag{13}$$

   for every $\mathcal{U} = \{U_i\}_{i\in\mathbb{I}_m} \in\ \mathcal{I}(\mathbf{k}\,,\,d)$.

2. Given a set $\mathcal{P} = \{P_j : j \in \mathbb{I}_m\} \subseteq \mathcal{M}_d(\mathbb{C})^+$, we denote by

$$\mathcal{P}' = \{P_j : j \in \mathbb{I}_m\}' = \{A \in \mathcal{M}_d(\mathbb{C}) : AP_j = P_j\,A \quad \text{for every} \quad j \in \mathbb{I}_m\}\,. \tag{14}$$

   Note that $\mathcal{P}'$ is a (closed) unital selfadjoint subalgebra of $\mathcal{M}_d(\mathbb{C})$. Therefore,

$$\mathcal{P}' \neq \mathbb{C}\,I_d \iff \text{there exists a non-trivial orthogonal projection } Q \in \mathcal{P}'\ . \tag{15}$$

Lemma 4.5 ([25])**.** *Let* $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ *and* $\mathcal{P} = \{P_i\}_{i\in\mathbb{I}_m} \in \mathrm{Gr}\,(\mathbf{k}\,,\,d)$*. Denote by* $\tau = \sum_{i\in\mathbb{I}_m} v_i^2\,k_i$ *. Then the map* $S_{\mathbf{v}} : \mathrm{Gr}\,(\mathbf{k}\,,\,d) \to \mathcal{M}_d(\mathbb{C})^+_\tau$ *given by*

$$S_{\mathbf{v}}(\mathcal{Q}) = \sum_{i\in\mathbb{I}_m} v_i^2\ Q_i \quad \textit{for} \quad \mathcal{Q} = \{Q_i\}_{i\in\mathbb{I}_m} \in \mathrm{Gr}\,(\mathbf{k}\,,\,d) \tag{16}$$

is smooth and, if $\mathcal{P}$ *satisfies that* $\mathcal{P}' = \mathbb{C}\,I_d$ *, then*

1. The matrix $S_{\mathbf{v}}(\mathcal{P}) \in \mathcal{G}l(d)^+_\tau$.

2. The image of $S_{\mathbf{v}}$ contains an open neighborhood of $S_{\mathbf{v}}(\mathcal{P})$ in $\mathcal{M}_d(\mathbb{C})^+_\tau$.

3. Moreover, $S_{\mathbf{v}}$ has a smooth local cross section around $S_{\mathbf{v}}(\mathcal{P})$ towards $\mathcal{P}$. $\square$

4.6. The set $\mathcal{I}_0(\mathbf{k}\,,\,d) = \{\mathcal{W} \in \mathcal{I}(\mathbf{k}\,,\,d) : S_{\mathbf{v}} \circ \Phi(\mathcal{W}) \in \mathcal{G}l(d)^+\}$ is open in $\mathcal{I}(\mathbf{k}\,,\,d)$. Observe that its definition does not depend on the sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ of weights. Moreover, the map $\gamma : \mathcal{I}_0(\mathbf{k}\,,\,d) \to \mathcal{PRS}_{\mathbf{v}}$ given by

$$\gamma(\mathcal{W}) = \{v_i\, W_i^*\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}} \quad \text{for every} \quad \mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{I}_0(\mathbf{k}\,,\,d)\ , \tag{17}$$

is a homeomorphism. Hence, using this map γ we can endow $\mathcal{PRS}_{\mathbf{v}}$ with the differential structure which makes γ a diffeomorphism. With this structure, each space $\mathcal{PRS}_{\mathbf{v}}$ becomes a submanifold of $\mathcal{RS}$. It is in this sense in which the map $\text{RSO} : \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l(d)^+_\tau$ defined in Eq. (12) is smooth. Indeed, we have that

$$\text{RSO} = S_{\mathbf{v}} \circ \Phi \circ \gamma^{-1}\ , \tag{18}$$

where $\Phi : \mathcal{I}(\mathbf{k}\,,\,d) \to \text{Gr}\,(\mathbf{k}\,,\,d)$ is the smooth map defined in Lemma 4.4. Now we can give an answer to the problem posed in the beginning of this section. $\triangle$

Definition 4.7. Let $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ and $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}\,(m, \mathbf{k}, d)$. We say that the system $\mathcal{V}$ is **irreducible** if $C_{\mathcal{V}} \stackrel{\text{def}}{=} \{V_i^* V_i : i \in \mathbb{I}_m\}' = \mathbb{C}\, I_d$. $\triangle$

In Section 7 we show examples of reducible and irreducible systems. See also Remark 6.5.

Theorem 4.8. *Let $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ and $\tau = \sum\limits_{i\in\mathbb{I}_m} v_i^2\, k_i$. If we fix an* **irreducible** *system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}\,(m, \mathbf{k}, d)$, then the map $\text{RSO} : \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l(d)^+_\tau$ defined in Eq. (12) has a smooth local cross section around $S_{\mathcal{V}}$ which sends $S_{\mathcal{V}}$ to $\mathcal{V}$.*

Proof. We have to prove that there exists an open neighborhood A of $S_{\mathcal{V}}$ in $\mathcal{G}l(d)^+_\tau$ and a smooth map $\rho : A \to \mathcal{PRS}_{\mathbf{v}}$ such that $\text{RSO}\,(\,\rho(S)\,) = S$ for every $S \in A$ and $\rho(S_{\mathcal{V}}) = \mathcal{V}$.

Denote by $P_i = P_{R(V_i^*)}$ for every $i \in \mathbb{I}_m$, and consider the system

$$\gamma^{-1}(\mathcal{V}) = \mathcal{U} = \{U_i\}_{i\in\mathbb{I}_m} \in \mathcal{I}(\mathbf{k}\,,\,d) \quad \text{given by} \quad U_i = v_i^{-1}\, V_i^* \in I(k_i\,,\,d) \ \ i \in \mathbb{I}_m\ .$$

Observe that $\Phi(\mathcal{U}) = \mathcal{P} = \{P_i\}_{i\in\mathbb{I}_m} \in \text{Gr}\,(\mathbf{k}\,,\,d)$ and $S_{\mathbf{v}}(\mathcal{P}) = S_{\mathcal{V}}$. By our hypothesis, we know that $\mathcal{P}' = \{V_i^* V_i : i \in \mathbb{I}_m\}' = \mathbb{C}\, I_d$. Let $\alpha : A \to \text{Gr}\,(\mathbf{k}\,,\,d)$ be the smooth section for the map $S_{\mathbf{v}} : \text{Gr}\,(\mathbf{k}\,,\,d) \to \mathcal{M}_d(\mathbb{C})^+_\tau$ given by Lemma 4.5. Hence A is an open neighborhood of $S_{\mathcal{V}} = S_{\mathbf{v}}(\mathcal{P})$ in $\mathcal{G}l(d)^+_\tau$, and $\alpha(S_{\mathcal{V}}) = \mathcal{P}$.

Take now the cross section $\beta : B \to \mathcal{I}(\mathbf{k}\,,\,d)$ for the map $\Phi : \mathcal{I}(\mathbf{k}\,,\,d) \to \text{Gr}\,(\mathbf{k}\,,\,d)$ given by Lemma 4.4, such that B is an open neighborhood of $\mathcal{P}$ in $\text{Gr}\,(\mathbf{k}\,,\,d)$, and that $\beta(\mathcal{P}) = \mathcal{U}$.

Finally we recall the diffeomorphism $\gamma : \mathcal{I}_0(\mathbf{k}\,,\,d) \to \mathcal{PRS}_{\mathbf{v}}$ defined in Eq. (17), where $\mathcal{I}_0(\mathbf{k}\,,\,d) = \{\mathcal{W} \in \mathcal{I}(\mathbf{k}\,,\,d) : S_{\mathbf{v}} \circ \Phi(\mathcal{W}) \in \mathcal{G}l(d)^+\}$ is an open subset of $\mathcal{I}(\mathbf{k}\,,\,d)$ such that $\mathcal{U} \in \mathcal{I}_0(\mathbf{k}\,,\,d)$. Note that $\gamma(\mathcal{U}) = \mathcal{V}$. Changing the first neighborhood A by some smaller open set, we can define the announced smooth cross section for the map RSO by

$$\rho = \gamma \circ \beta \circ \alpha : A \subseteq \mathcal{G}l(d)^+_\tau \to \mathcal{PRS}_{\mathbf{v}}\ .$$

Following our previous steps, we see that $\rho(S_{\mathcal{V}}) = \mathcal{V}$ and that

$$\text{RSO} \stackrel{(18)}{=} S_{\mathbf{v}} \circ \Phi \circ \gamma^{-1} \implies \text{RSO}(\rho(S)\,) = S \quad \text{for every} \quad S \in A\ . \qquad \square$$

Remark 4.9. In order to compute "local" minimizers for different functions defined on $\mathcal{RS}$ or some of its subsets, we shall consider two different (pseudo) metrics: Given $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$ and $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$, we recall the (punctual) metric defined in 4.1:

$$d_P(\mathcal{V},\, \mathcal{W}) = \left(\sum_{i\in\mathbb{I}_m} \|V_i - W_i\|_2^2 \right)^{1/2} = \|T_\mathcal{V} - T_\mathcal{W}\|_2 = \|T_\mathcal{V}^* - T_\mathcal{W}^*\|_2 \ .$$

We consider also a pseudo-metric defined by $d_S(\mathcal{V},\, \mathcal{W}) = \|S_\mathcal{V} - S_\mathcal{W}\|$.

Let $A \subseteq \mathcal{RS}$ and $f : A \to \mathbb{R}$ a continuous map. Fix $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in A$. Since the map $\mathcal{V} \mapsto S_\mathcal{V}$ is continuous, it is easy to see that if $\mathcal{V}$ is a local d_S minimizer of f over A, then $\mathcal{V}$ is also a local d_P minimizer. The converse needs not to be true.

Nevertheless, it is true under some assumptions: Theorem 4.8 shows that if $\mathcal{V}$ is a local d_P minimizer of $f : \mathcal{PRS}_\mathbf{v} \to \mathbb{R}$, in order to assure that $\mathcal{V}$ is also a local d_S minimizer it suffices to assume that $\{V_i^*V_i : i \in \mathbb{I}_m\}' = \mathbb{C}\, I_d$, i.e. that $\mathcal{V}$ is irreducible. $\triangle$

5 Spectral pictures

Recall that $(\mathbb{R}_+^d)^\downarrow$ is the set of vectors $\mu \in \mathbb{R}_+^d$ with non negative and decreasing entries. If all the entries are positive (i.e., if $\mu_d > 0$), we write $\mu \in (\mathbb{R}_{>0}^d)^\downarrow$. Given $S \in \mathcal{M}_d(\mathbb{C})^+$, we write $\lambda(S) \in (\mathbb{R}_+^d)^\downarrow$ the decreasing vector of eigenvalues of S, counting multiplicities. We denote by $S^\dagger$ the Moore-Penrose pseudo-inverse of S. We shall also use the following notations:

1. Given $x \in \mathbb{C}^n$ then $D(x) \in \mathcal{M}_d(\mathbb{C})$ denotes the diagonal matrix with main diagonal x.
2. If $d \le n$ and $y \in \mathbb{C}^d$, we write $(y\,,\, 0_{n-d}) \in \mathbb{C}^n$, where 0_{n-d} is the zero vector of $\mathbb{C}^{n-d}$. In this case, we denote by $D_n(y) = D\big(\,(y\,,\, 0_{n-d})\,\big) \in \mathbb{C}^n$.

Given $\mathcal{A} \subseteq \mathcal{M}_d(\mathbb{C})^+$ we consider its **spectral picture**:

$$\Lambda(\mathcal{A}) = \{\lambda(A) : \ A \in \mathcal{A}\} \subseteq (\mathbb{R}_+^d)^\downarrow \ ,$$

We say that $\Lambda(\mathcal{A})$ **determines** $\mathcal{A}$ whenever $A \in \mathcal{A}$ if and only if $\lambda(A) \in \Lambda(\mathcal{A})$. It is easy to see that this happens if and only if the set $\mathcal{A}$ is saturated with respect to unitary equivalence.

5.1 The set of dual RS's

Definition 5.1. Let $\mathcal{V} \in \mathcal{RS}$. We denote by

$$\Lambda(\mathcal{D}(\mathcal{V})\,) = \{\lambda(S_\mathcal{W}) : \ \mathcal{W} \in \mathcal{D}(\mathcal{V})\} \subseteq (\mathbb{R}_{>0}^d)^\downarrow \ , \tag{19}$$

that is, the spectral picture of the set of all dual RS's for $\mathcal{V}$. $\triangle$

The following result gives a characterization of $\Lambda(\mathcal{D}(\mathcal{V}))$.

Theorem 5.2. *Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$ and $\mu \in (\mathbb{R}_{>0}^d)^\downarrow$. We denote by $n = \operatorname{tr} \mathbf{k}$. Then the following conditions are equivalent:*

1. *The vector $\mu \in \Lambda(\mathcal{D}(\mathcal{V}))$.*

2. *There exists an orthogonal projection $P \in \mathcal{M}_n(\mathbb{C})$ such that* $\operatorname{rk} P = d$ *and*

$$\lambda\,(P\,D_n(\mu)\,P) = \big(\,\lambda(S_{\mathcal{V}}^{-1}), 0_{n-d}\,\big) = \lambda(G_{\mathcal{V}}^{\dagger})\ , \tag{20}$$

where $G_{\mathcal{V}} = T_{\mathcal{V}}\,T_{\mathcal{V}}^* \in \mathcal{M}_n(\mathbb{C})^+$ *is the Gram matrix of* $\mathcal{V}$.

Proof. Let $\mathcal{W} \in \mathcal{D}(\mathcal{V})$ with $\lambda(S_{\mathcal{W}}) = \mu$. Then $T_{\mathcal{W}}^*\,T_{\mathcal{V}} = I$ and

$$G_{\mathcal{V}}\,G_{\mathcal{W}}\,G_{\mathcal{V}} = T_{\mathcal{V}}\,(T_{\mathcal{V}}^*\,T_{\mathcal{W}})\,(T_{\mathcal{W}}^*\,T_{\mathcal{V}})\,T_{\mathcal{V}}^* = T_{\mathcal{V}}\,T_{\mathcal{V}}^* = G_{\mathcal{V}} \implies Q\,G_{\mathcal{W}}\,Q = G_{\mathcal{V}}^{\dagger}\ , \tag{21}$$

where $Q = G_{\mathcal{V}}\,G_{\mathcal{V}}^{\dagger} = P_{R(T_{\mathcal{V}})}$. Note that $\operatorname{rk} Q = \operatorname{rk} T_{\mathcal{V}} = d$, since $\mathcal{V}$ is a RS. Also

$$\lambda(G_{\mathcal{W}}) = \lambda(T_{\mathcal{W}}\,T_{\mathcal{W}}^*) = (\lambda(T_{\mathcal{W}}^*\,T_{\mathcal{W}}), 0_{n-d}) = (\lambda(S_{\mathcal{W}}), 0_{n-d}) = (\mu\,,\,0_{n-d})\ .$$

Then there exists $U \in \mathcal{U}(n)$ such that

$$U^*\,D(\mu\,,\,0_{n-d})\,U = U^*\,D_n(\mu)\,U = U^*\ D_n\big(\,\lambda(S_{\mathcal{W}})\,\big)\ U = T_{\mathcal{W}}\,T_{\mathcal{W}}^* = G_{\mathcal{W}}\ . \tag{22}$$

Let $P = U\,Q\,U^*$. Note that $\operatorname{rk} P = \operatorname{rk} Q = d$. Using (21) and (22) we get the item 2 :

$$\lambda\,(P\,D_n(\mu)\,P) = \lambda(U\,Q\,U^*\,D_n(\mu)\,U\,Q\,U^*) \overset{(22)}{=} \lambda(Q\,G_{\mathcal{W}}\,Q) \overset{(21)}{=} \lambda(G_{\mathcal{V}}^{\dagger}) = (\lambda(S_{\mathcal{V}}^{-1})\,,\,0_{n-d})\ .$$

Conversely, assume that there exists the projection $P \in \mathcal{M}_n(\mathbb{C})^+$ of item 2. Observe that there always exists $\mathcal{U} \in \mathcal{RS}$ such that $\lambda(S_{\mathcal{U}}) = \lambda(T_{\mathcal{U}}^*\,T_{\mathcal{U}}) = \mu$. Then

$$\lambda(G_{\mathcal{U}}) = \lambda(T_{\mathcal{U}}\,T_{\mathcal{U}}^*) = (\mu\,,\,0_{n-d}) \in (\mathbb{R}_+^n)^{\downarrow}\ .$$

Let $V \in \mathcal{U}(n)$ such that $V^*\,G_{\mathcal{U}}\,V = D_n(\mu)$. Denote by $Q = VPV^*$. Then we get that

$$\lambda(Q\,G_{\mathcal{U}}\,Q) = \lambda(P\,V^*\,G_{\mathcal{U}}\,V\,P) = \lambda\,(P\,D_n(\mu)\,P) \overset{(20)}{=} (\lambda(S_{\mathcal{V}}^{-1}), 0_{n-d}) = \lambda(G_{\mathcal{V}}^{\dagger})\ . \tag{23}$$

Then there exists $W \in \mathcal{U}(n)$ such that $W^*\,(Q\,G_{\mathcal{U}}\,Q)\,W = G_{\mathcal{V}}^{\dagger}$. Observe that

$$\operatorname{rk} Q = d \quad \text{and} \quad W^*(R(Q)\,) \supseteq R(G_{\mathcal{V}}^{\dagger}) = R(G_{\mathcal{V}}) = R(T_{\mathcal{V}}) \implies W^*QW = P_{R(T_{\mathcal{V}})}\ .$$

Moreover, $G_{\mathcal{V}}\,G_{\mathcal{V}}^{\dagger} = G_{\mathcal{V}}^{\dagger}\,G_{\mathcal{V}} = P_{R(G_{\mathcal{V}})} = P_{R(T_{\mathcal{V}})} = W^*QW$. Then

$$\begin{aligned} G_{\mathcal{V}} &= G_{\mathcal{V}}\,G_{\mathcal{V}}^{\dagger}\,G_{\mathcal{V}} = G_{\mathcal{V}}\,(W^*\,Q\,G_{\mathcal{U}}\,Q\,W)\,G_{\mathcal{V}} \\ &= G_{\mathcal{V}}\,P_{R(G_{\mathcal{V}})}\,(W^*\,G_{\mathcal{U}}\,W)\,P_{R(G_{\mathcal{V}})}\,G_{\mathcal{V}} = G_{\mathcal{V}}\,(W^*\,G_{\mathcal{U}}\,W)\,G_{\mathcal{V}}\ . \end{aligned}$$

We can rewrite this fact as $T_{\mathcal{V}}\big(\,T_{\mathcal{V}}^*\,W^*T_{\mathcal{U}}\,T_{\mathcal{U}}^*\,W\,T_{\mathcal{V}}\,\big)\,T_{\mathcal{V}}^* = T_{\mathcal{V}}\,T_{\mathcal{V}}^*$. Since $T_{\mathcal{V}}^*$ is surjective,

$$(T_{\mathcal{V}}^*\,W^*T_{\mathcal{U}})\,(T_{\mathcal{U}}^*\,W\,T_{\mathcal{V}}) = I_{\mathcal{H}} \implies V_d = T_{\mathcal{U}}^*\,W\,T_{\mathcal{V}} \in \mathcal{U}(d)\ . \tag{24}$$

Finally, take $\mathcal{W} = \{P_{\mathcal{K}_i}\,W\,T_{\mathcal{U}}\,V_d\}_{i\in\mathbb{I}_m} \in L(m, \mathbf{k}, d)$. Observe that

$$S_{\mathcal{W}} = \sum_{i\in\mathbb{I}_m} V_d^*\,T_{\mathcal{U}}^*\,W^*\,P_{\mathcal{K}_i}\,W\,T_{\mathcal{U}}\,V_d = V_d^*\,T_{\mathcal{U}}^*\,T_{\mathcal{U}}\,V_d = V_d^*\,S_{\mathcal{U}}\,V_d \in \mathcal{G}l(d)^+\ .$$

Then $\mathcal{W} \in \mathcal{RS}$ and $\lambda(S_{\mathcal{W}}) = \lambda(S_{\mathcal{U}}) = \mu$. Similarly, $T_{\mathcal{W}} = W\,T_{\mathcal{U}}\,V_d$. By Eq. (24), we deduce that $T_{\mathcal{W}}^*\,T_{\mathcal{V}} = V_d^*\,T_{\mathcal{U}}^*\,W\,T_{\mathcal{V}} = V_d^*\,V_d = I_{\mathcal{H}}$, so that $\mathcal{W} \in \mathcal{D}(\mathcal{V})$. $\square$

Remark 5.3. Let $\mathcal{V} \in \mathcal{RS}$ and $\mu \in (\mathbb{R}^d_{>0})^\downarrow$ as in Theorem 5.2. It turns out that condition (20) can be characterized in terms of interlacing inequalities.

More explicitly, let us denote by

$$\gamma = \mu^\uparrow \in (\mathbb{R}^d_{>0})^\uparrow \quad , \qquad \text{so that} \qquad \gamma_i = \mu_{d-i+1} \quad \text{for every} \quad i \in \mathbb{I}_d \ .$$

Similarly, we denote by $\rho = \lambda(S_\mathcal{V}^{-1})^\uparrow = (\lambda_i(S_\mathcal{V})^{-1})_{i\in\mathbb{I}_m} \in (\mathbb{R}^d_{>0})^\uparrow$. K. Fan and G. Pall showed that the existence of a projection P satisfying (20) is equivalent to the following inequalities:

1. $\mu_{d-i+1} = \gamma_i \geq \rho_i = \lambda_i(S_\mathcal{V})^{-1}$ for every $i \in \mathbb{I}_d$.
2. If $n = \text{tr } \mathbf{k} < 2\,d$ and we denote $r = 2\,d - n \in \mathbb{N}$, then

$$\gamma_i \ \leq \ \rho_{i+n-d} = \lambda_{i+n-d}(S_\mathcal{V})^{-1} = \lambda_{2\,d-n-i+1}(S_\mathcal{V}^{-1}) \quad \text{if} \quad 1 \leq i \leq r \ .$$

This fact together with Theorem 5.2 give a complete description of the spectral picture of the RS operators $S_\mathcal{W}$ for every $\mathcal{W} \in \mathcal{D}(\mathcal{V})$, which we write as follows. △

Corollary 5.4. *Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$, $n = \text{tr } \mathbf{k}$ and fix $\mu \in (\mathbb{R}^d_{>0})^\downarrow$. Then, the set $\Lambda(\mathcal{D}(\mathcal{V}))$ can be characterized as follows:*

1. *If $n \geq 2\,d$, we have that*

$$\mu \in \Lambda(\mathcal{D}(\mathcal{V})) \iff \mu_j \geq \lambda_j(S_\mathcal{V}^{-1}) = \lambda_{d-j+1}(S_\mathcal{V})^{-1} \quad \text{for every} \quad j \in \mathbb{I}_d \ . \tag{25}$$

2. *If $n < 2\,d$, then $\mu \in \Lambda(\mathcal{D}(\mathcal{V})) \iff \mu$ satisfies* (25) *and also the following conditions:*

$$\mu_i^\uparrow = \mu_{d-i+1} \ \leq \ \lambda_{i+n-d}(S_\mathcal{V})^{-1} = \lambda_{2\,d-n-i+1}(S_\mathcal{V}^{-1}) \quad \text{for every} \quad i \leq 2\,d-n \ . \tag{26}$$

Proof. It is a direct consequence of Theorem 5.2 and the Fan-Pall inequalities described in Remark 5.3. □

Corollary 5.5. *Let $\mathcal{V} \in \mathcal{RS}$. Then $\Lambda(\mathcal{D}(\mathcal{V}))$ is a convex set.*

Proof. It is clear that the inequalities given in Eqs. (25) and (26) are preserved by convex combinations. Observe that also the set $(\mathbb{R}^d_{>0})^\downarrow$ is convex. □

Corollary 5.6. *Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$. If $\mathcal{W} \in \mathcal{D}(\mathcal{V})$ then*

$$\text{RSP}\,(\mathcal{W}) \ \stackrel{def}{=} \ \text{tr } S_\mathcal{W}^2 \geq \text{tr } S_\mathcal{V}^{-2} = \sum_{i=1}^d \lambda(S_\mathcal{V})_i^{-2} = \text{RSP}\,(\mathcal{V}^\#) \ . \tag{27}$$

Moreover, $\mathcal{V}^\#$ is the unique *element of $\mathcal{D}(\mathcal{V})$ which attains the lower bound in* (27).

Proof. The inequality given in Eq. (27) is a direct consequence of (25). With respect to the uniqueness of $\mathcal{V}^\#$, fix another $\mathcal{W} \in \mathcal{D}(\mathcal{V})$. Then the equalities $T_\mathcal{W}^* T_\mathcal{V} = T_{\mathcal{V}^\#}^* T_\mathcal{V} = I$ imply that $T_\mathcal{W}^* = T_{\mathcal{V}^\#}^* + A$, for some $A \in L(\mathcal{K},\mathcal{H})$ that satisfies $A\,T_\mathcal{V} = 0$. With respect to $\mathcal{V}^\#$, note that $R(T_{\mathcal{V}^\#}) = R(T_\mathcal{V}\,S_\mathcal{V}^{-1}) = R(T_\mathcal{V}) \subseteq \ker A$, so that also $A\,T_{\mathcal{V}^\#} = 0$. Thus,

$$\begin{aligned} \text{tr } S_\mathcal{W} \ &= \|T_{\mathcal{V}^\#}^* + A\|_2^2 = \text{tr } \big(T_{\mathcal{V}^\#}^* T_{\mathcal{V}^\#}\big) + \text{tr } \big(AA^*\big) + 2 \text{ Re tr } \big(A\,T_{\mathcal{V}^\#}\big) \\ &= \text{tr } S_{\mathcal{V}^\#} + \|A\|_2^2 \ . \end{aligned} \tag{28}$$

On the other hand, if the lower bound in Eq. (27) is attained $\mathcal{W}$, using (25) we can deduce that $\lambda(S_\mathcal{W}) = \lambda(S_{\mathcal{V}^\#})$. Then also $\text{tr } S_\mathcal{W} = \text{tr } S_{\mathcal{V}^\#}$. But the previous equality forces that in this case $A = 0$ and hence $\mathcal{W} = \mathcal{V}^\#$. □

5.2 RS operators of projective systems

In this section we shall fix the parameters $(m, \mathbf{k}, d)$ and the sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ of weights. Now we give some new notations: First, recall that the set of projective RS's with fixed set of weights $\mathbf{v}$ is

$$\mathcal{PRS}_\mathbf{v} = \mathcal{PRS}_\mathbf{v}(m, \mathbf{k}, d) = \Big\{ \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS} : \ \|V_i\|_{sp} = v_i \text{ for every } i \in \mathbb{I}_m \Big\} .$$

We consider the set of operators $S_\mathcal{V}$ for $\mathcal{V} \in \mathcal{PRS}_\mathbf{v}$ and its spectral picture:

$$\mathcal{OP}_\mathbf{v} \stackrel{\text{def}}{=} \{S_\mathcal{V} : \ \mathcal{V} \in \mathcal{PRS}_\mathbf{v}\} \quad \text{and} \quad \Lambda(\mathcal{OP}_\mathbf{v}) \stackrel{\text{def}}{=} \{\lambda(S) : S \in \mathcal{OP}_\mathbf{v}\} \subseteq (\mathbb{R}^d_{>0})^\downarrow . \tag{29}$$

We shall give a characterization of the set $\Lambda(\mathcal{OP}_\mathbf{v})$ in terms of the Horn-Klyachko's theory of sums of hermitian matrices. In order to do this we shall describe briefly the basic facts about the spectral characterization obtained by Klyachko [21] and Fulton [18]. Let

$$\mathcal{K}^r_d = \big\{(j_1, \ldots, j_r) \in (\mathbb{I}_d)^r : \ j_1 < j_2 \ldots < j_r\big\} .$$

For $J = (j_1, \ldots, j_r) \in \mathcal{K}^r_d$, define the associated partition $\lambda(J) = (j_r - r, \ldots, j_1 - 1)$. For $r \in \mathbb{I}_{d-1}$ denote by $LR^r_d(m)$ the set of $(m+1)$-tuples $(J_0, \ldots, J_m) \in (\mathcal{K}^r_d)^{m+1}$, such that the Littlewood-Richardson coefficient of the associated partitions $\lambda(J_0), \ldots, \lambda(J_m)$ is positive, i.e. one can generate the Young diagram of $\lambda(J_0)$ from those of $\lambda(J_1), \ldots, \lambda(J_m)$ according to the Littlewood-Richardson rule (see [18]).

The theorem of Klyachko gives a characterization of the spectral picture of the set of all sums of m matrices in $\mathcal{H}(d)$ with fixed given spectra, in terms on a series of inequalities involving the $(m+1)$-tuples in $LR^r_d(m)$ (see [21] for a detailed formulation). We give a description of this result in the particular case where these m matrices are multiples of projections:

Lemma 5.7. Fix the parameters $(m, \mathbf{k}, d)$ and $\mathbf{v} \in \mathbb{R}^m_{>0}$ $\mu \in (\mathbb{R}^m_+)^\downarrow$. Then there exists a sequence $\{P_i\}_{i\in\mathbb{I}_m} \in \text{Gr}\,(\mathbf{k}\,,\,d)$ such that $\mu = \lambda\big(\sum_{i\in\mathbb{I}_m} v_i^2\, P_i\big)$ if and only if

$$\text{tr}\ \mu \ = \ \textstyle\sum_{i\in\mathbb{I}_m} v_i^2\, k_i \quad \text{and} \quad \sum_{i\in J_0} \mu_i \ \le \ \sum_{i\in\mathbb{I}_m} v_i^2\, |\, J_i \cap \mathbb{I}_{k_i}\,| \ , \tag{30}$$

for every $r \in \mathbb{I}_{d-1}$ and every $(m+1)$-tuple $(J_0, \ldots, J_m) \in LR^r_d(m)$. □

Proposition 5.8. *Fix the parameters $(m, \mathbf{k}, d)$ and the vector $\mathbf{v} \in \mathbb{R}^m_{>0}$ of weights. Fix also a positive matrix $S \in \mathcal{G}l(d)^+$. Then,*

$$S \in \mathcal{OP}_\mathbf{v} \iff \lambda(S) \in \Lambda(\mathcal{OP}_\mathbf{v}) \iff \lambda(S) \ \text{ satisfies Eq. (30)} .$$

Proof. The set $\mathcal{OP}_\mathbf{v} \subseteq \mathcal{G}l(d)^+$ is saturated by unitary equivalence. Indeed, if $\mathcal{V} \in \mathcal{PRS}_\mathbf{v}$ and $U \in \mathcal{U}(d)$, then $\mathcal{V} \cdot U \stackrel{\text{def}}{=} \{V_i\,U\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_\mathbf{v}$ and $U^*S_\mathcal{V}U = S_{\mathcal{V}\cdot U} \in \mathcal{OP}_\mathbf{v}$. This shows the first equivalence. On the other hand, using Lemma 4.4 and Eq. (17), we can assure that an ordered vector $\mu \in \Lambda(\mathcal{OP}_\mathbf{v})$ if and only if $\mu_d > 0$ and there exists a sequence of projections $\mathcal{P} = \{P_i\}_{i\in\mathbb{I}_m} \in \text{Gr}\,(\mathbf{k}\,,\,d)$ such that $\mu = \lambda(S_\mathbf{v}(\mathcal{P})) = \lambda\big(\sum_{i\in\mathbb{I}_m} v_i^2\, P_i\big)$. Hence, the second equivalence follows from Lemma 5.7. □

Corollary 5.9. *For every set $(m, \mathbf{k}, d)$ of parameters and every vector $\mathbf{v} \in \mathbb{R}^m_{>0}$ of weights,*

1. *The set $\Lambda(\mathcal{OP}_\mathbf{v})$ is convex.*

2. *Its closure* $\overline{\Lambda(\mathcal{OP}_{\mathbf{v}})}$ *is compact.*

3. *A vector* $\mu \in \overline{\Lambda(\mathcal{OP}_{\mathbf{v}})} \setminus \Lambda(\mathcal{OP}_{\mathbf{v}}) \iff \mu_d = 0$. *In other words,*

$$\overline{\Lambda(\mathcal{OP}_{\mathbf{v}})} \ \cap \ \mathbb{R}^m_{>0} = \Lambda(\mathcal{OP}_{\mathbf{v}}) \ . \tag{31}$$

Proof. Denote by $\mathcal{M}$ the set of vectors $\lambda \in (\mathbb{R}^d_+)^{\downarrow}$ which satisfies Eq. (30). It is clear that $\mathcal{M}$ is compact and convex. But Proposition 5.8 assures that $\Lambda(\mathcal{OP}_{\mathbf{v}}) = \mathcal{M} \cap \mathbb{R}^d_{>0} \subseteq \mathcal{M}$. This proves items 2 and 3. Item 1 follows by the fact that also $\mathbb{R}^d_{>0}$ is convex. □

Remark 5.10. With the notations of Corollary 5.9, actually $\overline{\Lambda(\mathcal{OP}_{\mathbf{v}})} = \mathcal{M}$. This fact is not obvious from the inequalities of Eq. (30), but can be deduced using Lemma 5.7. Indeed, it is clear that if $\mathcal{P} \in \mathrm{Gr}\,(\mathbf{k}\,,\,d)$ and $S_{\mathbf{v}}(\mathcal{P}) \notin \mathcal{G}l\,(d)^+$, then $S_{\mathbf{v}}(\mathcal{P})$ can be approximated by matrices $S_{\mathbf{v}}(\mathcal{Q})$ for sequences $\mathcal{Q} \in \mathrm{Gr}\,(\mathbf{k}\,,\,d)$ such that $S_{\mathbf{v}}(\mathcal{Q}) > 0$. Using Lemma 4.4 and Eq. (17), this means that these matrices $S_{\mathbf{v}}(\mathcal{Q}) \in \mathcal{OP}_{\mathbf{v}}$. △

6 Joint potential of projective RS's

Fix the parameters $(m, \mathbf{k}, d)$. We consider the set of dual pairs associated to $\mathcal{PRS}_{\mathbf{v}}$:

$$\mathcal{DP}_{\mathbf{v}} = \mathcal{DP}_{\mathbf{v}}\,(m, \mathbf{k}, d) \ \stackrel{\text{def}}{=} \ \Big\{ (\mathcal{V},\, \mathcal{W}) \in \mathcal{PRS}_{\mathbf{v}} \times \mathcal{RS} \ : \ \mathcal{W} \in \mathcal{D}(\mathcal{V}) \Big\} \ .$$

We consider on $\mathcal{DP}_{\mathbf{v}}$ the joint potential: Given $(\mathcal{V}\,,\,\mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$, let

$$\mathrm{RSP}\,(\mathcal{V}, \mathcal{W}) \ \stackrel{\text{def}}{=} \ \mathrm{RSP}\,(\mathcal{V}) + \mathrm{RSP}\,(\mathcal{W}) = \mathrm{tr}\ S_{\mathcal{V}}^2 + \mathrm{tr}\ S_{\mathcal{W}}^2 \in \mathbb{R}_{>0} \ . \tag{32}$$

We shall describe the structure of the minimizers of the joint potential both from a spectral and a geometrical point of view. We will denote by

$$p_{\mathbf{v}} = p_{\mathbf{v}}(m, \mathbf{k}, d) \ \stackrel{\text{def}}{=} \ \inf \{ \, \mathrm{RSP}\,(\mathcal{V}, \mathcal{W}) \ : \ (\mathcal{V},\, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}} \ \} \ . \tag{33}$$

Proposition 6.1. *For every set* $(m, \mathbf{k}, d)$ *of parameters, the following properties hold:*

1. *The infimum* $p_{\mathbf{v}}$ *in Eq.* (33) *is actually a* **minimum**.

2. *Let* $\tau = \sum_{i \in \mathbb{I}_m} v_i^2\, k_i$. *For every pair* $(\mathcal{V},\, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ *we have that*

$$\mathrm{RSP}\,(\mathcal{V},\, \mathcal{W}) \geq p_{\mathbf{v}} \geq \frac{\tau^4 + d^4}{d\,\tau^2} \ , \tag{34}$$

3. *This lower bound is attained if and only if* $\mathcal{V}$ *is tight* ($S_{\mathcal{V}} = \frac{\tau}{d}\, I_d$) *and* $\mathcal{W} = \frac{d}{\tau}\, \mathcal{V} = \mathcal{V}^{\#}$.

Proof. Given $(\mathcal{V},\, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$, Corollary 5.6 asserts that $\mathrm{RSP}\,(\mathcal{V},\, \mathcal{V}^{\#}) \leq \mathrm{RSP}\,(\mathcal{V},\, \mathcal{W})$ and also that equality holds only if $\mathcal{W} = \mathcal{V}^{\#}$. Thus

$$p_{\mathbf{v}} = \inf_{\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}} \mathrm{RSP}\,(\mathcal{V}, \mathcal{V}^{\#}) \stackrel{(5)}{=} \inf_{\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}} \ \sum_{i=1}^{d} \lambda_i(S_{\mathcal{V}})^2 + \lambda_i(S_{\mathcal{V}})^{-2} \ . \tag{35}$$

Consider the **strongly convex** map $F : \mathbb{R}^d_{>0} \to \mathbb{R}_{>0}$ given by $F(x) = \sum_{i=1}^d x_i^2 + x_i^{-2}$, for $x \in \mathbb{R}^d_{>0}$. Observe that $\mathrm{RSP}\,(\mathcal{V}, \mathcal{V}^{\#}) = F(\lambda(S_{\mathcal{V}})\,)$ for every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$. By Corollary 5.9

we know that $\Lambda(\mathcal{OP}_{\mathbf{v}})$ is convex subset of $(\mathbb{R}^d_{>0})^{\downarrow}$, and it becomes also compact under a restriction of the type $\lambda_d \geq \varepsilon$ (for any $\varepsilon > 0$). Since a strongly convex function defined in a compact convex set attains its local (and therefore global) minima at a unique point, it follows that there exists a unique $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \mathbf{k}, d) \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ such that

$$F(\,\lambda_{\mathbf{v}}\,) = \min_{\lambda \in \Lambda_{\mathbf{v}}(m,\mathbf{k},d)} F(\lambda) = p_{\mathbf{v}}\ . \tag{36}$$

This proves item 1. Moreover, using Lagrange multipliers it is easy to see that the restriction of F to the set $(\mathbb{R}^d_{>0})_\tau := \{\mathbf{x} \in \mathbb{R}^d_{>0} \,:\, \operatorname{tr}(\mathbf{x}) = \tau\,\}$ reaches its minimum in $\mathbf{x} = \frac{\tau}{d} \cdot \mathbb{1}$. Since $\Lambda(\mathcal{OP}_{\mathbf{v}}) \subset (\mathbb{R}^d_{>0})_\tau$ we get that

$$\operatorname{RSP}(\mathcal{V},\, \mathcal{V}^{\#}) = F(\lambda(S_{\mathcal{V}})\,) \geq F(\frac{\tau}{d} \cdot \mathbb{1}) = \frac{\tau^4 + d^4}{d\,\tau^2} \qquad \text{for every} \qquad \mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}\ ,$$

and this lower bound is attained if and only if $\lambda(S_{\mathcal{V}}) = \frac{\tau}{d} \cdot \mathbb{1}_d$. Note that in this case $S_{\mathcal{V}} = \frac{\tau}{d}\, I_d$, and therefore $\mathcal{V}^{\#} = \frac{d}{\tau}\, \mathcal{V}$. $\square$

Recall that we use in $\mathcal{RS}$ the metric $d_P(\mathcal{V},\, \mathcal{W}) = (\,\sum\limits_{i \in \mathbb{I}_m} \|V_i - W_i\|_2^2\,)^{1/2} = \|T^*_{\mathcal{V}} - T^*_{\mathcal{W}}\|_2$ and the pseudometric $d_S(\mathcal{V},\, \mathcal{W}) = \|S_{\mathcal{V}} - S_{\mathcal{W}}\|$ for pairs $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m}$ and $\mathcal{W} = \{W_i\}_{i \in \mathbb{I}_m} \in \mathcal{RS}$.

Lemma 6.2. *If a pair $(\mathcal{V},\, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is local d_P-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$, then $\mathcal{W} = \mathcal{V}^{\#}$.*

Proof. We have shown in Eq. (9) that, since $\mathcal{W} \in \mathcal{D}(\mathcal{V})$, then $T^*_{\mathcal{W}} = T^*_{\mathcal{V}^\#} + A$, for some $A \in L(\mathcal{K}, \mathcal{H})$ such that $A\,T_{\mathcal{V}} = A\,T_{\mathcal{V}^\#} = 0 \in L(\mathcal{H})$. Recall from Remark 2.6 that the set $\mathcal{D}(\mathcal{V})$ is convex. Then the line segment $\mathcal{W}_t = t\mathcal{W} + (1-t)\mathcal{V}^{\#} \in \mathcal{D}(\mathcal{V})$ satisfies that $T^*_{\mathcal{W}_t} = T^*_{\mathcal{V}^\#} + tA$ for every $t \in [0, 1]$. Then, as in Eq. (28), $S_{\mathcal{W}_t} = S_{\mathcal{V}^\#} + t^2\, AA^*$ and

$$K(t) \stackrel{\text{def}}{=} \operatorname{RSP}(\mathcal{V}\,,\, \mathcal{W}_t) = \operatorname{RSP}(\mathcal{V}\,,\, \mathcal{V}^{\#}) + t^4 \operatorname{tr}(AA^*)^2 + 2\,t^2 \operatorname{tr}\, T_{\mathcal{V}^\#} AA^* T^*_{\mathcal{V}^\#}\ ,$$

for every $t \in [0, 1]$. Observe that $K(1) = \operatorname{RSP}(\mathcal{V}\,,\, \mathcal{W})$. But taking one derivative of K, one gets that if $A \neq 0$ then K is strictly increasing near $t = 1$, which contradicts the local d_P-minimality for $(\mathcal{V}\,,\, \mathcal{W})$. Therefore $T^*_{\mathcal{W}_t} = T^*_{\mathcal{V}^\#}$ and $\mathcal{W} = \mathcal{V}^{\#}$. $\square$

Theorem 6.3. *For every set $(m, \mathbf{k}, d)$ of parameters there exists $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \mathbf{k}, d) \in (\mathbb{R}^d_{>0})^{\downarrow}$ such that the following conditions are equivalent for pair $(\mathcal{V},\, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$:*

1. *$(\mathcal{V},\, \mathcal{W})$ is local d_S-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$.*
2. *$(\mathcal{V},\, \mathcal{W})$ is global minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$.*
3. *It holds that $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$ and $\mathcal{W} = \mathcal{V}^{\#}$.*

Proof. Take the vector $\lambda_{\mathbf{v}}$ defined in Eq. (36). In the proof of Proposition 6.1 we have already seen that a pair $(\mathcal{V},\, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is a global minimizer for RSP $\iff \mathcal{W} = \mathcal{V}^{\#}$ and $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$. This means that $2 \iff 3$.

Suppose now that $(\mathcal{V},\, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is a local d_S-minimizer. By Remark 4.9 we know that it is also a local d_P-minimizer and by Lemma 6.2 we have that $\mathcal{W} = \mathcal{V}^{\#}$. In this case, denote $\lambda = \lambda(S_{\mathcal{V}})$ and take $U \in \mathcal{U}(d)$ such that $U^* D_\lambda U = S_{\mathcal{V}}$. Consider the segment line

$$h(t) = t\,\lambda_{\mathbf{v}} + (1-t)\,\lambda \quad \text{ for every } \quad t \in [0, 1]\ .$$

Then $h(t) \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ for every $t \in [0,1]$, since $\Lambda(\mathcal{OP}_{\mathbf{v}})$ is a convex set (Corollary 5.9). Consider the continuous curve $S_t = U^* D_{h(t)} U$ in $\mathcal{OP}_{\mathbf{v}}$ and a (not necessarily continuous) curve $\mathcal{V}_t \in \mathcal{PRS}_{\mathbf{v}}$ such that $S_0 = S_{\mathcal{V}}$, $\mathcal{V}_0 = \mathcal{V}$ and $S_{\mathcal{V}_t} = S_t$ for every $t \in [0,1]$. Nevertheless, since the curve S_t is continuous, we can assure that the map $t \mapsto \mathcal{V}_t$ is d_S-continuous.

Finally, we can consider the map $G : [0,1] \to \mathbb{R}$ given by

$$G(t) = \text{RSP}\,(\mathcal{V}_t\,,\, \mathcal{V}_t^{\#}) = \text{tr }\, S_t^2 + \text{tr }\, S_t^{-2} = \sum_{i=1}^{d} h_i(t)^2 + h_i(t)^{-2} = F(h(t)\,)$$

for $t \in [0,1]$, where F is the map defined after Eq. (35). Observe that $G(0) = \text{RSP}\,(\mathcal{V}\,,\, \mathcal{V}^{\#})$ and $G(1) = p_{\mathbf{v}}$, by Eq. (36). Then G has local minima at $t = 0$ and $t = 1$. By computing the second derivative of G in terms of the Hessian of F, we deduce that G must be constant, because otherwise it would be strictly convex. From this fact we can see that the map h is also constant, so that $\lambda_{\mathbf{v}} = \lambda$. Therefore $(\mathcal{V}, \mathcal{W}) = (\mathcal{V}\,,\, \mathcal{V}^{\#})$ is a global minimizer. $\square$

Recall that a system $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}$ is **irreducible** if $C_{\mathcal{V}} = \{V_i^* V_i : i \in \mathbb{I}_m\}' = \mathbb{C}\, I_d$.

Lemma 6.4. *Fix the set $(m, \mathbf{k}, d)$ of parameters and the weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$. Assume that $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is irreducible. Then the following conditions are equivalent:*

1. *The pair $(\mathcal{V}\,,\, \mathcal{V}^{\#})$ is local d_P-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$.*
2. *The pair $(\mathcal{V}\,,\, \mathcal{V}^{\#})$ is global minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$.*
3. *The system $\mathcal{V}$ is tight, i.e. $S_{\mathcal{V}} = \frac{\tau}{d}\, I_d$.*

Therefore in this case the vector $\lambda_{\mathbf{v}}$ of Theorem 6.3 is $\lambda_{\mathbf{v}} = \frac{\tau}{d}\, \mathbb{1}_d$.

Proof. Since $C_{\mathcal{V}} = \mathbb{C}\, I_d$, we can apply Theorem 4.8. Then the map $\text{RSO} : \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l(d)^+_\tau$ defined in Eq. (12) has a smooth local cross section around $S_{\mathcal{V}}$ which sends $S_{\mathcal{V}}$ to $\mathcal{V}$. Assume that there exists no $\sigma \in \mathbb{R}_{>0}$ such that $S_{\mathcal{V}} = \sigma\, I_d$. In this case there exist $\alpha, \beta \in \sigma(S_{\mathcal{V}})$ such that $\beta > \alpha > 0$. Consider the map $g : [0, \frac{\beta-\alpha}{2}] \to \mathbb{R}_{>0}$ given by

$$g(t) = (\alpha + t)^2 + (\alpha + t)^{-2} + (\beta - t)^2 + (\beta - t)^{-2}\ .$$

Then $g'(0) = 2(\alpha - \beta) - 2(\frac{1}{\beta} - \frac{1}{\alpha}) < 0$, which shows that we can construct a continuous curve $M : [\,0\,,\, \varepsilon\,] \to \mathcal{G}l(d)^+_\tau$ such that $M(0) = S_{\mathcal{V}}$ and

$$\text{tr }\, M(t)^2 + \text{tr }\, M(t)^{-2} \;<\; \text{tr }\, S_{\mathcal{V}}^2 + \text{tr }\, S_{\mathcal{V}}^{-2} = \text{RSP}\,(\mathcal{V}\,,\, \mathcal{V}^{\#}) \quad \text{for every} \quad t \in (0\,,\, \varepsilon\,]\ .$$

Hence, using the continuous local cross section mentioned before, we can construct a d_P-continuous curve $\mathcal{M} : [0\,,\, \delta\,] \to \mathcal{PRS}_{\mathbf{v}}$ such that $\text{RSO} \circ \mathcal{M} = M$, $\mathcal{M}(0) = \mathcal{V}$ and

$$\text{RSP}\,(\mathcal{M}(t)\,,\, \mathcal{M}(t)^{\#}) = \text{tr }\, M(t)^2 + \text{tr}\, M(t)^{-2} \;<\; \text{RSP}\,(\mathcal{V}\,,\, \mathcal{V}^{\#}) \quad \text{for} \quad t \in (0\,,\, \delta\,]\ .$$

This shows that $(\mathcal{V}\,,\, \mathcal{V}^{\#})$ is not a local d_P-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$. We have proved that $1 \implies 3$. Note that $3 \implies 2$ follows from (34) and $2 \implies 1$ is trivial. $\square$

Remark 6.5. It is easy to see that, if the parameters $(m\,,\, \mathbf{k}\,,\, d)$ allow the existence of at least one irreducible projective RS, then the set of irreducible systems becomes open and dense in $\mathcal{PRS}_{\mathbf{v}}\,(m\,,\, \mathbf{k}\,,\, d)$. Nevertheless, it is not usual that the minimizers are irreducible, even if they are tight (see Remark 6.7 and Examples 7.1 and 7.2).

On the other hand, if the system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is reducible, there exists a system $\mathcal{Q} = \{Q_j\}_{j\in\mathbb{I}_p}$ of minimal projections of the unital C^*-algebra $C_{\mathcal{V}}$ (with $p > 1$). This means that

- Each $Q_j \in C_{\mathcal{V}}$, and $Q_j^2 = Q_j^* = Q_j$.
- $\mathcal{Q}$ is a system of projections: $Q_j\,Q_k = 0$ if $j \neq k$ and $\sum_{j\in\mathbb{I}_p} Q_j = I_{\mathcal{H}}$.
- Minimality: The algebra $C_{\mathcal{V}}$ has no proper sub projection of any Q_j .

By compressing the system $\mathcal{V}$ to each subspace $\mathcal{H}_j = R(Q_j)$ in the obvious way, it can be shown that every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is an "orthogonal sum" of irreducible subsystems.

Another system of projections associated with $\mathcal{V}$ are the spectral projections of $S_{\mathcal{V}}$: If $\sigma(S_{\mathcal{V}}) = \{\sigma_1 , \dots , \sigma_r\}$, we denote these projections by

$$P_{\sigma_j} = P_{\sigma_j}(S_{\mathcal{V}}) \stackrel{\text{def}}{=} P_{\ker (S-\sigma_j\, I_d)} \in \mathcal{M}_d(\mathbb{C})^+ \ , \quad \text{for} \quad j \in \mathbb{I}_r \ .$$

Recall that $S_{\mathcal{V}}\,P_{\sigma_j} = \sigma_j\,P_{\sigma_j}$ and $\sum_{j=1}^r P_{\sigma_j} = I_d$, so that $S_{\mathcal{V}} = \sum_{j=1}^r \sigma_j\,P_{\sigma_j}$. △

Theorem 6.6. *Fix $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}_{>0}^m$. Let $(\mathcal{V}, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ be a d_P-local minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$ with $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$. Then*

1. *The RS operator $S_{\mathcal{V}} \in C_{\mathcal{V}} = \{V_i^*V_i :\ i \in \mathbb{I}_m\}'$.*
2. *If $\sigma(S_{\mathcal{V}}) = \{\sigma_1, \dots , \sigma_r\}$, then also $P_{\sigma_i} = P_{\sigma_i}(S_{\mathcal{V}}) \in C_{\mathcal{V}}$ for every $i \in \mathbb{I}_r$.*

Proof. Recall that $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}} \subseteq \mathcal{RS}$ and hence $0 \notin \sigma(S_{\mathcal{V}})$. On the other hand, we have already seen in Lemma 6.2 that $\mathcal{W}$ must be $\mathcal{V}^{\#}$. Let $\mathcal{Q} = \{Q_j\}_{j\in\mathbb{I}_p}$ be a system of minimal projections of the unital C^*-algebra $C_{\mathcal{V}}$, as in Remark 6.5.

Fix $j \in \mathbb{I}_p$ and denote by $\mathcal{S}_j = R(Q_j)$. For every $i \in \mathbb{I}_m$ put $\mathcal{T}_i = V_i(\mathcal{S}_j) \subseteq \mathcal{K}_i$, $t_i = \dim \mathcal{T}_i$ and $W_i = V_iQ_j \in L(\mathcal{H}_j , \mathcal{T}_i)$. Since $Q_j \in C_{\mathcal{V}}$ then each matrix $v_i^{-1}\,W_i^*$ is an isometry, so that the compression of $\mathcal{V}$ given by $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}\,(m\,,\,\mathbf{t}\,,\,s_j)$, where $\mathbf{t} = (t_1 , \dots , t_m)$ and $s_j = \dim \mathcal{S}_j$. Recall that $S_{\mathcal{V}}$ commutes with Q_j . This implies that $\mathcal{W}^{\#}$ is the same type of compression to $\mathcal{RS}_{\mathbf{v}}\,(m\,,\,\mathbf{t}\,,\,s_j)$ of the system $\mathcal{V}^{\#}$.

A straightforward computation shows that the pair $(\mathcal{W}, \mathcal{W}^{\#}) \in \mathcal{DP}_{\mathbf{v}}\,(m\,,\,\mathbf{t}\,,\,s_j)$ is still a d_P-local minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}\,(m\,,\,\mathbf{t}\,,\,s_j)$. Indeed, the key argument is that one can "complete" other systems in $\mathcal{PRS}_{\mathbf{v}}\,(m\,,\,\mathbf{t}\,,\,s_j)$ near $\mathcal{W}$ (and acting in $\mathcal{S}_j$) with the fixed orthogonal complement $\{V_i(I_d - Q_j)\}_{i\in\mathbb{I}_m}$, getting systems in $\mathcal{PRS}_{\mathbf{v}}\,(m\,,\,\mathbf{k}\,,\,d)$ near $\mathcal{V}$. It is easy to see that all the computations involved in the joint potential work independently on each orthogonal subsystem. This shows the minimality of $(\mathcal{W}, \mathcal{W}^{\#})$.

Observe that $W_i^*W_i = Q_jV_i^*V_iQ_j = V_i^*V_iQ_j$ for every $i \in \mathbb{I}_m$. Therefore, the minimality of Q_j in $C_{\mathcal{V}}$ shows that the system $\mathcal{W}$ satisfies that $C_{\mathcal{W}} = \mathbb{C}\,I_{\mathcal{S}_j}$. Hence, we can apply Lemma 6.4 on $\mathcal{S}_j$, and get that $S_{\mathcal{W}} = \alpha_j\,I_{\mathcal{S}_j}$ for some $\alpha_j > 0$. But when we return to $L(\mathcal{H})$, we get that $S_{\mathcal{V}}\,Q_j = \sum_{i\in\mathbb{I}_m} V_i^*V_iQ_j = \sum_{i\in\mathbb{I}_m} W_i^*W_i = S_{\mathcal{W}} = \alpha_j\,Q_j$. In particular, $\alpha_j \in \sigma(S_{\mathcal{V}})$.

We have proved that for every $j \in \mathbb{I}_p$ there exists $\alpha_j \in \sigma(S_{\mathcal{V}})$ such that $S_{\mathcal{V}}\,Q_j = \alpha_j\,Q_j$ and hence each projector $Q_j \leq P_{\alpha_j} = P_{\alpha_j}(S_{\mathcal{V}})$. Using that $\sum_{j\in\mathbb{I}_p} Q_j = I_d$ we see that each

$$P_{\sigma_k} = \sum_{j\in J_k} Q_j \in C_{\mathcal{V}} \ , \quad \text{where} \quad J_k = \{j \in \mathbb{I}_p :\ \alpha_j = \sigma_k\} \ . \tag{37}$$

Therefore also $S_{\mathcal{V}} = \sum_{k\in\mathbb{I}_r} \sigma_k\,P_{\sigma_k} \in C_{\mathcal{V}}$. □

Remark 6.7. Theorem 6.6 assures that if $(\mathcal{V}, \mathcal{V}^{\#})$ is a d_P-local minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$, then $\mathcal{V}$ is an orthogonal sum of **tight** systems in the following sense:

If $\sigma(S_{\mathcal{V}}) = \{\sigma_1, \dots, \sigma_r\}$, and we denote $\mathcal{H}_j = R(P_{\sigma_j}) = \ker(S - \sigma_j I_d)$ for every $j \in \mathbb{I}_r$, then $\mathcal{H} = \bigoplus_{j\in\mathbb{I}_r} \mathcal{H}_j$. By Theorem 6.6 each $P_{\sigma_j} \in C_{\mathcal{V}}$. Then, putting $d_j = \dim \mathcal{H}_j$,

$$\mathcal{K}_{i,j} = V_i(\mathcal{H}_j) \subseteq \mathcal{K}_i \quad , \quad k_{i,j} = \dim \mathcal{K}_{i,j} \quad \text{and} \quad \mathbf{k}^j = (k_{1,j}, \dots, k_{m,j}) \quad ,$$

for every $i \in \mathbb{I}_m$ and $j \in \mathbb{I}_r$, we can define the the tight compression of $\mathcal{V}$ to each $\mathcal{H}_j$:

$$\mathcal{V}^j = \{V_i P_{\sigma_j}\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}(m, \mathbf{k}^j, d_j) \quad \text{for} \quad j \in \mathbb{I}_r .$$

Indeed, since $P_{\sigma_j} \in C_{\mathcal{V}}$ then $\mathcal{V}^j$ is projective. Also $S_{\mathcal{V}^j} = S_{\mathcal{V}} P_{\sigma_j} = \sigma_j P_{\sigma_j}$, which means that $\mathcal{V}^j$ is σ_j - tight. Observe that the decomposition of each $\mathcal{V}^j$ into irreducible tight systems (as in Remark 6.5) follows from the orthogonal decomposition of $\mathcal{H}_j$ given in Eq. (37).

In particular, every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ such that $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$ (the unique vector of Theorem 6.3) must have this structure, because in this case $(\mathcal{V}, \mathcal{V}^{\#})$ is a d_S (hence also d_P) local minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$. Observe that the structure of all global minimizers $\mathcal{V}$ is almost the same: Since $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$, the number r of tight components, the sizes d_j and the tight constants σ_j for each space $\mathcal{H}_j$ coincide for every such minimizer $\mathcal{V}$.

Note that, if such a $\mathcal{V}$ is not tight, then it can not be irreducible. On the other hand, its dual $\mathcal{V}^{\#}$ can only be projective if $V_i P_{\sigma_j} = 0$ or V_i for every $i \in \mathbb{I}_m$ and $j \in \mathbb{I}_r$. △

7 Examples and conclusions

The following two examples are about irreducible systems.

Example 7.1. Let $d = k_1 + k_2$ and $\mathbf{k} = (k_1, k_2)$. Assume that $k_1 > k_2$. We shall see that, in this case, there is no irreducible (Riesz) systems in $\mathcal{PRS}(2, \mathbf{k}, d)$. Observe that the situation is the same whatever the weights (v_1, v_2) are.

Indeed, if $\mathcal{V} = (V_1, V_2) \in \mathcal{PRS}_{\mathbb{1}}(2, \mathbf{k}, d)$, let $\mathcal{S}_i = R(V_i^*)$ and $P_i = P_{\mathcal{S}_i} = V_i^* V_i$ for $i = 1, 2$. Then $\mathbb{C}^d = \mathcal{S}_1 \oplus \mathcal{S}_2$ (not necessarily orthogonal). Observe that $\dim S_1 = \dim S_2^{\perp} = k_1$ and $2k_1 > d$. Hence $\mathcal{T} = \mathcal{S}_1 \cap \mathcal{S}_2^{\perp} \neq \{0\}$. Since $P = P_{\mathcal{T}} \leq P_1$ and $P \leq I_d - P_2$, then $P \in C_{\mathcal{V}}$ and $0 \neq P \neq I_d$. Therefore $C_{\mathcal{V}} \neq \mathbb{C} I_d$.

In particular, if the decomposition $\mathbb{C}^d = \mathcal{S}_1 \oplus \mathcal{S}_2$ is orthogonal, then $S_{\mathcal{V}} = P_1 + P_2 = I_d$. So, in this case $\mathcal{V}$ is tight and reducible. △

Example 7.2. If $m \geq d$ and $\mathbf{k} = \mathbb{1}_m$, then $\mathcal{PRS}(m, \mathbf{k}, d)$ is the set of m-vector frames for the space $\mathbb{C}^d$. In this case $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}$ is reducible $\iff$ there exists $J \subseteq \mathbb{I}_m$ such that $\emptyset \neq J \neq \mathbb{I}_m$ and the subspaces $\operatorname{span}\{f_i : i \in J\}$ and $\operatorname{span}\{f_j : j \notin J\}$ are orthogonal.

Indeed, if $A = A^*$, then $A \in C_{\mathcal{F}} \iff$ every f_i is an eigenvector of A. But different eigenvalues of A must have orthogonal subspaces of eigenvectors. Observe that in this case the set of irreducible systems is an open and dense subset of $\mathcal{PRS}_{\mathbf{v}}$, since it is the intersection of $2^m - 2$ open dense sets (one for each fixed nontrivial $J \subseteq \mathbb{I}_m$). △

7.3. Minimizers and majorization: Theorem 6.3 states that there exists a vector $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \mathbf{k}, d) \in (\mathbb{R}^d_{>0})^{\downarrow}$ such that a system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}(m, \mathbf{k}, d)$ satisfies that $(\mathcal{V}, \mathcal{V}^{\#})$ is a global minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$ if and only if $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$. This vector is found as the unique minimizer of the map $F(\lambda) = \sum_{i=1}^d \lambda_i^2 + \lambda_i^{-2}$ on the convex set $\Lambda(\mathcal{OP}_{\mathbf{v}})$.

In all the examples where $\lambda_{\mathbf{v}}$ could be explicitly computed, it satisfied a stronger condition, in terms of majorization (see [5, Cap. II] for definitions and basic properties). We shall see that in these examples there is a vector $\lambda \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ such that

$$\lambda \prec \lambda(S_{\mathcal{V}}) \quad \text{for every} \quad \mathcal{V} \in \mathcal{PRS}_{\mathbf{v}} \quad (\text{the symbol } \prec \text{ means majorization}) . \tag{38}$$

Observe that such a vector $\lambda \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ must be the unique minimizer for F on $\Lambda(\mathcal{OP}_{\mathbf{v}})$, since the map F is permutation invariant and convex. Hence $\lambda = \lambda_{\mathbf{v}}$. Moreover, those cases where $\lambda_{\mathbf{v}}$ satisfies Eq. (38) have some interesting properties regarding the structure of minimizers of the joint potential. For example, that $\lambda_{t\mathbf{v}}(m, \mathbf{k}, d) = t^2 \lambda_{\mathbf{v}}(m, \mathbf{k}, d)$ for $t > 0$, a fact that is not evident at all from the properties of these vectors. $\triangle$

Conjecture 7.4. For every set of parameters $(m, \mathbf{k}, d)$ and $\mathbf{v} \in \mathbb{R}^d_{>0}$, the vector $\lambda_{\mathbf{v}}(m, \mathbf{k}, d)$ of Theorem 6.3 satisfies the majorization minimality of Eq. (38) on $\Lambda(\mathcal{OP}_{\mathbf{v}})$. $\triangle$

Example 7.5. Given $\mathbf{v} = \mathbf{v}^{\downarrow} \in \mathbb{R}^m_{>0}$ and $d \leq m$, the d-irregularity of $\mathbf{v}$ is the index

$$r \;=\; r_d(\mathbf{v}) \;\stackrel{\text{def}}{=}\; \max \left\{ j \in \mathbb{I}_{d-1} \,:\, (d-j)\, v_j^2 \;>\; \textstyle\sum_{i=j+1}^m v_i^2 \right\} ,$$

or $r = 0$ if this set is empty. In [24, Prop. 2.3] (see also [2, Prop. 4.5]) it is shown that for any set of parameters $(m\,,\, \mathbb{1}_m\,,\, d)$ and every $\mathbf{v} = \mathbf{v}^{\downarrow} \in \mathbb{R}^m_{>0}$, there is $c \in \mathbb{R}$ such that

$$\lambda_{\mathbf{v}}(m, d) \;\stackrel{\text{def}}{=}\; (v_1^2\,,\, \dots\,,\, v_r^2\,,\, c\,\mathbb{1}_{d-r}) \in \Lambda(\mathcal{OP}_{\mathbf{v}}(m, \mathbb{1}_m\,,\, d))$$

and it satisfies Eq. (38). Therefore $\lambda_{\mathbf{v}}(m, d) = \lambda_{\mathbf{v}}(m, \mathbb{1}_m\,,\, d)$ by 7.3. Thus, in the case of vector frames, Conjecture 7.4 is known to be true. $\triangle$

In the following examples we shall compute explicitly the the vector $\lambda_{\mathbf{v}}$ and the global minimizers of the joint potential in $\mathcal{PRS}_{\mathbf{v}}$. Since we shall use Eq. (38) as our main tool (showing Conjecture 7.4 in these cases), we need a technical result about majorization, similar to [23, Lemma 2.2]. Recall that the symbol $\prec_w$ means weak majorization.

Lemma 7.6. Let $\alpha\,,\, \gamma \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$ and $b \in \mathbb{R}$ such that $b \leq \min_{k \in \mathbb{I}_n} \gamma_k$. Then, if

$$\operatorname{tr}(\gamma\,,\, b\,\mathbb{1}_m) \;\leq\; \operatorname{tr}(\alpha\,,\, \beta) \quad \text{and} \quad \gamma \prec_w \alpha \;\implies\; (\gamma\,,\, b\,\mathbb{1}_m) \prec_w (\alpha\,,\, \beta) .$$

Observe that we are not assuming that $(\alpha\,,\, \beta) = (\alpha\,,\, \beta)^{\downarrow}$.

Proof. Let $h = \operatorname{tr} \beta$ and $\rho = \frac{h}{m}\,\mathbb{1}_m$. Then it is easy to see that

$$\textstyle\sum_{i \in \mathbb{I}_k} (\gamma^{\downarrow}\,,\, b\,\mathbb{1}_m)_i \;\leq\; \sum_{i \in \mathbb{I}_k} (\alpha^{\downarrow}\,,\, \rho)_i \;\leq\; \sum_{i \in \mathbb{I}_k} (\alpha^{\downarrow}\,,\, \beta^{\downarrow})_i \quad \text{for every} \quad k \in \mathbb{I}_{n+m} .$$

Since $(\gamma^{\downarrow}\,,\, b\,\mathbb{1}_m) = (\gamma\,,\, b\,\mathbb{1}_m)^{\downarrow}$, we can conclude that $(\gamma\,,\, b\,\mathbb{1}_m) \prec_w (\alpha\,,\, \beta)$. $\square$

Example 7.7. Assume that $\operatorname{tr} \mathbf{k} = d$. Then the elements of $\mathcal{PRS}_{\mathbf{v}}(m\,,\, \mathbf{k}\,,\, d)$ are **Riesz** systems. Assume that the weights are ordered in such a way that $\mathbf{v} = \mathbf{v}^{\downarrow}$. We shall see that the vector $\lambda = (v_1^2\,\mathbb{1}_{k_1}\,,\, \dots\,,\, v_m^2\,\mathbb{1}_{k_m}) \prec \lambda(S_{\mathcal{V}})$ for every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}(m\,,\, \mathbf{k}\,,\, d)$. Hence λ satisfies Eq. (38), and $\lambda_{\mathbf{v}} = \lambda$ by 7.3.

Indeed, given $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}$, consider the projections $P_i = v_i^{-2}\, V_i^* V_i$ and denote by $\mathcal{S}_i = R(P_i)$ for every $i \in \mathbb{I}_m$. Then $S_{\mathcal{V}} = \sum_{i \in \mathbb{I}_m} v_i^2\, P_i$ and $\mathbb{C}^d = \bigoplus_{i \in \mathbb{I}_m} \mathcal{S}_i$ where the direct sum is not necessarily orthogonal. Let

$$\mathcal{S} = \textstyle\bigoplus_{i \in \mathbb{I}_{m-1}} \mathcal{S}_i \subseteq \mathbb{C}^d \quad , \quad P = P_{\mathcal{S}} \quad \text{and} \quad Q = I_d - P = P_{\mathcal{S}^{\perp}} .$$

Consider the restriction $A = \sum_{i=1}^{m-1} v_i^2 \, P_i \in L(\mathcal{S})^+$. It is well known that the pinching matrix

$$M = P \, S_{\mathcal{V}} \, P + Q \, S_{\mathcal{V}} \, Q = \begin{array}{c} \left[\begin{array}{cc} A + v_m^2 \, PP_2P & 0 \\ 0 & v_m^2 \, QP_mQ \end{array} \right] \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \end{array}$$

satisfies that $\lambda(M) \prec \lambda(S_{\mathcal{V}})$. Using an inductive argument on m (the case $m = 1$ is trivial), for the Riesz system $\mathcal{V}_0 = \{V_i\big|_{\mathcal{S}}\}_{i \in \mathbb{I}_{m-1}}$ (for $\mathcal{S}$) such that $S_{\mathcal{V}_0} = A$, we can assure that

$$\gamma = (v_1^2 \, \mathbb{1}_{k_1} \, , \, \dots \, , \, v_{m-1}^2 \, \mathbb{1}_{k_{m-1}}) \prec \lambda(A) \prec_w \lambda\big(A + v_m^2 \, PP_2P \big) = \alpha \quad \text{in} \quad \mathbb{R}^{d-k_m} \ .$$

Since $v_m \leq v_{m-1}$, Lemma 7.6 assures that $\lambda = (\gamma \, , \, v_m^2 \, \mathbb{1}_{k_m}) \prec (\alpha \, , \beta) = \lambda(M)$, where $\beta = \lambda(v_m^2 \, QP_mQ) \in \mathbb{R}^{k_m}$. Hence, we have proved that $\lambda \prec \lambda(S_{\mathcal{V}})$.

Recall a system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is a minimizer if and only if $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}} = \lambda$. Now, it is easy to see that $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$ if and only if the projections P_i are mutually orthogonal. $\triangle$

Example 7.8. Assume that the parameters $(m \, , \mathbf{k} \, , d)$ satisfy that

$$m = 2 \quad \text{and} \quad \operatorname{tr} \mathbf{k} = k_1 + k_2 > d \quad , \quad \text{but} \quad k_1 \neq d \neq k_2 \ .$$

Fix $\mathbf{v} = (v_1 \, , v_2)$ with $v_1 \geq v_2$. For the space $\mathcal{PRS}_{\mathbf{v}} \, (2 \, , \mathbf{k} \, , d)$ the vector $\lambda_{\mathbf{v}}$ of Theorem 6.3 and all the global minimizers of the joint potential can be computed: Denote by

$$r_0 = k_1 + k_2 - d \quad , \quad r_1 = k_1 - r_0 \quad \text{and} \quad r_2 = k_2 - r_0 \ .$$

We shall see that the vector $\mu = (\, (v_1^2 + v_2^2) \, \mathbb{1}_{r_0} \, , \, v_1^2 \, \mathbb{1}_{r_1} \, , \, v_2^2 \, \mathbb{1}_{r_2})$ satisfies Eq. (38), so that $\lambda_{\mathbf{v}}(2 \, , \mathbf{k} \, , d) = \mu$ by 7.3. Moreover, the minimizers are those systems $\mathcal{V} = (V_1 \, , V_2) \in \mathcal{PRS}_{\mathbf{v}}$ such that the two projections $P_i = v_i^{-2} \, V_i^* V_i$ (for $i = 1, 2$) commute.

Indeed, if $\mathcal{S}_i = R(P_i) = R(V_i^*)$ for $i = 1, 2$, then $\mathcal{M}_0 = \mathcal{S}_1 \cap \mathcal{S}_2$ has $\dim \mathcal{M}_0 = r_0$. Also $\mathcal{M}_i = \mathcal{S}_i \ominus \mathcal{M}_0$ have $\dim \mathcal{M}_i = r_i$ for $i = 1, 2$. Hence $\mathbb{C}^d = \mathcal{M}_0 \perp (\mathcal{M}_1 \oplus \mathcal{M}_2)$ and

$$S_{\mathcal{V}} = v_1^2 \, P_1 + v_1^2 \, P_2 = (v_1^2 + v_2^2) \, P_{\mathcal{M}_0} + v_1^2 \, P_{\mathcal{M}_1} + v_1^2 \, P_{\mathcal{M}_2} \ .$$

Note that $\mathcal{M}_1 \perp \mathcal{M}_2 \iff P_1 \, P_2 = P_2 \, P_1 = P_{\mathcal{M}_0}$. In this case $\lambda(S_{\mathcal{V}}) = \mu$. Otherwise, still $S_{\mathcal{V}}\big|_{\mathcal{M}_0} = (v_1^2 + v_2^2) \, I_{\mathcal{M}_0}$ and $S_{\mathcal{V}}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathcal{M}_1 \oplus \mathcal{M}_2$. Hence, if we denote by $T = S_{\mathcal{V}}\big|_{\mathcal{M}_1 \oplus \mathcal{M}_2} = (v_1^2 \, P_{\mathcal{M}_1} + v_1^2 \, P_{\mathcal{M}_2})\big|_{\mathcal{M}_1 \oplus \mathcal{M}_2} \in \mathcal{G}l(\mathcal{M}_1 \oplus \mathcal{M}_2)^+$, then $\|T\|_{sp} \leq v_1^2 + v_2^2$ and

$$S_{\mathcal{V}} = \left[\begin{array}{cc} (v_1^2 + v_2^2) \, I_{r_0} & 0 \\ 0 & T \end{array} \right] \begin{array}{c} \mathcal{M}_0 \\ \mathcal{M}_0^\perp \end{array} \quad \text{with} \quad \lambda(S_{\mathcal{V}}) = (\, (v_1^2 + v_2^2) \, \mathbb{1}_{r_0} \, , \, \lambda(T) \,) \in (\mathbb{R}_{>0}^d)^\downarrow \ .$$

Using Example 7.7 for the space $\mathcal{M}_1 \oplus \mathcal{M}_2$, we can deduce that $(v_1^2 \, \mathbb{1}_{r_1} \, , \, v_2^2 \, \mathbb{1}_{r_2}) \prec \lambda(T)$. Therefore also $\mu = (\, (v_1^2 + v_2^2) \, \mathbb{1}_{r_0} \, , \, v_1^2 \, \mathbb{1}_{r_1} \, , \, v_2^2 \, \mathbb{1}_{r_2}) \prec (\, (v_1^2 + v_2^2) \, \mathbb{1}_{r_0} \, , \, \lambda(T) \,) = \lambda(S_{\mathcal{V}})$. $\triangle$

Example 7.9. Let $m = 3$, $d = 4$, $\mathbf{k} = (3 \, , 2 \, , 2)$ and $\mathbf{v} = \mathbb{1}_3$. Denote by $\mathcal{E} = \{e_i : i \in \mathbb{I}_4\}$ the canonical basis of $\mathbb{C}^4$. Then $\lambda_{\mathbb{1}}(3 \, , \mathbf{k} \, , 4) = (2 \, , 2 \, , \frac{3}{2} \, , \frac{3}{2} \,)$ and a minimizer is given by any system $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_3} \in \mathcal{PRS}_{\mathbb{1}}$ such that the subspaces $\mathcal{S}_i = R(V_i^*)$ for $i \in \mathbb{I}_3$ are

$$\mathcal{S}_1 = \operatorname{span}\{e_1 \, , e_2 \, , e_3\} \quad , \quad \mathcal{S}_2 = \operatorname{span}\big\{ \, e_1 \, , w_2 \, \big\} \quad \text{and} \quad \mathcal{S}_3 = \operatorname{span}\big\{ \, e_2 \, , w_3 \, \big\} \ ,$$

where $w_2 = \frac{-e_3}{2} + \frac{\sqrt{3} \, e_4}{2}$ and $w_3 = \frac{-e_3}{2} - \frac{\sqrt{3} \, e_4}{2}$. The fact that $\lambda(\mathcal{S}_{\mathcal{V}}) = (2 \, , 2 \, , \frac{3}{2} \, , \frac{3}{2} \,)$ for such a system $\mathcal{V}$ is a direct computation. On the other hand, if $\mathcal{W} = \{W_i\}_{i \in \mathbb{I}_3} \in \mathcal{PRS}_{\mathbb{1}}(3 \, , \mathbf{k} \, , 4)$, then there exist unit vectors $x_2 \in R(W_1^*) \cap R(W_2^*)$ and $x_3 \in R(W_1^*) \cap R(W_3^*)$.

Denote by $\mathcal{T} = \text{span}\{x_2\,,\,x_3\}$. If $\dim \mathcal{T} = 1$ then $\lambda_1(S_\mathcal{W}) \geq \langle S_\mathcal{W}\, x_2\,,\,x_2\rangle = 3$ and $\lambda_2(S_\mathcal{W}) \geq 1$. If $\dim \mathcal{T} = 2$, using that $\mathcal{T} \subseteq R(W_1^*)$ and $x_i \in R(W_i^*)$ for $i = 2,\, 3$, we get

$$\lambda_1(S_\mathcal{W}) + \lambda_2(S_\mathcal{W}) \;\; \geq \;\; \textstyle\sum_{i\in \mathbb{I}_3} \operatorname{tr}\left(P_\mathcal{T}\, W_i^* W_i\, P_\mathcal{T}\right) \geq \operatorname{tr} P_\mathcal{T} + \operatorname{tr} P_{\text{span}\{x_2\}} + \operatorname{tr} P_{\text{span}\{x_3\}} = 4\ .$$

In any case, we have shown that $(2\,,\,2) \prec_w \alpha = (\lambda_1(S_\mathcal{W})\,,\,\lambda_2(S_\mathcal{W})\,)$. Therefore, using Lemma 7.6 we get that $(2\,,\,2\,,\,\frac{3}{2}\,,\,\frac{3}{2}\,) \prec \lambda(S_\mathcal{W})$. Now, apply 7.3.

The minimizers $\mathcal{V} \in \mathcal{PRS}_\mathbf{v}$ such that $\lambda(S_\mathcal{V}) = (2\,,\,2\,,\,\frac{3}{2}\,,\,\frac{3}{2}\,)$ have some interestig properties. For example they are the sum of two tight systems, $\mathcal{V}^\#$ is not projective, and the involved projections do not commute. More precisely, the cosine of the Friedrich angles of their images are $c(\mathcal{S}_i\,,\,\mathcal{S}_j) = \frac{1}{2}$ for every $i \neq j$. $\triangle$

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