STRONG SOLUTIONS FOR TWO-DIMENSIONAL NONLOCAL CAHN-HILLIARD-NAVIER-STOKES SYSTEMS

Sergio Frigeri
Dipartimento di Matematica *F. Enriques*
Università degli Studi di Milano
Milano I-20133, Italy
sergio.frigeri@unimi.it

Maurizio Grasselli
Dipartimento di Matematica *F. Brioschi*
Politecnico di Milano
Milano I-20133, Italy
maurizio.grasselli@polimi.it

Pavel Krejčí
Czech Academy of Sciences
CZ-11567 Praha 1, Czech Republic
krejci@math.cas.cz

September 4, 2013

Abstract

A well-known diffuse interface model for incompressible isothermal mixtures of two immiscible fluids consists of the Navier-Stokes system coupled with a convective Cahn-Hilliard equation. In some recent contributions the standard Cahn-Hilliard equation has been replaced by its nonlocal version. The corresponding system is physically more relevant and mathematically more challenging. Indeed, the only known results are essentially the existence of a global weak solution and the existence of a suitable notion of global attractor for the corresponding dynamical system defined without uniqueness. In fact, even in the two-dimensional case, uniqueness of weak solutions is still an open problem. Here we take a step forward in the case of regular potentials. First we prove the existence of a (unique) strong solution in two dimensions. Then we show that any weak solution regularizes in finite time uniformly with respect to bounded sets of initial data. This result allows us to deduce that the global attractor is the union of all the bounded complete trajectories which are strong solutions. We also demonstrate that each trajectory converges to a single equilibrium, provided that the potential is real analytic and the external forces vanish.

Keywords: Navier-Stokes equations, nonlocal Cahn-Hilliard equations, regular potentials, incompressible binary fluids, strong solutions, global attractors, convergence to equilibrium, Łojasiewicz-Simon inequality.

AMS Subject Classification 2010: 35Q30, 37L30, 45K05, 76D03, 76T99.

1 Introduction

The evolution of an incompressible mixture of two immiscible fluids can be described through a diffuse interface model (cf., e.g., [18, 20, 22, 25] and their references). Assuming that the temperature variations are negligible, taking the density is equal to one, and suppose the viscosity ν to be constant, the model H (see [21]) reduces to the so-called Cahn-Hilliard-Navier-Stokes system

$$\begin{aligned}
&\varphi_t + u \cdot \nabla\varphi = \nabla \cdot (\kappa \nabla \mu),\\
&\mu = -\Delta\varphi + F'(\varphi),\\
&u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla\pi = \mu\nabla\varphi + h(t),\\
&\mathrm{div}(u) = 0,
\end{aligned}$$

in $\Omega \times (0,\infty)$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain. Here u denotes the (average) velocity and φ is the difference of the two fluid concentrations. Moreover, $\kappa > 0$ is the mobility coefficient, F is a suitable double well potential density, π the pressure and h a given external (non-gradient) force.

The existing theoretical literature (see, for instance, [1, 2, 5, 12, 13, 14, 26, 29]) can be summarized by saying that all the results known for the Navier-Stokes system can be extended to the Cahn-Hilliard-Navier-Stokes one, with some additional technical difficulties when, for instance, F is a singular (i.e. logarithmic) potential and/or the mobility κ depends on φ and vanishes at pure phases (cf. [1, 5]). However, we recall that the Cahn-Hilliard equation has a phenomenological nature (cf. [6]). Instead, a rigorous derivation from a microscopic model yields a nonlocal equation (see [16, 17]). In this case the chemical potential μ has the following form

$$\mu = a\varphi - J * \varphi + F'(\varphi),$$

where $*$ denotes the convolution product over Ω, $J : \mathbb{R}^d \to \mathbb{R}$ is a sufficiently smooth interaction kernel such that $J(x) = J(-x)$ and $a(x) = \int_\Omega k(x-y)dy$. Motivated by this fact, in [7] we have introduced and analyzed the following nonlocal Cahn-Hilliard-Navier-Stokes system

$$\varphi_t + u \cdot \nabla\varphi = \Delta\mu, \tag{1.1}$$

$$\mu = a\varphi - J * \varphi + F'(\varphi), \tag{1.2}$$

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla\pi = \mu\nabla\varphi + h(t), \tag{1.3}$$

$$\mathrm{div}(u) = 0, \tag{1.4}$$

endowed with boundary and initial conditions

$$\frac{\partial\mu}{\partial n} = 0, \quad u = 0 \quad \text{on } \partial\Omega \times (0, T) \tag{1.5}$$

$$u(0) = u_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{1.6}$$

For such a problem we have proven first the existence of a global weak solution satisfying an energy inequality (equality in dimension two) for a regular potential F (see [7]). Then in [9] we have established the existence of a global attractor for the generalized semiflow ($d = 2$) and a trajectory attractor ($d = 3$). Similar results have recently been extended to singular potentials of logarithmic type (cf. [10]). However, an important issue has been left open: the uniqueness of weak solutions in dimension two. This is well known for the standard local models and it suggests that the present model is more difficult to handle. The main reason seems to be the poorer regularity of φ which makes the capillarity term (i.e. the Korteweg force) $\mu\nabla\varphi$ difficult to handle (see [7]). Here we are not able to address this issue but we come close. More precisely, we prove the existence of a (unique) strong solution and the regularization in finite time of any weak solution. The latter is uniform with respect to bounded set of initial data so that, as a by-product, we deduce that the global attractor we found in [9] is smooth. More precisely, it is the union of all the bounded complete trajectories which are strong solutions to (1.1)-(1.6). Finally, taking advantage of the regularization property, we show that any weak trajectory does converge to a unique equilibrium (cf. [15, 23, 24] for nonlocal Cahn-Hilliard equations).

2 Notation and known results

We set $H := L^2(\Omega)$ and $V := H^1(\Omega)$. For every $f \in V'$ we denote by $\overline{f}$ the average of f over Ω, i.e., $\overline{f} := |\Omega|^{-1}\langle f, 1\rangle$. Here $|\Omega|$ is the Lebesgue measure of Ω. We assume that $\partial\Omega$ is smooth enough.

Then we introduce the Hilbert spaces

$$V_0 := \{v \in V : \overline{v} = 0\}, \qquad V_0' := \{f \in V' : \overline{f} = 0\},$$

and the operator $A : V \to V'$, $A \in \mathcal{L}(V, V')$, defined by

$$\langle Au, v\rangle := \int_\Omega \nabla u \cdot \nabla v \qquad \forall u, v \in V.$$

We recall that A maps V onto V_0' and the restriction of A to V_0 maps V_0 onto V_0' isomorphically. Further, we denote by $\mathcal{N} : V_0' \to V_0$ the inverse map defined by

$$A\mathcal{N}f = f, \quad \forall f \in V_0' \qquad \text{and} \qquad \mathcal{N}Au = u, \quad \forall u \in V_0.$$

As is well known, for every $f \in V_0'$, $\mathcal{N}f$ is the unique solution with zero mean value of the Neumann problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

In addition, we have

$$\langle Au, \mathcal{N}f\rangle = \langle f, u\rangle, \qquad \forall u \in V, \quad \forall f \in V_0', \tag{2.1}$$

$$\langle f, \mathcal{N}g\rangle = \langle g, \mathcal{N}f\rangle = \int_\Omega \nabla(\mathcal{N}f) \cdot \nabla(\mathcal{N}g), \qquad \forall f, g \in V_0'. \tag{2.2}$$

We consider the canonical Hilbert spaces for the Navier-Stokes equations with no-slip boundary condition (see, e.g., [28])

$$G_{div} := \overline{\{u \in C_0^\infty(\Omega)^d : \ \mathrm{div}(u) = 0\}}^{L^2(\Omega)^d}, \quad V_{div} := \{u \in H_0^1(\Omega)^d : \ \mathrm{div}(u) = 0\}.$$

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the scalar product on both H and G_{div}, respectively. Instead, V_{div} is endowed with the scalar product

$$(u, v)_{V_{div}} = (\nabla u, \nabla v), \qquad \forall u, v \in V_{div}.$$

We shall also need to introduce the Stokes operator S with no-slip boundary condition. More precisely, $S : D(S) \subset G_{div} \to G_{div}$ is defined as $S := -P\Delta$ with domain $D(S) = H^2(\Omega)^d \cap V_{div}$, where $P : L^2(\Omega)^d \to G_{div}$ is the Leray projector. Notice that we have

$$(Su, v) = (u, v)_{V_{div}} = (\nabla u, \nabla v), \qquad \forall u \in D(S), \quad \forall v \in V_{div},$$

and $S^{-1}: G_{div} \to G_{div}$ is a self-adjoint compact operator in G_{div}. Thus, according with classical results, S possesses a sequence of eigenvalues $\{\lambda_j\}$ with $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_j \to \infty$, and a family $\{w_j\} \subset D(S)$ of eigenfunctions which is orthonormal in G_{div}. Let us also recall Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|\nabla u\|^2, \qquad \forall u \in V_{div}.$$

The trilinear form b which appears in the weak formulation of the Navier-Stokes equations is defined as follows

$$b(u, v, w) = \int_\Omega (u \cdot \nabla) v \cdot w, \qquad \forall u, v, w \in V_{div},$$

and the associated bilinear operator B from $V_{div} \times V_{div}$ into V'_{div} is defined by

$$\langle B(u, v), w \rangle := b(u, v, w), \qquad \forall u, v, w \in V_{div}.$$

We shall set $B(u, u) := Bu$, for all $u \in V_{div}$. We recall that we have

$$b(u, w, v) = -b(u, v, w), \qquad \forall u, v, w \in V_{div}, \tag{2.3}$$

and that the following estimates hold in dimension two

$$|b(u, v, w)| \leq c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2}, \quad \forall u, v, w \in V_{div}, \tag{2.4}$$

$$|b(u, v, w)| \leq c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|^{1/2}\|Sv\|^{1/2}\|w\|, \quad \forall u \in V_{div},\ v \in D(S),\ w \in G_{div}. \tag{2.5}$$

If X is a Banach space and $\tau \in \mathbb{R}$, we shall denote by $L^p_{tb}(\tau, \infty; X)$, $1 \leq p < \infty$, the space of functions $f \in L^p_{loc}([\tau, \infty); X)$ that are translation bounded in $L^p_{loc}([\tau, \infty); X)$, i.e. such that

$$\|f\|^p_{L^p_{tb}(\tau,\infty;X)} := \sup_{t \geq \tau} \int_t^{t+1} \|f(s)\|^p_X ds < \infty. \tag{2.6}$$

We shall use the following lemma. Its simple proof is given below for the reader's convenience.

Lemma 1. *Let $f \in L^{p_1}(\tau, \infty; X)$ with $f_t \in L^{p_2}_{tb}(\tau, \infty; X)$, where $1 \leq p_1 < \infty$, $1 < p_2 \leq \infty$, $\tau \in \mathbb{R}$ and X is a reflexive Banach space. Then $f(t) \to 0$ in X as $t \to \infty$.*

Proof. We argue by contradiction. Suppose there exist a sequence $\{t_n\}$ with $t_n \to \infty$ and a constant $\sigma > 0$ such that $\|f(t_n)\|_X \geq \sigma$, for all n. Set $\tau_n := t_n + 1/n$. Since $f \in L^{p_1}(\tau, \infty; X)$ with $1 \leq p_1 < \infty$, then, by possibly extracting a subsequence, for every n there exists $t'_n \in [t_n, \tau_n]$ such that $\|f(t'_n)\|_X \leq \sigma/2$. We therefore get a contradiction, since, denoting by $p'_2 \in [1, \infty)$ the conjugate of p_2,

$$0 < \frac{\sigma}{2} \leq \|f(t'_n) - f(t_n)\|_X \leq \int_{t_n}^{t'_n} \|f_t(s)\|_X ds \leq \|f_t\|_{L^{p_2}_{tb}(\tau,\infty;X)} \frac{1}{n^{p'_2}} \to 0.$$

□

We also report the uniform Gronwall lemma which will be useful in the sequel (see, e.g., [27]).

Lemma 2. *Let Φ be an absolutely continuous nonnegative function on $[\tau, \infty)$ and ω_1, ω_2 two nonnegative locally summable functions on $[\tau, \infty)$ satisfying*

$$\frac{d}{dt}\Phi(t) \leq \omega_1(t)\Phi(t) + \omega_2(t), \qquad \text{for a.e. } t \in [\tau, \infty), \tag{2.7}$$

and such that

$$\int_t^{t+1} \omega_i(s)ds \leq a_i, \quad i = 1, 2, \qquad \int_t^{t+1} \Phi(s)ds \leq a_3, \tag{2.8}$$

for all $t \geq \tau$, where a_1, a_2, a_3 are some nonnegative constants. Then

$$\Phi(t + 1) \leq (a_2 + a_3)e^{a_1}, \qquad \forall t \geq \tau. \tag{2.9}$$

We now summarize the main results of [7]. They are concerned with the existence of dissipative weak solutions and the validity of the energy identity and of a dissipative estimate in dimension two.

The assumptions on J and F are listed below

(H1) $J \in W^{1,1}(\mathbb{R}^d), \quad J(x) = J(-x), \quad a \geq 0 \quad \text{a.e. in } \Omega.$

(H2) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + a(x) \geq c_0, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

(H3) $F \in C^2(\mathbb{R})$ and there exist $c_1 > 0$, $c_2 > 0$ and $q > 0$ such that

$$F''(s) + a(x) \geq c_1|s|^{2q} - c_2, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

(H4) There exist $c_3 > 0$, $c_4 \geq 0$ and $r \in (1, 2]$ such that

$$|F'(s)|^r \leq c_3|F(s)| + c_4, \qquad \forall s \in \mathbb{R}.$$

Remark 1. Assumption $J \in W^{1,1}(\mathbb{R}^d)$ can be weakened. Indeed, it can be replaced by $J \in W^{1,1}(B_\delta)$, where $B_\delta := \{z \in \mathbb{R}^d : |z| < \delta\}$ with $\delta := \text{diam}(\Omega)$, or also by (see, e.g., [4])

$$\sup_{x\in\Omega} \int_\Omega \big(|J(x - y)| + |\nabla J(x - y)|\big)dy < \infty.$$

The above assumptions allow to prove the following result (see [7])

Theorem 1. *Let $h \in L^2_{loc}([0,\infty); V'_{div})$, $u_0 \in G_{div}$, $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\Omega)$ and suppose that (H1)-(H4) are satisfied. Then, for every given $T > 0$, there exists a weak solution $[u, \varphi]$ to (1.1)–(1.6) such that*

$$u \in L^\infty(0,T; G_{div}) \cap L^2(0,T; V_{div}), \quad \varphi \in L^\infty(0,T; L^{2+2q}(\Omega)) \cap L^2(0,T; V), \tag{2.10}$$

$$u_t \in L^{4/3}(0,T; V'_{div}), \quad \varphi_t \in L^{4/3}(0,T; V'), \qquad d = 3, \tag{2.11}$$

$$u_t \in L^2(0,T; V'_{div}), \quad d = 2, \tag{2.12}$$

$$\varphi_t \in L^2(0,T; V'), \quad d = 2 \quad \text{or} \quad d = 3 \text{ and } q \geq 1/2, \tag{2.13}$$

and satisfying the energy inequality

$$\mathcal{E}(u(t), \varphi(t)) + \int_0^t \Big(\nu \|\nabla u\|^2 + \|\nabla \mu\|^2\Big) d\tau \leq \mathcal{E}(u_0, \varphi_0) + \int_0^t \langle h(\tau), u \rangle d\tau, \tag{2.14}$$

for every $t > 0$, where we have set

$$\mathcal{E}(u(t), \varphi(t)) = \frac{1}{2}\|u(t)\|^2 + \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\varphi(x,t) - \varphi(y,t))^2 dx dy + \int_\Omega F(\varphi(t)).$$

If $d = 2$, then any weak solution satisfies the energy identity

$$\frac{d}{dt}\mathcal{E}(u, \varphi) + \nu \|\nabla u\|^2 + \|\nabla \mu\|^2 = \langle h(t), u \rangle, \tag{2.15}$$

In particular we have $u \in C([0,\infty); G_{div})$, $\varphi \in C([0,\infty); H)$ and $\int_\Omega F(\varphi) \in C([0,\infty))$. Furthermore, if $d = 2$ and $h \in L^2_{tb}(0,\infty; V'_{div})$, then any weak solution satisfies also the dissipative estimate

$$\mathcal{E}(u(t), \varphi(t)) \leq \mathcal{E}(u_0, \varphi_0) e^{-kt} + F(m_0)|\Omega| + K, \qquad \forall t \geq 0, \tag{2.16}$$

where $m_0 = (\varphi_0, 1)$ and k, K are two positive constants which are independent of the initial data, with K depending on Ω, ν, J, F and $\|h\|_{L^2_{tb}(0,\infty; V'_{div})}$.

Remark 2. All the previous results hold for a viscosity ν depending on ϕ which is sufficiently smooth and bounded from above and from below (see [7], cf. also [9, 10]). Here we assume ν to be constant just to avoid further technicalities in the sequel.

3 Strong solutions in two dimensions

In this section we state and prove our main result, namely the existence of a (global) strong solution to (1.1)–(1.6) and its uniqueness. More precisely, we have

Theorem 2. *Let $h \in L^2_{loc}([0,\infty); G_{div})$, $u_0 \in V_{div}$, $\varphi_0 \in V \cap L^\infty(\Omega)$ and suppose that (H1)-(H4) are satisfied. Then, for every given $T > 0$, there exists a weak solution $[u, \varphi]$ such that*

$$u \in L^\infty(0,T; V_{div}) \cap L^2(0,T; H^2(\Omega)^2), \quad \varphi \in L^\infty(\Omega \times (0,T)) \cap L^\infty(0,T; V), \tag{3.1}$$

$$u_t \in L^2(0,T; G_{div}), \quad \varphi_t \in L^2(0,T; H). \tag{3.2}$$

Furthermore, suppose in addition that $F \in C^3(\mathbb{R})$ and that $\varphi_0 \in H^2(\Omega)$. Then, system (1.1)–(1.4) admits a unique strong solution on $[0,T]$ satisfying (3.1), (3.2) and also

$$\varphi \in L^\infty(0,T; W^{1,p}(\Omega)), \quad 2 \le p < \infty, \tag{3.3}$$

$$\varphi_t \in L^\infty(0,T; H) \cap L^2(0,T; V). \tag{3.4}$$

If $J \in W^{2,1}(\mathbb{R}^2)$, we have in addition

$$\varphi \in L^\infty(0,T; H^2(\Omega)). \tag{3.5}$$

Moreover, let $[u_{0i}, \varphi_{0i}, h_i] \in V_{div} \times H^2(\Omega) \times L^2_{loc}([0,\infty); G_{div})$, $i = 1,2$, be two sets of data and denote by $[u_i, \varphi_i]$ the corresponding solutions. Then, there exists a positive constant Λ which is a continuous and increasing function of the norms of the data of two solutions and which also depends on T, F, J, Ω, ν, such that the following continuous dependence estimate holds

$$\begin{aligned}
&\|u_2(t) - u_1(t)\|^2 + \|\varphi_2(t) - \varphi_1(t)\|^2_{V_0'} \\
&+ \int_0^t \|\nabla u_2(\tau) - \nabla u_1(\tau)\|^2 d\tau + \int_0^t \|\varphi_2(\tau) - \varphi_1(\tau)\|^2 d\tau \\
&\le \Lambda\Big(\|u_{02} - u_{01}\|^2 + \|\varphi_{02} - \varphi_{01}\|^2_{V_0'} + \|h_2 - h_1\|^2_{L^2(0,T;G_{div})}\Big),
\end{aligned} \tag{3.6}$$

for every $t \in [0,T]$.

Remark 3. The regularity properties (3.1)–(3.5) imply that

$$u \in C([0,\infty); V_{div}), \quad \varphi \in C([0,T]; V) \cap C_w([0,T]; H^2(\Omega))$$

Actually, we have also $\varphi \in C([0,T]; H^\delta(\Omega))$ for every $\delta \in [0,2)$. Recall that the time continuity of the velocity field into V_{div} is a consequence of the fact that $u \in C_w([0,\infty); V_{div})$ and of the following differential identity

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \nu\|Su\|^2 + (Bu, Su) = (\mu\nabla\varphi, Su) + (h, Su), \tag{3.7}$$

which is deduced by testing equation (1.3) by Su.

Remark 4. If the condition $\varphi_0 \in L^\infty(\Omega)$ in the first part of Theorem 2 is removed, a boundedness estimate for the order parameter φ can still be recovered. In particular, it can be proved (see [15, Lemma 2.10]) that for every $t_0 > 0$ there exists a constant $\overline{C}_{m,t_0} > 0$, where m is such that $|\overline{\varphi}_0| \le m$, such that

$$\sup_{t\ge 2t_0} \|\varphi(t)\|_{L^\infty(\Omega)} \le \overline{C}_{m,t_0}.$$

Moreover, (3.1)–(3.4) still hold provided the time interval $(0,T)$ is replaced by $(2t_0,T)$, for every $T > 2t_0$.

Remark 5. In Theorem 2 condition $J \in W^{2,1}(\mathbb{R}^2)$ is actually needed to ensure the regularity property $\varphi \in L^\infty(0,T;H^2(\Omega))$ only.

Proof. We shall carry out the proof by providing some formal regularization estimates. The argument can be made rigorous by means, e.g., of a Faedo-Galerkin approximation technique (see [7] for details).

We first observe that the property $\varphi \in L^\infty(\Omega \times (0,T))$ can be obtained by exploiting the same argument used in [4, Theorem 2.1]. Indeed, by multiplying (1.1) by $\varphi|\varphi|^{p-1}$ and integrating on Ω the resulting equation, the contribution of the convective term vanishes due to the incompressibility condition (1.4) and the proof of [4, Theorem 2.1] entails

$$\sup_{t\in(0,T)} \|\varphi(t)\|_{L^\infty(\Omega)} \le \overline{C}, \tag{3.8}$$

where the constant $\overline{C}$ depends on the initial conditions, in particular on $\|u_0\|$, on $\|\varphi_0\|_{L^\infty(\Omega)}$ and on T (see [4, Estimate (2.28)]). Furthermore, if $h \in L^2_{tb}(0,\infty;G_{div})$ then, thanks to the dissipative estimate (2.16), we have $\sup_{t\ge 0}\|\varphi(t)\|_{L^{2+2q}(\Omega)} \le \overline{C}$, the constant $\overline{C}$ being dependent on the initial data and on h only. Hence, due to [4, Estimate (2.28)], the constant $\overline{C}$ in (3.8) does not depend on T.

As far as the regularity of the velocity u is concerned, notice that, since the Korteweg-force term $\mu\nabla\varphi \in L^2(0,T;L^2(\Omega)^2)$, then by applying [28, Theorem 3.10], we immediately obtain $(3.1)_1$ and $(3.2)_2$.

Henceforth we shall denote by c a positive constant which depends only on J, F and Ω, while $\overline{c}$ will denote a positive constant depending on J, F, Ω and also on the initial conditions u_0 and φ_0 (in particular on $\|\nabla u_0\|$ and on $\|\varphi_0\|_{L^\infty(\Omega)}$). The values of both c and $\overline{c}$ may possibly vary from line to line, even within the same estimate. We shall divide the proof into three main steps.

Step 1. Estimate of φ_t in $L^2(0,T;H)$

We multiply (1.1) by μ_t in H and get

$$\int_\Omega \varphi_t\mu_t + \int_\Omega (u\cdot\nabla\varphi)\mu_t + \frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2$$

$$= \int_\Omega (a + F''(\varphi))\varphi_t^2 - (\varphi_t, J * \varphi_t) + \int_\Omega (u \cdot \nabla\varphi)\mu_t + \frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 = 0. \tag{3.9}$$

Now, we have

$$\begin{aligned} \Big| \int_\Omega (u \cdot \nabla\varphi)\mu_t \Big| &= \Big| \int_\Omega (u \cdot \nabla\varphi)(a\varphi_t - J * \varphi_t + F''(\varphi)\varphi_t) \Big| \\ &\leq \frac{c_0}{4}\|\varphi_t\|^2 + \bar{c}\|u\|_{H^2}^2\|\nabla\varphi\|^2, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} |(\varphi_t, J * \varphi_t)| &= |(-u \cdot \nabla\varphi + \Delta\mu, J * \varphi_t)| \\ &\leq |(u \cdot \nabla\varphi, J * \varphi_t)| + |(\nabla\mu, \nabla J * \varphi_t)| \\ &\leq \frac{c_0}{4}\|\varphi_t\|^2 + c\|u\|_{H^2}^2\|\nabla\varphi\|^2 + c\|\nabla\mu\|^2. \end{aligned} \tag{3.11}$$

Plugging (3.10) and (3.11) into (3.9), using assumption (H2) and integrating the resulting estimate in time between 0 and t, we obtain

$$\frac{1}{2}\|\nabla\mu\|^2 + \frac{c_0}{2}\int_0^t \|\varphi_t\|^2 d\tau \leq \frac{1}{2}\|\nabla\mu_0\|^2 + \int_0^t \bar{c}\|u\|_{H^2}^2\|\nabla\varphi\|^2 d\tau + c\int_0^t \|\nabla\mu\|^2 d\tau, \tag{3.12}$$

and on account of the following

$$\|\nabla\mu\|^2 \geq \frac{c_0^2}{4}\|\nabla\varphi\|^2 - c\|\varphi\|^2, \tag{3.13}$$

from (3.12) we are led to the differential inequality

$$\|\nabla\mu\|^2 \leq \|\nabla\mu_0\|^2 + \bar{c}_T + \int_0^t m(\tau)\|\nabla\mu(\tau)\|^2 d\tau, \qquad \forall t \in [0, T], \tag{3.14}$$

where $m := c\left(\|u\|_{H^2}^2 + 1\right) \in L^1(0, T)$, for all $T > 0$. Thus the standard Gronwall lemma gives

$$\nabla\mu \in L^\infty(0, T; H), \qquad \forall T > 0, \tag{3.15}$$

so that, using also (3.12), we infer

$$\varphi \in L^\infty(0, T; V), \qquad \varphi_t \in L^2(0, T; H), \qquad \forall T > 0. \tag{3.16}$$

This concludes the proof of (3.1) and (3.2).

Step 2. Estimate of φ_t in $L^\infty(0, T; H)$

We differentiate (1.1) with respect to time and multiply the resulting identity in H by μ_t. This yields

$$\int_\Omega \varphi_{tt}\mu_t + \int_\Omega \mu_t u_t \cdot \nabla\varphi + \int_\Omega \mu_t u \cdot \nabla\varphi_t + \|\nabla\mu_t\|^2 = 0, \tag{3.17}$$

and, due to (1.4), we obtain

$$\int_\Omega \varphi_{tt}\mu_t + \|\nabla\mu_t\|^2 = \int_\Omega \varphi_t u\cdot\nabla\mu_t + \int_\Omega \varphi u_t\cdot\nabla\mu_t, \tag{3.18}$$

which entails

$$\int_\Omega \varphi_{tt}\mu_t + \frac{1}{2}\|\nabla\mu_t\|^2 \le \int_\Omega (\varphi_t^2 u^2 + \varphi^2 u_t^2). \tag{3.19}$$

Observe now that

$$\begin{aligned}
&\int_\Omega \varphi_{tt}\mu_t = \int_\Omega \varphi_{tt}(a\varphi_t - J*\varphi_t + F''(\varphi)\varphi_t)\\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega a\varphi_t^2 - (J*\varphi_t, -u_t\cdot\nabla\varphi - u\cdot\nabla\varphi_t + \Delta\mu_t) + \int_\Omega F''(\varphi)\varphi_t\varphi_{tt}\\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega (a+F''(\varphi))\varphi_t^2 - (\nabla J*\varphi_t, u_t\varphi) - (\nabla J*\varphi_t, u\varphi_t)\\
&+ (\nabla J*\varphi_t, \nabla\mu_t) - \frac{1}{2}\int_\Omega F'''(\varphi)\varphi_t^3.
\end{aligned} \tag{3.20}$$

On the other hand we have

$$\begin{aligned}
&|(\nabla J*\varphi_t, u_t\varphi)| \le \|\nabla J\|_{L^1}\|u_t\|\|\varphi\|_{L^\infty}\|\varphi_t\| \le \frac{1}{2}\|u_t\|^2\|\varphi_t\|^2 + \overline{c},\\
&|(\nabla J*\varphi_t, u\varphi_t)| \le \|\nabla J\|_{L^1}\|u\|_{L^\infty}\|\varphi_t\|^2 \le c\|u\|_{H^2}\|\varphi_t\|^2,\\
&|(\nabla J*\varphi_t, \nabla\mu_t)| \le \frac{1}{4}\|\nabla\mu_t\|^2 + \|\nabla J\|_{L^1}^2\|\varphi_t\|^2.
\end{aligned}$$

Therefore from (3.19) we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (a+F''(\varphi))\varphi_t^2 + \frac{1}{4}\|\nabla\mu_t\|^2 &\le c(\|u\|_{H^2}^2 + \|u\|_{H^2} + \|u_t\|^2 + 1)\|\varphi_t\|^2\\
&+ \|\varphi\|_{L^\infty}^2\|u_t\|^2 + \frac{1}{2}\int_\Omega F'''(\varphi)\varphi_t^3 + \overline{c}.
\end{aligned} \tag{3.21}$$

The integral term containing φ_t^3 can be estimated by means of Gagliardo-Nirenberg inequality in dimension two, that is,

$$\left|\frac{1}{2}\int_\Omega F'''(\varphi)\varphi_t^3\right| \le \overline{c}\|\varphi_t\|_{L^3}^3 \le \overline{c}(\|\varphi_t\|^3 + \|\varphi_t\|^2\|\nabla\varphi_t\|) \le \frac{c_0^2}{32}\|\nabla\varphi_t\|^2 + \overline{c}\|\varphi_t\|^4 + \overline{c}. \tag{3.22}$$

We now need to estimate $\nabla\varphi_t$ in terms of $\nabla\mu_t$. In order to do that, let us first control $\nabla\varphi$ in terms of $\nabla\mu$ in L^p, for every $2 \le p < \infty$. We then take the gradient of $\mu = a\varphi - J*\varphi + F'(\varphi)$, multiply it by $\nabla\varphi|\nabla\varphi|^{p-2}$ and integrate the resulting identity on Ω. We get

$$\int_\Omega \nabla\varphi|\nabla\varphi|^{p-2}\cdot\nabla\mu = \int_\Omega (a+F''(\varphi))|\nabla\varphi|^p + \int_\Omega (\varphi\nabla a - \nabla J*\varphi)\cdot\nabla\varphi|\nabla\varphi|^{p-2},$$

and so, by (H2), we find

$$c_0\|\nabla\varphi\|_{L^p}^p \le \|\nabla\varphi\|_{L^p}^{p-1}\|\nabla\mu\|_{L^p} + (\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})\|\varphi\|_{L^p}\|\nabla\varphi\|_{L^p}^{p-1}$$
$$\le \frac{c_0}{2}\|\nabla\varphi\|_{L^p}^p + c\|\nabla\mu\|_{L^p}^p + c(\|\nabla a\|_{L^\infty} + \|\nabla J\|_{L^1})^p\|\varphi\|_{L^p}^p.$$

We therefore obtain

$$\|\nabla\varphi\|_{L^p} \le c\|\nabla\mu\|_{L^p} + \overline{c}, \tag{3.23}$$

with $\overline{c}$ depending also on p. We now see that the L^p-norm of $\nabla\mu$ can be estimated in terms of the L^2-norm of φ_t. Indeed, using once more the two dimensional Gagliardo-Nirenberg inequality, we infer

$$\begin{aligned}
&\|\nabla\mu\|_{L^p} \le c\|\nabla\mu\|^{2/p}\|\nabla\mu\|_{H^1}^{1-2/p}\\
&\le c\|\nabla\mu\|^{2/p}\|\mu\|_{H^2}^{1-2/p} \le c\|\nabla\mu\|^{2/p}(\|\Delta\mu\|^{1-2/p} + \|\mu\|^{1-2/p})\\
&\le \overline{c}(\|\varphi_t\|^{1-2/p} + \|u\cdot\nabla\varphi\|^{1-2/p} + 1)\\
&\le \overline{c}(\|\varphi_t\|^{1-2/p} + \|u\|_{L^q}^{1-2/p}\|\nabla\varphi\|_{L^p}^{1-2/p} + 1),
\end{aligned}$$

where $p^{-1} + q^{-1} = 1/2$ and where we have taken into account (3.15) and the fact that the H^2-norm of μ is equivalent to the L^2- norm of $-\Delta\mu + \mu$, due to (1.5). By (3.23) we therefore deduce the desired estimate

$$\|\nabla\mu\|_{L^p} \le \overline{c}(1 + \|\varphi_t\|^{1-2/p}). \tag{3.24}$$

We now take the gradient of μ_t and multiply it in L^2 by $\nabla\varphi_t$. We get

$$\begin{aligned}
&\int_\Omega \nabla\mu_t\cdot\nabla\varphi_t = \int_\Omega (a + F''(\varphi))|\nabla\varphi_t|^2\\
&+ \int_\Omega (\nabla a\varphi_t - \nabla J * \varphi_t)\cdot\nabla\varphi_t + \int_\Omega F'''(\varphi)\varphi_t\nabla\varphi\cdot\nabla\varphi_t.
\end{aligned} \tag{3.25}$$

Observe that we have

$$\begin{aligned}
&\Big|\int_\Omega F'''(\varphi)\varphi_t\nabla\varphi\cdot\nabla\varphi_t\Big| \le c\|\varphi_t\|_{L^3}\|\nabla\varphi\|_{L^6}\|\nabla\varphi_t\|\\
&\le \overline{c}\Big(\|\varphi_t\| + \|\varphi_t\|^{2/3}\|\nabla\varphi_t\|^{1/3}\Big)\Big(1 + \|\varphi_t\|^{2/3}\Big)\|\nabla\varphi_t\|\\
&\le \overline{c}\Big(\|\varphi_t\|^{5/3}\|\nabla\varphi_t\| + \|\varphi_t\|^{4/3}\|\nabla\varphi_t\|^{4/3} + \|\varphi_t\|^{2/3}\|\nabla\varphi_t\|^{4/3} + \|\varphi_t\|\|\nabla\varphi_t\|\Big)\\
&\le \frac{c_0}{4}\|\nabla\varphi_t\|^2 + \overline{c}\|\varphi_t\|^4 + \overline{c},
\end{aligned} \tag{3.26}$$

Thus from (3.25) and (3.26) and using also (H2), we deduce

$$\frac{1}{c_0}\|\nabla\mu_t\|^2 + \frac{c_0}{4}\|\nabla\varphi_t\|^2 \ge \|\nabla\mu_t\|\|\nabla\varphi_t\| \ge c_0\|\nabla\varphi_t\|^2 - \frac{c_0}{4}\|\nabla\varphi_t\|^2 - \overline{c}\|\varphi_t\|^2$$

$$- \frac{c_0}{4}\|\nabla\varphi_t\|^2 - \overline{c}\|\varphi_t\|^4 - \overline{c},$$

so that

$$\frac{4}{c_0^2}\|\nabla\mu_t\|^2 \geq \|\nabla\varphi_t\|^2 - \overline{c}\|\varphi_t\|^4 - \overline{c}. \tag{3.27}$$

We now go back to (3.21). By combining (3.22) and (3.27) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (a + F''(\varphi))\varphi_t^2 + \frac{1}{8}\|\nabla\mu_t\|^2 \leq \alpha(t)\|\varphi_t\|^2 + \overline{c}\|\varphi_t\|^4 + \beta(t) + \overline{c}, \tag{3.28}$$

where $\alpha := c(\|u\|_{H^2}^2 + \|u\|_{H^2} + \|u_t\|^2 + 1)$ and $\beta := \|\varphi\|_{L^\infty}^2\|u_t\|^2$. We have $\alpha,\, \beta \in L^1(0,T)$. From (3.28) we can easily infer the desired estimate. Indeed, let us multiply (3.28) by $(1 + \int_\Omega (a + F''(\varphi))\varphi_t^2)^{-1}$ and get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\log\Big(1 + \int_\Omega (a + F''(\varphi))\varphi_t^2\Big) \leq \frac{1}{c_0}\alpha(t) + \frac{\overline{c}\Big(\int_\Omega \varphi_t^2\Big)^2}{1 + \int_\Omega (a + F''(\varphi))\varphi_t^2} + \beta(t) + \overline{c} \\
&\leq \frac{1}{c_0}\alpha(t) + \beta(t) + \overline{c}\|\varphi_t\|^2 + \overline{c}.
\end{aligned}$$

Integrating this last inequality between 0 and $t \in (0,T)$ and using the second of (3.16) and the fact that $\varphi_t(0) \in H$ (since $\varphi_0 \in H^2(\Omega)$) we therefore deduce that

$$\varphi_t \in L^\infty(0,T;H), \qquad \forall T > 0. \tag{3.29}$$

In particular, on account of (3.23) and (3.24), we also have

$$\nabla\mu, \nabla\varphi \in L^\infty(0,T;L^p(\Omega)), \qquad \forall T > 0, \qquad 2 \leq p < \infty. \tag{3.30}$$

Furthermore, by integrating (3.28) between 0 and $t \in [0,T]$ and using (3.27) and (3.29), we also get

$$\varphi_t \in L^2(0,T;V). \tag{3.31}$$

By comparison in (1.1) we can finally obtain estimates for μ and φ in $L^\infty(0,T;H^2(\Omega))$. Indeed, we have

$$\|\Delta\mu\| \leq \|\varphi_t\| + c\|\nabla u\|\|\nabla\varphi\|_{L^p}, \tag{3.32}$$

which implies that $\Delta\mu \in L^\infty(0,T;L^2(\Omega))$, thanks to (3.29) and (3.30). Recalling (1.5) and the smoothness of $\partial\Omega$, we also have

$$\mu \in L^\infty(0,T;H^2(\Omega)). \tag{3.33}$$

Apply now the second derivative operator $\partial_{ij}^2 := \frac{\partial^2}{\partial x_i \partial x_j}$ to (1.2), multiply the resulting identity by $\partial_{ij}^2\varphi$ and integrate on Ω. Using the assumption $J \in W^{2,1}(\mathbb{R}^2)$, we get

$$\int_\Omega \partial_{ij}^2\mu\partial_{ij}^2\varphi = \int_\Omega (a + F''(\varphi))(\partial_{ij}^2\varphi)^2 + \int_\Omega (\partial_i a\partial_j\varphi + \partial_j a\partial_i\varphi)\partial_{ij}^2\varphi$$
$$+ \int_\Omega (\varphi\partial_{ij}^2 a - \partial_{ij}^2 J * \varphi)\partial_{ij}^2\varphi + \int_\Omega F'''(\varphi)\partial_i\varphi\partial_j\varphi\partial_{ij}^2\varphi.$$

From this identity, by means of (H2) and (3.30) it is easy to obtain

$$\|\partial_{ij}^2\mu\|^2 \geq \frac{c_0^2}{4}\|\partial_{ij}^2\varphi\|^2 - \bar{c}. \tag{3.34}$$

Such estimate together with (3.33) entail

$$\varphi \in L^\infty(0,T;H^2(\Omega)). \tag{3.35}$$

Step 3. Continuous dependence and uniqueness of strong solutions
Let us consider two strong solutions $z_1 := [u_1, \varphi_1]$ and $z_2 := [u_2, \varphi_2]$ corresponding to initial data $z_{01} := [u_{01}, \varphi_{01}]$ and $z_{02} := [u_{02}, \varphi_{02}]$ and to external forces h_1 and h_2, respectively. Taking the difference between the variational formulation of (1.1) and (1.2) written for each solution and setting $u := u_2 - u_1$, $\varphi := \varphi_2 - \varphi_1$, $\mu := \mu_2 - \mu_1$ and $h := h_2 - h_1$, we have

$$\langle u_t, v\rangle + \nu(\nabla u, \nabla v) + b(u_2, u_2, v) - b(u_1, u_1, v) = -(\varphi_2\nabla\mu_2, v) + (\varphi_1\nabla\mu_1, v) + (h, v) \tag{3.36}$$

$$\langle \varphi_t, \psi\rangle + (\nabla\mu, \nabla\psi) = (u_2\varphi_2, \nabla\psi) - (u_1\varphi_1, \nabla\psi), \tag{3.37}$$

for every $v \in V_{div}$ and every $\psi \in V$. Let us choose $v = u$ and $\psi = \mathcal{N}\varphi$ and sum the first resulting identity to the second one multiplied by γ, where the positive constant γ will be suitably chosen. After some easy calculations we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|\nabla u\|^2 + b(u_2, u_2, u) - b(u_1, u_1, u) + \frac{\gamma}{2}\frac{d}{dt}\|\varphi\|_{V_0'}^2 + \gamma(\varphi, \mu)$$
$$= -(\varphi\nabla\mu_2, u) - (\varphi_1\nabla\mu, u) + \gamma(u_2, \varphi\nabla\mathcal{N}\varphi) + \gamma(u, \varphi_1\nabla\mathcal{N}\varphi) + (h, u). \tag{3.38}$$

Notice that

$$\gamma(\varphi, \mu) = \gamma(\varphi, a\varphi - J * \varphi + F'(\varphi_2) - F'(\varphi_1)) \geq c_0\gamma\|\varphi\|^2 - \gamma(\varphi, J * \varphi)$$
$$\geq c_0\gamma\|\varphi\|^2 - \gamma\|\varphi\|_{V_0'}\|J\|_V\|\varphi\| \geq c_0\gamma\|\varphi\|^2 - \|\varphi\|^2 - c\gamma^2\|\varphi\|_{V_0'}^2. \tag{3.39}$$

Furthermore, as far as the first two terms on the right hand side of (3.38) are concerned, we have

$$|(\varphi\nabla\mu_2, u)| \leq \|\varphi\|\|\nabla\mu_2\|_{L^4}\|u\|_{L^4} \leq \frac{\nu}{4}\|\nabla u\|^2 + c\|\nabla\mu_2\|_{L^4}^2\|\varphi\|^2, \tag{3.40}$$

$$|(\varphi_1\nabla\mu, u)| = |(\mu\nabla\varphi_1, u)| \le \|\mu\|\|\nabla\varphi_1\|_{L^4}\|u\|_{L^4} \le \frac{\nu}{4}\|\nabla u\|^2 + c\|\nabla\varphi_1\|^2_{L^4}\|\varphi\|^2, \qquad (3.41)$$

where we have used the bound

$$\|\mu\| = \|a\varphi - J*\varphi + F'(\varphi_2) - F'(\varphi_1)\| \le 2\|a\|_{L^\infty}\|\varphi\| + c\|\varphi\| \le c\|\varphi\|.$$

The last two terms on the right hand side of (3.38) can be estimated as follows

$$|\gamma(u_2, \varphi\nabla\mathcal{N}\varphi)| \le \gamma\|u_2\|_{L^\infty}\|\varphi\|\|\nabla\mathcal{N}\varphi\| \le c\gamma\|u_2\|_{H^2}\|\varphi\|\|\varphi\|_{V_0'}$$
$$\le \|\varphi\|^2 + c\gamma^2\|u_2\|^2_{H^2}\|\varphi\|^2_{V_0'}, \qquad (3.42)$$
$$|\gamma(u, \varphi_1\nabla\mathcal{N}\varphi)| \le \frac{\gamma}{2}\|u\|^2 + \frac{\gamma}{2}\|\varphi_1\|^2_{L^\infty}\|\varphi\|^2_{V_0'}. \qquad (3.43)$$

Consider the trilinear forms on the left hand side of (3.38). By (2.4) we have

$$b(u_2, u_2, u) - b(u_1, u_1, u) = b(u, u_1, u) \le c\|u\|\|\nabla u_1\|\|\nabla u\|$$
$$\le \frac{\nu}{4}\|\nabla u\|^2 + c\|\nabla u_1\|^2\|u\|^2 \qquad (3.44)$$

Plugging (3.39)–(3.44) into (3.38) we get

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \gamma\|\varphi\|^2_{V_0'}\Big) + \frac{\nu}{8}\|\nabla u\|^2 + \gamma c_0\|\varphi\|^2 \le c(1 + \|\nabla\varphi_1\|^2_{L^4} + \|\nabla\mu_2\|^2_{L^4})\|\varphi\|^2$$
$$+ c\gamma(\gamma\|u_2\|^2_{H^2} + \|\varphi_1\|^2_{L^\infty} + \gamma)\|\varphi\|^2_{V_0'} + c(\gamma + \|\nabla u_1\|^2)\|u\|^2 + \frac{2}{\nu\lambda_1}\|h\|^2. \qquad (3.45)$$

Thanks to (3.30), we can now choose $\gamma = \gamma_*$ such that

$$\Gamma_* := c_0\gamma_* - c(1 + \|\nabla\varphi_1\|^2_{L^\infty(0,T;L^4(\Omega))} + \|\nabla\mu_2\|^2_{L^\infty(0,T;L^4(\Omega))}) > 0.$$

Hence from (3.45) we deduce

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + \gamma_*\|\varphi\|^2_{V_0'}\Big) + \frac{\nu}{8}\|\nabla u\|^2 + \Gamma_*\|\varphi\|^2 \le \eta(t)\Big(\|u\|^2 + \gamma_*\|\varphi\|^2_{V_0'}\Big) + \frac{2}{\nu\lambda_1}\|h\|^2, \quad (3.46)$$

where

$$\eta := c(\|\nabla u_1\|^2 + \gamma_*\|u_2\|^2_{H^2} + \|\varphi_1\|^2_{L^\infty} + \gamma_*) \in L^1(0,T), \qquad \forall T > 0.$$

The standard Gronwall lemma then yields

$$\|u(t)\|^2 + \gamma_*\|\varphi(t)\|^2_{V_0'} \le e^{2\int_0^t \eta(s)ds}\Big(\|u_0\|^2 + \gamma_*\|\varphi_0\|^2_{V_0'} + \frac{4}{\nu\lambda_1}\|h\|^2_{L^2(0,t;G_{div})}\Big), \qquad (3.47)$$

for every $t \in [0,T]$, where we have set $u_0 := u_{02} - u_{01}$ and $\varphi_0 := \varphi_{02} - \varphi_{01}$. By integrating (3.46) between 0 and t and taking (3.47) into account, we also get

$$\frac{\nu}{4}\int_0^t \|\nabla u\|^2 d\tau + 2\Gamma_*\int_0^t \|\varphi\|^2 d\tau$$
$$\le \Big(\|u_0\|^2 + \gamma_*\|\varphi_0\|^2_{V_0'} + \frac{4}{\nu\lambda_1}\|h\|^2_{L^2(0,t;G_{div})}\Big)\Big(1 + 2e^{2\int_0^t \eta(s)ds}\int_0^t \eta(s)ds\Big), \qquad (3.48)$$

for every $t \in [0,T]$. Finally, by combining (3.47) and (3.48), we obtain (3.6). □

Remark 6. It is not difficult to see that the φ component of the strong solution to system (1.1)–(1.5) satisfies

$$\varphi \in C([0,\infty); H^2(\Omega)). \tag{3.49}$$

Indeed, by combining (3.17)–(3.20) and taking into account the regularity properties of the strong solution, we can see that $\int_\Omega \big(a + F''(\varphi)\big)\varphi_t^2$ is absolutely continuous on $[0,\infty)$. Using (H2) and the fact that $\varphi \in C([0,\infty); C(\overline{\Omega}))$ (see Remark 3) we get $\|\varphi_t\|^2 \in C([0,\infty))$. Now, (3.33) and $\mu_t \in L^2_{loc}([0,\infty); V)$ imply that $\mu \in C([0,\infty); V)$ and, by using (3.33) again, we also have $\mu \in C_w([0,\infty); H^2(\Omega))$ so that $\Delta\mu \in C_w([0,\infty); H)$. Moreover, since $u \in C([0,\infty); L^4(\Omega))$ and $\nabla\varphi \in C_w([0,\infty); L^4(\Omega))$ (cf. Remark 3), then we have $u \cdot \nabla\varphi \in C_w([0,\infty); H)$. Thus from (1.1) we deduce that $\varphi_t \in C_w([0,\infty); H)$ and, on account of the continuity of $t \mapsto \|\varphi_t(t)\|$, then $\varphi_t \in C([0,\infty); H)$. Recall now that $\nabla\varphi \in C([0,\infty); H^\epsilon(\Omega)^2)$, for every $\epsilon \in [0,1)$ (cf. Remark 3). Then, choosing $\epsilon \in [1/2,1)$, we have $\nabla\varphi \in C([0,\infty); L^4(\Omega)^2)$. Thus $u \cdot \nabla\varphi \in C([0,\infty); H)$ and so (1.1) yields $\Delta\mu \in C([0,\infty); H)$ which entails $\mu \in C([0,\infty); H^2(\Omega))$. This and the assumption $J \in W^{2,1}(\mathbb{R}^2)$ allow us to deduce (3.49).

4 Uniform estimates and the global attractor

In this section we establish some uniform in time regularization estimates by exploiting the results proved in the previous section. As a consequence we deduce a regularity property for the global attractor of the dynamical system generated by (1.1)–(1.5) whose existence has been shown in [9].

Proposition 1. *Let $h \in L^2_{tb}(0,\infty; G_{div})$, $u_0 \in V_{div}$, $\varphi_0 \in V \cap L^\infty(\Omega)$ and suppose that (H1)-(H4) are satisfied. Then, the weak solution $[u,\varphi]$ of Theorem 2 satisfies*

$$u \in L^\infty(0,\infty; V_{div}) \cap L^2_{tb}(0,\infty; H^2(\Omega)^2), \quad \varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^\infty(0,\infty; V), \tag{4.1}$$

$$u_t \in L^2_{tb}(0,\infty; G_{div}), \quad \varphi_t \in L^2_{tb}(0,\infty; H). \tag{4.2}$$

Furthermore, suppose in addition that $F \in C^3(\mathbb{R})$ and that $\varphi_0 \in H^2(\Omega)$. Then, the unique strong solution of Theorem 2 satisfies (4.1), (4.2) *and, in addition,*

$$\varphi \in L^\infty(0,\infty; W^{1,p}(\Omega)), \qquad 2 \le p < \infty, \tag{4.3}$$

$$\varphi_t \in L^\infty(0,\infty; H) \cap L^2_{tb}(0,\infty; V). \tag{4.4}$$

If $J \in W^{2,1}(\mathbb{R}^2)$, we also have

$$\varphi \in L^\infty(0,\infty; H^2(\Omega)). \tag{4.5}$$

Moreover, there exists a constant $\Lambda_1 = \Lambda_1(m)$, depending on m (and on F, J, Ω, ν), such that, for every initial data $z_0 := [u_0, \varphi_0] \in V_{div} \times H^2(\Omega)$, with $|\overline{\varphi}_0| \leq m$, there exists a time $t^* := t^*(\mathcal{E}(z_0)) \geq 0$ such that the strong solution corresponding to z_0 satisfies

$$\|\nabla u(t)\| + \|\varphi(t)\|_{H^2(\Omega)} + \int_t^{t+1} \|u(s)\|_{H^2(\Omega)^2} \leq \Lambda_1(m), \qquad \forall t \geq t^*. \tag{4.6}$$

Proof. Let us first notice that, setting $z(t) := [u(t), \varphi(t)]$ and $z_0 := [u_0, \varphi_0]$, by integrating the energy identity (2.15) between t and $t+1$ we have

$$\mathcal{E}(z(t+1)) + \int_t^{t+1} \Big(\frac{\nu}{2}\|\nabla u\|^2 + \|\nabla \mu\|^2\Big) d\tau \leq \mathcal{E}(z(t)) + \frac{1}{2\nu\lambda_1}\int_t^{t+1} \|h\|^2 d\tau. \tag{4.7}$$

Therefore, using also the dissipative estimate (2.16), we get

$$\int_t^{t+1} \Big(\frac{\nu}{2}\|\nabla u\|^2 + \|\nabla \mu\|^2\Big) d\tau \leq \mathcal{E}(z_0)e^{-kt} + F(m)|\Omega| + K \tag{4.8}$$

where the constant K depends on $\|h\|_{L^2_{tb}(0,\infty;G_{div})}$ and on F, J, Ω, ν. Notice that the initial energy $\mathcal{E}(z_0)$ can be estimated as

$$\mathcal{E}(z_0) \leq \frac{1}{2}\|u_0\|^2 + M\|\varphi_0\|^2 + \int_\Omega F(\varphi_0), \qquad M := \sup_{x\in\Omega}\int_\Omega |J(x-y)|dy.$$

From (4.8), setting $\Lambda_0(m) := F(m)|\Omega| + K + 1$, we deduce that there exists a time $t_0 = t_0(\mathcal{E}(z_0)) > 0$, given e.g. by $t_0 = \frac{1}{k}\log(\mathcal{E}(z_0) + c)$, where $\mathcal{E}(z_0) + c > 1$, such that

$$\int_t^{t+1} \Big(\frac{\nu}{2}\|\nabla u\|^2 + \|\nabla \mu\|^2\Big) d\tau \leq \Lambda_0(m), \qquad \forall t \geq t_0. \tag{4.9}$$

We now establish the uniform in time version of estimates $(3.1)_1$ and $(3.2)_1$ for the velocity field. To this aim, notice first that (2.5) implies (see also [28, Lemma 3.8])

$$\|Bu\| \leq c\|u\|^{1/2}\|\nabla u\|\|Su\|^{1/2}, \qquad \forall u \in D(S) = H^2(\Omega)^2 \cap V_{div}.$$

Therefore, by splitting the term (Bu, Su) on the left hand side of (3.7) and using the estimate above, we get the following differential inequality

$$\frac{d}{dt}\|\nabla u\|^2 + \nu\|Su\|^2 \leq \frac{3}{\nu}\|\mu\nabla\varphi\|^2 + \frac{3}{\nu}\|h\|^2 + \sigma\|\nabla u\|^2, \tag{4.10}$$

where $\sigma(t) := c_\nu\|u\|^2\|\nabla u\|^2$. Now, recalling Remark 4 (see also the proof of [15, Lemma 2.10]), the assumption $h \in L^2_{tb}(0,\infty;G_{div})$ and the dissipative estimate (2.16), we know that there exists a constant $C_0(m) > 0$ depending on m, and a time $t_1 = t_1(\mathcal{E}(z_0))$ depending on $\mathcal{E}(z_0)$ such that

$$\sup_{t\geq t_1}\|\varphi(t)\|_{L^\infty(\Omega)} \leq C_0(m). \tag{4.11}$$

Therefore we have $\sup_{t\geq t_1}\|\mu(t)\|_{L^\infty(\Omega)} \leq C_1(m)$. Then, using also (4.9) and (3.13), we get

$$\int_t^{t+1} \|\mu(\tau)\nabla\varphi(\tau)\|^2 d\tau \leq C_2(m), \qquad \int_t^{t+1} \sigma(\tau)d\tau \leq C_3(m), \tag{4.12}$$

for all $t \geq t_2 := \max\{t_0, t_1\}$. Therefore, (4.8) and (4.12) allow us to apply Lemma 2 to the differential inequality (4.10) and we deduce that

$$\|\nabla u(t)\|^2 \leq C_4(m) := \frac{2}{\nu}\Big(2C_2(m) + 2\|h\|^2_{L^2_{tb}(0,\infty;G_{div})} + \Lambda_0(m)\Big)e^{C_3(m)}, \tag{4.13}$$

for all $t \geq t_3 := t_2 + 1$. Furthermore, by integrating (4.10) between t and $t+1$, for $t \geq t_3$, we obtain

$$c\nu \int_t^{t+1} \|u(s)\|^2_{H^2(\Omega)} ds \leq C_5(m) := (1 + C_3(m))C_4(m) + \frac{4}{\nu}\Big(C_2(m) + \|h\|^2_{L^2_{tb}(0,\infty;G_{div})}\Big), \tag{4.14}$$

for all $t \geq t_3$, where we have also used [28, Lemma 3.7]. Estimates (4.13) and (4.14) in particular imply $(4.1)_1$.

Now, let us write (1.3) in the form $u_t = -Bu - \nu Su + \mu\nabla\varphi + h$ and observe that, owing to [28, Lemma 3.8] (or (2.5)), we have

$$\int_t^{t+1} \|Bu(s)\|^4 ds \leq \int_t^{t+1} \|u(s)\|^2\|\nabla u(s)\|^4\|Su(s)\|^2 ds \leq C_6(m) := \frac{c}{\nu\lambda_1}C_4^3(m)C_5(m),$$

for all $t \geq t_3$, and hence

$$\int_t^{t+1} \|u_t(s)\|^2 ds \leq C_7(m) := c\Big(C_6^{1/2}(m) + \nu C_5(m) + C_2(m) + \|h\|^2_{L^2_{tb}(0,\infty;G_{div})}\Big), \tag{4.15}$$

for all $t \geq t_3$. Note that (4.15) entails $(4.2)_1$.

We are now in a position to get uniform in time regularization estimates for φ_t first in $L^2_{tb}(\tau,\infty;H)$ and then in $L^\infty(\tau,\infty;H)$, for some $\tau > 0$.

Let us note first that, by combining (3.9)–(3.11) and taking (4.11) into account, we obtain the following differential inequality, for all $t > t_1$,

$$\frac{d}{dt}\|\nabla\mu\|^2 + c_0\|\varphi_t\|^2 \leq (C_8(m)\|u\|^2_{H^2} + c)\|\nabla\mu\|^2 + C_9(m)\|u\|^2_{H^2}\|\varphi\|^2. \tag{4.16}$$

Observe that (cf. (4.14))

$$\int_t^{t+1} (C_8(m)\|u(s)\|^2_{H^2} + c)ds \leq C_{10}(m) := \frac{1}{c\nu}C_5(m)C_8(m) + c, \tag{4.17}$$

$$\int_t^{t+1} C_9(m)\|u(s)\|^2_{H^2}\|\varphi(s)\|^2 ds \leq C_{11}(m) := \frac{|\Omega|}{c\nu}C_0^2(m)C_5(m)C_9(m), \tag{4.18}$$

for all $t \geq t_3$. Then, using (4.9) and (4.17), (4.18), we can apply the uniform Gronwall lemma to (4.16) in $[t_3, \infty)$ and get

$$\|\nabla\mu(t)\|^2 \leq C_{12}(m) := (C_{11}(m) + \Lambda_0(m))e^{C_{10}(m)}, \qquad \forall t \geq t_4 := t_3 + 1. \tag{4.19}$$

Now, by integrating (4.16) between t and $t+1$, for $t \geq t_4$, we also deduce

$$c_0 \int_t^{t+1} \|\varphi_t(s)\|^2 ds \leq C_{13}(m) := (1 + C_{10}(m))C_{12}(m) + C_{11}(m), \qquad \forall t \geq t_4. \tag{4.20}$$

Estimates (4.19) and (4.20) imply, in particular, $(4.1)_2$ and $(4.2)_2$, respectively.

Let us now consider estimate (3.28). Set

$$\Phi(t) := \frac{1}{2}\int_\Omega (a + F''(\varphi(t)))\varphi_t^2(t),$$

and notice that, on account of (4.11), we have

$$\frac{c_0}{2}\|\varphi_t(t)\|^2 \leq \Phi(t) \leq C_{14}(m)\|\varphi_t(t)\|^2, \qquad \forall t \geq t_1. \tag{4.21}$$

Then, by arguing as in the previous section and taking (4.11) into account, we easily see that (3.28) can be rewritten as follows

$$\frac{d}{dt}\Phi(t) + \frac{1}{8}\|\nabla\mu_t\|^2 \leq \omega(t)\Phi(t) + \beta(t) + C_{15}(m), \qquad \forall t \geq t_1, \tag{4.22}$$

where $\omega(t) := \alpha(t) + C_{16}(m)\Phi(t)$, and α, β the same as in (3.28). Then, by using (4.21), (4.20), (4.14) and (4.15), we have

$$\int_t^{t+1} \Phi(s)ds \leq C_{17}(m) := \frac{1}{c_0}C_{13}(m)C_{14}(m), \tag{4.23}$$

$$\int_t^{t+1} \omega(s)ds \leq C_{18}(m) := c\Big(\frac{1}{\nu}C_5(m) + C_7(m) + C_{16}(m)C_{17}(m) + 1\Big), \tag{4.24}$$

$$\int_t^{t+1} \beta(s)ds \leq C_{19}(m) := C_0^2(m)C_7(m), \tag{4.25}$$

for all $t \geq t_4$. By applying once more the uniform Gronwall lemma to (4.22) in the interval $[t_4, \infty)$, we deduce

$$\|\varphi_t(t)\|^2 \leq C_{20}(m) := \frac{2}{c_0}\Big(C_{15}(m) + C_{17}(m) + C_{19}(m)\Big)e^{C_{18}(m)}, \tag{4.26}$$

for all $t \geq t_5 := t_4 + 1$. Then, by integrating (4.22) between t and $t+1$, for $t \geq t_5$, and using (4.21), (4.26) and (3.27) (written with a constant $C_{21}(m)$ in place of $\bar{c}$, for $t \geq t_1$, due to (4.11)), we also find

$$\int_t^{t+1} \|\nabla\varphi_t(s)\|^2 ds \leq C_{22}(m) := \frac{32}{c_0}\Big(C_{14}C_{20}C_{18} + C_{19} + C_{15}\Big) + \big(1 + C_{20}^2\big)C_{21}, \tag{4.27}$$

for all $t \geq t_5$, where all C_i depend on m. Observe that estimates (4.26) and (4.27) yield (4.4).

Furthermore, owing to (3.23) and (3.24), we also have

$$\|\nabla\varphi(t)\|_{L^p(\Omega)^2} \leq C_{23}(m), \qquad \forall t \geq t_5, \qquad 2 \leq p < \infty \tag{4.28}$$

Finally, on account of (3.32), (4.26) and (4.13), we obtain

$$\|\mu(t)\|_{H^2} \leq c\| - \Delta\mu(t) + \mu(t)\| \leq C_{24}(m) := c\big(C_1(m) + C_{20}^{1/2}(m) + C_4^{1/2}(m)C_{23}(m)\big), \tag{4.29}$$

for all $t \geq t_5$, and recalling (3.34), provided that $J \in W^{2,1}(\mathbb{R}^2)$, we get

$$\|\varphi(t)\|_{H^2} \leq C_{25}(m), \qquad \forall t \geq t_5. \tag{4.30}$$

Estimates (4.28) and (4.30) yield (4.3). □

Let us now recall the main result about the existence of the global attractor for weak solutions to system (1.1)–(1.5) in the autonomous case (cf. [9]). Since the weak solutions to system (1.1)–(1.5) are not known to be unique but the energy identity holds, the existence of the global attractor is achieved by using J.M. Ball's approach based on the notion of generalized semiflows (cf. [3], to which we refer for the main definitions and results).

We assume that h is time independent, i.e., $h \in G_{div}$, and, for $m \geq 0$ fixed, we introduce the metric space

$$\mathcal{X}_m := G_{div} \times \mathcal{Y}_m, \tag{4.31}$$

where

$$\mathcal{Y}_m := \{\varphi \in H \,:\, F(\varphi) \in L^1(\Omega),\ |(\varphi, 1)| \leq m\}, \tag{4.32}$$

The space $\mathcal{X}_m$ is endowed with the metric

$$\mathbf{d}(z_2, z_1) = \|u_2 - u_1\| + \|\varphi_2 - \varphi_1\| + \Big|\int_\Omega F(\varphi_2) - \int_\Omega F(\varphi_1)\Big|^{1/2}, \quad \forall z_1, z_2 \in \mathcal{X}_m,$$

where $z_1 := [u_1, \varphi_1]$ and $z_2 := [u_2, \varphi_2]$.

Suppose that (H1)–(H4) are satisfied and that $h \in G_{div}$. Let $\mathcal{G}_m$ be the set of all weak solutions to system (1.1)–(1.6) from $[0, \infty)$ to $\mathcal{X}_m$ given by Theorem 1 and corresponding to all initial data $z_0 \in \mathcal{X}_m$. Then, in [9, Prop. 3 and Thm. 3] it is proved that $\mathcal{G}_m$ is a generalized semiflow on $\mathcal{X}_m$ (i.e., $\mathcal{G}_m$ satisfies conditions (H1)–(H4) from [3] in the space $\mathcal{X}_m$) which possesses a (unique) global attractor $\mathcal{A}_m$.

Take $z_0 \in \mathcal{X}_m$ and consider a weak solution $z := [u, \varphi] \in C([0, \infty); \mathcal{X}_m)$ corresponding to z_0. From (2.14), written with $t = \tau$, we know that for every $\tau > 0$ there exists $t_\tau \in (0, \tau]$

such that $z(t_\tau) \in V_{div} \times V$. Thanks to Remark 4, we can also assume that $\varphi(t_\tau) \in L^\infty(\Omega)$. We can therefore write the differential inequality (4.16) in $[t_\tau, \infty)$ and, by integrating (4.16) between t_τ and $t > t_\tau$, we can see that there exists $s_\tau \in (t_\tau, t]$ such that $\varphi_t(s_\tau) \in H$ and hence $\varphi(s_\tau) \in H^2(\Omega)$. Summing up, introducing the (complete) metric space

$$\mathcal{X}_m^1 := V_{div} \times \mathcal{Y}_m^1, \qquad \mathcal{Y}_m^1 := \{\varphi \in H^2(\Omega) : \ |(\varphi, 1)| \le m\}, \tag{4.33}$$

endowed with the metric

$$\mathbf{d_1}(z_2, z_1) = \|\nabla u_2 - \nabla u_1\| + \|\varphi_2 - \varphi_1\|_{H^2(\Omega)}, \quad \forall z_1, z_2 \in \mathcal{X}_m^1,$$

then, for every $\tau > 0$, there exists $s_\tau \in (0, \tau]$ such that $z(s_\tau) \in \mathcal{X}_m^1$ and starting from the time s_τ the weak solution corresponding to z_0 becomes a (unique) strong solution $z \in C([s_\tau, \infty); \mathcal{X}_m^1)$ (cf. Remarks 3 and 6). Such a solution satisfies the dissipative estimate (4.6) in $[s_\tau, \infty)$. Let us consider a bounded in $\mathcal{X}_m$ subset $B \subset \mathcal{X}_m$. Choosing $\tau = 1$ for every $z_0 \in B$, then every weak solution z starting from $z_0 \in B$ becomes (at a certain time $s_1 \in (0, 1]$ depending on z_0 and on the weak solution considered from z_0) a strong solution satisfying (4.6) in $[1, \infty)$. We therefore deduce that there exists a time $t^* = t^*(B) \ge 1$ such that

$$z(t) \in \mathcal{B}_1(\Lambda_1(m)), \qquad \forall t \ge t^*, \tag{4.34}$$

where $\mathcal{B}_1(\Lambda_1(m))$ is the closed ball in $\mathcal{X}_m^1$ given by

$$\mathcal{B}_1(\Lambda_1(m)) := \{w \in \mathcal{X}_m^1 : \ \mathbf{d_1}(w, 0) \le \Lambda_1(m)\}.$$

This fact immediately implies that $\mathcal{A}_m \subset \mathcal{B}_1$. Indeed, we have $\text{dist}_{\mathcal{X}_m^1}(T(t)\mathcal{A}_m, \mathcal{B}_1) = \text{dist}_{\mathcal{X}_m^1}(\mathcal{A}_m, \mathcal{B}_1) = 0$, which implies $\mathcal{A}_m \subset \overline{\mathcal{B}_1}^{\mathcal{X}_m^1} = \mathcal{B}_1$. We recall that the multivalued evolution map $T(t)$ is defined, for every $t \ge 0$ and every subset $E \subset \mathcal{X}_m$, as (cf. [3])

$$T(t)E := \{z(t) : \ z \in \mathcal{G}_m, \ z(0) \in E\}. \tag{4.35}$$

Summing up we have just proven the following regularity result for the global attractor

Theorem 3. *Let (H1)–(H4) be satisfied and assume that $h \in G_{div}$ is independent of time. Then the global attractor $\mathcal{A}_m$ of the generalized semiflow $\mathcal{G}_m$ associated with system* (1.1)–(1.5) *is such that*

$$\mathcal{A}_m \subset \mathcal{B}_1(\Lambda_1(m)).$$

Thus the global attractor is the union of all the bounded complete trajectories which are strong solutions to (1.1)-(1.6).

5 Convergence to equilibria

In this section we shall prove that every weak solution to system (1.1)–(1.6) converges to a stationary solution as $t \to \infty$, provided that F is real analytic and $h \equiv 0$.

Let us first consider the set of all stationary solutions z_∞ to system (1.1)–(1.5), namely the set of pairs $z_\infty := [0, \varphi_\infty] \in \mathcal{X}_m$ (for some $m \geq 0$), where φ_∞ solves the integral equation

$$a\varphi_\infty - J * \varphi_\infty + F'(\varphi_\infty) = \mu_\infty, \tag{5.1}$$

with some constant $\mu_\infty \in \mathbb{R}$ given necessarily by $\mu_\infty = \overline{F'(\varphi_\infty)}$. Therefore we introduce

$$\begin{aligned} \mathcal{E}_m \quad = \quad \Big\{ & z_\infty = [0, \varphi_\infty] : \quad \varphi_\infty \in H, \quad F(\varphi_\infty) \in L^1(\Omega), \quad |\overline{\varphi_\infty}| \leq m, \\ & a\varphi_\infty - J * \varphi_\infty + F'(\varphi_\infty) - \overline{F'(\varphi_\infty)} = 0 \quad \text{a.e. in } \Omega \Big\}. \end{aligned} \tag{5.2}$$

We point out that, by using an easy iteration argument from (5.1), on account that F' has polynomial growth, we can deduce that $\varphi_\infty \in L^\infty(\Omega)$. The structure of the stationary set is rather complicated. In particular, such a set may be a continuum (see [8] for an example and [19] where the author proves the existence of solutions φ_∞ to (5.7) with $\overline{\varphi}_\infty = 0$ in cylindrical bounded domains). It is also worth observing that to every stationary solution $z_\infty = [0, \varphi_\infty]$ there corresponds a stationary pressure π_∞ given by $\pi_\infty = \overline{F'(\varphi_\infty)}\varphi_\infty + c$, where $c \in \mathbb{R}$ is an arbitrary constant (cf. (1.3)).

We begin with the following preliminary but crucial result.

Lemma 3. *Assume that (H1)–(H4) are satisfied. Take $z_0 \in \mathcal{X}_m$ and let $z \in C([0,\infty); \mathcal{X}_m)$ be a weak solution corresponding to z_0. Then, we have*

$$\emptyset \neq \omega(z) \subset \mathcal{E}_m \tag{5.3}$$

and

$$u(t) \to 0 \qquad in \quad G_{div}, \qquad as \quad t \to \infty. \tag{5.4}$$

Furthermore, there exists a time $t^* = t^*(z_0)$ depending on z_0 such that the trajectory $\bigcup_{t \geq t^*} \{z(t)\}$ is precompact in $\mathcal{X}_m$.

Proof. From (2.14), by letting $t \to \infty$, we obtain that

$$u \in L^2(0, \infty; V_{div}). \tag{5.5}$$

On the other hand, from (1.3), written as $u_t = -Bu - \nu S u + \mu \nabla \varphi$, we get

$$\|u_t\|_{V'_{div}} \leq \nu \|\nabla u\| + c\|u\|\|\nabla u\| + \|\varphi\|_{L^\infty(\Omega)} \|\nabla \mu\|.$$

Now, (2.14) also implies that $u \in L^\infty(0,\infty;G_{div})$ and that $\nabla\mu \in L^2(0,\infty;H)$. Hence, on account of (4.11) as well, from the previous estimate we infer that

$$u_t \in L^2(\tau,\infty;V'_{div}), \tag{5.6}$$

for some $\tau > 0$. From (5.5) and (5.6) we deduce (5.4). Let us now take $\widetilde{z} \in \omega(z_0)$ arbitrary, with $\widetilde{z} := [\widetilde{u},\widetilde{\varphi}]$. Then, there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $u(t_n) \to \widetilde{u}$ in G_{div} and $\varphi(t_n) \to \widetilde{\varphi}$ in H. We get $\widetilde{u} = 0$ and, up to a subsequence,

$$\mu(t_n) \to \widetilde{\mu}, \qquad \text{a.e. in} \quad \Omega, \tag{5.7}$$

where $\widetilde{\mu} := a\widetilde{\varphi} - J*\widetilde{\varphi} + F'(\widetilde{\varphi})$. By integrating (4.22) between t and $t+1$ we easily deduce that $\nabla\mu_t \in L^2_{tb}(\tau,\infty;H)$ for some $\tau > 0$. Since we also have $\nabla\mu \in L^2(0,\infty;H)$, then Lemma 1 yields

$$\nabla\mu(t) \to 0 \qquad \text{in} \quad H, \qquad \text{as} \quad t \to \infty. \tag{5.8}$$

From (5.7) and (5.8) we easily deduce that $\widetilde{\mu}$=const almost everywhere in Ω, where the constant is necessarily given by $\overline{F'(\widetilde{\varphi})}$. Therefore $\widetilde{z} = [\widetilde{u},\widetilde{\varphi}] = [0,\widetilde{\varphi}] \in \mathcal{E}_m$ (note that $F(\widetilde{\varphi}) \in L^1(\Omega)$ is ensured by Fatou's lemma), and (5.3) is proven. Finally, the precompactness of the trajectory is an immediate consequence of (4.34). □

Remark 7. Lemma 3 yields in particular an existence result for equation (5.1).

We now recall the generalized Łojasiewicz-Simon inequality established in [11] which is the main tool for proving our convergence result.

Let V and W be Banach spaces embedded into a Hilbert space H and its dual H', respectively, with dense and continuous injections. Assume that the restriction of the Riesz map $R \in \mathcal{L}(H,H')$ to V is an isomorphism from V onto $W = R(V)$. Moreover, let $H = H_0 + H_1$, where $H_1 \subset V$ is a finite-dimensional subspace and H_0 is its orthogonal complement in H. Introduce the subspace of H'

$$H_0^0 := \big\{ g \in H' : \ \langle g,\varphi\rangle = 0 \ \text{for all}\ \varphi \in H_0 \big\}.$$

Then let

$$\mathcal{F} := \mathcal{G}_1 + \mathcal{G}_2,$$

where the functionals $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the following conditions

- $\mathcal{G}_1 : U \subset V \to \mathbb{R}$ is Fréchet differentiable on an open set U such that the Fréchet derivative $D\mathcal{G}_1 : U \to W$ is a real analytic operator which satisfies

$$\langle D\mathcal{G}_1(\varphi_2) - D\mathcal{G}_1(\varphi_1), \varphi_2 - \varphi_1\rangle \geq \alpha_1 \|\varphi_2 - \varphi_1\|_H^2, \tag{5.9}$$

$$\|D\mathcal{G}_1(\varphi_2) - D\mathcal{G}_1(\varphi_1)\|_{H'} \le \alpha_2 \|\varphi_2 - \varphi_1\|_H, \tag{5.10}$$

for all $\varphi_1, \varphi_2 \in U$ and for some constants $\alpha_1, \alpha_2 > 0$. Furthermore, the second Fréchet derivative $D^2\mathcal{G}_1(\varphi) \in \mathcal{L}(V, W)$ is assumed to be an isomorphism for all $\varphi \in U$.

- $\mathcal{G}_2 : H \to \mathbb{R}$ is assumed to be in the form

$$\mathcal{G}_2(\varphi) = \frac{1}{2}\langle \mathcal{K}\varphi, \varphi\rangle + \langle l, \varphi\rangle + \rho, \qquad \forall \varphi \in H,$$

where $\mathcal{K} \in \mathcal{L}(H, H')$ is a self-adjoint compact operator such that its restriction to V is a compact operator in $\mathcal{L}(V, W)$ and $l \in W$, $\rho \in \mathbb{R}$ are given.

The inequality we need is given by

Lemma 4 ([11])**.** *Let the previous assumptions be satisfied for the spaces V, W, H, H' and for the functional $\mathcal{F}$. Let $[\varphi_\infty, \mu_\infty] \in U \times H_0^0$ satisfy $D\mathcal{F}(\varphi_\infty) = \mu_\infty$. Then, there exist $\sigma, \lambda > 0$ and $\theta \in (0, 1/2]$ such that the following inequality holds*

$$|\mathcal{F}(\varphi) - \mathcal{F}(\varphi_\infty)|^{1-\theta} \le \lambda \inf\left\{\|D\mathcal{F}(\varphi) - \mu\|_{H'},\ \mu \in H_0^0\right\}, \tag{5.11}$$

for all $\varphi \in U$ satisfying $\varphi - \varphi_\infty \in H_0$ and $\|\varphi - \varphi_\infty\|_H \le \sigma$.

We can now state the main result of this section.

Theorem 4. *Assume that (H1)–(H4) are satisfied with F real analytic. Take $z_0 \in \mathcal{X}_m$ and let $z \in C([0,\infty); \mathcal{X}_m)$ be a weak solution corresponding to z_0. Then, there exists $z_\infty := [0, \varphi_\infty] \in \mathcal{E}_m$ with $\overline{\varphi}_\infty = \overline{\varphi}_0$ such that*

$$z(t) \to z_\infty \quad \text{in} \quad \mathcal{X}_m, \qquad \text{as} \quad t \to \infty. \tag{5.12}$$

Moreover, there exist some constants $\overline{\gamma} \ge 0$, $\theta \in [0, 1/2)$ and a time $\overline{t} > 0$ which depend on z_0 and z_∞ (and on the weak solution z originated from z_0) such that

$$\|u(t)\|_{V'_{div}} + \|\varphi(t) - \varphi_\infty\|_{V'} \le \overline{\gamma}\, t^{-\frac{\theta}{1-2\theta}}, \qquad \forall t > \overline{t}. \tag{5.13}$$

Proof. Our aim is to prove that $\varphi_t \in L^1(\tau, \infty; V')$, for some $\tau > 0$. This, together with (5.4) and with the precompactness of the trajectory in $G_{div} \times H$, will allow to deduce the convergence in $G_{div} \times H$ of a whole trajectory $z = [u, \varphi]$ originating from an initial datum $z_0 = [u_0, \varphi_0] \in \mathcal{X}_m$ to a stationary solution $z_\infty \in \mathcal{E}_m$ with $\overline{\varphi}_\infty = \overline{\varphi}_0$. Observe that if $z : [0, \infty) \to \mathcal{X}_m$ is a weak solution, then the convergence condition $z(t) \to z_\infty$ in $\mathcal{X}_m$ is equivalent to the condition $z(t) \to z_\infty$ in $G_{div} \times H$, since the convergence $\int_\Omega F(\varphi(t)) \to \int_\Omega F(\varphi_\infty)$ is ensured by (4.11) and Lebesgue's dominated convergence theorem.

The key point is the application of Lemma 4 to a suitable functional $\mathcal{F}$ which is, in our case, the energy functional E associated with the φ component of the solution, namely,

$$E(\varphi) = \frac{1}{2}\|\sqrt{a}\varphi\|^2 - \frac{1}{2}(\varphi, J * \varphi) + \int_\Omega F(\varphi). \tag{5.14}$$

More precisely, we set (cf. Lemma 4)

$$\begin{aligned}
&H := H' = L^2(\Omega), \qquad H_0 := \{\psi \in H : \overline{\psi} = 0\}, \qquad H_0^0 = \{\psi = \text{const}\},\\
&V = L^\infty(\Omega), \qquad W := R(V), \qquad \|f\|_W := \|R^{-1}f\|_V,\\
&\mathcal{G}_1(\psi) := \int_\Omega \Big(F(\psi) + \frac{1}{2}a\psi^2\Big), \qquad U = U_m := \{\psi \in V : |\psi(x)| < C_0(m), \ \text{ a.e. } \ x \in \Omega\},\\
&\mathcal{K}(\psi) := -J * \psi, \qquad l = \rho = 0,
\end{aligned} \tag{5.15}$$

where the positive constant $C_0(m)$ is the same as in (4.11).

All the assumptions of Lemma 4 are fulfilled. Indeed, $\mathcal{G}_1$ is Fréchet differentiable on the whole V with $D\mathcal{G}_1(\varphi) \in W$, for all $\varphi \in V$ given by

$$\langle D\mathcal{G}_1(\varphi), h\rangle = \int_\Omega \big(F'(\varphi) + a\varphi\big)h, \qquad \forall h \in V.$$

Furthermore, $D\mathcal{G}_1$ is a real analytic operator, since F is assumed real analytic, and we have

$$\begin{aligned}
&\langle D\mathcal{G}_1(\varphi_2) - D\mathcal{G}_1(\varphi_1), \varphi_2 - \varphi_1\rangle = \int_\Omega \big(F''(\eta\varphi_2 + (1-\eta)\varphi_1) + a\big)|\varphi_2 - \varphi_1|^2\\
&\geq c_0\|\varphi_2 - \varphi_1\|^2, \qquad \forall \varphi_1, \varphi_2 \in V,
\end{aligned}$$

thanks to (H2), where $\eta = \eta(x) \in (0,1)$. Hence (5.9) is satisfied (with $\alpha_1 = c_0$). As far as (5.10) is concerned, observe that $D\mathcal{G}_1$ is locally Lipschitz from V to H'. Indeed, we have

$$\|D\mathcal{G}_1(\varphi_2) - D\mathcal{G}_1(\varphi_1)\|_{H'} \leq \|F'(\varphi_2) - F'(\varphi_1)\| + a_\infty\|\varphi_2 - \varphi_1\| \leq \Gamma_m\|\varphi_2 - \varphi_1\|^2,$$

for all $\varphi_1, \varphi_2 \in U_m$, which yields (5.10) (with $\alpha_2 = \Gamma_m$). Moreover, the second Fréchet derivative is given by

$$\langle D^2\mathcal{G}_1(\varphi)h_1, h_2\rangle = \int_\Omega \big(F''(\varphi) + a\big)h_1h_2, \qquad \forall h_1, h_2 \in V,$$

for all $\varphi \in V$. Hence $D^2\mathcal{G}_1(\varphi) \in \mathcal{L}(V, W)$ is an isomorphism for all $\varphi \in U_m$. Finally, thanks to (H1), the convolution operator $\mathcal{K}$ is compact from H to H and also from V to W (due to the compact embedding $W^{1,\infty}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$). The Fréchet derivative of $\mathcal{F} = E$ is given by

$$DE(\varphi) = F'(\varphi) + a\varphi - J * \varphi = \mu, \tag{5.16}$$

and we have that $[\varphi_\infty, \mu_\infty] \in U_m \times H_0^0$ satisfy $DE(\varphi_\infty) = \mu_\infty$ iff $z_\infty := [0, \varphi_\infty] \in \mathcal{E}_m$ with $\varphi_\infty \in U_m$ and $\mu_\infty = \overline{F'(\varphi_\infty)}$. Therefore, taking $[\varphi_\infty, \mu_\infty] \in U_m \times H_0^0$ such that $DE(\varphi_\infty) = \mu_\infty$, Lemma 4 entails the existence of $\sigma, \lambda > 0$ and $\theta \in (0, 1/2]$ such that

$$|E(\varphi) - E(\varphi_\infty)|^{1-\theta} \leq \lambda \inf \big\{ \|\mu - \widetilde{\mu}\|, \ \widetilde{\mu} = \text{const} \big\} = \lambda \|\mu - \overline{\mu}\| \leq \lambda c_p \|\nabla \mu\|, \tag{5.17}$$

for all $\varphi \in U_m$ satisfying $\overline{\varphi} = \overline{\varphi_\infty}$ (i.e. $\varphi - \varphi_\infty \in H_0$) and $\|\varphi - \varphi_\infty\|_H \leq \sigma$, where c_p is the Poincaré-Wirtinger constant.

Now, let $z_0 \in \mathcal{X}_m$ and z be a weak solution corresponding to z_0. Take $z_\infty \in \omega(z)$ and let $\{t_n\}$ be a sequence such that $t_n \to \infty$ and $z(t_n) \to z_\infty$ in $\mathcal{X}_m$. Consider the function

$$\Phi(t) := \mathcal{E}(z(t)) - \mathcal{E}(z_\infty).$$

We have

$$\Phi'(t) = -\nu \|\nabla u\|^2 - \|\nabla \mu\|^2 \leq -c_\nu (\|\nabla u\| + \|\nabla \mu\|)^2 \leq 0, \qquad \text{for a.a. } t > 0, \tag{5.18}$$

where $c_\nu = \min\{1, \nu\}/2$. Since $\Phi(t_n) \to 0$ and Φ is non-increasing in $(0, \infty)$, then $\Phi(t) \to 0$, as $t \to \infty$ and $\Phi \geq 0$. Now, due to (5.4) and to (5.17) (notice that $2(1-\theta) > 1$), we have

$$\begin{aligned}\Phi^{1-\theta}(t) &= \Big(\frac{1}{2}\|u(t)\|^2 + E(\varphi(t)) - E(\varphi_\infty)\Big)^{1-\theta} \\ &\leq \|u(t)\|^{2(1-\theta)} + |E(\varphi(t)) - E(\varphi_\infty)|^{1-\theta} \\ &\leq c_\lambda \Big(\|\nabla u\| + \|\nabla \mu\| \Big), \end{aligned} \tag{5.19}$$

for all $t \geq t_0$, for some $t_0 > 0$, provided that $\|\varphi(t) - \varphi_\infty\| < \sigma$, where $c_\lambda = \max\{1/\sqrt{\lambda_1}, \lambda c_p\}$. Therefore, by combining (5.18) and (5.19) we get

$$-\frac{d}{dt}\Phi^\theta(t) = -\theta \Phi^{\theta-1}(t)\Phi'(t) \geq \frac{\theta c_\nu}{c_\lambda}\Big(\|\nabla u(t)\| + \|\nabla \mu(t)\| \Big), \tag{5.20}$$

provided that $\varphi(t) \in U_m$ with $\|\varphi(t) - \varphi_\infty\| < \sigma$ and $\overline{\varphi}(t) = \overline{\varphi}_\infty = \overline{\varphi}_0$. By means of a classical argument, together with equations (1.1) and (1.2), we can now deduce that $\varphi_t \in L^1(\tau, \infty; V')$. Indeed, for every $\delta \in (0, 1)$ there exists $N = N_\delta$ such that for all $n \geq N_\delta$ we have $\|u(t_n)\| < \delta$ and $\|\varphi(t_n) - \varphi_\infty\| < \delta$. Set

$$t^* = t^*(\delta) := \sup \big\{ t \geq t_N : \|u(s)\| < 1, \ \|\varphi(s) - \varphi_\infty\| < \sigma, \quad \forall s \in [t_N, t] \big\}. \tag{5.21}$$

Then, estimate (5.20) holds for all $t \in [t_N, t^*]$. By integrating it between t_N and t^* and possibly choosing a larger N we have

$$\int_{t_N}^{t^*} \Big(\|\nabla u(\tau)\| + \|\nabla \mu(\tau)\| \Big) d\tau \leq \frac{c_\lambda}{\theta c_\nu} \Phi^\theta(t_N) < \delta. \tag{5.22}$$

We now claim that there exists $\delta_* > 0$ such that $t^*(\delta_*) = \infty$. Indeed, suppose this is not true, i.e. $t^*(\delta) < \infty$ for all $\delta > 0$. Then, we have

$$\begin{aligned}\int_{t_N}^{t^*} \|u_t(\tau)\|_{V'_{div}} d\tau &\le \int_{t_N}^{t^*} \Big(\nu\|\nabla u(\tau)\| + c\|u(\tau)\|\|\nabla u(\tau)\| + \|\varphi(\tau)\|_{L^\infty(\Omega)}\|\nabla\mu(\tau)\|\Big) d\tau \\ &\le b_1 \int_{t_N}^{t^*} \Big(\|\nabla u(\tau)\| + \|\nabla\mu(\tau)\|\Big) d\tau \le b_1\delta, \end{aligned} \tag{5.23}$$

where $b_1 = \max\big\{\nu + c\Lambda_1(m)/\sqrt{\lambda_1}, C_0(m)\big\}$, and where N_δ is assumed large enough, i.e., such that $t_{N_\delta} \ge t_1(\mathcal{E}(z_0))$ (see (4.11)). Furthermore, we have

$$\begin{aligned}&\int_{t_N}^{t^*} \|\varphi_t(\tau)\|_{V'} d\tau \le \int_{t_N}^{t^*} \Big(\|\nabla\mu(\tau)\| + \|\varphi(\tau)\|_{L^\infty}\|u(\tau)\|\Big) d\tau \\ &\le b_2 \int_{t_N}^{t^*} \Big(\|\nabla u(\tau)\| + \|\nabla\mu(\tau)\|\Big) d\tau \le b_2\delta, \end{aligned} \tag{5.24}$$

where $b_2 = \max\big\{1, C_0(m)/\sqrt{\lambda_1}\big\}$. Therefore, we deduce

$$\|u(t^*)\|_{V'_{div}} \le \|u(t_N)\|_{V'_{div}} + \int_{t_N}^{t^*} \|u_t(\tau)\|_{V'_{div}} d\tau \le b_3\delta, \tag{5.25}$$

$$\|\varphi(t^*) - \varphi_\infty\|_{V'} \le \|\varphi(t_N) - \varphi_\infty\|_{V'} + \int_{t_N}^{t^*} \|\varphi_t(\tau)\|_{V'} d\tau \le b_4\delta, \tag{5.26}$$

where $b_3 = 1/\sqrt{\lambda_1} + b_1$ and $b_4 = 1 + b_2$. Let us now take a sequence $\{\delta_n\}$ such that $\delta_n \to 0$. Then, from definition (5.21), for every n at least one of the following two conditions holds

$$\|u(t^*(\delta_n))\| = 1, \qquad \|\varphi(t^*(\delta_n)) - \varphi_\infty\| = \sigma. \tag{5.27}$$

By possibly extracting a subsequence we have, e.g., $\|\varphi(t^*(\delta_n)) - \varphi_\infty\| = \sigma$. Writing (5.26) with $\delta = \delta_n$ and taking into account the precompactness of the trajectory in $G_{div} \times H$ we get a contradiction. Thus, for some $\delta_* > 0$ we have (setting $\bar{t} := t_{N_{\delta_*}}$)

$$\int_{\bar{t}}^{\infty} \Big(\|\nabla u(\tau)\| + \|\nabla\mu(\tau)\|\Big) d\tau < \delta_* < \infty, \tag{5.28}$$

so that

$$u \in L^1(\bar{t}, \infty; V_{div}), \qquad \nabla\mu \in L^1(\bar{t}, \infty; H). \tag{5.29}$$

This implies that $\varphi_t \in L^1(\bar{t}, \infty; V')$, due $(4.1)_2$ and to the estimate

$$\|\varphi_t\|_{V'} \le \|\nabla\mu\| + c\|\varphi\|_V\|\nabla u\|.$$

By using the precompactness of the trajectory in $G_{div} \times H$ again, we deduce that $\varphi(t) \to \varphi_\infty$ in H as $t \to \infty$. Therefore we have $z(t) \to z_\infty$ in $\mathcal{X}_m$ as $t \to \infty$. We now provide an estimate for the convergence rate in $V'_{div} \times V'$. Indeed, from (5.18) and (5.19) we deduce

$$\Phi'(t) \le -\frac{c_\nu}{c_\lambda^2}\Phi^{2(1-\theta)}(t), \qquad \forall t > \bar{t}$$

which yields by integration

$$\Phi(t) \le \Phi(0)\big\{1 + b_5\Phi^{1-2\theta}(0)t\big\}^{-\frac{1}{1-2\theta}}, \qquad \forall t > \bar{t}, \tag{5.30}$$

where $b_5 = c_\nu(1-2\theta)/c_\lambda^2$. On the other hand, by integrating (5.20) from $t \ge \bar{t}$ to ∞ we get

$$\int_t^\infty \Big(\|\nabla u(\tau)\| + \|\nabla\mu(\tau)\|\Big)d\tau = \frac{c_\lambda}{\theta c_\nu}\Phi^\theta(t), \qquad \forall t > \bar{t}. \tag{5.31}$$

Finally, we obtain

$$\|u(t)\|_{V'_{div}} \le \int_t^\infty \|u_t(\tau)\|_{V'_{div}}d\tau \le b_1\int_t^\infty \Big(\|\nabla u(\tau)\| + \|\nabla\mu(\tau)\|\Big)d\tau, \tag{5.32}$$

$$\|\varphi(t) - \varphi_\infty\|_{V'} \le \int_t^\infty \|\varphi_t(\tau)\|_{V'}d\tau \le b_2\int_t^\infty \Big(\|\nabla u(\tau)\| + \|\nabla\mu(\tau)\|\Big)d\tau. \tag{5.33}$$

By combining (5.30)–(5.33) we deduce the convergence rate estimate (5.13) with $\overline{\gamma} = (b_1+b_2)c_\lambda\theta^{-1}c_\nu^{-1}b_5^{-\theta/(1-2\theta)}$. □

Remark 8. By using standard interpolation inequalities one can deduce from (5.13) convergence rate estimates in stronger norms. Of course, the convergence exponent deteriorates.

Acknowledgments. The first author was supported by the FTP7-IDEAS-ERC-StG Grant ♯200497(BioSMA) and the FP7-IDEAS-ERC-StG Grant #256872 (EntroPhase). He is also grateful for the support received during his visit to the Institute of Mathematics of the Academy of Sciences of the Czech Republic in Prague (GAČR Grant P201/10/2315 and RVO: 67985840).

References

[1] H. Abels, *On a diffusive interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal. **194** (2009), 463-506.

[2] H. Abels, *Longtime behavior of solutions of a Navier-Stokes/Cahn-Hilliard system*, Proceedings of the Conference "Nonlocal and Abstract Parabolic Equations and their Applications", Bedlewo, Banach Center Publ. **86** (2009), 9-19.

[3] J.M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equation*, J. Nonlinear Sci. **7** (1997), 475-502 (Erratum, J. Nonlinear Sci. **8** (1998), 233).

[4] P.W. Bates, J. Han, *The Neumann boundary problem for a nonlocal Cahn-Hilliard equation*, J. Differential Equations **212** (2005), 235-277.

[5] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter formulation*, Asymptot. Anal. **20** (1999), 175-212.

[6] J.W. Cahn, J.E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys. **28** (1958), 258-267.

[7] P. Colli, S. Frigeri, M. Grasselli, *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Math. Anal. Appl. **386** (2012), 428-444.

[8] E. Feireisl, F. Issard-Roch, H. Petzeltová, *A non-smooth version of the Łojasiewicz-Simon theorem with applications to non-local phase-field systems*, J. Differential Equations **199** (2004), 1-21.

[9] S. Frigeri, M. Grasselli, *Global and trajectories attractors for a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Dynam. Differential Equations **24** (2012), 827-856.

[10] S. Frigeri, M. Grasselli, *Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials*, Dyn. Partial Differ. Equ. **9** (2012), 273-304.

[11] H. Gajewski, J.A. Griepentrog, *A descent method for the free energy of multicomponent systems*, Discrete Contin. Dyn. Syst. **15** (2005), 505-528.

[12] C.G. Gal, M. Grasselli, *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 401-436.

[13] C.G. Gal, M. Grasselli, *Trajectory attractors for binary fluid mixtures in 3D*, Chinese Ann. Math. Ser. B **31** (2010), 655-678.

[14] C.G. Gal, M. Grasselli, *Instability of two-phase flows: a lower bound on the dimension of the global attractor of the Cahn-Hilliard-Navier-Stokes system,* Phys. D **240** (2011), 629-635.

[15] C.G. Gal, M. Grasselli, *Longtime behavior of nonlocal Cahn-Hilliard equations*, arXiv:1207.4018.

[16] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits*, J. Statist. Phys. **87** (1997), 37-61.

[17] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. II. Phase motion*, SIAM J. Appl. Math. **58** (1998), 1707-1729.

[18] M.E. Gurtin, D. Polignone, J. Viñals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Meth. Appl. Sci. **6** (1996), 8-15.

[19] J. Han, *The Cauchy problem and steady state solutions for a nonlocal Cahn-Hilliard equation*, Electron. J. Differential Equations **113** (2004), 9 pp.

[20] M. Heida, J. Málek, K.R. Rajagopal, *On the development and generalizations of Cahn-Hilliard equations within a thermodynamic framework*, Z. Angew. Math. Phys. **63** (2012), 145-169.

[21] P.C. Hohenberg, B.I. Halperin, *Theory of dynamical critical phenomena*, Rev. Mod. Phys. **49** (1977), 435-479.

[22] D. Jasnow, J. Viñals, *Coarse-grained description of thermo-capillary flow*, Phys. Fluids **8** (1996), 660-669.

[23] S.-O. Londen, H. Petzeltová, Convergence of solutions of a non-local phase-field system, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), 653-670.

[24] S.-O. Londen, H. Petzeltová, Regularity and separation from potential barriers for a non-local phase-field system, J. Math. Anal. Appl. **379** (2011), 724-735.

[25] A. Morro, *Phase-field models of Cahn-Hilliard Fluids and extra fluxes*, Adv. Theor. Appl. Mech. **3** (2010), 409-424.

[26] V.N. Starovoitov, *The dynamics of a two-component fluid in the presence of capillary forces*, Math. Notes **62** (1997), 244-254.

[27] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, New York, 1997.

[28] R. Temam, Navier-Stokes Equations. Theory and Numerical Analysis, North-Holland, Amsterdam, 1984.

[29] L. Zhao, H. Wu, H. Huang, *Convergence to equilibrium for a phase-field model for the mixture of two viscous incompressible fluids*, Commun. Math. Sci. **7** (2009), 939-962.