Symmetries of Stochastic Differential Equations using Girsanov transformations

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Abstract. Aiming at enlarging the class of symmetries of an SDE, we introduce a family of stochastic transformations able to change also the underlying probability measure exploiting Girsanov Theorem and we provide new determining equations for the infinitesimal symmetries of the SDE. The well-defined subset of the previous class of measure transformations given by Doob transformations allows us to recover all the Lie point symmetries of the Kolmogorov equation associated with the SDE. This gives the first stochastic interpretation of all the deterministic symmetries of the Kolmogorov equation. The general theory is applied to some relevant stochastic models.

1. Introduction

The analysis of infinitesimal symmetries of (deterministic) ordinary and partial differential equations (ODEs and PDEs) is a classical and well-developed tool in studying differential equations providing a very powerful method to both compute some of their explicit solutions or to analyze their qualitative behavior (see [4, 38] for an introduction to the subject).

The application of Lie symmetry analysis techniques to stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs) has received a growing interest in recent years. Without claiming to be exhaustive, from the first studies on Brownian-motion-driven SDEs (see [7, 3, 24]) and Markov processes (see [26, 27, 36, 8]) many different notions of symmetries for Brownian-motion-driven SDEs has been proposed (see the symmetries with random time change [47, 45, 20], W-symmetry in [21, 23], weak symmetries in [15], random symmetries in [22, 25, 32] and symmetries of diffusions in [16]). Moreover, random transformations of semimartingales on Lie groups has been introduced (see [1]) and the notion of symmetry has been extended also to SDEs driven by general semimartingales (see [2, 35]). Furthermore the idea of reduction and reconstruction of differential equations admitting a solvable Lie algebra of symmetries has been generalized from the deterministic to the stochastic setting (see [35, 30] for strong symmetries and [14] for weak symmetries). Symmetries of SDEs arising from variational problems has been considered and Noether theorem and integration by quadratures have been generalized to variational stochastic problems (see [34, 41, 46]). Finally, a generalization of differential constraint method to SPDEs has been proposed in [12, 13], while in [2, 17] symmetries have been applied to the study of invariant numerical methods for SDEs.

In the case of Brownian-motion-driven SDEs or Lévy-process-driven SDEs, the solution process to the equation is a Markov process. This means that the mean value of a Markovian function of the process solves a deterministic linear PDE called Kolmogorov equation (whose adjoint is the also very well known Fokker-Planck equation). In this setting, a natural problem is the comparison between the set of Lie point symmetries of Kolmogorov (or Fokker-Planck) equation and the set of symmetries of the corresponding SDE (for the importance of Lie point symmetries of Kolmogorov equation in integrating the associated diffusion process see [11, 9, 10, 28, 29]). In the case of Brownian-motion-driven SDEs all the different notions of symmetry introduced for SDEs (strong symmetries, W symmetries, weak symmetries) have been proved to be also symmetries of the corresponding Kolmogorov equation, but the converse is false (see [16, 21, 24, 31, 33]). In this paper we provide an appropriate extension of the notion of symmetry of an SDE so that the converse of the previous result holds.

In order to do this we extend the approach first proposed in [15], where the concept of weak symmetry of an SDE has been introduced considering a three components stochastic transformation for the weak solutions to a Brownian-motion-driven SDE. Indeed, this stochastic transformation is composed by a deterministic transformation of the state variable X, a random time change and a random rotation of the Brownian motion W. Since the set of these stochastic transformations forms an infinite dimensional Lie group, it is possible to consider the associated Lie algebra, whose elements are called infinitesimal stochastic transformations. The infinitesimal stochastic transformations constitute the infinitesimal generators of the finite stochastic transfor-

mations. When the process $P_T(X, W)$, obtained transforming a weak solution (X, W) to an SDE through the stochastic transformation T, is again a (weak) solution to the same SDE, we say that T is a (finite) weak symmetry for the SDE. The same notion can be extended to the case of infinitesimal transformations, and this allows us to obtain explicitly a set of determining equations characterizing the infinitesimal symmetries of an SDE.

In the present paper we enlarge the previous class of stochastic transformations introducing a new component which represents the possibility of changing the underlying probability measure of the solution process (X, W) via the use of Girsanov theorem. We call this kind of transformations *extended stochastic transformations* and the corresponding symmetries *extended (weak) symmetries.*
The first result of the paper is, thus, the introduction of this new family of stochastic transformations and the study of their geometric properties (they form an infinite dimensional Lie group with associated Lie algebra).
The second main result is that for any infinitesimal Lie point symmetry of Kolmogorov equation there exists an infinitesimal extended (weak) symmetry of the corresponding SDE. Furthermore, the family of infinitesimal extended weak symmetries of the SDE is larger than the set of Lie point symmetries of the corresponding Kolmogorov equation. Indeed, we find that any extended weak symmetry of the SDE that is also a symmetry of the Kolmogorov equation is associated with a *Doob change of measure.* Since Doob transformations form a strict subset of the class of absolutely continuous changes of measure which can be obtained within a Girsanov approach, we prove that the extended weak symmetries of an SDE are in general more than the Lie point symmetries of the Kolmogorov equation. To the best of our knowledge, these results represent the first purely stochastic interpretation of the whole set of symmetries of Kolmogorov equations. In particular, for one-dimensional Brownian motion, our stochastic symmetry analysis permits us to individuate not only all the symmetries of the one-dimensional heat equation, but also a new infinite dimensional family of symmetries associated with the random time invariance of the process (see Remark 5.1). Analogously, for two-dimensional Brownian motion, besides the symmetries of the corresponding two-dimensional heat equation, we find two new infinite dimensional families of symmetries: the first one is related to random time invariance and the second one is related to random rotations invariance of the process (see Remark 5.2).

The plan of the paper is the following: in Section 2 we briefly recall the symmetry analysis of the Brownian-motion-driven SDEs given in [15]. In Section 3 we introduce the fourth transformation of an SDE which allows the change of the underlying probability measure into another equivalent probability measure and we show how the new global stochastic transformation acts on the solution process and on the SDE coefficients. We also provide a geometrical description of the group of stochastic transformations and we derive the new determining equations for the infinitesimal symmetries. In Section 4 we compare the extended weak symmetries of an SDE with the Lie point symmetries of the corresponding Kolmogorov equation, proving that the two notions coincide as long as we consider extend weak symmetries with a Doob change of measure. In Section 5 the theoretical results of the paper are applied to a list of SDEs which are important both from a theoretical point of view and for their applications. Finally in Section 6 we give a proof of the martingale property of

the exponential supermartingale in our framework which is necessary for rigorously applying Girsanov theorem.

2. A three components stochastic transformation of an SDE

Since in this paper we generalize the stochastic transformations introduced in [15] and [14] for Brownian-motion-driven SDEs, for the convenience of the reader we collect in this section the main facts and definitions that can be useful in the following. A more detailed exposition, as well as all the proofs, can be found in [15] and [14]. In the following, the Einstein summation convention on repeated indices is used.

Let M, M' be open subsets of $\mathbb{R}^n$. We consider the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t \subset \mathcal{F}$ is a filtration. We fix a finite time horizon $[0, \mathcal{T}]$ with $\mathcal{T} \geq 0$. Let $W = (W^1, \ldots, W^m) = (W^\alpha)$ be an $\mathcal{F}_t$-m-dimensional Brownian motion and let $\mu : M \to \mathbb{R}^n$ and $\sigma : M \to \text{Mat}(n, m)$ be two smooth functions.

Definition 2.1. *The process (X, W) solves (in a weak sense) the SDE with coefficients μ, σ (shortly solves the SDE (μ, σ)) if, for all $t \in [0, \mathcal{T}]$,*

$$X_t^i - X_0^i = \int_0^t \mu^i(X_s)\, ds + \int_0^t \sigma_\alpha^i(X_s)\, dW_s^\alpha \quad i = 1, \ldots, n.$$

When (X, W) solves the SDE (μ, σ) we write, as usual

$$\begin{aligned} dX_t &= \mu(X_t)\, dt + \sigma(X_t) \cdot dW_t \\ &= \mu dt + \sigma \cdot dW_t \end{aligned} \tag{1}$$

In order to ensure that the integrals in the above definition are well defined we suppose that the processes $(|\mu(X_s)|^{\frac{1}{2}})_{s \in [0, \mathcal{T}]}$ and $(\sigma(X_s))_{s \in [0, \mathcal{T}]}$ belong to the class $M_{loc}^2([0, \mathcal{T}])$, i.e. to the class of processes $(Y_s)_{s \in [0, \mathcal{T}]}$ that are progressively measurable and such that $\int_0^t Y_s^2(\omega)\, ds < \infty$ for almost every $\omega \in \Omega$ and $t \in [0, \mathcal{T}]$. We recall that, given an SDE (μ, σ), we can introduce the corresponding infinitesimal generator given by

$$L = \frac{1}{2}(\sigma\sigma^T)^{ij}\partial_i\partial_j + \mu^i\partial_i. \tag{2}$$

In the following we only consider autonomous SDEs, and we introduce three different transformations for their solution processes.

Spatial transformations

We call spatial transformation of an SDE a transformation Φ acting on the process component X through a generic diffeomorphism $\Phi : M \to M'$. Let us denote with $\nabla\Phi : M \to \text{Mat}(n, m)$ the Jacobian matrix

$$(\nabla\Phi)_j^i = \partial_j\Phi^i.$$

By using the well-known Itô formula (see, e.g., [43] Section 32 or [37] Chapter 4) it is easy to compute how the coefficients of the SDE change.

Proposition 2.2. *Given a diffeomorphism $\Phi : M \to M'$, if the process (X, W) is solution to the SDE (μ, σ), then the process $(\Phi(X), W)$ is solution to the SDE (μ', σ') with*

$$\begin{aligned} \mu' &= L(\Phi) \circ \Phi^{-1} \\ \sigma' &= (\nabla\Phi \cdot \sigma) \circ \Phi^{-1} \end{aligned}$$

We remark that the spatial transformation does not act on the Brownian motion.

Random time changes

In the following we consider only Markovian absolutely continuous random time changes of the solution process to an SDE through a smooth and strictly positive density $\eta : M \to \mathbb{R}_+$. We denote by H_η the transformation

$$t' = \int_0^t \eta_s(X_s)ds.$$

This time transformation necessarily acts on both components of the solution process (X, W). In particular, given a Brownian motion W_t and denoting by W'_t the stochastic process solution to

$$dW'_t = \sqrt{\eta(X_t)}dW_t,$$

it is possible to prove that $H_\eta(W')$ is also a Brownian motion.

The total action of the random time change transformation on the coefficients of an SDE is described by the following proposition.

Proposition 2.3. *Let $\eta : M \to \mathbb{R}_+$ be a smooth and strictly positive function and let (X, W) be a solution to the SDE (μ, σ). Then the process $(H_\eta(X), H_\eta(W'))$ is solution to the SDE (μ', σ') with*

$$\mu' = \frac{1}{\eta}\mu$$

$$\sigma' = \frac{1}{\sqrt{\eta}}\sigma.$$

Random rotations

Since Brownian motion is invariant with respect to random rotations (see [15] and [1] for the general concept of gauge transformations), and the solution to the SDE is understood in a weak sense, an additional possible transformation of the SDE consists in a random rotation of the driving Brownian motion.

By using the Lévy characterization of Brownian motion the following result states how an SDE is modified by a random rotation of its driving process.

Proposition 2.4. *Let $B : M \to SO(m)$ be a smooth function and let (X, W) be a solution to the SDE (μ, σ). Then (X, W'), where*

$$dW'_t = B(X_t) \cdot dW_t,$$

is a solution to the SDE (μ', σ') with

$$\mu' = \mu,$$
$$\sigma' = \sigma \cdot B^{-1}.$$

Stochastic transformations

Collecting the natural transformations introduced above, we can give the following definition.

Definition 2.5. *Given a diffeomorphism $\Phi : M \to M'$ and two smooth functions $B : M \to SO(m)$ and $\eta : M \to \mathbb{R}_+$, the triad $T = (\Phi, B, \eta)$ is called a* stochastic transformation.

We can distinguish two actions of the stochastic transformation $T = (\Phi, B, \eta)$: one on the solution processes and the other one on the coefficients of an SDE.

Definition 2.6. *If (X, W) is a solution process, we define the transformed process $P_T(X, W) = (X', W')$ as*

$$X' = \Phi(H_\eta(X))$$
$$W' = H_\eta(\tilde{W}),$$

where

$$\tilde{W}_t^\alpha = \int_0^t B_\beta^\alpha(X_s) dW_s^\beta.$$

Definition 2.7. *If (X, W) is a solution process of an SDE (μ, σ), we define the transformed SDE $E_T(\mu, \sigma) = (\mu', \sigma')$ as*

$$\mu' = \left(\frac{1}{\eta} L(\Phi)\right) \circ \Phi^{-1}$$
$$\sigma' = \left(\frac{1}{\sqrt{\eta}} \nabla\Phi \cdot \sigma \cdot B^T\right) \circ \Phi^{-1}.$$

By composing the previous propositions one can prove the following result.

Proposition 2.8. *If T is a stochastic transformation and (X, W) is a solution to the SDE (μ, σ), then $P_T(X, W)$ is a solution to the SDE $E_T(\mu, \sigma)$.*

Since the set of all stochastic transformations is a group with composition law given by

$$T' \circ T = (\Phi' \circ \Phi, (B' \circ \Phi) \cdot B, (\eta' \circ \Phi)\eta)$$

and unit $1_M = (id_M, I_m, 1)$, we can consider a one parameter group T_a of stochastic transformations such that $T_a \circ T_b = T_{a+b}$ and $T_0 = 1_M$ and the corresponding infinitesimal generator V.

Definition 2.9. *A triad $V = (Y, C, \tau)$, where Y is a vector field on M, $C : M \to \mathfrak{so}(m)$ and $\tau : M \to \mathbb{R}$ are smooth functions, is called an infinitesimal stochastic transformation if*

$$\begin{aligned} Y(x) =& \ \partial_a(\Phi_a(x))|_{a=0} \\ C(x) =& \ \partial_a(B_a(x))|_{a=0} \\ \tau(x) =& \ \partial_a(\eta_a(x))|_{a=0} \end{aligned} \tag{3}$$

On the other side, considering the triad (Y, C, τ) defined as above, the solution (Φ_a, B_a, η_a) to the equations

$$\begin{aligned} \partial_a(\Phi_a(x)) =& \ Y(\Phi_a(x)) \\ \partial_a(B_a(x)) =& \ C(\Phi_a(x)) \cdot B_a(x) \\ \partial_a(\eta_a(x)) =& \ \tau(\eta_a(x))\eta_a(x) \end{aligned} \tag{4}$$

with initial data $\Phi_0(x) = id_M(x)$, $B_0(x) = I_m$, $\eta_0(x) = 1$, is a one-parameter group of stochastic transformations.

Weak symmetries and determining equations

The introduction of a (global) stochastic transformation of the solution process to an SDE allows us to provide a good definition of (weak) symmetry for the SDE.

Definition 2.10. *A stochastic transformation T is a* (finite weak) symmetry *of an SDE (μ, σ) if, for every solution process (X, W), $P_T(X, W)$ is a solution process to the same SDE.*

The following characterization of the symmetry property in terms of the coefficients of the SDE holds.

Proposition 2.11. *A stochastic transformation $T = (\Phi, B, \eta)$ is a (finite weak) symmetry of an SDE (μ, σ) if and only if*

$$\begin{aligned} \mu &= \left(\tfrac{1}{\eta} L\left(\Phi\right)\right) \circ \Phi^{-1} \\ \sigma &= \left(\tfrac{1}{\sqrt{\eta}} \nabla \Phi \cdot \sigma \cdot B^{-1}\right) \circ \Phi^{-1}. \end{aligned} \tag{5}$$

An infinitesimal stochastic transformation V generating a one parameter group T_a is called an *infinitesimal symmetry* of the SDE (μ, σ) if T_a is a (finite) symmetry of (μ, σ).

A useful characterization of the infinitesimal symmetries of an SDE is provided by the following statement.

Theorem 2.12. *An infinitesimal stochastic transformation V is an (infinitesimal weak) symmetry of the SDE (μ, σ) if and only if V generates a one-parameter group on M and the following* determining equations *hold*

$$[Y, \sigma] = -\frac{1}{2} \tau \sigma - \sigma \cdot C \tag{6}$$

$$Y(\mu) - L(Y) = -\tau \mu, \tag{7}$$

where $[Y, \sigma]_\alpha^i = Y(\sigma_\alpha^i) - \partial_k(\Phi^i)\sigma_\alpha^k$.

The determining equations obtained in Theorem 2.12 provide a powerful tool for the explicit computation of the symmetries of an SDE. Indeed, they are definitely easier to solve than the equations introduced in Proposition 2.11 for finite symmetries.

3. An additional random transformation: a probability measure change

3.1. Extended stochastic transformations

In this section we introduce an additional transformation of an SDE and of its solution process given by the change of measure through Girsanov theorem. We propose here only a short introduction of the probabilistic theory of absolutely continuous predictable change of measure of Brownian motion. The interested reader is referred, e.g., to [43] Section 38 and to [37] Section 8.6 for an extended treatment of the subject.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space where an $(\mathcal{F}_t)$-Brownian motion W_t is defined. We fix a finite time horizon $I = [0, \mathcal{T}]$ with $\mathcal{T} > 0$ and we work with the filtration $\mathcal{F}_\mathcal{T} = \{\mathcal{F}_t, 0 < t < \mathcal{T}\}$.
Given a process $(\theta_s)_{s \in [0, \mathcal{T}]} \in M_{loc}^2[0, \mathcal{T}]$, we define the process $(Z_t)_{t \in [0, \mathcal{T}]}$ by setting

$$Z_t = Z_t(\theta) := \exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right\}. \tag{8}$$

An application of Itô formula gives

$$dZ_t = Z_t \theta_t dW_t,$$

which says that Z is a local martingale. Since Z_t is strictly positive, we have that Z_t is always a supermartingale. The process Z_t is called *exponential supermartingale* and can be used as a Radon-Nikodym derivative of a new probability measure $\mathbb{Q}$. This means that we can define on Ω the probability law $\mathbb{Q}_{\mathcal{F}_{\mathcal{T}}}(d\omega) := Z_{\mathcal{T}}(\omega) \mathbb{P}_{\mathcal{F}_{\mathcal{T}}}(d\omega)$, i.e. $\mathbb{Q}_{\mathcal{F}_{\mathcal{T}}}(A) := \int_A Z_{\mathcal{T}}(\omega) \, d\, \mathbb{P}_{\mathcal{F}_{\mathcal{T}}}(\omega), \forall A \in \mathcal{F}$.

We recall the fundamental result allowing the change of a probability measure into an equivalent one.

Theorem 3.1 (Girsanov's theorem)**.** *Let* $(Z_t)_{t \in [0,\mathcal{T}]}$ *be the exponential supermartingale defined in (8). If* $(Z_t)_{t \in [0,\mathcal{T}]}$ *is a* $\mathbb{P}$*-martingale, then the process* $\left(\hat{W}_t\right)_{t \in [0,\mathcal{T}]}$ *given by*

$$\hat{W}_t = W_t - \int_0^t \theta_s ds$$

is an $(\mathcal{F})_t$*-Brownian motion with respect to the probability measure* $\mathbb{Q}$*, where*

$$\frac{d\, \mathbb{Q}_{\mathcal{F}_{\mathcal{T}}}}{d\, \mathbb{P}_{\mathcal{F}_{\mathcal{T}}}} = Z_{\mathcal{T}}.$$

Proof. The proof can be found for example in [43], Theorem 38.5. $\qquad\square$

In order to better explain the probabilistic meaning of the previous theorem, we recall the following notion.

Definition 3.2. *Let* $\mathbb{P}$ *and* $\mathbb{Q}$ *two probability measures on a measurable space* $(\Omega, \mathcal{F})$. $\mathbb{Q}$ *is called* absolutely continuous *with respect to* $\mathbb{P}$*, and we write* $\mathbb{Q} \ll \mathbb{P}$*, if*

$$\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0 \text{ for all } A \in \mathcal{F}.$$

The measures $\mathbb{P}$ *and* $\mathbb{Q}$ *are* equivalent *if* $\mathbb{Q} \ll \mathbb{P}$ *and* $\mathbb{P} \ll \mathbb{Q}$*, that is if they have the same set of null events.*

Since the Radon-Nikodym derivative in Girsanov theorem is strictly positive, i.e. $\frac{d\, \mathbb{Q}_{\mathcal{F}_{\mathcal{T}}}}{d\, \mathbb{P}_{\mathcal{F}_{\mathcal{T}}}} = Z_{\mathcal{T}} > 0$, one can prove that the measure $\mathbb{Q}_{\mathcal{F}_{\mathcal{T}}}$ is actually equivalent to $\mathbb{P}_{\mathcal{F}_{\mathcal{T}}}$.

In order to be able to apply Girsanov theorem we need Z to be a true martingale and not only a local one. Establishing when the supermartingale Z_t is a (global) martingale is in general not an easy task. Instead of using the well known Novikov condition (see [42], Chapter VIII, Proposition 1.14), we introduce the following definition and lemma which allow us to achieve the same goal.

Definition 3.3. *A smooth SDE* (μ, σ) *is called* non explosive *if any solution* (X, W) *to* (μ, σ) *is defined for all times* $t \geq 0$*. A smooth vector field* h *is called* non explosive *for the non explosive SDE* (μ, σ) *if the SDE* $(\mu + \sigma \cdot h, \sigma)$ *is a non explosive SDE. A positive smooth function* η *is called* a non explosive time change *for the non explosive SDE* (μ, σ) *if the SDE* $\left(\frac{\mu}{\eta}, \frac{\sigma}{\sqrt{\eta}}\right)$ *is non explosive.*

Lemma 3.4. *Suppose that the equation* (μ, σ) *is a non explosive SDE, let* $h : M \to \mathbb{R}^n$ *be a smooth non explosive vector field, and suppose that* (X, W) *is a solution to the SDE* (μ, σ)*. Then the exponential supermartingale* Z_t *associated with* $\theta_t = h(X_t)$ *is a (global) martingale.*

Proof. The present lemma is a special case of [19, Theorem 9.1.4] or of [44], proved under very general conditions. In Section 6, following [44], we propose a proof specialized to our non explosiveness hypothesis. □

Remark 3.5. *It is simple to see that, if (μ, σ) is a non explosive SDE and $T = (\Phi, B, 1)$ is a stochastic transformation, then also $P_T(\mu, \sigma)$ is non explosive.*

Theorem 3.6. *Let (X, W) be a solution to the non explosive SDE (μ, σ) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let h be a smooth non explosive vector field for (μ, σ). Then $(X, \hat{W})$ is a solution to the SDE $(\mu', \sigma') = (\mu + \sigma \cdot h, \sigma)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where*

$$\hat{W}_t = -\int_0^t h(X_s)ds + W_t$$

$$\frac{\mathrm{d}\,\mathbb{Q}_{\mathcal{F}_\mathcal{T}}}{\mathrm{d}\,\mathbb{P}_{\mathcal{F}_\mathcal{T}}} = \exp \int_0^\mathcal{T} h_\alpha(X_s)dW_s^\alpha - \frac{1}{2}\int_0^\mathcal{T} \sum_{\alpha=1}^m (h_\alpha(X_s))^2 ds.$$

Proof. The proof is a simple application of Girsanov theorem and Lemma 3.4. □

We define a new (finite) stochastic transformation including, besides the three transformations described in the previous section, also a change of the underlying probability measure.

Definition 3.7. *Let $\Phi : M \to M'$ be a diffeomorphism and let $B : M \to SO(m)$, $\eta : M \to \mathbb{R}_+$ and $h : M \to \mathbb{R}^m$ be smooth functions. We call $T := (\Phi, B, \eta, h)$ a (weak finite) extended stochastic transformation.*

Since we cannot apply the previous transformation to any SDE, we give the following definition.

Definition 3.8. *Let $T = (\Phi, B, \eta, h)$ be an extended stochastic transformation. If the pair (X, W) is a continuous stochastic process where X takes values in M and W is an m-dimensional Brownian motion in the space $(\Omega, \mathcal{F}, \mathbb{P})$ and (X, W) is a solution to the non explosive SDE (μ, σ) for which h and η are non explosive, we can define the process $P_T(X, W) = (X', W')$, where X' takes values in M' and W' is a Brownian motion in the space $(\Omega, \mathcal{F}, \mathbb{Q})$. The process components are given by*

$$X' = \Phi(H_\eta(X))$$
$$d\tilde{W}_t = \sqrt{\eta(X_t)} B(X_t)(dW_t - h(X_t)\,dt)$$
$$W' = H_\eta\left(\tilde{W}\right)$$
$$\frac{\mathrm{d}\,\mathbb{Q}_{\mathcal{F}_\mathcal{T}}}{\mathrm{d}\,\mathbb{P}_{\mathcal{F}_\mathcal{T}}} = \exp\left(\int_0^\mathcal{T} h_\alpha(X_s)dW_s^\alpha - \frac{1}{2}\int_0^\mathcal{T} \sum_{\alpha=1}^m (h_\alpha(X_s))^2 ds\right).$$

We call $P_T(X, W)$ the transformed process *of (X, W) with respect to T and we call the function P_T the* process transformation *associated with T.*

The previous definition is necessary if we are interested in defining the transformation P_T on the set of solutions (X, W) to the SDE (μ, σ). If we focus only on the SDE identified with the pair of smooth functions (μ, σ) and on the action E_T of the stochastic transformation T, the previous definition is no more necessary and we can define the transformed SDE $E_T(\mu, \sigma)$ without making any request on the non explosiveness of the process (X, W).

Definition 3.9. *Let* $T = (\Phi, B, \eta, h)$ *be an extended stochastic transformation. If* (μ, σ) *is an SDE on* M *then we define* $E_T(\mu, \sigma) := (E_T(\mu), E_T(\sigma))$ *the SDE on* M' *defined as*

$$E_T(\mu) = \left(\frac{1}{\eta}\left[L(\Phi) + \nabla\Phi \cdot \sigma \cdot h\right]\right) \circ \Phi^{-1}$$

$$E_T(\sigma) = \left(\frac{1}{\sqrt{\eta}}\nabla\Phi \cdot \sigma \cdot B^{-1}\right) \circ \Phi^{-1}.$$

We call $E_T(\mu, \sigma)$ *the* transformed SDE *of* (μ, σ) *with respect to* T *and we call the map* E_T *the* SDE transformation *associated with* T.

We stress that a specific choice on the order according to which the transformations are applied was performed. First the change of measure and then the rotation together with the random time change. While the diffusion coefficient does not change, the form of the transformed drift strongly depends on this choice.

Theorem 3.10. *Let* $T = (\Phi, B, \eta, h)$ *be an extended stochastic transformation and let* (X, W) *be a solution to the non explosive SDE* (μ, σ) *such that* $E_T(\mu, \sigma)$ *is non explosive. Then* $P_T(X, W)$ *is solution to the SDE* $E_T(\mu, \sigma)$.

Proof. We show how the SDE (μ, σ) changes when we apply the transformations in the order specified above. For simplicity we omit the explicit dependence on the values of the process. The starting equation is

$$dX_t = \mu dt + \sigma \cdot dW_t$$

We apply first the measure change obtaining

$$dX'_t = (\mu + \sigma \cdot h)\, dt + \sigma \cdot dW'_t$$

and then the rotation of the Brownian motion and the random time change, obtaining

$$dX''_t = \frac{1}{\eta}(\mu + \sigma \cdot h)\, dt + \frac{1}{\sqrt{\eta}}\sigma \cdot B^{-1} \cdot dW''_t$$

$$= \frac{1}{\eta}\tilde{\mu} dt + \tilde{\sigma} \cdot dW''_t$$

where we set $\tilde{\mu} = \mu + \sigma \cdot h$ and $\tilde{\sigma} = \frac{1}{\sqrt{\eta}}\sigma \cdot B^{-1}$. Finally, we apply the spatial transformation considering only the drift (since the equations involving σ coincide with the analogous equations obtained considering non extended stochastic transformation), and we get

$$E_T(\mu) = \left(\frac{1}{\eta}\left[\frac{1}{2}\left(\sigma \cdot \sigma^T\right)^{ij}\partial_i\partial_j\Phi + \tilde{\mu}^i\partial_i\Phi\right]\right) \circ \Phi^{-1}$$

$$= \left(\frac{1}{\eta}\left[L(\Phi) + \nabla\Phi \cdot \sigma \cdot h\right]\right) \circ \Phi^{-1}.$$

Exploiting Definition 3.9 and Theorem 3.10, this concludes the proof. $\square$

In the following, in order to simplify notations, we refer to extended stochastic transformations just as stochastic transformations.

3.2. Stochastic transformations group and associated infinitesimal transformations

Let $G = SO(m) \times \mathbb{R}_+ \ltimes \mathbb{R}^m$ be the group of rototranslations with a scaling factor whose elements $g = (B, \eta, h)$ can be identified with the matrices of the form

$$\begin{pmatrix} \sqrt{\eta} \cdot B^{-1} & h \\ 0 & 1 \end{pmatrix}$$

If we consider the trivial principal bundle $\pi : M \times G \to M$ with structure group G, we can define the following action of G on $M \times G$

$$R_{g_2} : M \times G \to M \times G$$
$$(x, g_1) \to (x, g_1 \cdot g_2)$$

which leaves M invariant, where $g_1 \cdot g_2$ denotes the standard product $g_1 \cdot g_2 = (B_1, \eta_1, h_1) \cdot (B_2, \eta_2, h_2) = (B_2 B_1, \eta_2 \eta_1, \sqrt{\eta_1} B_1^{-1} h_2 + h_1)$. If we consider another trivial principal bundle $\pi' : M' \times G \to M'$, we say that a diffeomorphism $F : M \times G \to M' \times G$ between the two principal bundles $M \times G$ and $M' \times G$ is an *isomorphism* if F preserves the structures of principal bundles of both $M \times G$ and $M' \times G$, i.e. there exists a diffeomorphism $\Phi : M \to M'$ such that

$$F \circ \pi' = \pi \circ \Phi$$
$$F \circ R_g = R_g \circ F$$

for any $g \in G$. It is easy to check that an isomorphism of the previous form is completely determined by its value on (x, e), where e is the unit element of G. Therefore, there is a natural identification between a stochastic transformation $T = (\Phi, B, \eta, h)$ and the isomorphism F_T such that $F_T(x, e) = (\Phi(x), g)$ where $g = (B, \eta, h)$. In particular we can exploit the natural composition of isomorphisms in order to define a composition of stochastic transformations in the following way: if $T_1 = (\Phi_1, B_1, \eta_1, h_1)$ and $T_2 = (\Phi_2, B_2, \eta_2, h_2)$ then

$$T_2 \circ T_1 = \left(\Phi_2 \circ \Phi_1, (B_2 \circ \Phi_1) \cdot B_1, (\eta_2 \circ \Phi_1) \eta_1, \sqrt{\eta_1} B_1^{-1} \cdot (h_2 \circ \Phi_1) + h_1 \right).$$

Moreover we can consider the inverse transformation of $T = (\Phi, B, \eta, h)$ given by

$$T^{-1} = \left(\Phi^{-1}, \left(B \circ \Phi^{-1} \right)^{-1}, \left(\eta \circ \Phi^{-1} \right)^{-1}, -\frac{1}{\sqrt{\eta}} B \cdot h \circ \Phi^{-1} \right).$$

The following theorem, generalizing a result obtained in [15] for stochastic transformations without the change of probability measure, shows that the identification of stochastic transformations with the isomorphisms of a suitable trivial principal bundle has a deep probabilistic counterpart in terms of SDE and process transformations.

Theorem 3.11. *Let T_1, T_2 be two stochastic transformations, let (μ, σ) be a non explosive SDE such that $E_{T_1}(\mu, \sigma)$ and $E_{T_2}(E_{T_1}(\mu, \sigma))$ are non explosive and let (X, W) be a solution to the SDE (μ, σ) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ we have*

$$P_{T_2}(P_{T_1}(X, W)) = P_{T_2 \circ T_1}(X, W)$$
$$E_{T_2}(E_{T_1}(\mu, \sigma)) = E_{T_2 \circ T_1}(\mu, \sigma).$$

Proof. The proof of this result without the change of probability measure can be found in [15], hence we have just to add the part related to the last component of the

stochastic transformation.

Applying the two transformations to the Brownian motion, according to the fixed specified order, we obtain

$$
\begin{aligned}
dW_t' &= \sqrt{\eta_1} B_1 \cdot (dW_t - h_1 dt) \\
dW_t'' &= \sqrt{\eta_2} B_2 \cdot (dW_t' - h_2 dt') \\
&= \sqrt{\eta_2} B_2 \cdot (\sqrt{\eta_1} B_1 (dW_t - h_1 dt) - \eta_1 h_2 dt) \\
&= \sqrt{\eta_2 \eta_1} B_2 \cdot B_1 \cdot \left(dW_t - \left(h_1 + \sqrt{\eta_1} B_1^{-1} \cdot h_2 \right) dt \right) \\
&= \sqrt{\tilde{\eta}} \tilde{B} \cdot \left(dW_t - \tilde{h} dt \right),
\end{aligned}
$$

where

$$
\tilde{h} = \sqrt{\eta_1} B_1^{-1} \cdot h_2 + h_1.
$$

Noting that, after the first transformation T_1, the state variable is $\Phi(X)$, we obtain the statement. $\qquad \square$

Since the set of stochastic transformations is a group with respect to the composition $\circ$, we can consider the one parameter group $T_a = (\Phi_a, B_a, \eta_a, h_a)$ and the corresponding infinitesimal transformation $V = (Y, C, \tau, H)$ obtained in the usual way

$$
\begin{aligned}
Y(x) &= \partial_a (\Phi_a(x))|_{a=0} \\
C(x) &= \partial_a (B_a(x))|_{a=0} \\
\tau(x) &= \partial_a (\eta_a(x))|_{a=0} \\
H(x) &= \partial_a (h_a(x))|_{a=0}.
\end{aligned}
$$

On the other hand, given $V = (Y, C, \tau, H)$, where Y is a vector field on M, $C : M \to \mathfrak{so}(m)$, $\tau : M \to \mathbb{R}$ and $H : M \to \mathbb{R}^m$ are smooth functions, we can reconstruct the one parameter transformation group T_a exploiting the following relations

$$
\begin{aligned}
\partial_a (\Phi_a(x)) &= Y(\Phi_a(x)) \\
\partial_a (B_a(x)) &= C(\Phi_a(x)) \cdot B_a(x) \\
\partial_a (\eta_a(x)) &= \tau(\Phi_a(x)) \eta_a(x)
\end{aligned}
$$

with initial data $\Phi_0(x) = id_M(x)$, $B_0(x) = I_m$, $\eta_0(x) = 1$. Moreover, using Theorem 3.11 and the properties of the flow, we obtain

$$
\begin{aligned}
h_{b+a}(x) &= \sqrt{\eta_a(x)} B_a^{-1}(x) \cdot h_b(\Phi_a(x)) + h_a(x) \\
\partial_b (h_{b+a}(x)) &= \sqrt{\eta_a(x)} B_a^{-1}(x) \cdot \partial_b (h_b(\Phi_a(x))) \\
\partial_b (h_{b+a}(x))|_{b=0} = \partial_a (h_a(x)) &= \sqrt{\eta_a(x)} B_a^{-1}(x) \cdot H(\Phi_a(x))
\end{aligned}
$$

subjected to the initial condition $h_0(x) = 0$.

The notions of composition and one parameter group of stochastic transformations allow us to consider the action of a finite stochastic transformation on an infinitesimal one, as well as the Lie brackets between two infinitesimal stochastic transformations. Let $T = (\Phi, B, \eta, h)$ be a finite stochastic transformation and let $V = (Y, C, \tau, H)$ be an infinitesimal stochastic transformation with associated one

parameter group given by T_a. Then there exists an infinitesimal stochastic transformation $T_*(V)$, called *push forward of V through the transformation T*, generating the one parameter group given by $T \circ T_a \circ T^{-1}$ and having the following form

$$
\begin{aligned}
T_*(V) = \quad & \left(\Phi_*(Y), (B \cdot C \cdot B^{-1} + Y(B) \cdot B^{-1}) \circ \Phi^{-1}, \left(\tau + \tfrac{Y(\eta)}{\eta} \right) \circ \Phi^{-1}, \right. \\
& \left. \left(-\tfrac{1}{\sqrt{\eta}} B \cdot \left(-\tfrac{\tau}{2} + C \right) \cdot h + \tfrac{1}{\sqrt{\eta}} B \cdot H + \tfrac{1}{\sqrt{\eta}} B \cdot Y(h) \right) \circ \Phi^{-1} \right),
\end{aligned}
$$

where $\Phi_*(Y)$ denotes the push forward of the vector field Y. Using the previous expression we can define the Lie brackets between two infinitesimal stochastic transformations $V_1 = (Y_1, C_1, \tau_1, H_1)$ and $V_2 = (Y_2, C_2, \tau_2, H_2)$ which is

$$
\begin{aligned}
[V_1, V_2] = \quad & ([Y_1, Y_2], Y_1(C_2) - Y_2(C_1) - \{C_1, C_2\}, Y_1(\tau_2) - Y_2(\tau_1) \\
& Y_1(H_2) - Y_2(H_1) - \left(-\tfrac{\tau_1}{2} + C_1 \right) \cdot H_2 + \left(-\tfrac{\tau_2}{2} + C_2 \right) \cdot H_1).
\end{aligned}
$$

3.3. Determining equations

In this section we provide the new *determining equations* (including the probability measure change) for the infinitesimal symmetries of an SDE.

Proposition 3.12. *An (extended) stochastic transformation $T = (\Phi, B, \eta, h)$ is a (finite) symmetry of the non explosive SDE (μ, σ) if and only if*

$$
\mu = \left(\frac{1}{\eta} \left[L(\Phi) + \nabla \Phi \cdot \sigma \cdot h \right] \right) \circ \Phi^{-1}
$$

$$
\sigma = \left(\frac{1}{\sqrt{\eta}} \nabla \Phi \cdot \sigma \cdot B^{-1} \right) \circ \Phi^{-1}.
$$

Proof. The proof is a simple generalization of the proof of Proposition 2.11 (see [15]) using Theorem 3.10 in replacement of Proposition 2.8. It is important to note that since $E_T(\mu, \sigma) = (\mu, \sigma)$, in order to use Lemma 3.4, we need only to require that (μ, σ) is non explosive. $\qquad\square$

The characterization expressed by Proposition 2.11 for the coefficients $E_T(\mu)$ and $E_T(\sigma)$ under the symmetry hypothesis still holds because the new probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ (see Definition 3.2).

Theorem 3.13. *Let $V = (Y, C, \tau, H)$ be an infinitesimal stochastic transformation. Then V is an infinitesimal symmetry of the SDE (μ, σ) if and only if V generates a one parameter group defined on M and we have*

$$
\begin{aligned}
Y(\mu) - L(Y) - \sigma \cdot H + \tau\mu &= 0 \\
[Y, \sigma] + \tfrac{1}{2}\tau\sigma + \sigma \cdot C &= 0.
\end{aligned}
$$

Proof. We prove only that the property of V to be an infinitesimal symmetry is a sufficient condition for the validity of determining equations. The one parameter group $T_a = (\Phi_a, B_a, \eta_a, h_a)$ associated with the infinitesimal transformation $V = (Y, C, \tau, H)$ is a symmetry if and only if the following equations are satisfied

$$
\mu = \left(\frac{1}{\eta_a} \left[L(\Phi_a) + \nabla \Phi_a \cdot \sigma \cdot h_a \right] \right) \circ \Phi_a^{-1},
$$

$$
\sigma = \left(\frac{1}{\sqrt{\eta_a}} \nabla \Phi_a \cdot \sigma \cdot B_a^{-1} \right) \circ \Phi_a^{-1}.
$$

The first equation is equivalent to

$$L\left(\Phi_a\right) + \nabla\Phi_a \cdot \sigma \cdot h_a = \eta_a \mu\left(\Phi_a\right).$$

Deriving both sides with respect to the parameter a and then evaluating them in $a = 0$, we obtain

$$L\left(\partial_a\left(\Phi_a\right)|_{a=0}\right) + \partial_a\left(\nabla\Phi_a \cdot \sigma\right)|_{a=0} \cdot h_0 + \nabla\Phi_0 \cdot \sigma \cdot \partial_a\left(h_a\right)|_{a=0} =$$
$$= \partial_a\left(\eta_a\right)|_{a=0} \mu\left(\Phi_0\right) + \eta_0 \, \partial_a\left(\mu\left(\Phi_a\right)\right)|_{a=0}$$

and therefore the first determining equation

$$L\left(Y\right) + \sigma \cdot H = \tau\mu + Y\left(\mu\right).$$

The second equation, not depending on the measure change, coincides with the analogous equation obtained in Theorem 2.12 and this concludes the proof. □

4. Doob transformations and symmetries of Kolmogorov equation

4.1. Lie point symmetries of Kolmogorov equation

In the previous section we showed how the determining equations modify if we include in the stochastic transformation the change of the probability measure underlying the SDE. A natural question arising in this setting is the comparison between the determining equations for the infinitesimal symmetries of an SDE introduced in Theorem 3.13 and the determining equations for the symmetries of the associated Kolmogorov equation.

The symmetries of the Kolmogorov equations usually involves transformations explicitly depending on time that would be lost if we consider only autonomous transformations for our system. For this reason we have to extend our analysis in order to permit also non autonomous transformations. In order to achieve this, we use the standard trick, well known in the literature of dynamical system, for transforming a non autonomous system into an autonomous one. Indeed, given an SDE of the form (1), we introduce a trivial extra component $dZ_t = dt$, i.e., we consider the system

$$\begin{aligned} dX_t^i &= \mu^i(X_t)dt + \sigma_\alpha^i(X_t)dW_t^\alpha \\ dZ_t &= dt \end{aligned} \tag{9}$$

In this setting, the Kolmogorov equation associated with (1) can be written as $L(u) = 0$, where L is the infinitesimal generator associated with (9), i.e.

$$L(u) = A^{ij}\partial_{x^i x^j}u + \mu\partial_{x^i}u + \partial_z u = 0, \tag{10}$$

and $A = \frac{1}{2}\sigma \cdot \sigma^T$. In the following we always suppose that the matrix A has constant rank.

Equation (10) is a second order partial differential equation in the independent variables (x^i, z) and in the dependent variable u describing the behavior of the mean value of regular functions of the solution process X_t. More precisely, a solution $u(x, z)$ to equation (10) is of the form $\mathbb{E}[f(X_\mathcal{T})|X_z = x] = u(x, z)$, with $z \in [0, \mathcal{T}]$ and $u(x, \mathcal{T}) = f(x)$ (see, e.g., [37] Section 8.1).

Since (10) is a PDE, it can be seen as a submanifold of the second order jet bundle $J^2(\mathbb{R}^{n+1}, \mathbb{R})$ described by the equation $L(u) = 0$ (see [38] Chapter 2). In this setting a Lie point infinitesimal symmetry for (10) is a vector field Ξ on $J^0(\mathbb{R}^{n+1}, \mathbb{R})$ such that

$$\Xi^{(2)}(L(u))|_{L(u)=0} = 0, \tag{11}$$

where $\Xi^{(2)}$ denotes the second order prolongation of the vector field Ξ. If we rewrite (11) for a vector field of the form

$$\Xi = m(x,z)\partial_z + \phi^i(x,z)\partial_{x^i} + \Psi(x,z,u)\partial_u \tag{12}$$

we get the following determining equations for the Lie point infinitesimal symmetries of (10)

$$\begin{align}
\Psi(x,z,u) &= -k(x,z)u + k_0(x,z) \tag{13}\\
L(k_0) &= 0 \tag{14}\\
L(k) &= 0 \tag{15}\\
L(\phi) - \Xi(\mu) + 2A \cdot \nabla k - L(m)\mu &= 0 \tag{16}\\
L(m)A + \Xi(A) - \nabla\phi \cdot A - A \cdot (\nabla\phi)^T &= 0 \tag{17}\\
A \cdot \nabla m &= 0 \tag{18}
\end{align}$$

We remark that the function k_0 only appears in the first two conditions and corresponds to the trivial symmetry $k_0(x,t)\frac{\partial}{\partial u}$ which takes into account the superposition principle for the solutions to any linear PDEs. Therefore, later on, we omit k_0 (and the corresponding condition $L(k_0) = 0$) in the determining equations.

The comparison between the determining equations for the Lie point infinitesimal symmetries of the PDE (10) and the determining equation for the stochastic symmetries for the SDE (9) arising in Theorem 3.13 is not straightforward, first of all due to the different nature of the involved objects. Indeed, when we look for the infinitesimal symmetries of the PDE we look for a vector field of the form (12), while when we search symmetries for SDE (9), we deal with an infinitesimal transformation $V = (Y, C, \tau, H)$, where Y is a vector field on $\mathbb{R}^{n+1}$, $C : \mathbb{R}^{n+1} \to \mathfrak{so}(m)$, $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$ and $H : \mathbb{R}^{n+1} \to \mathbb{R}^m$ are smooth functions.

In the following, in order to compare the determining equations arising in these two different settings we restrict the class of transformations for the probability measure, introducing the notion of Doob transformations.

4.2. Doob transformations

The Doob transformation is a special kind of change of measure, and so a particular case of Girsanov transformation, in general related with Markov processes. Originally introduced by Doob (see [18]) it has been generalized in many directions (see [43] for the continuous case and [6] and references therein for an overview on the topic).

In this paper we only consider the original Doob transformation, which can be seen as a transformation of the infinitesimal generator L of a given diffusion process. In the conservative case (i.e. when the probability is conserved at each time) this transformation assumes the following simple form:

$$L^\varphi = \varphi^{-1} L \varphi$$

with φ a strictly positive function on the state space M such that $\varphi(X_t)$ is a martingale. The new diffusion process corresponding to the infinitesimal generator L^φ has a path measure $\mathbb{Q}$ (i.e. a measure on the pathspace of the process trajectories) which is absolutely continuous with respect to the original path measure $\mathbb{P}$ with density given by the following Radon-Nikodym derivative on the time interval $[0, \mathcal{T}]$:

$$\frac{d\mathbb{Q}_{\mathcal{F}_{\mathcal{T}}}}{d\mathbb{P}_{\mathcal{F}_{\mathcal{T}}}} = \varphi^{-1}(X_0)\varphi(X_{\mathcal{T}}).$$

In the following we fix a non explosive SDE (μ, σ) and a solution (X, W) to it and we give a simplified definition of Doob transformation adapted to our setting. Aiming to make more explicit the link between Girsanov transformations and Doob ones we introduce the auxiliary function $\mathfrak{h}(x) := \log(\varphi(x))$.

Definition 4.1. *Given a smooth function $h : M \to \mathbb{R}^m$ which is non explosive with respect to (μ, σ), we say that h is a* Doob transformation characterized by the smooth function $\mathfrak{h} : M \to \mathbb{R}$ *if the measure $\mathbb{Q}$ generated by the Girsanov transformation h is such that*

$$\frac{d\,\mathbb{Q}_{\mathcal{F}_\mathcal{T}}}{d\,\mathbb{P}_{\mathcal{F}_\mathcal{T}}} = \exp\left(\mathfrak{h}(X_\mathcal{T}) - \mathfrak{h}(X_0)\right).$$

Remark 4.2. *Using the martingale property of $\varphi(X_t)$ and the fact that $\mathfrak{h}(X_t) = \log(\varphi(X_t))$ we have that*

$$\frac{d\,\mathbb{Q}_{\mathcal{F}_t}}{d\,\mathbb{P}_{\mathcal{F}_t}} = \exp\left(\mathfrak{h}(X_t) - \mathfrak{h}(X_0)\right),$$

for any $t \leq \mathcal{T}$.

The next result provides suitable conditions on the functions h and $\mathfrak{h}$ in order to guarantee that a Girsanov transformation is also a Doob one.

Proposition 4.3. *Let $h : M \to \mathbb{R}^m$ be a non explosive smooth function associated with a Girsanov transformation of a non explosive SDE (μ, σ). If $\mathfrak{h} : M \to \mathbb{R}^m$ is a smooth function satisfying*

$$h_\alpha(x) \qquad = \sigma^i_\alpha(x)\partial_{x^i}(\mathfrak{h})(x) \tag{19}$$

$$\frac{1}{2}\sum_{\alpha=1}^m (h_\alpha(x))^2 = -L(\mathfrak{h})(x), \tag{20}$$

then h is also a Doob transformation.

Proof. By equating the Radon-Nikodym derivative given in Theorem 3.6 with the one given in Definition 4.1 we obtain the equality

$$\int_0^\mathcal{T} h_\alpha(X_s)dW_s^\alpha - \frac{1}{2}\int_0^\mathcal{T}\sum_{\alpha=1}^m (h_\alpha(X_s))^2 ds = \mathfrak{h}(X_\mathcal{T}) - \mathfrak{h}(X_0). \tag{21}$$

Applying Itô formula to the function $\mathfrak{h}$ we get

$$\mathfrak{h}(X_\mathcal{T}) - \mathfrak{h}(X_0) = \int_0^\mathcal{T} L\mathfrak{h}(X_t)dt + \int_0^\mathcal{T} \nabla\mathfrak{h}(X_t)\sigma(X_t)dW_t. \tag{22}$$

The proof follows by the uniqueness of canonical semimartingale decomposition of continuous processes (see, e.g. Section 31 and Definition 31.3 in [43]) and by uniqueness of martingale representation theorem for processes adapted to Brownian filtration (see, e.g., Theorem 36.1 in [43]). $\qquad\square$

Remark 4.4. *It is important to note that the fact that a change of measure h is a Doob transformation strongly depends on the coefficients of the SDE (μ, σ). Indeed, the right hand side of equation (19) contains σ and the right hand side of equation (20) involves, through the operator L, both μ and σ.*

Relations (19) and (20) allow us to derive also the conditions under which a stochastic transformation $T = (\Phi, B, \eta, h)$ involves the Doob transformation associated with $\mathfrak{h}$. Using relations (19) and (20) it is indeed possible to write a condition such that the one-parameter group T_a of the stochastic transformations is generated by an infinitesimal stochastic transformation $V = (Y, C, \tau, H)$. Indeed if in equations (19) and (20) both h_a and $\mathfrak{h}_a$ depend on a parameter a and we take the derivative with respect to that parameter in $a = 0$ (when $h_0 = 0$ as well as $\mathfrak{h}_0 = 0$), we obtain that there exists a function $k = \partial_a(\mathfrak{h}_a)|_{a=0}$ such that

$$H_\alpha(x) = \sigma_\alpha^i(x)\partial_{x^i}(k)(x) \qquad (23)$$
$$0 \quad = L(k). \qquad (24)$$

Using the previous characterization we can state the following theorem.

Theorem 4.5. *An infinitesimal stochastic transformation $V = (Y, C, \tau, H)$ is a symmetry of the SDE (μ, σ) involving only Doob transformations with respect to (μ, σ) if and only if V generates a one parameter group of transformations and there exists a smooth function k such that the following equations hold*

$$\begin{aligned}
H - \sigma^T \cdot \nabla k &= 0 \\
Y(\mu) - L(Y) - \sigma \cdot \sigma^T \cdot \nabla k + \tau\mu &= 0 \\
[Y, \sigma] + \tfrac{1}{2}\tau\sigma + \sigma \cdot C &= 0 \\
L(k) &= 0.
\end{aligned}$$

Proof. The proof is an easy consequence of the determining equations in Theorem 3.13 and equations (23) and (24). $\qquad \square$

4.3. Relation between determining equations for PDEs and SDEs

In this section we compare the determining equations for the symmetries of an SDE of the form (9) with the determining equations for Lie point infinitesimal symmetries of the corresponding Kolmogorov equation (10). In particular, we prove the following result.

Theorem 4.6. *Let $V = (Y, C, \tau, H)$ be an infinitesimal symmetry of Doob type for the SDE (9), with $Y = m(x, z)\frac{\partial}{\partial z} + \phi^i(x, z)\frac{\partial}{\partial x^i}$ and $H = \sigma^T \cdot \nabla k$. Then the vector field $\Xi = Y - k(x, z)u\frac{\partial}{\partial u}$ is a Lie point infinitesimal symmetry for the associated Kolmogorov equation (10).*

Proof. Since V is a symmetry for (9), Theorem 4.5 provides the following determining equations for V

$$\begin{aligned}
L(k) &= 0 & (25) \\
L(m) &= \tau & (26) \\
L(\phi) - Y(\mu) + \sigma \cdot \sigma^T \cdot \nabla k - \tau\mu &= 0 & (27) \\
Y(\sigma) - \nabla\phi \cdot \sigma + \frac{1}{2}\tau\sigma + \sigma \cdot C &= 0 & (28) \\
\sigma^T \cdot \nabla m &= 0 & (29)
\end{aligned}$$

We want to prove that, if $A = \frac{1}{2}\sigma \cdot \sigma^T$ is of constant rank, the vector field $\Xi = Y - ku\frac{\partial}{\partial u}$ satisfies the determining equation for the Kolmogorov PDE (10).

The first condition, i.e. $L(k) = 0$, is the same in both the sets of determining equations and, since μ^i do not depend on u, $\Xi(\mu^i) = Y(\mu^i)$ and equations (26) and (27) imply

(16). Moreover, multiplying on the left (29) by σ, we immediately get (18). In order to prove that (17) holds, we consider the right multiplication of (28) by σ^T and we get, using $\sigma \cdot \sigma^T = 2A$,

$$Y(\sigma)\sigma^T - 2\nabla\phi \cdot A + \tau A + \sigma \cdot C \cdot \sigma^T = 0 \qquad (30)$$

Hence, considering the semi-sum of (30) and its transposed and using the fact that C is an antisymmetric matrix, we get

$$Y(A) + \tau A - \nabla\phi \cdot A - A \cdot (\nabla\phi)^T = 0.$$

Since A does not depend on u, $Y(A) = \Xi(A)$ and, using (26), we get (17) $\qquad \square$

Now we prove the converse of the previous theorem, showing that with any infinitesimal Lie point symmetry of the Kolmogorov equation it is possible to associate an infinitesimal Doob symmetry of the corresponding SDE.

Theorem 4.7. *Let* $\Xi = m(x,z)\frac{\partial}{\partial z} + \phi^i(x,z)\frac{\partial}{\partial x^i} - k(x,z)u\frac{\partial}{\partial u}$ *be a Lie point symmetry for the PDE (10), with* $A = \frac{1}{2}\sigma \cdot \sigma^T$ *of constant rank. Then there exist smooth functions* $C : \mathbb{R}^{n+1} \to \mathfrak{so}(m)$ *and* $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$*, such that* $V = (Y, C, \tau, H)$*, with* $Y = \Xi + k(x,z)u\frac{\partial}{\partial u}$ *and* $H = \sigma^T \cdot \nabla k$*, is a Doob infinitesimal symmetry for the SDE (9).*

Proof. As in the proof of Theorem 4.6, (25) and (15) coincide and, in order to satisfy (26) and (27) we have to choose $\tau = L(m)$ and exploit the fact that $\Xi(\mu^i) = Y(\mu^i)$, since μ^i do not depend on u. Moreover, the hypothesis of constant rank for A ensures that (18) implies (29). Hence, we have to choose the matrix C in order to satisfy (28). Considering the left product of (28) with σ^T, we get the following expression for C, where, once again, we exploit the fact that A is of constant rank

$$C = (\sigma^T \cdot \sigma)^{-1}\sigma^T\nabla\phi \cdot \sigma - (\sigma^T \cdot \sigma)^{-1}\sigma^T Y(\sigma) - \frac{1}{2}\tau I.$$

The last step is to prove that the matrix C defined above is antisymmetric. Following the same line of the proof of Theorem 4.6, if we consider the sum of (30) with its transpose, we find

$$Y(\sigma \cdot \sigma^T) + 2\tau A - 2\nabla\phi \cdot A - 2A \cdot (\nabla\phi)^T + \sigma \cdot (C + C^T) \cdot \sigma^T = 0$$

and, using (17), we get $\sigma \cdot (C + C^T) \cdot \sigma^T = 0$. Therefore, since A is of constant rank, $C + C^T = 0$, i.e. C is an antisymmetric matrix, and this concludes the proof. $\qquad \square$

Remark 4.8. *Theorems 4.6 and 4.7 show that there exists a one-to-one correspondence between infinitesimal symmetries of Doob type of the SDE and Lie point infinitesimal symmetries of the corresponding Kolmogorov equation. On the other hand, if we allow more general changes of the probability measure of Girsanov type, the family of symmetries of the SDE introduced in this paper is wider than the family of the symmetries for the corresponding Kolmogorov equation. The new (non Doob type) symmetries can belong to two different families.*
The first family consists of non Doob symmetries such that the function k satisfying $H = \sigma^T\nabla k$ does exist, but does not satisfies $L(k) = 0$. We call these kind of symmetries almost Doob symmetries.
The second family of non Doob symmetries contains symmetries $V = (Y, C, \tau, H)$ such that does not exist any function k for which $H = \sigma^T\nabla k$. We remark that the second family is not empty only for SDEs driven by m dimensional Brownian motions with

$m > 1$, while for SDEs driven by one-dimensional Brownian motions only almost Doob symmetries exist. In the next section we provide some examples of this fact and, in particular, we find almost Doob symmetries for one-dimensional Brownian motion, Ornstein-Uhlenbeck process and CIR model. Moreover, we show that two-dimensional Brownian motion admits non Doob infinitesimal symmetries of both the types described above.

5. Examples

In this last section we propose some non trivial examples, in order to show that the introduction of the probability measure change allows us to obtain a larger number of symmetries with respect to the methods available in the previous literature.

5.1. One-dimensional Brownian motion

Let us consider the following SDE

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 \\ 0 \end{pmatrix} dW_t^1$$

corresponding to a one-dimensional Brownian motion, where the second component, admitting solution $Z_t = t$, has been introduced in order to include in our setting also time dependent transformations.

If we are interested in finding symmetries of Doob type for this SDE, we have to look for

$$V = (Y, C, \tau, H) = \left(\begin{pmatrix} f(x,z) \\ m(x,z) \end{pmatrix}, 0, \tau(x,z), H_1(x,z) \right) \tag{31}$$

where $H_1 = k_x$ and the functions k, f, m, τ satisfy the following determining equations

$$\begin{aligned} \tfrac{1}{2}k_{xx} + k_z &= 0 \\ \tfrac{1}{2}f_{xx} + f_z + k_x &= 0 \\ \tfrac{1}{2}m_{xx} + m_z &= \tau \\ f_x &= \tfrac{1}{2}\tau \\ m_x &= 0. \end{aligned}$$

In order to solve this system, we look for a function $k(x,z)$ satisfying the first equation. If we chose $k = c_1 + c_2 x + c_3(x^2 - z)$, we can solve the remaining equations and we find

$$\begin{aligned} f(x,z) &= -c_2 z - 2c_3 xz + c_4 x + c_5 \\ m(x,z) &= -2c_3 z^2 + 2c_4 z + c_6 \\ \tau(x,z) &= -4c_3 z + 2c_4 \end{aligned}$$

Therefore, the Doob symmetries for the one-dimensional Brownian motion associated with the function $k = c_1 + c_2 x + c_3(x^2 - z)$ are

$$V_1 = \left(\begin{pmatrix} z \\ 0 \end{pmatrix}, 0, 0, -1 \right)$$

$$V_2 = \left(\begin{pmatrix} 2xz \\ 2z^2 \end{pmatrix}, 0, 4z, -2x \right)$$

$$V_3 = \left(\begin{pmatrix} x \\ 2z \end{pmatrix}, 0, 2, 0 \right)$$

$$V_4 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0, 0, 0 \right)$$

$$V_5 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0, 0, 0 \right)$$

Hence, exploiting Theorem 4.6 and adding the symmetry V_6 corresponding to $H_1 = 0$, we recover, in our framework, the following generators of the well known six dimensional algebra of Lie point symmetries for the Kolmogorov equation associated with one dimensional Brownian motion

$$
\begin{aligned}
\Xi_1 &= z\partial_x + xu\partial_u \\
\Xi_2 &= 2xz\partial_x + 2z^2\partial_z + u(x^2 - z)\partial_u \\
\Xi_3 &= x\partial_x + 2z\partial_z \\
\Xi_4 &= \partial_x \\
\Xi_5 &= \partial_z \\
\Xi_6 &= u\partial_u.
\end{aligned}
$$

Let us now consider the determining equations for general symmetries (not necessarily of Doob type) of the one-dimensional Brownian motion. In this case the determining equations provided by Theorem 3.13 are

$$
\begin{aligned}
\tfrac{1}{2}f_{xx} + f_z + H_1 &= 0 \\
\tfrac{1}{2}m_{xx} + m_z &= \tau \\
f_x &= \tfrac{1}{2}\tau \\
m_x &= 0.
\end{aligned}
$$

Here we have no conditions on H_1 and we can find the following infinite-dimensional family of almost Doob symmetries (depending on an arbitrary deterministic function of the time $\alpha(z)$)

$$\tilde{V}_\alpha = \left(\begin{pmatrix} \tfrac{1}{2}\alpha(z)x \\ \int \alpha(z)\,dz \end{pmatrix}, 0, \alpha(z), -\frac{1}{2}x\alpha'(z) \right).$$

We remark that, even if H satisfies the condition $H = H_1 = \sigma^T \nabla k$ for $k = -\frac{1}{4}x^2\alpha'(z)$, we cannot ensure that

$$\frac{1}{2}k_{xx} + k_z = 0$$

since this condition holds only if $\alpha'(z) = 0$. Therefore, for general (non constant) functions $\alpha(z)$, $\tilde{V}_\alpha$ is not a Doob symmetry and Theorem 4.6 does not apply. This means that we cannot associate with $\tilde{V}_\alpha$ any symmetry of the Kolmogorov PDE $\frac{1}{2}u_{xx} + u_z = 0$.

Remark 5.1. *The symmetries V_α are related with the* random time invariance *of Brownian motion (see Section 2). Indeed, if we consider the deterministic time change $t' = f(t)$, then $W'_{t'} = \int_0^{t'} \sqrt{f'(f^{-1}(s))}dW_{f^{-1}(s)}$ is a new Brownian motion (see [15]). The symmetry $\tilde{V}_\alpha$ provides a generalization of the previous fact, considering $\tilde{W}_{t'} = \sqrt{f'(f^{-1}(t'))}W_{f^{-1}(t')}$ instead of the integral $\int_0^{t'} \sqrt{f'(f^{-1}(s))}dW_{f^{-1}(s)}$. Obviously, since $W'_{t'}$ is a $\mathbb{P}$-Brownian motion, $\tilde{W}_{t'}$ cannot be a $\mathbb{P}$-Brownian motion. But, since $\tilde{V}_\alpha$ is a symmetry, using Girsanov transformation we have that $\tilde{W}_{t'}$ is a $\mathbb{Q}$-Brownian motion, where $\mathbb{Q}$ has density with respect to $\mathbb{P}$ given by*

$$\frac{d\mathbb{Q}_{\mathcal{F}_\mathcal{T}}}{d\mathbb{P}_{\mathcal{F}_\mathcal{T}}} = \exp\left(\int_0^{\mathcal{T}} \frac{f''(s)}{2\sqrt{f(s)}} W_s dW_s - \frac{1}{2}\int_0^{\mathcal{T}} \left(\frac{f''(s)W_s}{2\sqrt{f(s)}} \right)^2 ds \right).$$

5.2. Ornstein-Uhlenbeck model

Let us consider the Ornstein-Uhlenbeck process, solution to the following SDE

$$\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} aX_t + b \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 \\ 0 \end{pmatrix} dW_t^1$$

where, as in the previous example, the second component has been introduced in order to include in our framework time dependent transformations. In this case, if we look for a Doob type symmetries of the form (31) and we consider $H_1 = k_x$, the determining equations are

$$\begin{aligned}
\tfrac{1}{2}k_{xx} + (ax+b)\,k_x + k_z &= 0 \\
\tfrac{1}{2}f_{xx} + (ax+b)\,f_x + f_z &= af + \tau\,(ax+b) - k_x \\
\tfrac{1}{2}m_{xx} + (ax+b)\,m_x + m_z &= \tau \\
f_x &= \tfrac{1}{2}\tau \\
m_x &= 0.
\end{aligned} \tag{32}$$

With a suitable ansatz we can find a solution to the first equation of the form

$$k\,(x,z) = c_1 + c_2\,(ax+b)\,e^{-az} + c_3\left(ax^2 + 2bx + \frac{a+2b^2}{2a}\right)e^{-2az}$$

so that

$$H_1\,(x,z) = c_2 a e^{-az} + 2c_3\,(ax+b)\,e^{-2az}.$$

The remaining determining equations read

$$\begin{aligned}
-(ax+b)\,f_x + f_z &= af - c_2 a e^{-az} - 2c_3(ax+b)e^{-2az} \\
m_z &= \tau \\
\tau &= 2f_x \\
m_x &= 0,
\end{aligned} \tag{33}$$

whose solution is

$$\begin{aligned}
f(x,z)) &= \tfrac{c_3}{2a}e^{-2az}(ax+b) + \tfrac{c_4}{2a}e^{2az}(ax+b) + \tfrac{c_2}{2}e^{-az} + c_5 e^{az} \\
m(x,z) &= -\tfrac{c_3}{2a}e^{-2az} + \tfrac{c_4}{2a}e^{2az} + c_6 \\
\tau(x,z) &= c_3 e^{-2az} + c_4 e^{2az}.
\end{aligned}$$

Therefore we have the following family of infinitesimal symmetries of Doob type for the Ornstein-Uhlenbeck SDE

$$V_1 = \left(\begin{pmatrix} \tfrac{1}{2}e^{-az} \\ 0 \end{pmatrix}, 0, 0, ae^{-az}\right)$$

$$V_2 = \left(\begin{pmatrix} \tfrac{1}{2a}e^{-2az}(ax+b) \\ -\tfrac{1}{2a}e^{-2az} \end{pmatrix}, 0, e^{-2az}, 2(ax+b)e^{-2az}\right)$$

$$V_3 = \left(\begin{pmatrix} \tfrac{1}{2a}e^{2az}(ax+b) \\ \tfrac{1}{2a}e^{2az} \end{pmatrix}, 0, e^{2az}, 0\right)$$

$$V_4 = \left(\begin{pmatrix} e^{az} \\ 0 \end{pmatrix}, 0, 0, 0\right)$$

$$V_5 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0, 0, 0\right).$$

Using Theorem 4.6, we can associate with V_i the generators for the symmetry algebra of the Kolmogorov equation associated with the Ornstein-Uhlenbeck SDE,

where the vector field Ξ_6 corresponds to the trivial symmetry V_6 associated with $H_1 = 0$.

$$
\begin{aligned}
\Xi_1 &= \tfrac{1}{2}e^{-az}\partial_x - u(ax+b)e^{-az}\partial_u \\
\Xi_2 &= \tfrac{1}{2a}e^{-2az}(ax+b)\partial_x - \tfrac{1}{2a}e^{-2az}\partial_z - ue^{-2az}(ax^2 + 2bx + \tfrac{a+2b^2}{2a})\partial_u \\
\Xi_3 &= \tfrac{1}{2a}e^{2az}(ax+b)\partial_x + \tfrac{1}{2a}e^{2az}\partial_z \\
\Xi_4 &= e^{az}\partial_x \\
\Xi_5 &= \partial_z \\
\Xi_6 &= u\partial_u.
\end{aligned}
$$

Let us now consider the determining equations for general symmetries (not necessarily of Doob type) for the Ornstein-Uhlenbeck process. In this case the determining equations provided by Theorem 3.13 are

$$
\begin{aligned}
\tfrac{1}{2}f_{xx} + (ax+b)\,f_x + f_z &= af + \tau\,(ax+b) - H_1 \\
\tfrac{1}{2}m_{xx} + (ax+b)\,m_x + m_z &= \tau \\
f_x &= \tfrac{1}{2}\tau \\
m_x &= 0.
\end{aligned}
\tag{34}
$$

Solving these equations we find the following symmetries

$$
\tilde{V}_1 = \left(\begin{pmatrix} \tfrac{1}{2}(\tfrac{1}{a} + e^{2az})x - \tfrac{1}{2}\tfrac{b}{a^2} + \tfrac{1}{2}\tfrac{b}{a}e^{2az} \\ \tfrac{z}{a} + \tfrac{1}{2a}e^{2az} \end{pmatrix}, 0, \tfrac{1}{a} + e^{2az}, x \right)
$$

$$
\tilde{V}_2 = \left(\begin{pmatrix} \tfrac{x}{2a} - \tfrac{b}{2a^2} + e^{az} \\ \tfrac{z}{a} \end{pmatrix}, 0, \tfrac{1}{a}, x \right).
$$

We remark that $\tilde{V}_1$ and $\tilde{V}_2$ are not symmetries of Doob type, since we have $H_1 = x = \sigma^T \nabla k$, for $k = \tfrac{1}{2}x^2 + k_1(z)$, but $L(k) \neq 0$.

5.3. CIR model

In this section we look for the symmetries of the CIR model

$$
\begin{pmatrix} dX_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} aX_t + b \\ 1 \end{pmatrix} dt + \begin{pmatrix} \sigma_0\sqrt{X_t} \\ 0 \end{pmatrix} dW_t^1,
$$

which is widely used in mathematical finance to describe the behavior of the interest rates (see [5] Chapter 3). We remark that CIR model is also important for the fact that closed formulas for its Laplace transform and its transition probability are known. In particular, the advantages of the knowledge of Lie point symmetries of the Kolmogorov equation associated with CIR model and their relationship with closed formulas is discussed in [11].

If we consider infinitesimal transformations V of the form (31) and $H_1 = \sigma_0\sqrt{x}k_x$, the determining equations for the symmetries of Doob type for the CIR model are

$$
\begin{aligned}
\tfrac{1}{2}\sigma_0^2 x k_{xx} + (ax+b)\,k_x + k_z &= 0 \\
\tfrac{1}{2}\sigma_0^2 x f_{xx} + (ax+b)\,f_x + f_z &= af + (ax+b)\tau - \sigma_0^2 x k_x \\
\tfrac{1}{2}\sigma_0^2 x m_{xx} + (ax+b)\,m_x + m_z &= \tau \\
\sigma_0\sqrt{x}f_x - \tfrac{1}{2}\sigma_0\tfrac{1}{\sqrt{x}}f &= \tfrac{1}{2}\sigma_0\sqrt{x}\tau \\
\sigma_0\sqrt{x}\,m_x &= 0.
\end{aligned}
$$

The first step to solve these equations is to find a function k satisfying the first determining equation. If we chose

$$
k(x,z) = c_1(ax+b)e^{-az} + c_2
$$

we can solve the remaining determining equations and we obtain

$$
\begin{aligned}
f(x,z) &= \tfrac{1}{2}c_1\sigma_0^2 x e^{-az} + c_3 x e^{az} \\
m(x,z) &= -\tfrac{1}{2a}c_1\sigma_0^2 e^{-az} + \tfrac{c_3}{a}e^{az} + c_4 \\
\tau(x,z) &= \tfrac{1}{2}c_1\sigma_0^2 e^{-az} + c_3 e^{az}
\end{aligned}
$$

corresponding to the infinitesimal symmetries

$$
V_1 = \left(\begin{pmatrix} \frac{\sigma_0^2}{2} x e^{-az} \\ -\frac{\sigma_0^2}{2a} e^{-az} \end{pmatrix}, 0, \frac{\sigma_0^2}{2}e^{-az}, \sigma_0 a\sqrt{x}e^{-az} \right)
$$

$$
V_2 = \left(\begin{pmatrix} x e^{az} \\ \frac{1}{a} e^{az} \end{pmatrix}, 0, e^{az}, 0 \right)
$$

$$
V_3 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0, 0, 0 \right).
$$

Once again we can exploit Theorem 4.6 in order to find the Lie point infinitesimal symmetries for the Kolmogorov equation associated with the CIR model, i.e.

$$
\begin{aligned}
\Xi_1 &= \frac{\sigma_0^2}{2} x e^{-az}\partial_x - \frac{\sigma_0^2}{2a}e^{-az}\partial_z - u(ax+b)e^{-az}\partial_u \\
\Xi_2 &= x e^{az}\partial_x + \frac{1}{a}e^{az}\partial_z \\
\Xi_3 &= \partial_z \\
\Xi_4 &= u\partial_u
\end{aligned}
$$

where the last symmetry, as usual, corresponds to the choice $H_1 = 0$. They form the generators of all non trivial symmetries of Kolmgorov CIR model equation (see [11]).

Let us now consider the determining equations for general symmetries (not necessarily of Doob type) for the CIR model. In this case the determining equations provided by Theorem 3.13 are

$$
\begin{aligned}
\tfrac{1}{2}\sigma_0^2 x f_{xx} + (ax+b)f_x + f_z &= af + (ax+b)\tau - \sigma_0\sqrt{x}H_1 \\
\tfrac{1}{2}\sigma_0^2 x m_{xx} + (ax+b)m_x + m_z &= \tau \\
\sigma_0\sqrt{x}f_x - \tfrac{1}{2}\sigma_0\frac{1}{\sqrt{x}}f &= \tfrac{1}{2}\sigma_0\sqrt{x}\tau \\
\sigma_0\sqrt{x}\,m_x &= 0,
\end{aligned}
$$

and we can find the solution

$$
\begin{aligned}
f(x,z) &= c_1\sqrt{x} + c_2\frac{\sigma_0}{a}x \\
m(x,z) &= c_2\frac{\sigma_0}{a}z \\
\tau(x,z) &= c_2\frac{\sigma_0}{a} \\
H_1 &= c_1\left(\frac{\sigma_0}{8x} - \frac{b}{2\sigma_0 x} + \frac{a}{2\sigma_0}\right) + c_2\sqrt{x}
\end{aligned}
$$

corresponding to the infinitesimal symmetries

$$
\tilde{V}_1 = \left(\begin{pmatrix} \sqrt{x} \\ 0 \end{pmatrix}, 0, 0, \frac{\sigma_0}{8x} - \frac{b}{2\sigma_0 x} + \frac{a}{2\sigma_0} \right)
$$

$$
\tilde{V}_2 = \left(\begin{pmatrix} \frac{\sigma_0}{a}x \\ \frac{\sigma_0}{a}z \end{pmatrix}, 0, \frac{\sigma_0}{a}, \sqrt{x} \right).
$$

We remark that $\tilde{V}_1$ and $\tilde{V}_2$ are not symmetries of Doob type, since we have $H_1 = \sqrt{x} = \sigma^T\nabla k_i$, for $k_1 = -\frac{1}{4\sqrt{x}} + \frac{b}{\sigma_0^2\sqrt{x}} + \frac{a\sqrt{x}}{\sigma_0^2}$ and $k_2 = \frac{x}{\sigma_0}$ but, as in the previous cases, k_i do not satisfy $L(k_i) = 0$.

5.4. Two-dimensional Brownian motion

Let us consider the following SDE

$$\begin{pmatrix} dX_t \\ dY_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}$$

corresponding to a two-dimensional Brownian motion. As in the previous examples, the third component $dZ_t = dt$ has been introduced in order to include time dependent transformations in our setting.

When we look for Doob type infinitesimal symmetries for this SDE, we have to find an infinitesimal transformation $V = (Y, C, \tau, H)$

$$V = \left(\begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ m(x,y,z) \end{pmatrix}, \begin{pmatrix} 0 & c(x,y,z) \\ -c(x,y,z) & 0 \end{pmatrix}, \tau(x,y,z), \begin{pmatrix} H_1(x,y,z) \\ H_2(x,y,z) \end{pmatrix} \right)$$

where $H_1 = k_x, H_2 = k_y$ and the functions k, f, g, m, τ, c satisfy the following determining equations

$$\begin{aligned} \tfrac{1}{2}k_{xx} + \tfrac{1}{2}k_{yy} + k_z &= 0 \\ \tfrac{1}{2}f_{xx} + \tfrac{1}{2}f_{yy} + f_z + k_x &= 0 \\ \tfrac{1}{2}g_{xx} + \tfrac{1}{2}g_{yy} + g_z + k_y &= 0 \\ \tfrac{1}{2}m_{xx} + \tfrac{1}{2}m_{yy} + m_z &= \tau \\ f_x &= \tfrac{1}{2}\tau \\ f_y &= c \\ g_x &= -c \\ g_y &= \tfrac{1}{2}\tau \\ m_x &= 0 \\ m_y &= 0. \end{aligned}$$

If we chose $k = c_1 + c_2 x + c_3 y + c_4(x^2 + y^2 - 2z)$, we can solve the previous system and we find

$$\begin{aligned} f(x,y,z) &= -2c_4 xz + \tfrac{1}{2}c_5 x - c_2 z + c_6 y + c_7 \\ g(x,y,z) &= -2c_4 yz + \tfrac{1}{2}c_5 y - c_3 z - c_6 x + c_8 \\ m(x,y,z) &= -2c_4 z^2 + c_5 z + c_9 \\ \tau(x,y,z) &= -4c_4 z + c_5 \\ c(x,y,z) &= c_6. \end{aligned}$$

Therefore, the infinitesimal symmetries for the two-dimensional Brownian motion associated with the Doob function $k = c_1 + c_2 x + c_3 y + c_4(x^2 + y^2 - 2z)$ are

$$V_1 = \left(\begin{pmatrix} -z \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$V_2 = \left(\begin{pmatrix} 0 \\ -z \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$V_3 = \left(\begin{pmatrix} -2xz \\ -2yz \\ -2z^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, -4z, \begin{pmatrix} 2x \\ 2y \end{pmatrix} \right)$$

$$V_4 = \left(\begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y \\ z \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$V_5 = \left(\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$V_6 = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$V_7 = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$V_8 = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right).$$

Using Theorem 4.6, we can associate with any V_i a corresponding symmetry of the Kolmogorov equation

$$\frac{1}{2}u_{xx} + \frac{1}{2}u_{yy} + u_z = 0. \tag{35}$$

In this way we recover, adding the trivial symmetry V_9 corresponding to $H_1 = H_2 = 0$, the following generators for the well known nine-dimensional Lie algebra of Lie point symmetries of (35)

$$
\begin{aligned}
\Xi_1 &= z\frac{\partial_x}{+} xu\partial_u \\
\Xi_2 &= z\partial_y + yu\partial_u \\
\Xi_3 &= 2xz\partial_x + 2yz\partial_y + 2z^2\partial_z + u(x^2 + y^2 - 2z)\partial_u \\
\Xi_4 &= \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + z\partial_z \\
\Xi_5 &= y\partial_x - x\partial_y \\
\Xi_6 &= \partial_x \\
\Xi_7 &= \partial_y \\
\Xi_8 &= \partial_z \\
\Xi_9 &= u\partial_u.
\end{aligned}
$$

Let us now consider the determining equations for general symmetries (not necessarily of Doob type) of the two-dimensional Brownian motion. In this case the determining equations provided by Theorem 3.13 are

$$
\begin{aligned}
\tfrac{1}{2}f_{xx} + \tfrac{1}{2}f_{yy} + f_z + H_1 &= 0 \\
\tfrac{1}{2}g_{xx} + \tfrac{1}{2}g_{yy} + g_z + H_2 &= 0 \\
\tfrac{1}{2}m_{xx} + \tfrac{1}{2}m_{yy} + m_z &= \tau \\
f_x &= \tfrac{1}{2}\tau \\
f_y &= c \\
g_x &= -c \\
g_y &= \tfrac{1}{2}\tau \\
m_x &= 0 \\
m_y &= 0.
\end{aligned}
$$

Here we have no conditions on the components of the vector field H and we can find two infinite-dimensional families of symmetries (each one depending on an arbitrary function of z) which are not of Doob type.

The first family is the generalization of the almost Doob symmetries already found for the one-dimensional Brownian motion, i.e.

$$\tilde{V}_\alpha = \left(\left(\begin{array}{c} \frac{1}{2}\alpha(z)x \\ \frac{1}{2}\alpha(z)y \\ \int \alpha(z)\,dz \end{array} \right), \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \alpha(z), \left(\begin{array}{c} -\frac{1}{2}x\alpha'(z) \\ -\frac{1}{2}y\alpha'(z) \end{array} \right) \right).$$

We remark that, despite H satisfies the condition $H = \sigma^T \nabla k$ for $k = -\frac{1}{4}\alpha'(z)(x^2 + y^2)$, we cannot ensure that

$$\frac{1}{2}k_{xx} + \frac{1}{2}k_{yy} + k_z = 0$$

since this condition is satisfied only if $\alpha'(z) = 0$. Therefore, since for general (non constant) functions $\alpha(z)$, $\tilde{V}_\alpha$ is not a Doob symmetry, Theorem 4.6 does not apply and we cannot associate with $\tilde{V}_\alpha$ any symmetries of the Kolmogorov PDE (35).

The second family of non Doob-type symmetries is given by

$$\tilde{V}_\beta = \left(\left(\begin{array}{c} \beta(z)y \\ -\beta(z)x \\ 0 \end{array} \right), \left(\begin{array}{cc} 0 & \beta(z) \\ -\beta(z) & 0 \end{array} \right), 0, \left(\begin{array}{c} -y\beta'(z) \\ x\beta'(z) \end{array} \right) \right)$$

and does not have an analogous for the one-dimensional Brownian motion. In fact, in this case the vector field H does not satisfy the condition $H = \sigma^T \nabla k = \nabla k$ for any k.

Remark 5.2. *The presence of the symmetries $\tilde{V}_\beta$ is related to the invariance with respect to random rotations of Brownian motion (see Section 2) in the same way as $\tilde{V}_\alpha$ is related to the invariance of Brownian motion with respect to random time changes (see Remark 5.1). Indeed if $B(t)$ is a smooth deterministic function of time taking values in $O(2)$ we have that $\tilde{W}_t = B(t) \cdot W_t$ is a Brownian motion with respect to the new probability measure $\mathbb{Q}$ having density*

$$\frac{d\,\mathbb{Q}_{\mathcal{F}_\mathcal{T}}}{d\,\mathbb{P}_{\mathcal{F}_\mathcal{T}}} = \exp\left(-\int_0^\mathcal{T} W_s^T \cdot B'(s)^T \cdot dW_s - \frac{1}{2}\int_0^\mathcal{T} |B'(s) \cdot W_s|^2 ds \right).$$

Therefore, the introduction of the notion of stochastic transformation for an SDE allows us to enlarge the class of symmetries, showing that the stochastic approach is successful in providing a wider family of symmetries which would be lost in a purely deterministic approach throughout the Kolmogorov equation associated with the SDE.

6. Proof of Lemma 3.4

In this section we give a proof of Lemma 3.4, specializing to our setting a general result obtained in [44]. Since in Lemma 3.4 we consider a smooth SDE (μ, σ) which is non explosive and a smooth function $h : M \to \mathbb{R}^n$ such that $(\mu + \sigma \cdot h, \sigma)$ is a non explosive SDE, in order to provide a rigorous proof, first of all we have to suitably extend the filtered probability space $(\Omega, \mathcal{M}_t \subset \mathcal{F}, \mathbb{P})$. Considering on $\mathbb{R}^n \cup \{\infty\}$ and $\mathbb{R}^m \cup \{\infty\}$ the topology given by the one point compactification of the standard topology of $\mathbb{R}^n, \mathbb{R}^m$, we take

$$\Omega \subset C^0(\mathbb{R}_+, (\mathbb{R}^n \cup \{\infty\}) \times (\mathbb{R}^m \cup \{\infty\}))$$

as the space containing the functions for which $\omega(t) = \infty$, for every t such that there exists a $t' \leq t$ for which $\omega(t') = \infty$, i.e. the space of continuous functions which once arrived at ∞ they remain at ∞. The probability law $\mathbb{P}$ is such that, if $X_t^i(\omega) = \omega^i(t)$ and $W^\alpha(\omega)_t = \omega^{n+\alpha}(t)$ $(i = 1, ..., n, \alpha = 1, ..., m)$, then (X, W) is a solution to the SDE (μ, σ) and the filtration $\mathcal{M}_t$ is the right continuous modification of the natural filtration generated by (X, W) (in general $\mathcal{M}_t$ is not completed as $\mathcal{F}_t$). Denoting by $\hat{\tau} = \inf\{t : X_t = \infty\}$ the explosion time of the process X, the hypotheses of non explosiveness in Lemma 3.4 ensure that $\hat{\tau} = \infty$ $\mathbb{P}$-almost surely.

In this setting, we consider the process M given by

$$M_t := \int_0^t h(X_s)^T \cdot dW_s \qquad \forall t \geq 0,$$

with quadratic variation given by

$$[M, M]_t := \int_0^t h(X_s)^T \cdot h(X_s) ds \qquad \forall t \geq 0,$$

and we introduce the non negative process Z defined by

$$Z_t := \exp\left(M_t - \frac{1}{2}[M, M]_t\right) \qquad \forall t \geq 0.$$

By applying Ito-formula one can easily see that the process Z is a non negative continuous $\mathbb{P}$-local martingale. In order to limit the quadratic variation we consider the stopping times defined as

$$\tau_n := \inf\{t \geq 0 | [M, M]_t > n\},$$

where $n \in \mathbb{N}_0$. The stopping times τ_n are increasing with respect to $n \in \mathbb{N}_0$ and we write $\tau = \sup_{n \in \mathbb{N}_0} \tau_n = \lim_{n \uparrow \infty} \tau_n$. Novikov condition (see, e.g., [42, Chapter VIII, Proposition 1.14]) ensures that the stopped $\mathbb{P}$-local martingale Z^{τ_n} is a uniformly integrable $\mathbb{P}$-martingale $\forall n \in \mathbb{N}_0$.

Proposition 6.1. *There exists a probability measure $\mathbb{Q}$ such that for any $n \in \mathbb{N}_0$*

$$\frac{d\,\mathbb{Q}_{\mathcal{M}_{\tau_n}}}{d\,\mathbb{P}_{\mathcal{M}_{\tau_n}}} = Z_\infty^{\tau_n}.$$

Proof. For any stopping time η, let $\mathcal{M}_{\eta-}$ be the σ-algebra defined as

$$\mathcal{M}_{\eta-} := \sigma(X_0) \vee \sigma(\{A \cap \{\eta > t\} | A \in \mathcal{M}_t, t \geq 0\}).$$

Considering the sequence of probability measures $\{\mathbb{Q}_n\}_{n \in \mathbb{N}_0}$ given by $d\,\mathbb{Q}_n = Z_\infty^{\tau_n} d\,\mathbb{P}$, and observing that

$$\mathbb{Q}_n(A) = \mathbb{Q}_m(A) \qquad \forall A \in \mathcal{M}_{(\tau_n \wedge \tau_m)-} \text{ and } \forall m, n \in \mathbb{N},$$

we get that the set function

$$\mathbb{Q} : \bigcup_{n \in \mathbb{N}} \mathcal{M}_{\tau_n-} \longrightarrow [0, 1]$$

$$A \longmapsto \mathbb{Q}_n(A) \qquad \forall A \in \mathcal{M}_{\tau_n-}$$

is well defined.

Using [39, Theorem V.4.1], we can extend the family of compatible probabilities $\mathbb{Q}_n$

to a probability measure $\mathbb{Q}$ on $\bigvee_{n \in \mathbb{N}} \mathcal{M}_{\tau_n -} = \mathcal{M}_{\tau -}$. Moreover, by [40, Theorem E.2] we can further extend $\mathbb{Q}$ to a probability measure on $(\Omega, \mathcal{M})$, with $\mathcal{M} := \bigvee_{n \in \mathbb{N}} \mathcal{M}$ and, with a slight abuse of notation, we denote it again by $\mathbb{Q}$.

If we fix $n \in \mathbb{N}_0$ and $A \in \mathcal{M}_{\tau_n}$, we note that $\tau_n(\omega) < \tau_{n+1}(\omega)$ on $\{\tau_n < \infty\}$, since $[M, M]_t(\omega)$ is absolutely continuous and it is $\mathbb{P}$-almost surely finite by the hypothesis of non explosiveness of h. This then yields

$$\mathbb{Q}(A) = \mathbb{Q}(A \cap \{\tau_n < \infty\}) + \mathbb{Q}(A \cap \{\tau_n = \infty\})$$
$$= \mathbb{E}_{\mathbb{P}}[Z_\infty^{\tau_{n+1}} \chi_{A \cap \{\tau_n < \infty\}}] + \mathbb{E}_{\mathbb{P}}[Z_\infty^{\tau_n} \chi_{A \cap \{\tau_n = \infty\}}] = \mathbb{E}_{\mathbb{P}}[Z_\infty^{\tau_n} \chi_A]$$

since $A \cap \{\tau_n = \infty\} \in \mathcal{M}_{\tau_n -}$. The statement then follows. $\qquad \square$

Remark 6.2. *The probability measure $\mathbb{Q}$ constructed in Proposition 6.1 is in general non unique. An important consequence of the proof of Lemma 3.4 is that if both (μ, σ) and $(\mu + \sigma \cdot h, \sigma)$ are non explosive, then $\mathbb{Q}$ is unique.*

Introducing

$$\hat{W}_t = W_t^\tau - \int_0^{t \wedge \tau} h(X_s) ds,$$

by Girsanov theorem $\hat{W}$ is a Brownian motion till the stopping time τ. We have that (X, W) is a solution to the SDE $(\mu + \sigma \cdot h, \sigma)$ under any probability $\mathbb{Q}$ (given by Proposition 6.1) till the stopping time τ.

Remark 6.3. *It is clear that $\hat{\tau}$ is the explosion time of both the solution processes X of the SDEs (μ, σ) and $(\mu + \sigma \cdot h, \sigma)$. We have that $\hat{\tau} \leq \tau$. Indeed, since h is smooth and X_t is a continuous process, the only possibility for which $\tau \neq \infty$ is when the process X_t diverges to infinity. Since the converse is in general not true, i.e. X_t can go to infinity and $[M, M]_t$ can remain bounded, we have $\hat{\tau} \leq \tau$. This implies that $(X, \hat{W})$ is a solution to the SDE $(\mu + \sigma \cdot h, \sigma)$ till its explosion stopping time $\hat{\tau}$ for any probability $\mathbb{Q}$ built using Proposition 6.1.*

Proof of Lemma 3.4. By Remark 6.3 we have $\hat{\tau} \leq \tau$, which implies that

$$\{\tau < \infty\} \subseteq \{\hat{\tau} < \infty\}.$$

For any probability measure $\mathbb{Q}$ constructed using Proposition 6.1, in particular by monotonicity, we obtain that

$$\mathbb{Q}(\tau < \infty) \leq \mathbb{Q}(\hat{\tau} < \infty) = 0,$$

where we used the fact that $(\mu + \sigma \cdot h, \sigma)$ is non explosive. This means that, $\forall t \geq 0$ and for all $A \in \mathcal{M}_t$ with $\mathbb{P}(A) = 0$, we have

$$\mathbb{Q}(A) = \lim_{n \uparrow \infty} \mathbb{Q}(A \cap \{\tau_n > t\}) \leq \lim_{n \uparrow \infty} \mathbb{E}_{\mathbb{P}}[Z_\infty^{\tau_n} \chi_A] = 0.$$

Hence $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{M}_t$, $\forall t \geq 0$. Moreover, if we consider the $\mathbb{P}$-martingale R defined by

$$R_t := \frac{d\,\mathbb{Q}_{\mathcal{M}_t}}{d\,\mathbb{P}_{\mathcal{M}_t}} \qquad \forall t \geq 0,$$

we have that $R_t^{\tau_n} = Z_t^{\tau_n}$, $\forall n \in \mathbb{N}$ and $\mathbb{P}$-almost surely. Since $\lim_{n \uparrow \infty} \tau_n = \tau$ and $\tau = \infty$ $\mathbb{P}$-almost surely, taking the limit $n \uparrow \infty$, we get $R_t = Z_t$. Thus, Z is a $\mathbb{P}$-martingale. $\qquad \square$

Remark 6.4. *Since a posteriori the measure $\mathbb{Q}$ is absolutely continuous with respect to the measure $\mathbb{P}$ we can consider the measure $\mathbb{Q}$ as defined on the whole completion $\mathcal{F}$ of $\mathcal{M}$.*

Acknowledgments

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