# On the improvement of the Hardy inequality due to singular magnetic fields

Luca Fanelli,[a] David Krejčiřík,[b] Ari Laptev[c] and Luis Vega[d]

a) *Dipartimento di Matematica, SAPIENZA Università di Roma, P. le Aldo Moro 5, 00185 Roma; fanelli@mat.uniroma1.it.*

b) *Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 12000 Prague 2, Czechia; david.krejcirik@fjfi.cvut.cz.*

c) *Department of Mathematics, Imperial College London, Huxley Building, 180 Queens Gate, London SW7 2AZ, United Kingdom; a.laptev@imperial.ac.uk*

d) *Departamento de Matemáticas, Universidad del Pais Vasco, Aptdo. 644, 48080 Bilbao, & Basque Center for Applied Mathematics (BCAM), Alameda Mazarredo 14, 48009 Bilbao, Spain; luis.vega@ehu.es & lvega@bcamath.org.*

11 July 2018

**Abstract**

We establish magnetic improvements upon the classical Hardy inequality for two specific choices of singular magnetic fields. First, we consider the Aharonov-Bohm field in all dimensions and establish a sharp Hardy-type inequality that takes into account both the dimensional as well as the magnetic flux contributions. Second, in the three-dimensional Euclidean space, we derive a non-trivial magnetic Hardy inequality for a magnetic field that vanishes at infinity and diverges along a plane.

## 1 Introduction

The subcriticality of the Laplacian in $\mathbb{R}^d$ for $d \geq 3$ can be quantified by means of the classical Hardy inequality

$$-\Delta \geq \left(\frac{d-2}{2}\right)^2 \frac{1}{r^2} \tag{1}$$

valid in the sense of quadratic forms in $L^2(\mathbb{R}^d)$, where $-\Delta$ is the standard self-adjoint realisation of the Laplacian in $L^2(\mathbb{R}^d)$ and $r$ is the distance to the origin of $\mathbb{R}^d$. On the other hand, the Laplacian is critical in $\mathbb{R}$ and $\mathbb{R}^2$ in the sense that the spectrum of the shifted operator $-\Delta + V$ starts below zero whenever the operator of multiplication $V$ is bounded, non-positive and non-trivial. In quantum mechanics, interpreting $-\Delta$ as the Hamiltonian of a free electron, the Hardy inequality (1) can be interpreted as the uncertainty principle with important consequences for the stability of atoms and molecules.

Inequality (1) goes back to 1920 [6] and it is well known that it is optimal in the sense that the dimensional constant is the best possible and no other non-negative term could be added on the right-hand side of (1). A much more recent observation is that adding any magnetic field leads to an improved Hardy inequality, including dimension $d = 2$. A variant of this statement is the magnetic Hardy inequality

$$(-i\nabla + A)^2 - \left(\frac{d-2}{2}\right)^2 \frac{1}{r^2} \geq \frac{c_{d,B}}{1 + r^2 \log^2(r)}\,, \tag{2}$$

valid in the sense of quadratic forms in $L^2(\mathbb{R}^d)$ for $d \geq 2$. Here $A : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector potential and $c_{d,B}$ is a non-negative constant that depends only on the dimension $d$ and the magnetic field $B = \mathrm{d}A$; the constant $c_{d,B}$ is positive if, and only if, the field $B$ is not identically equal to zero. This inequality was first proved by Laptev and Weidl in 1999 [9] for $d = 2$ under an extra flux condition, in which case the lower bound holds with a better weight (without the logarithm) on the right-hand side of (2). A general version of (2) is due to Cazacu and Krejčiřík [3], but we also refer to [10], [1], [2], [7, Sec. 6] and [5] for previous related works.

The principal motivation of the present paper is our curiosity about the structure of the magnetic improvement on the right-hand side of (2). Our first result deals with the Aharonov-Bohm potential

$$A_\alpha(x, y, z_1, \dots, z_{d-2}) := \alpha \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0, \dots, 0 \right), \qquad \alpha \in \mathbb{R}, \tag{3}$$

where $(x, y, z_1, \dots, z_{d-2}) \in \mathbb{R}^d$. We abbreviate $z := (z_1, \dots, z_{d-2}) \in \mathbb{R}^{d-2}$ and denote by $\rho(x, y, z) := \sqrt{x^2 + y^2}$ the distance of a point $(x, y, z) \in \mathbb{R}^d$ to the subspace $\{x = y = 0\} \subset \mathbb{R}^d$ of dimension $d-2$. Let us also recall that $r(x, y, z) := \sqrt{x^2 + y^2 + |z|^2}$ denotes the distance of $(x, y, z) \in \mathbb{R}^d$ to the origin of $\mathbb{R}^d$. Because of the singularity of $A_\alpha$ at the origin, it is important to specify the self-adjoint realisation of the associated magnetic Laplacian; we customarily understand $(-i\nabla + A_\alpha)^2$ as the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R}^d \setminus \{\rho = 0\})$.

**Theorem 1.** *Let $A_\alpha$ be given by* (3)*. For every $\alpha \in \mathbb{R}$, one has*

$$(-i\nabla + A_\alpha)^2 - \left(\frac{d-2}{2}\right)^2 \frac{1}{r^2} \geq \frac{\operatorname{dist}(\alpha, \mathbb{Z})^2}{\rho^2} \tag{4}$$

*in the sense of quadratic forms in $L^2(\mathbb{R}^d)$ with $d \geq 2$.*

This theorem in dimension $d = 2$ is due to Laptev and Weidl [9]. The novelty of Theorem 1 consists in the present extension to the higher dimensions, $d \geq 3$. The result is optimal in the sense that that the constants appearing in (4) are the best possible and no other non-negative term could be added on the right-hand side of the inequality. In other words, subtracting the right-hand side from the left-hand side, the obtained operator would be critical. Notice also that the flux-type condition $\alpha \notin \mathbb{Z}$ is necessary to have the subcriticality of the operator on the left-hand side of (4). Indeed, $(-i\nabla + A_\alpha)^2$ is unitarily equivalent to the magnetic-free Laplacian whenever $\alpha \in \mathbb{Z}$, so in this case the criticality of the operator on the left-hand side of (4) follows from the optimality of the classical Hardy inequality (1).

Because of the special form of the vector potential (3), the operator $(-i\nabla + A_\alpha)^2$ admits a natural decomposition with respect to the variables $(x, y) \in \mathbb{R}^2$ and $z \in \mathbb{R}^{d-2}$. However, it is important to stress that (4) does not follow as a result of this separation of variables. In fact, while the right-hand side of (4) is the two-dimensional contribution coming from the angular component of the magnetic Laplacian in the $(x, y)$-plane, the second (dimensional) term on the left-hand side of the inequality is a contribution coming from both the radial component of the magnetic Laplacian in the $(x, y)$-plane as well as the Laplacian in the $z$-space.

The feature of the Aharonov-Bohm potential (3) is that its singularity is supported on a manifold of codimension two. Our next interest lies in a vector potential with a singularity supported on a hyperplane. In this case, we restrict our attention to the the three-dimensional toy model

$$A_\beta(x, y, z) := \beta \left( \frac{y}{z^2}, 0, 0 \right), \qquad \beta \in \mathbb{R}, \tag{5}$$

where $(x, y, z) \in \mathbb{R}^3$. The seemingly simple choice (5) for the vector potential is of course very special, but on the other hand the model is intrinsically three-dimensional in the sense that no reduction to lower dimensions via a separation of variables is available.

We understand the magnetic Laplacian $-\Delta_\beta := (-i\nabla + A_\beta)^2$ corresponding to (5) with $\beta \in \mathbb{R} \setminus \{0\}$ as the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R}^d \setminus \{z = 0\})$. Because of the strong singularity of $A_\beta$ on the plane $\{z = 0\}$, the unperturbed version of $-\Delta_\beta$ is not the Laplacian $-\Delta$ in $\mathbb{R}^3$ but rather the Dirichlet Laplacian in $\mathbb{R}^3 \setminus \{z = 0\}$ that we denote by $-\Delta_0$. More specifically, the singularity of the potential requires that the functions from the operator domain of $-\Delta_\beta$ vanish on $\{z = 0\}$, representing thus certain confinement of the electron to one of the two half-spaces $\{z > 0\}$ or $\{z < 0\}$. The unperturbed operator $-\Delta_0$ satisfies the Hardy inequality

$$-\Delta_0 \geq \frac{1}{4} \frac{1}{z^2}, \tag{6}$$

which is optimal (in the same way as (1) is optimal for $-\Delta$). Notice that the distance to the origin in (1) is replaced by the distance to the plane $\{z = 0\}$ in (6). Our next result shows that there is always a specific improvement whenever $\beta \neq 0$.

**Theorem 2.** *Let $A_\beta$ be given by* (5)*. For every $\beta \in \mathbb{R}$, one has*

$$- \Delta_\beta - \frac{1}{4}\frac{1}{z^2} \geq |\beta| \left(1 + \frac{y^2}{z^2}\right) \frac{1}{z^2} \tag{7}$$

*in the sense of quadratic forms in $L^2(\mathbb{R}^3)$.*

The first term on the right-hand side of (7) is easy to obtain, while the second term is much less obvious. Our strategy to identify the second improvement is based on a test-function argument, which we believe to be of independent interest. Contrary to Theorem 1, we do not know whether the inequality of Theorem 2 is optimal if $\beta \neq 0$.

The rest of the paper naturally splits into two independent sections. In Section 2 we quickly prove Theorem 1, while Theorem 2 is established in a longer Section 3.

# 2 The Aharonov-Bohm field

In this section we exclusively consider the vector potential $A_\alpha$ from (3) with any $d \geq 2$.

## 2.1 Preliminaries

For every $\alpha \in \mathbb{R}$, we introduce the magnetic Laplacian $(-i\nabla + A_\alpha)^2$ as the self-adjoint non-negative operator in $L^2(\mathbb{R}^d)$ associated with the quadratic form

$$Q_\alpha[\psi] := \|(\nabla + iA_\alpha)\psi\|^2 , \qquad \mathsf{D}(Q_\alpha) := \overline{C_0^\infty(\mathbb{R}^d \setminus \{\rho = 0\})}^{|||\cdot|||} ,$$

where $\|\cdot\|$ denotes the usual norm of $L^2(\mathbb{R}^d)$ and

$$|||\psi||| := \sqrt{\|(\nabla + iA_\alpha)\psi\|^2 + \|\psi\|^2} .$$

By the diamagnetic inequality, we have the form domain inclusion $\mathsf{D}(Q_\alpha) \subset W_0^{1,2}(\mathbb{R}^d \setminus \{\rho = 0\}) = W^{1,2}(\mathbb{R}^d)$, where the equality follows from the fact that the subset $\{\rho = 0\} \subset \mathbb{R}^d$ is a polar set (*cf.* [4, Sec. VIII.6]). Using the special structure (5) of the potential $A_\beta$, we have

$$Q_\alpha[\psi] = \int_{\mathbb{R}^d} \left( \left| \left(-i\partial_x - \frac{\alpha\, y}{x^2 + y^2}\right) \psi \right|^2 + \left| \left(-i\partial_y + \frac{\alpha\, x}{x^2 + y^2}\right) \psi \right|^2 + |\nabla_z \psi|^2 \right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z ,$$

where $z = (z_1, \dots, z_{d-2})$ is a $(d-2)$-dimensional coordinate. In the sense of distributions,

$$(-i\nabla + A_\alpha)^2 = \Big( - i\partial_x - \frac{\alpha\, y}{x^2 + y^2}\Big)^2 + \Big( - i\partial_y + \frac{\alpha\, x}{x^2 + y^2}\Big)^2 - \Delta_z ,$$

where $-\Delta_z$ is the usual (distributional) Laplacian in the $z$ variables.

If $\alpha = 0$, then $\mathsf{D}(Q_0) = W^{1,2}(\mathbb{R}^d)$ and the operator associated with $Q_0$ is just the standard self-adjoint realisation of the Laplacian $-\Delta$ in $L^2(\mathbb{R}^d)$. More generally, if $\alpha \in \mathbb{Z}$ then $(-i\nabla + A_\alpha)^2$ is unitarily equivalent to the (magnetic-free) Laplacian $-\Delta$. This can be seen as follows. Passing to the polar coordinates in the $(x, y)$-plane, *i.e.* writing $(x, y) = (\rho \cos\varphi, \rho \sin\varphi)$ with $\rho \in (0, \infty)$ and $\varphi \in (0, 2\pi]$, we have the obvious unitary equivalences

$$\begin{aligned} (-i\nabla + A_\alpha)^2 &\cong -\rho^{-1}\, \partial_\rho\, \rho\, \partial_\rho + \frac{(-i\partial_\varphi + \alpha)^2}{\rho^2} - \Delta_z \\ &\cong \bigoplus_{m \in \mathbb{Z}} \left( -\rho^{-1}\, \partial_\rho\, \rho\, \partial_\rho + \frac{\nu_m^2}{\rho^2} \right) - \Delta_z . \end{aligned} \tag{8}$$

Here $\nu_m := m + \alpha$ are the eigenvalues of the one-dimensional operator $-i\partial_\varphi + \alpha$ in $L^2([0,2\pi))$, subject to periodic boundary conditions. The corresponding set of eigenfunctions read $\{e^{im\varphi}\}_{m\in\mathbb{Z}}$. If $\alpha$ is an integer, then the direct sum is indistinguishable from the usual partial-wave decomposition of the Laplacian $-\Delta$.

Finally, let us notice that the spectrum of $(-i\nabla + A_\alpha)^2$ equals the semiaxis $[0,\infty)$ for every real $\alpha$.

## 2.2 The improved Hardy inequality

Let $\psi \in C_0^\infty(\mathbb{R}^d \setminus \{\rho = 0\})$, a core of $Q_\alpha$. Employing the polar coordinates in the $(x,y)$-plane as in (8) and writing $\phi(\rho,\varphi,z) =: \psi(\rho\cos\varphi, \rho\sin\varphi, z)$, we have

$$\begin{aligned} Q_\alpha[\psi] &= \int_{\mathbb{R}^{d-2}} \int_0^{2\pi} \int_0^\infty \left( |\partial_\rho\phi|^2 + \frac{|\partial_\varphi\phi + i\alpha\phi|^2}{\rho^2} + |\nabla_z\phi|^2 \right) \rho \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}z \\ &\geq \int_{\mathbb{R}^{d-2}} \int_0^{2\pi} \int_0^\infty \left( |\partial_\rho\phi|^2 + \frac{\operatorname{dist}(\alpha,\mathbb{Z})^2}{\rho^2} |\phi|^2 + |\nabla_z\phi|^2 \right) \rho \, \mathrm{d}\rho \, \mathrm{d}\varphi \, \mathrm{d}z \,, \end{aligned} \tag{9}$$

where we omit to specify the arguments of $\phi$ and abuse a bit the notation for $\rho$. This inequality explains the quantity on the right-hand side of (4). To obtain the dimensional term on the left-hand side of (4), we write

$$\phi(\rho,\varphi,z) = f(\rho,\varphi,z)\, (\rho^2 + |z|^2)^{-(d-2)/4} \,,$$

which is in fact the definition of the new test function $f$. Notice that $f(0,\varphi,z) = 0$ for all $\varphi \in [0,2\pi)$ and $z \in \mathbb{R}^{d-2}$. A straightforward computation employing an integration by parts yields

$$\begin{aligned} \int_{\mathbb{R}^{d-2}} \int_0^\infty \left( |\partial_\rho\phi|^2 + |\nabla_z\phi|^2 \right) \rho \, \mathrm{d}\rho \, \mathrm{d}z &= \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^{d-2}} \int_0^\infty \frac{|\phi|^2}{\rho^2 + |z|^2} \rho \, \mathrm{d}\rho \, \mathrm{d}z \\ &\quad + \int_{\mathbb{R}^{d-2}} \int_0^\infty \left( |\partial_\rho f|^2 + |\nabla_z f|^2 \right) (\rho^2 + |z|^2)^{-(d-2)/2} \rho \, \mathrm{d}\rho \, \mathrm{d}z \\ &\geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^{d-2}} \int_0^\infty \frac{|\phi|^2}{\rho^2 + |z|^2} \rho \, \mathrm{d}\rho \, \mathrm{d}z \,. \end{aligned} \tag{10}$$

Estimates (9) and (10) yield (4), after coming back to the Cartesian coordinates and noticing that $\rho^2 + |z|^2 = r^2$. This concludes the proof of Theorem 1.

The present proof also explains why the inequality (4) is optimal. Indeed, the inequality (9) is sharp in the sense that it is achieved by any function of the form $\phi(\rho,\varphi,z) = g(\rho,z)e^{im\varphi}$, where $m \in \mathbb{Z}$ is chosen in such a way that it minimises the distance $\operatorname{dist}(\alpha,\mathbb{Z})$ (so that $e^{im\varphi}$ is an eigenfunction corresponding to the lowest eigenvalue $\operatorname{dist}(\alpha,\mathbb{Z})^2$ of the operator $(-i\partial_\varphi + \alpha)^2$). The other inequality (9) is not achieved by a non-trivial $\phi$, but it is also sharp in the following sense. For any function $f$ depending only on $r$, we have

$$\begin{aligned} \int_{\mathbb{R}^{d-2}} \int_0^\infty \left( |\partial_\rho f|^2 + |\nabla_z f|^2 \right) (\rho^2 + |z|^2)^{-(d-2)/2} \rho \, \mathrm{d}\rho \, \mathrm{d}z &= \int_{\mathbb{S}_+^{d-2}} \int_0^\infty |\partial_r f|^2 \rho \, \mathrm{d}r \, \mathrm{d}\sigma \\ &\leq |\mathbb{S}_+^{d-2}| \int_0^\infty |\partial_r f|^2 r \, \mathrm{d}r \,, \end{aligned}$$

where $\mathbb{S}_+^{d-2} := \mathbb{S}^{d-2} \cap \{\rho > 0\}$ and $\mathbb{S}^{d-2}$ is the unit sphere in the $(\rho,z)$-half-space. It is well known that there exists a sequence of functions $\{f_n\}_{n=1}^\infty \subset C_0^\infty((0,\infty))$ such that

$$f_n(r) \xrightarrow[n\to\infty]{} 1 \text{ pointwise} \qquad \text{and} \qquad \int_0^\infty |\partial_r f_n(r)|^2 r \, \mathrm{d}r \xrightarrow[n\to\infty]{} 0 \,,$$

for the integral corresponds to the radial part of the two-dimensional Laplacian.

# 3 The confining field

The organisation of this section dealing with the vector potential (5) is as follows. In Section 3.1 we rigorously introduce the corresponding magnetic Laplacian $-\Delta_\beta$ as a self-adjoint operator in $L^2(\mathbb{R}^3)$ and state its basic spectral properties. The elementary part of Theorem 2 (*i.e.* just the first term on the right-hand side of (7)) is established in Section 3.2. The complete proof of Theorem 2 is given in the remaining Sections 3.3 and 3.4.

## 3.1 Preliminaries

For every $\beta \in \mathbb{R}$, we introduce the magnetic Laplacian $-\Delta_\beta$ the self-adjoint non-negative operator in $L^2(\mathbb{R}^3)$ associated with the quadratic form

$$Q_\beta[\psi] := \|(\nabla + iA_\beta)\psi\|^2 \,, \qquad \mathsf{D}(Q_\beta) := \overline{C_0^\infty(\mathbb{R}^3 \setminus \{z=0\})}^{|||\cdot|||} \,,$$

where $\|\cdot\|$ denotes the usual norm of $L^2(\mathbb{R}^3)$ and

$$|||\psi||| := \sqrt{\|(\nabla + iA_\beta)\psi\|^2 + \|\psi\|^2} \,.$$

By the diamagnetic inequality, we have the form domain inclusion $\mathsf{D}(Q_\beta) \subset W_0^{1,2}(\mathbb{R}^3 \setminus \{z=0\})$. Using the special structure (5) of the potential $A_\beta$, we have

$$Q_\beta[\psi] = \int_{\mathbb{R}^3} \left( \left| \left( -i\partial_x + \frac{\beta}{z^2} y \right) \psi \right|^2 + |\partial_y \psi|^2 + |\partial_z \psi|^2 \right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and, in the sense of distributions,

$$-\Delta_\beta = \Big( -i\partial_x + \frac{\beta}{z^2} y \Big)^2 - \partial_y^2 - \partial_z^2 = -\Delta - 2i\frac{\beta}{z^2} y\, \partial_x + \frac{\beta^2}{z^4} y^2 \,, \tag{11}$$

where $-\Delta$ is the usual (distributional) Laplacian in the $(x,y,z)$ variables.

Notice that $-\Delta_0$ (*i.e.* $\beta = 0$) is just the Dirichlet Laplacian in $\mathbb{R}^3 \setminus \{z=0\}$ for which the form domain equality $\mathsf{D}(Q_0) = W_0^{1,2}(\mathbb{R}^3 \setminus \{z=0\})$ holds. The spectrum of $-\Delta_0$ is well known, $\sigma(-\Delta_0) = [0,\infty)$. Moreover, using the classical one-dimensional Hardy inequality

$$\forall u \in W_0^{1,2}(\mathbb{R} \setminus \{0\}) \,, \qquad \int_{\mathbb{R}} |u'(z)|^2 \,\mathrm{d}z \geq \frac{1}{4} \int_{\mathbb{R}} \frac{|u(z)|^2}{z^2} \,\mathrm{d}z \,, \tag{12}$$

and Fubini's theorem, it follows that $-\Delta_0$ is subcritical in the sense that the three-dimensional Hardy inequality

$$\forall \psi \in W_0^{1,2}(\mathbb{R}^3 \setminus \{z=0\}) \,, \qquad \int_{\mathbb{R}^3} |\nabla \psi(x,y,z)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{|\psi(x,y,z)|^2}{z^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \,,$$

holds. This is the precise statement of (6).

It is not difficult to see that the spectrum of $-\Delta_\beta$ coincides with the spectrum of the unperturbed operator $-\Delta_0$ (as well as of the usual Laplacian without the extra Dirichlet condition).

**Proposition 1.** *For every $\beta \in \mathbb{R}$, one has*

$$\sigma(-\Delta_\beta) = [0,\infty) \,.$$

*Proof.* The inclusion $\sigma(-\Delta_\beta) \subset [0,\infty)$ follows trivially because of the non-negativity of $-\Delta_\beta$. The opposite inclusion $\sigma(-\Delta_\beta) \supset [0,\infty)$ can be established by the Weyl criterion, by choosing the singular sequence localised at the infinity of the cone $\{|y| < |z|\}$, where the terms in (11) containing $\beta$ can be made arbitrarily small. We omit the details. □

## 3.2 The elementary Hardy inequality

Now we turn to the elementary part of the Hardy inequality (7).

**Proposition 2.** *For every $\beta \in \mathbb{R}$, one has*

$$\forall \psi \in \mathsf{D}(Q_\beta) \,, \qquad Q_\beta[\psi] \geq \left( \frac{1}{4} + |\beta| \right) \int_{\mathbb{R}^3} \frac{|\psi(x,y,z)|^2}{z^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \,. \tag{13}$$

*Proof.* Since the result will be re-proved in the following subsection, here we provide just a sketchy proof.

Writing

$$-\Delta_\beta = \underbrace{\left(-i\partial_x + \frac{\beta}{z^2}y\right)^2 - \partial_y^2}_{-\Delta'_\beta} - \partial_z^2\,, \tag{14}$$

we notice that $-\Delta'_\beta$ is the magnetic Laplacian in $L^2(\mathbb{R}^2)$ corresponding to the two-dimensional vector potential

$$A'_\beta(x,y) := \frac{\beta}{z^2}\,(y,0)\,,$$

which depends parametrically on $z$ (and $\beta$). The corresponding two-dimensional magnetic field is constant

$$B'_\beta(x,y) := \operatorname{rot} A'_\beta(x,y) = -\frac{\beta}{z^2}\,. \tag{15}$$

The operator $-\Delta'_\beta$ is the celebrated Landau Hamiltonian.

The spectral problem for $-\Delta'_\beta$ is explicitly solvable. The easiest way how to see it is to perform a partial Fourier transform with respect to the $x$-variable, which yields a unitary equivalence

$$-\Delta'_\beta \cong \left(\xi + \frac{\beta}{z^2}y\right)^2 - \partial_y^2\,, \tag{16}$$

where $\xi \in \mathbb{R}$ is the dual variable to $x$. Noticing that the right-hand side of (16) is the Hamiltonian of a shifted harmonic oscillator (the shift can be handled as yet another unitary transform), we get the familiar formula (the natural numbers $\mathbb{N}$ contain zero in our convention)

$$\sigma(-\Delta'_\beta) = \frac{2|\beta|}{z^2}\left(\mathbb{N} + \frac{1}{2}\right), \qquad \beta \neq 0\,.$$

Each point in the spectrum is an eigenvalue of infinite multiplicity (Landau levels). (If $\beta = 0$, then $\sigma(-\Delta'_\beta) = [0,\infty)$.) In particular,

$$\inf \sigma(\Delta'_\beta) = \frac{|\beta|}{z^2}$$

(which is trivially valid also for $\beta = 0$).

Using the last result in (14), we get

$$-\Delta_\beta \geq -\partial_z^2 + \frac{|\beta|}{z^2} \geq \frac{1}{4z^2} + \frac{|\beta|}{z^2}\,, \tag{17}$$

which is the desired result (13). The second estimate in (17) follows from the classical Hardy inequality (12), by noticing that the form core consists of functions that vanish on the plane $\{z = 0\}$. □

**Remark 1.** Proposition 2 can be alternatively proved also by a standard commutator trick (see, *e.g.*, [2, Sec. 2.4]). Let us denote by $\Pi_j := -i\partial_j + (A_\beta)_j$ with $j \in \{1,2,3\} \cong \{x,y,z\}$ the $j^{\text{th}}$ component of the magnetic gradient. Then one has the identity

$$\forall C_0^\infty(\mathbb{R}^3 \setminus \{z=0\})\,, \qquad \|\Pi_j\psi\|^2 + \|\Pi_k\psi\|^2 = \|(\Pi_j \pm i\Pi_k)\psi\|^2 \pm \langle \psi, (B_\beta)_{jk}\psi\rangle\,, \tag{18}$$

for any pair $j,k \in \{1,2,3\}$, where $(B_\beta)_{jk} := \partial_j(A_\beta)_k - \partial_k(A_\beta)_j$ are the coefficients of the magnetic tensor $B_\beta := \mathrm{d}A_\beta$ and $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\mathbb{R}^3)$. In our case (5), we have

$$B_\beta = \beta \begin{pmatrix} 0 & -1/z^2 & 2y/z^3 \\ 1/z^2 & 0 & 0 \\ -2y/z^3 & 0 & 0 \end{pmatrix}.$$

Using the formula (18) with $(j,k) := (1,2)$, one therefore obtains, for every $\psi \in C_0^\infty(\mathbb{R}^3 \setminus \{z=0\})$,

$$Q_\beta[\psi] = \|\Pi_1\psi\|^2 + \|\Pi_2\psi\|^2 + \|\Pi_3\psi\|^2 \geq \pm\beta \left\|\frac{\psi}{z}\right\|^2 + \|\partial_z\psi\|^2 \geq \left(\pm\beta + \frac{1}{4}\right)\left\|\frac{\psi}{z}\right\|^2,$$

where the last inequality is due to the classical Hardy inequality (12). Since the obtained result holds with either the plus or minus sign, we arrive at (13) for every $\psi \in C_0^\infty(\mathbb{R}^3 \setminus \{z=0\})$. By density, the result extends to all $\psi \in \mathsf{D}(Q_\beta)$.

### 3.3 Quantification of the elementary Hardy inequality

Our next goal is to show that (13) can still be improved. To do so, we have to employ the terms we neglected in the crude estimates (17).

Let $\psi \in C_0^\infty(\mathbb{R}^3 \setminus \{z=0\})$, a core of $Q_\beta$. The function is implicitly assumed to depend on the space variables $(x,y,z) \in \mathbb{R}^3$ and for brevity we omit to specify the arguments in the integrals below.

First of all, let us perform the partial Fourier transform with respect to the $x$-variable as in (16):

$$Q_\beta[\psi] = \int_{\mathbb{R}^3} \left( \left| \left( \xi + \frac{\beta}{z^2} y \right) \hat{\psi} \right|^2 + |\partial_y \hat{\psi}|^2 + |\partial_z \hat{\psi}|^2 \right) \mathrm{d}\xi \, \mathrm{d}y \, \mathrm{d}z \,. \tag{19}$$

Notice that the transformed function $\hat{\psi} = \hat{\psi}(\xi, y, z)$ still vanishes in a neighbourhood of $\{z = 0\}$.

In the second step, we make the change of test function

$$\hat{\psi}(\xi, y, z) = \sqrt{|z|}\, \phi(\xi, y, z) \tag{20}$$

to single out the first term on the right-hand side of (13). Putting (20) into (19) and integrating by parts with respect to $z$, we arrive at

$$Q_\beta[\psi] - \frac{1}{4} \left\| \frac{\psi}{z} \right\|^2 = \int_{\mathbb{R}^3} \left( \left| \left( \xi + \frac{\beta}{z^2} y \right) \phi \right|^2 + |\partial_y \phi|^2 + |\partial_z \phi|^2 \right) |z| \, \mathrm{d}\xi \, \mathrm{d}y \, \mathrm{d}z \,. \tag{21}$$

From now on, let us assume that $\beta$ is non-zero. Then

$$\eta(\xi, y, z) := \exp\left( -\frac{|\beta|}{2z^2} (y - y_0)^2 \right) \qquad \text{with} \qquad y_0 := -\frac{z^2 \xi}{\beta}$$

is an eigenfunction of the operator on the right-hand side of (16) corresponding to the lowest eigenvalue. In this two-dimensional context, the variable $z$ is understood as a parameter and $\xi$ gives rise to the degeneracies. In the third step, we make the change of test function

$$\phi(\xi, y, z) = \eta(\xi, y, z)\, \varphi(\xi, y, z) \tag{22}$$

to single out the second term on the right-hand side of (13). Putting (22) into (21) and integrating by parts with respect to $y$, we arrive at

$$Q_\beta[\psi] - \left( \frac{1}{4} + |\beta| \right) \left\| \frac{\psi}{z} \right\|^2 = \int_{\mathbb{R}^3} \left( |\partial_y \varphi|^2 + |\partial_z \phi|^2 \right) \eta^2(\xi, y, z)\, |z| \, \mathrm{d}\xi \, \mathrm{d}y \, \mathrm{d}z \,. \tag{23}$$

Since the right-hand side of (23) is non-negative, we have just re-proved (13).

**Remark 2.** If the operator associated with the form on the right-hand side of (23) had compact resolvent (which is not true), we would immediately arrive at a (local) improved Hardy inequality (*cf.* [8]). In our case, however, we have to proceed more carefully.

### 3.4 A test-function argument

Having (23) at our disposal, the next step is to use the contribution of the term containing $|\partial_z \phi|^2$. First of all, notice that

$$\partial_z \phi = \partial_z \varphi + \varphi \, \frac{|\beta|}{z^3} (y^2 - y_0^2) \,.$$

The main trick is to pick an arbitrary function $f : \mathbb{R}^3 \to \mathbb{R}$ and write

$$\begin{aligned}
|\partial_z \phi|^2 &= \left| \partial_z \varphi - f\varphi + \varphi \left( \frac{|\beta|}{z^3}(y^2 - y_0^2) + f \right) \right|^2 \\
&= |\partial_z \varphi - f\varphi|^2 + |\varphi|^2 \left( \frac{|\beta|}{z^3}(y^2 - y_0^2) + f \right)^2 + 2\Re \left[ (\partial_z \bar{\varphi} - f\bar{\varphi})\varphi \left( \frac{|\beta|}{z^3}(y^2 - y_0^2) + f \right) \right] \\
&= |\partial_z \varphi - f\varphi|^2 + |\varphi|^2 \left( \frac{\beta^2}{z^6}(y^2 - y_0^2)^2 - f^2 \right) + (\partial_z |\varphi|^2) \left( \frac{|\beta|}{z^3}(y^2 - y_0^2) + f \right) \,.
\end{aligned}$$

Putting this expression into (23) and integrating by parts with respect to $z$, we arrive at

$$Q_\beta[\psi] - \left(\frac{1}{4} + |\beta|\right) \left\|\frac{\psi}{z}\right\|^2 = \int_{\mathbb{R}^3} \left(|\partial_y \varphi|^2 + |\partial_z \varphi - f\varphi|^2 + V_f |\varphi|^2\right) \eta^2(\xi, y, z)\, |z| \, \mathrm{d}\xi \, \mathrm{d}y \, \mathrm{d}z \,, \tag{24}$$

where

$$V_f(\xi, y, z) := -\partial_z f - \frac{1}{z} f - 2\frac{|\beta|}{z^3}(y^2 - y_0^2) f - \frac{\beta^2}{z^6}(y^2 - y_0^2)^2 + 2\frac{|\beta|}{z^4}(y^2 + y_0^2) \,. \tag{25}$$

Now we may play with many possible choices of $f$ to try to make the potential $V$ positive and employ the following consequence of (24):

$$Q_\beta[\psi] - \left(\frac{1}{4} + |\beta|\right) \left\|\frac{\psi}{z}\right\|^2 \geq \int_{\mathbb{R}^3} V_f \, |\hat{\psi}|^2 \, \mathrm{d}\xi \, \mathrm{d}y \, \mathrm{d}z \,.$$

In particular, the special choice

$$f(\xi, y, z) := \frac{1}{2} \frac{|\beta|}{z^3}(y_0^2 - y^2) \tag{26}$$

leads to a spectacularly simple expression

$$V_f(\xi, y, z) = \frac{|\beta|}{z^4}(y^2 + y_0^2) \,. \tag{27}$$

Neglecting the second term on the right-hand side of (27) and performing the inverse Fourier transform in the $\xi$-variable, we therefore obtain the following Hardy inequality

$$Q_\beta[\psi] - \left(\frac{1}{4} + |\beta|\right) \left\|\frac{\psi}{z}\right\|^2 \geq |\beta| \int_{\mathbb{R}}^3 \frac{y^2}{z^4} \, |\psi|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{28}$$

which coincides with (7) after the extension of $\psi \in C_0^\infty(\mathbb{R}^3 \setminus \{z = 0\})$ to all $\mathsf{D}(Q_\beta)$. This concludes the proof of Theorem 2. ☐

**Remark 3.** Keeping both terms on the right-hand side of (27) and performing the inverse Fourier transform in the $\xi$-variable, we have established an interesting operator inequality

$$-\Delta_\beta \geq \frac{-\partial_x^2}{|\beta|} + \left(\frac{1}{4} + |\beta| + |\beta| \frac{y^2}{z^2}\right) \frac{1}{z^2} \,.$$

## Acknowledgment

The research of D.K. was partially supported by the GACR grant No. 18-08835S and by FCT (Portugal) through project PTDC/MAT-CAL/4334/2014. The research of L.V. was partially supported by ERCEA Advanced Grant 669689-HADE, MTM2014-53145-P, and IT641-13.

# References

[1] B. Alziary, J. Fleckinger-Pellé, and P. Takáč, *Eigenfunctions and Hardy inequalities for a magnetic Schrodinger operator in* $\mathbb{R}^2$, Math. Meth. Appl. Sci. **26** (2003), 1093–1136.

[2] A. Balinsky, A. Laptev, and A. Sobolev, *Generalized Hardy inequality for the magnetic Dirichlet forms*, J. Stat. Phys. **116** (2004), 507–521.

[3] C. Cazacu and D. Krejčiřík, *The Hardy inequality and the heat equation with magnetic field in any dimension*, Comm. Partial Differential Equations **41** (2016), 1056–1088.

[4] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford University Press, Oxford, 1987.

[5] T. Ekholm and F. Portmann, *A magnetic contribution to the Hardy inequality*, J. Math. Phys. **55** (2014), 022101.

[6] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Zeit. **6** (1920), 314–317.

[7] H. Kovařík, *Eigenvalue bounds for two-dimensional magnetic Schrödinger operators*, J. Spectr. Theory **1** (2011), 363–387.

[8] D. Krejčiřík, *Twisting versus bending in quantum waveguides*, Analysis on Graphs and its Applications, Cambridge, 2007 (P. Exner et al., ed.), Proc. Sympos. Pure Math., vol. 77, Amer. Math. Soc., Providence, RI, 2008, pp. 617–636.

[9] A. Laptev and T. Weidl, *Hardy inequalities for magnetic Dirichlet forms*, Oper. Theory Adv. Appl. **108** (1999), 299–305.

[10] T. Weidl, *A remark on Hardy type inequalities for critical Schrödinger operators with magnetic fields*, Oper. Theory Adv. Appl. **110** (1999), 345–352.