UNIQUENESS PROPERTIES OF SOLUTIONS TO THE BENJAMIN-ONO EQUATION AND RELATED MODELS.

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ABSTRACT. We prove that if u_1, u_2 are solutions of the Benjamin-Ono equation defined in $(x,t) \in \mathbb{R} \times [0,T]$ which agree in an open set $\Omega \subset \mathbb{R} \times [0,T]$, then $u_1 \equiv u_2$. We extend this uniqueness result to a general class of equations of Benjamin-Ono type in both the initial value problem and the initial periodic boundary value problem. This class of 1-dimensional non-local models includes the intermediate long wave equation. Finally, we present a slightly stronger version of our uniqueness results for the Benjamin-Ono equation.

1. Introduction

We consider the initial value problem (IVP) for the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + u\partial_x u = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

BO

where $u = u(x,t)$ is a real-valued function, and $\mathcal{H}$ denotes the Hilbert transform

$$\begin{aligned} \mathcal{H}f(x) &:= \frac{1}{\pi}\,\text{p.v.}\Big(\frac{1}{\text{x}} * \text{f}\Big)(\text{x}) \\ &:= \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy = (-i\,\text{sgn}(\xi)\widehat{f}(\xi))^{\vee}(x) \end{aligned} \tag{1.2}$$

H

The BO equation was first deduced by Benjamin [3] and Ono [35] as a model for long internal gravity waves in deep stratified fluids. Later, it was shown to be a completely integrable system (see [2], [6] and references therein). In particular, real solutions of the IVP (1.1)

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satisfy infinitely many conservation laws, which provide an a priori estimate for the $H^{n/2}$-norm, $n \in \mathbb{Z}^+$.

The problem of finding the minimal regularity measured in the Sobolev scale $H^s(\mathbb{R})$, $s \in \mathbb{R}$, required to guarantee that the IVP (1.1) is locally or globally well-posed (WP) in $H^s(\mathbb{R})$ has been extensively studied, see [1], [12], [36], [20], [17], [39], [5] and [11] where global WP was established in $H^0(\mathbb{R}) = L^2(\mathbb{R})$, (for further details and results regarding the well-posedness of the IVP (1.1) we refer to [29] and to [10] for a different proof of the result in [11]).

We remark that a result established in [33] (see also [21]) implies that no well-posedness result in $H^s(\mathbb{R}), s \in \mathbb{R}$, for the IVP (1.1) can be established by using solely a contraction principle argument.

It was first shown in [12] and [13] that polynomial decay of the data may not be preserved by the solution flow of the BO equation. The results in [12] and [13] which present some unique continuation properties of the BO equation have been extended to fractional order weighted Sobolev spaces and have shown to be optimal in [7] and [8]. More precisely, using the notation

$$Z_{s,r} := H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \;\; \dot{Z}_{s,r} = Z_{s,r} \cap \{f \in L^1(\mathbb{R}) : \widehat{f}(0) = 0\},$$

with $s, r > 0$ one has the results :

(i) [7] The IVP (1.1) is locally WP in $Z_{s,r}$ for $s \geq r \in [1, 5/2)$ and if $u \in C([0,T] : Z_{5/2,2})$ is a solution of (1.1) s.t. $u(\cdot, t_j) \in Z_{5/2,5/2}$, $j = 1, 2$ with $t_1,\ t_2 \in [0,T]$, $t_1 \neq t_2$, then $u \in C([0,T] : \dot{Z}_{5/2,2})$.

(ii) [7] The IVP (1.1) is locally WP in $\dot{Z}_{s,r}$ $s \geq r \in [5/2, 7/2)$.

(iii) [7] If $u \in C([0,T] : \dot{Z}_{7/2,3})$ is a solution of (1.1) s.t. $\exists\, t_1,\ t_2,\ t_3 \in [0,T]$, $t_1 < t_2 < t_3$ with $u(\cdot, t_j) \in Z_{7/2,7/2}$, $j = 1, 2, 3$, then $u \equiv 0$.

(iv) [8] The IVP (1.1) has solutions $u \in C([0,T] : \dot{Z}_{7/2,3})$, $u \not\equiv 0$, for which $\exists\, t_1,\ t_2, \in [0,T]$, $t_1 < t_2$, with $u(\cdot, t_j) \in Z_{7/2,7/2}$, $j = 1, 2$.

Our first main result in this work is the following theorem:

TH1 **Theorem 1.1.** *Let* $u_1,\ u_2$ *be solutions to the IVP* (1.1) *for* $(x,t) \in \mathbb{R} \times [0,T]$ *such that*

$$u_1,\ u_2 \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-2}(\mathbb{R})), \quad s > 5/2. \quad (1.3)$$ m1

If there exists an open set $\Omega \subset \mathbb{R} \times [0,T]$ *such that*

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega, \tag{1.4}$$ H1a

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.5}$$ result1

In particular, if u_1 *vanishes in* Ω*, then* $u_1 \equiv 0$.

rr1 **Remark 1.2.** *(i) Under the same hypotheses, Theorem 1.1 applies to solutions of the generalized BO equation*

$$\partial_t u - \mathcal{H}\partial_x^2 u + \partial_x f(u) = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.6}$$ gBO

with $f : \mathbb{R} \to \mathbb{R}$ *smooth enough and* $f(0) = 0$*. In particular, it applies for* $f(u) = u^k,\ k = 2,3,4,...$ *for which the well posedness of the associated IVP was considered in* [1], [18], [17], [19], [40], [41], *see also* [25].

(ii) The hypothesis (1.3) *guarantees that the solutions satisfy the equation* (1.1) *point-wise, which will be required in our proof.*

(iii) A similar result to that described in Theorem 1.1 for the IVP associated to the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + \partial_x u^k = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R},\ k = 2,3,...., \tag{1.7}$$ gKdV

was established in [38]*, and for some evolution equations of Schrödinger type in* [16]*. In both cases, their proofs are based on appropriate forms of the so called Carleman estimates. Our proof of Theorem 1.1 is elementary and relies on simple properties of the Hilbert transform as a boundary value of analytic functions.*

(iv) We observe that the unique continuation in (iii) before the statement of Theorem 1.1 applies to a single solution of the BO equation but not to any two solutions as in Theorem 1.1. This is due to the fact that the argument in the proof there depends upon the whole symmetry structure of the BO equation.

(v) Theorem 1.1 can be seen as a corollary of the following linear result whose proof is exactly the one given below for Theorem 1.1 :
Assume that $k,\, j \in \mathbb{Z}^+ \cup \{0\}$ *and that*

$$a_m : \mathbb{R} \times [0,T] \to \mathbb{R},\ m = 0,1,...,k, \quad \textit{and} \quad b : \mathbb{R} \times [0,T] \to \mathbb{R}$$

are continuous functions with $b(\cdot)$ *never vanishing on* $(x,t)\in\mathbb{R}\times[0,T]$, *and consider the IVP*

$$\begin{cases} \partial_t w - b(x,t)\,\mathcal{H}\partial_x^j w + \displaystyle\sum_{m=0}^{k} a_m(x,t)\partial_x^m w = 0, \\ \\ w(x,0) = w_0(x). \end{cases} \tag{1.8}$$ general

TH3 **Theorem 1.3.** *Let*

$$w \in C([0,T]:H^s(\mathbb{R}))\cap C^1((0,T):H^{s-2}(\mathbb{R})), \quad s > \max\{k;j\}+1/2,$$

be a solution to the IVP (1.8). *If there exists an open set* $\Omega\subset\mathbb{R}\times[0,T]$ *such that*

$$w(x,t) = 0, \quad (x,t)\in\Omega, \tag{1.9}$$ HA1

then,

$$w(x,t) = 0 \quad (x,t)\in\mathbb{R}\times[0,T]. \tag{1.10}$$ result22

ole **Remark 1.4.** *(i) In particular, applying Theorem 1.3 to the difference of two solutions* u_1, u_2 *of the Burgers-Hilbert (BH) equation (see* [4]*)*

$$\partial_t u - \mathcal{H}u + u\partial_x u = 0, \qquad (x,t)\in\mathbb{R}\times\mathbb{R}, \tag{1.11}$$ BH

one sees that the result in Theorem 1.1, with $s > 3/2$, *holds for the IVP associated to the BH equation* (1.11).

(ii) The result of Theorem 1.1 extends to solutions of the initial periodic boundary value problem (IPBVP) associated to the generalized BO equation

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + \partial_x f(u) = 0, \qquad (x,t)\in\mathbb{S}^1\times\mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.12}$$ gBO-PBVP

with $f(\cdot)$ *as in part (i) of this remark. More precisely :*

TH2 **Theorem 1.5.** *Let* $u_1,\ u_2$ *be solutions of the IPBVP* (1.12) *in* $(x,t)\in\mathbb{S}^1\times[0,T]$ *such that*

$$u_1,\ u_2 \in C([0,T]:H^s(\mathbb{S}^1))\cap C^1((0,T):H^{s-2}(\mathbb{S}^1)),\ s > 5/2. \tag{1.13}$$ m2

If there exists an open set $\Omega\subset\mathbb{S}^1\times[0,T]$ *such that*

$$u_1(x,t) = u_2(x,t), \quad (x,t)\in\Omega, \tag{1.14}$$ HB1

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t)\in\mathbb{S}^1\times[0,T]. \tag{1.15}$$ result

In particular, if u_1 *vanishes in* Ω, *then* $u_1\equiv 0$.

Remark 1.6. *The well-posedness of the initial IPBVP* (1.12) *has been studied in* [26], [27] *and* [32].

Next, we consider the Intermediate Long Wave (ILW) equation

$$\partial_t u - \mathcal{L}_\delta \partial_x^2 u + \frac{1}{\delta}\partial_x u + u\partial_x u = 0, \qquad (x,t) \in \mathbb{R}\times\mathbb{R}, \tag{1.16}$$ ILW

where $u = u(x,t)$ is a real-valued function, $\delta > 0$ and

$$\mathcal{L}_\delta f(x) := -\frac{1}{2\delta}\,\text{p.v.}\int \coth\left(\frac{\pi(x-y)}{2\delta}\right) f(y)\,dy. \tag{1.17}$$ T

Note that $\mathcal{L}_\delta$ is a multiplier operator with $\partial_x \mathcal{L}_\delta$ having symbol

$$\sigma(\partial_x\mathcal{L}_\delta) = \widehat{\partial_x\mathcal{L}_\delta} = 2\pi\xi\coth(2\pi\delta\xi). \tag{1.18}$$ symbol

The ILW equation (1.16) describes long internal gravity waves in a stratified fluid with finite depth represented by the parameter δ, see [24], [14], [15].

Also, the ILW equation has been proven to be complete integrable, see [22] and [23].

In [1] it was proven that solutions of the ILW as $\delta \to \infty$ (deep-water limit) converge to solutions of the BO equation with the same initial data.

Also, in [1] it was shown that if $u_\delta(x,t)$ denotes the solution of the ILW equation (1.16), then

$$v_\delta(x,t) = \frac{3}{\delta}\, u_\delta\big(x, \frac{3}{\delta}t\big) \tag{1.19}$$ scaleKdV

converges as $\delta \to 0$ (shallow-water limit) to the solution of the KdV equation, i.e. (1.7) with $k = 2$, with the same initial data.

For further comments on general properties of the ILW equation we refer to the recent survey [37] and references therein.

The well-posedness of the IVP associated to the ILW equation (1.16) was studied in [1] and more recently in [34].

Our next theorem extends the result in Theorem 1.1 to solution of the IVP associated to the ILW(1.16):

TH5 **Theorem 1.7.** *Let $u_1,\ u_2$ be solutions to* (1.16) *in $(x,t) \in \mathbb{R}\times[0,T]$ such that*

$$u_1,\ u_2 \in C([0,T]:H^s(\mathbb{R})) \cap C^1((0,T):H^{s-2}(\mathbb{R})), \quad s > 5/2. \tag{1.20}$$ m1a

If there exists an open set $\Omega \subset \mathbb{R}\times[0,T]$ such that

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \Omega, \tag{1.21}$$ H1

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.22}$$ result2

In particular, if u_1 *vanishes in* Ω*, then* $u_1 \equiv 0$.

Remark 1.8. *The observations in (i) and (v) in Remark 1.2 and (ii) in Remark 1.4 apply, after some simple modifications, to the ILW equation* (1.16)*.*

Next, we present the following slight improvement of Theorem 1.1 and Theorem 1.5 :

TH10 **Theorem 1.9.** *Let* $u_1,\ u_2$ *be solutions to* (1.1) *in* $(x,t) \in \mathbb{R} \times [0,T]$ *such that*

$$u_1,\ u_2 \in C([0,T] : H^s(\mathbb{R})) \cap C^1((0,T) : H^{s-2}(\mathbb{R})), \quad s > 5/2. \tag{1.23}$$ m10

If there exists an open set $I \subset \mathbb{R},\ 0 \in I$ *such that*

$$u_1(x,0) = u_2(x,0), \qquad x \in I, \tag{1.24}$$ H10

and for each $N \in \mathbb{Z}^+$

$$\int_{|x|\le R} |\partial_t u_1(x,0) - \partial_t u_2(x,0)|^2 dx \le c_N\, R^N \qquad as \qquad R \downarrow 0, \tag{1.25}$$ H15

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{1.26}$$ result11

TH11 **Theorem 1.10.** *Let* $u_1,\ u_2$ *be solutions of the IPBVP* (1.12) *in* $(x,t) \in \mathbb{S}^1 \times [0,T] \simeq \mathbb{R}/\mathbb{Z} \times [0,T]$ *such that*

$$u_1,\ u_2 \in C([0,T] : H^s(\mathbb{S}^1)) \cap C^1((0,T) : H^{s-2}(\mathbb{S}^1)),\ s > 5/2. \tag{1.27}$$ m22

If there exists an open set $I \subset [-1/2, 1/2]$ *with* $0 \in I$ *such that*

$$u_1(x,0) = u_2(x,0), \quad x \in I, \tag{1.28}$$ HB12

and for each $N \in \mathbb{Z}^+$

$$\int_{|x|\le R} |\partial_t u_1(x,0) - \partial_t u_2(x,0)|^2 d\theta \le c_N\, R^N \qquad as \qquad R \downarrow 0, \tag{1.29}$$ H20

then,

$$u_1(x,t) = u_2(x,t), \quad (x,t) \in \mathbb{S}^1 \times [0,T]. \tag{1.30}$$ result23

Remark 1.11. *It will be clear form our proof of Theorem 1.9 that a similar argument provides the proof of Theorem 1.10 which will be omitted.*

The rest of this paper is organized as follows : section 2 contains some preliminary estimates required for Theorem 1.1 as well as its proof. It also includes the modification needed to extend the argument in the proof of Theorem 1.1 from the IVP to the IPBVP to prove Theorem 1.5. Section 3 contains the proof of Theorem 1.7, and section 4 consists of the proof of Theorem 1.9.

2. Proof of Theorem 1.1

To prove Theorem 1.1 we need the following result from complex analysis whose proof follows directly from Schwarz reflection principle:

pro1 **Proposition 2.1.** *Let $I \subseteq \mathbb{R}$ be an open interval, $b \in (0, \infty]$ and*

$$D_b = \{z = x + iy \in \mathbb{C} : 0 < y < b\}, \quad L = \{x + i0 \in \mathbb{C} : x \in I\}. \tag{2.1}$$ sets

Let $F : D_b \cup L \to \mathbb{C}$ be a continuous function such that $F\big|_{D_b}$ is analytic. If $F\big|_L \equiv 0$, then $F \equiv 0$.

As a consequence we have

col1 **Corollary 2.2.** *Let $f \in H^s(\mathbb{R})$, $s > 1/2$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$ such that*

$$f(x) = \mathcal{H}f(x) = 0, \qquad \forall\, x \in I,$$

then $f \equiv 0$.

Proof. Denoting $U = U(x, y)$ the harmonic extension of f to the upper half-plane D, one sees that its harmonic conjugate $V = V(x, y)$ has boundary value $V(x, 0) = \mathcal{H}f(x)$ with

$$\widehat{(f + i\mathcal{H}f)}(\xi) = 2\,\chi_{[0,\infty)}(\xi)\,\widehat{f}(\xi), \qquad \widehat{f} \in L^1(\mathbb{R}). \tag{2.2}$$ HT

Thus, $F := U + iV$ is continuous on $\overline{D}_\infty$ and analytic on D_∞ with $F\big|_L \equiv 0$. Hence, Proposition 2.1 yields the desired result

□

Proof of Theorem 1.1 . Defining $w(x, t) = (u_1 - u_2)(x, t)$ one has that

$$\partial_t w - \mathcal{H}\partial_x^2 w + \partial_x u_2\, w + u_1\, \partial_x w = 0, \quad (x, t) \in \mathbb{R} \times [0, T]. \tag{2.3}$$ eq1

By hypotheses (1.3) and (1.21) there exist open intervals I, $J \subset \mathbb{R}$ such that

$$\begin{aligned} w(x, t) &= \partial_x w(x, t) \\ &= \partial_t w(x, t) = \partial_x^2 w(x, t) = 0, \qquad (x, t) \in I \times J \subset \Omega. \end{aligned} \tag{2.4}$$ zeros1

Thus, the equation (2.3) tells us

$$\mathcal{H}\partial_x^2 w(x, t) = 0, \quad (x, t) \in I \times J \subset \Omega. \tag{2.5}$$ zeros2

Combining (2.4) and (2.5) and fixing $t^* \in J$ it follows that

$$\partial_x^2 w(x,t^*) = \mathcal{H}\partial_x^2 w(x,t^*) = 0, \quad x \in I, \tag{2.6}$$ zeros3

with $\partial_x^2 w(\cdot,t^*),\ \mathcal{H}\partial_x^2 w(\cdot,t^*) \in H^s(\mathbb{R}),\ s > 1/2$.

Therefore, using Corollary 2.2 one has that $\partial_x^2 w(\cdot,t^*) \equiv 0$ which implies that $w(\cdot,t^*) \equiv 0$ and completes the proof.

□

To extend the previous argument to prove Theorem 1.5 we need the following result from complex analysis :

pro2 **Proposition 2.3.** *Let $J \subset [-\pi,\pi]$ be an open non-empty interval and*

$B_1(0) = \{z = x+iy \in \mathbb{C} : |z| < 1\},\ \ A = \{z \in \mathbb{C} : |z| = 1,\ \arg(z) \in J\}$.

Let $F : B_1(0) \cup A \to \mathbb{C}$ be a continuous function such that $F\big|_{B_1(0)}$ is analytic.

If $F\big|_A \equiv 0$, then $F \equiv 0$.

Proof. The proof follows from Proposition 2.1 by considering $F_o T(z)$ where T is a fractional linear transformation mapping the upper half-plane to the unit disk $B_1(0)$.

□

3. Proof of Theorem 1.7

First, we shall prove the following result :

col11 **Corollary 3.1.** *Let $f \in H^s(\mathbb{R})$, $s > 3/2$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$ such that*

$$f(x) = \mathcal{L}_\delta \partial_x f(x) = 0, \qquad \forall\, x \in I,$$

with $\mathcal{L}_\delta$ as in (1.17), (1.18), *then $f \equiv 0$.*

Proof. We define

$$F(x) = \partial_x f(x) + i\mathcal{L}_\delta \partial_x f(x), \quad x \in \mathbb{R}, \tag{3.1}$$ a1

and consider its Fourier transform

$$\begin{aligned}
\widehat{F}(\xi) &= (\partial_x f \widehat{+ i\mathcal{L}_\delta} \partial_x f)(\xi)\\
&= 2\pi i\xi(1+\coth(2\pi\delta\xi))\,\widehat{\mathrm{f}}(\xi)\\
&= 2\pi i\xi\Big(1 + \frac{e^{2\pi\delta\xi} + e^{-2\pi\delta\xi}}{e^{2\pi\delta\xi} - e^{-2\pi\delta\xi}}\Big)\,\widehat{f}(\xi)\\
&= -4\pi i\xi\,\frac{e^{4\pi\delta\xi}}{1 - e^{4\pi\delta\xi}}\,\widehat{f}(\xi)
\end{aligned} \tag{3.2}$$ a2

We observe that by considering $\partial_x f$ with $f \in H^s(\mathbb{R})$, $s > 3/2$, one cancels the singularity of F at $\xi = 0$ introduced by $\coth(\xi)$.

By hypothesis and (3.2) one concludes that $\widehat{F} \in L^1(\mathbb{R})$ and has exponential decay for $\xi < 0$. Hence,

$$F(x) = \int_{-\infty}^{\infty} e^{2\pi i \xi x}\, \widehat{F}(\xi)\, d\xi \tag{3.3}$$ a3

has an analytic extension

$$F(x+iy) = \int_{-\infty}^{\infty} e^{2\pi i \xi (x+iy)}\, \widehat{F}(\xi)\, d\xi \tag{3.4}$$ a4

to the strip

$$D_{2\delta} = \{z = x+iy \in \mathbb{C} \,:\, 0 < y < 2\delta\}$$

with F continuous on

$$\{z = x+iy \,:\, 0 \leq y < 2\delta\}$$

from the hypothesis on f. Now, Proposition 2.1 leads the desired result.

□

Proof of Theorem 1.7. Once Corollary 3.1 is available the proof of Theorem 1.7 is similar to that given for Theorem 1.1, therefore it will be omitted.

□

4. Proof of Theorem 1.9

To prove Theorem 1.9 we need an auxiliary lemma:

lemma1 **Lemma 4.1.** *Let $f \in L^2(\mathbb{R})$ be a real valued function. If there exists an open set $I \subset \mathbb{R}$, $0 \in I$, such that*

$$f(x,0) = 0, \qquad x \in I, \tag{4.1}$$ H12

and for each $N \in \mathbb{Z}^+$

$$\int_{|x|\leq R} |\mathcal{H}f(x)|^2 dx \leq c_N\, R^N \qquad \text{as} \qquad R \downarrow 0, \tag{4.2}$$ H11

then,

$$f(x) = 0, \qquad x \in \mathbb{R}. \tag{4.3}$$ result10

Proof. Consider the analytic function $F = F(x+iy)$ defined in $\mathbb{R} \times (0,\infty)$ with boundary values

$$F(x+i0) = -\mathcal{H}f(x) + i f(x).$$

Since $F\big|_I$ is real we can use Schwarz reflexion principle to find $\widetilde{F}$ analytic in $I \times (-\infty, \infty)$ with $\widetilde{F} = F$ on $I \times [0, \infty)$.

We observe : $\Re\, \widetilde{F}(x + i0) = \mathcal{H}f(x),\ x \in I$ with $\mathcal{H}f\big|_I \in C^{\infty}$, by the support property of f, and by assumption (4.2) $\partial_x^j \mathcal{H}f(0) = 0$, $j \in \mathbb{Z}^+ \cup \{0\}$. Hence

$$\frac{\partial^j}{\partial z^j}\, \widetilde{F}(0,0) = 0 \qquad j = 0, 1, 2, ...$$

which completes the proof. □

Proof of Theorem 1.9. Defining $w(x,t) = (u_1 - u_2)(x,t)$ it follows that

$$\partial_t w - \mathcal{H}\partial_x^2 w + \partial_x u_1\, w + u_2\, \partial_x w = 0, \quad (x,t) \in \mathbb{R} \times [0,T]. \tag{4.4}$$ eq11

Since $w(x,0) = 0,\ x \in I$, one has that $\partial_x^j w(x,0) = 0,\ x \in I$, $j \in \mathbb{Z}^+ \cup \{0\}$, and using (4.4)

$$\mathcal{H}\partial_x^2 w(x,0) = \partial_t w(x,0)$$

We now apply the hypothesis (4.2) and Lemma 4.1 to conclude that $\partial_x^2 w(x,0) = 0,\ x \in \mathbb{R}$.

□

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