Multiplicity of singularities is not a bi-Lipschitz invariant

Lev Birbrair[1], Alexandre Fernandes[2], J. Edson Sampaio, Misha Verbitsky[3]

Abstract. It was conjectured that multiplicity of a singularity is bi-Lipschitz invariant. We disprove this conjecture, constructing examples of bi-Lipschitz equivalent complex algebraic singularities with different values of multiplicity.

Contents

1 Introduction

The famous multiplicity conjecture, stated by Zariski in 1971 (see [Z2]), is formulated as follows: if two germs of complex analytic hypersurfaces are ambient topolological equivalent, then they have the same multiplicity. It was proved by Zariski [Z1] for germs of plane analytic curves. The results of Pham-Teisser in [PT] show that this result can be extended in the following "metric" way: if the two germs of complex analytic curves in n-dimensional space are bi-Lipschitz equivalent with respect to the outer metric, then the germs of the space curves have the same multiplicity. Comte in [Co] proves that the multiplicity of complex analytic germs (not necessarily codimension 1 sets) is invariant under bi-Lipschitz homeomorphism with Lipschitz constants close enough to 1 (this is a severe assumption). These results motivated the following question, closely related to the multiplicity conjecture:

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is the multiplicity of a germ of analytic singularity a bi-Lipschitz invariant? This question was stated as a conjecture in [BFS].

The Lipschitz Regularity Theorem in [BFLS] shows that if the multiplicity of a complex analytic germ is equal to one, then it is a bi-Lipschitz invariant. Namely, if a germs of an analytic set is bi-Lipschitz equivalent to a smooth germ, then it is smooth itself. Later, Fernandes and Sampaio in [FS] give a positive answer to this question for surfaces in 3-dimensional space with respect to the ambient bi-Lipschitz equivalence. More recently, Neumann and Pichon ([NP]) showed that the multiplicity is an invariant under bi-Lipschitz equivalence, for germs of normal surface singularities. Another important result in [FS] is the following: in order to prove (or disprove) the bi-Lipschitz invariance of the multiplicity, it is enough to prove it for the algebraic cones, i.e. for the algebraic sets, defined by homogeneous polynomials. In [BFS] the authors show that the conjecture has a positive answer for 1 or 2 dimensional complex analytic sets.

The present paper shows that the multiplicity of complex algebraic sets is not a bi-Lipschitz invariant for the sets of dimension bigger or equal to three. Moreover, we show that there exists an infinite family of $3-$dimensional germs, such that all of them are bi-Lipschitz equivalent, but they have different multiplicities. The idea of the construction is to consider the complex cones over different embeddings of $\mathbb{C}P^1 \times \mathbb{C}P^1$ to complex projective spaces. Using the topology of Smale-Barden manifolds, we show that all the links of such singularities are diffeomorphic. That is why the germs of the corresponding cones are bi-Lipschitz equivalent. From the other hand, the multiplicities of the cones at the origin may be explicitly calculated in terms of the embeddings.

2 Smale-Barden manifolds

The classification of 5-manifolds is due to S. Smale ([S]) and D. Barden ([B]).

Definition 2.1: A simply connected, compact, oriented 5-manifold is called **Smale-Barden manifold**.

The Smale-Barden manifolds are uniquely determined by their second Stiefel-Whitney class and the linking form.

Theorem 2.2: ([B]) Let X, X' be two Smale-Barden manifolds. Assume that $H^2(X) = H^2(X')$ and this isomorphism is compatible with the linking form and preserves the second Stiefel-Whitney class. Then X is diffeomorphic to X'. ■

Corollary 2.3: There exists only two Smale-Barden manifolds M with $H^2(M) = \mathbb{Z}$, the product $S^2 \times S^3$ and the total space of a non-trivial S^3-bundle over S^2 (see [C] for an introduction to Barden theory, where this manifold is formally introduced).

Proof: Indeed, the linking form on $\mathbb{Z}$ vanishes, therefore the manifold is uniquely determined by the Stiefel-Whitney class $w_2(M)$. Hence we have only two possibilities: $w_2(M) = 0$ and $w_2(M) \neq 0$. ■

In early 2000-ies, the classification of 5-manifolds attracted interest coming from algebraic geometry, in the context of Sasakian geometry and geometry of generalized Seifert manifolds ([K1], [K2]). In the present paper we are interested in S^1-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Proposition 2.4: Let $\pi : M \longrightarrow B$ be a simply connected 5-manifold obtained as a total space of an S^1-bundle L over $B = \mathbb{C}P^1 \times \mathbb{C}P^1$. Then $H^2(M)$ is torsion-free, and M is diffeomorphic to $S^2 \times S^3$.

Proof. Step 1: Universal coefficients formula gives an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_1(M;\mathbb{Z}),\mathbb{Z}) \to H^2(M;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(M;\mathbb{Z}),\mathbb{Z}) \to 0.$$

This implies that $H^2(M;\mathbb{Z})$ is torsion-free.

Step 2: Consider the following exact sequence of homotopy groups

$$0 \longrightarrow \pi_2(M) \longrightarrow \pi_2(B) \xrightarrow{\ \phi\ } \pi_1(S^1) \longrightarrow \pi_1(M) \longrightarrow 0$$

Since $\pi_1(M) = 0$, the map ϕ, representing the first Chern class of L, is surjective. This exact sequence becomes

$$0 \longrightarrow \pi_2(M) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

giving $\pi_2(M) = \mathbb{Z}$, and $H^2(M) = \mathbb{Z}$ because $H^2(M)$ is torsion-free.

Step 3: To deduce Proposition 2.4 from the Smale-Barden classification, it remains to show that $w_2(M) = 0$. However, $w_2(M) = \pi^*(\omega_2(B))$ ([K2, Lemma 36]), and the latter clearly vanishes, because $w_2(S^2) = 0$. ■

3 Multiplicity of homogeneous singularities

Given a projective variety $X \subset \mathbb{C}P^n$, the **projective cone** of X is the union of all 1-dimensional subspaces $l \subset \mathbb{C}^{n+1}$ such that l, interpreted as a point in $\mathbb{C}P^n$, belongs to X. The **link** of $C(X)$ is an intersection of $C(X)$ with a unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$.

Let A be complex algebraic set of $\mathbb{C}^{n+1}$ and $x \in A$. The **multiplicity of** A **at** x, denoted by $mult(A, x)$, is defined to be the multiplicity of the maximal ideal of the local ring $\mathcal{O}_{A,x}$. Given a projective variety $X \subset \mathbb{C}P^n$, we see that multiplicity of the projective cone $C(X)$ at the origin $0 \in \mathbb{C}^{n+1}$ coincides with degree of X (see [Ch], subsection 11.3).

Next, we shall be interested in the following geometric situation. Let $X \subset \mathbb{C}P^n$ be a variety isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. Then the Picard group of $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ is isomorphic to $\mathbb{Z}^2$, and a line bundle is determined by its *bidegree*. We shall denote a line bundle of bidegree a, b by $\mathcal{O}(a, b)$.

Proposition 3.1: Let $X \subset \mathbb{C}P^n$ be a variety isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, and $S \subset C(X)$ be its link. Assume that and $\mathcal{O}(1)\big|_X = \mathcal{O}(a, b)$. Then X has degree $2ab$. If, in addition, a and b are relatively prime, the link of $C(X)$ is diffeomorphic to $S^2 \times S^3$.

Proof. Step 1: Since $c_1(\mathcal{O}(a, b))^2 = 2ab$, and degree of a subvariety $X \subset \mathbb{C}P^n$ is its intersection with the top power of $\mathcal{O}(1)\big|_X$, one has $\deg X = 2ab$.

Step 2: Consider the homotopy exact sequence

$$0 \longrightarrow \pi_2(S) \longrightarrow \pi_2(X) \xrightarrow{\ \phi\ } \pi_1(S^1) \longrightarrow \pi_1(S) \longrightarrow \pi_1(X) \longrightarrow 0$$

for the circle bundle $\pi : S \longrightarrow X$. Since the map ϕ represents the first Chern class of $\mathcal{O}(1)\big|_X$, it is obtained as a quotient of $\mathbb{Z}^2$ by a subgroup generated by (a, b), and this map is surjective because a and b are relatively prime. Then

$\pi_1(S) = \pi_1(X) = 0$, and Proposition 2.4 implies that S is diffeomorphic to $S^2 \times S^3$. ■

4 Lipschitz invariance of singularities

Let $X \subset \mathbb{C}^n$ be a complex variety. The induced metric from the Euclidean distance on $\mathbb{C}^n$ gives a distance on X; it is called **the outer metric** on X.

Definition 4.1: Let $X \subset \mathbb{C}^n$ and $X' \subset \mathbb{C}^{n'}$ be complex varieties equipped with the outer metrics, $x \in X, x' \in X'$ marked points. We say that (X, x) is **bi-Lipschitz equivalent** to (X', x') if there exist a neighborhoods U of x in $\mathbb{C}^n$ and U' of x in $\mathbb{C}^n$, and a bi-Lipschitz homeomorphism of $X \cap U$ to $X' \cap U'$ mapping x to x'.
Definition 4.2: Let $X, X' \subset \mathbb{C}^n$ be complex varieties equipped with the outer metrics, $x \in X, x' \in X'$ marked points. We say that (X, x) is **ambient bi-Lipschitz equivalent** to (X', x') if there exists a bi-Lipschitz equivalence of a neighbourhood U of x in $\mathbb{C}^n$ and a neighbourhood U' of x in $\mathbb{C}^n$ mapping $X \cap U$ to $X' \cap U'$ and x to x'.

Actually, the two definitions above do not coincide. The ambient bi-Lipschitz equivalence implies bi-Lipschitz equivalence, but the examples presented in [BG] show that the converse does not hold true in general.

As it was already mentioned in Introduction, it was conjectured in [BFS] that the multiplicity is a bi-Lipschitz invariant. We prove that this is false. Here is the main result of this paper.

Theorem 4.3: For each $n \geq 3$, there exists a family $\{Y_i\}_{i\in\mathbb{Z}}$ of n-dimensional complex algebraic varieties $Y_i \subset \mathbb{C}^{n_i+1}$ such that:

(a) for each pair $i \neq j$, the germs at the origin of $Y_i \subset \mathbb{C}^{n_i+1}$ and $Y_j \subset \mathbb{C}^{n_j+1}$ are bi-Lipschitz equivalent, but $(Y_i, 0)$ and $(Y_j, 0)$ have different multiplicity.

(b) for each pair $i \neq j$, there are n-dimensional complex algebraic varieties $Z_{ij}, \widetilde{Z}_{ij} \subset \mathbb{C}^{n_i+n_j+2}$ such that $(Z_{ij}, 0)$ and $(\widetilde{Z}_{ij}, 0)$ are ambient bi-Lipschitz equivalent, but $mult(Z_{ij}, 0) = mult(Y_i, 0)$ and $mult(\widetilde{Z}_{ij}, 0) = mult(Y_j, 0)$ and, in particular, they have different multiplicity.

Proof. Let $\{p_i\}_{i\in\mathbb{Z}}$ be the family of odd prime numbers. For each $i \in \mathbb{Z}$, let L_i be a very ample bundle on $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ of bidegree $(2, p_i)$. Let X_i be projective variety obtained by the embedding of the very ample bundle L_i. Consider the link of the singularity $S_i := C(X_i) \cap S^{2m_i+1}$, where S^{2m_i+1} is the unit sphere centered in $0 \in \mathbb{C}^{m_i+1}$. Then, for each pair $i \neq j$ the links S_i, S_j are diffeomorphic to $S^2 \times S^3$ (Proposition 3.1). In particular, S_i to S_j are bi-Lipschitz homeomorphic. Since a bi-Lipschitz map from S_i to S_j induces a bi-Lipschitz map of their cones, then the affine cones $(C(X_i), 0)$ and $(C(X_j), 0)$ are bi-Lipschitz equivalent, but $mult(C(X_i), 0) = 4p_i$ and $mult(C(X_j), 0) = 4p_j$ (Proposition 3.1). Thus, if for each $i \in \mathbb{Z}$ we define $Y_i := C(X_i) \times \mathbb{C}^{n-3}$, then we have that the family $\{Y_i\}_{i\in\mathbb{Z}}$ satisfies the item (a), since $mult(Y_i, 0) = mult(C(X_i), 0) = 4p_i$, for all $i \in \mathbb{Z}$.

Concerning to the item (b), let $\phi_{ij} : Y_i \to Y_j$ be a bi-Lipschitz homeomorphism such that $\phi_{ij}(0) = 0$. Let $\widetilde{\phi_{ij}} : \mathbb{C}^{n_i+1} \to \mathbb{C}^{n_j+1}$ (resp. $\widetilde{\psi_{ij}} : \mathbb{C}^{n_j+1} \to \mathbb{C}^{n_i+1}$) be a Lipschitz extension of ϕ_{ij} (resp. $\psi_{ij} = \phi_{ij}^{-1}$) (see [Ki], [M] and [W]). Let us define $\varphi, \psi : \mathbb{C}^{n_i+1} \times \mathbb{C}^{n_j+1} \to \mathbb{C}^{n_i+1} \times \mathbb{C}^{n_j+1}$ as follows:

$$\varphi(x, y) = (x - \widetilde{\psi_{ij}}(y + \widetilde{\phi_{ij}}(x)), y + \widetilde{\phi_{ij}}(x))$$

and

$$\psi(z, w) = (z + \widetilde{\psi_{ij}}(w), w - \widetilde{\phi_{ij}}(z + \widetilde{\psi_{ij}}(w))).$$

It easy to verify that $\psi = \varphi^{-1}$ and since φ and ψ are composition of Lipschitz maps, they are also Lipschitz maps. Moreover, if $Z_{ij} = Y_i \times \{0\}$ and $\widetilde{Z}_{ij} = \{0\} \times Y_j$, we obtain that $\varphi(Z_{ij}) = \widetilde{Z}_{ij}$ (see [Sa]). Therefore, $(Z_{ij}, 0)$ and $(\widetilde{Z}_{ij}, 0)$ are bi-Lipschitz equivalent, but $mult(Z_{ij}, 0) = mult(Y_i, 0)$ and $mult(\widetilde{Z}_{ij}, 0) = mult(Y_j, 0)$ and, in particular, they have different multiplicity. ∎

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Misha Verbitsky
Instituto Nacional de Matemática Pura e Aplicada (IMPA)
Estrada Dona Castorina, 110
Jardim Botânico, CEP 22460-320
Rio de Janeiro, RJ - Brasil
also:
Laboratory of Algebraic Geometry,
National Research University Higher School of Economics,
Department of Mathematics, 6 Usacheva street, Moscow, Russia.

Lev Bribrair, Alexandre Fernandes and J. Edson Sampaio
Departamento de Matemática, Universidade Federal do Ceará,
Rua Campus do Pici, s/n, Bloco 914
Pici, CEP 60440-900
Fortaleza-CE, Brazil