MULTIPLICITY, REGULARITY AND BLOW-SPHERICAL EQUIVALENCE OF COMPLEX ANALYTIC SETS

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ABSTRACT. This paper is devoted to study multiplicity and regularity of complex analytic sets. We present an equivalence for complex analytical sets, named blow-spherical equivalence and we obtain several applications with this new approach. For example, we reduce to homogeneous complex algebraic sets a version of Zariski's multiplicity conjecture in the case of blow-spherical homeomorphism, we give some partial answers to the Zariski's multiplicity conjecture, we show that a blow-spherical regular complex analytic set is smooth and we give a complete classification of the complex analytic curves.

1. INTRODUCTION

Recently, L. Birbrair, A. Fernandes and V. Grandjean in [7] (see also [6]) defined a new equivalence, named blow-spherical equivalence, to study some properties of subanalytic sets such as, for example, to generalize thick-thin decomposition of normal complex surface singularity germs introduced in [14]. With the aim to study multiplicity and regularity as well as to present some classifications of complex analytic sets, we define a weaker variation of the equivalence presented in [7], namely also blow-spherical equivalence. Roughly speaking, two subset germs of Euclidean spaces are called blow-spherical equivalent, if their spherical modifications are homeomorphic and, in particular, this homeomorphism induces a homeomorphism between their tangent links. This equivalence, essentially, lives between topological equivalence and subanalytic bi-Lipschitz equivalence. We obtain several applications with this new approach, for example, we get a reduction for a version of Zariski's multiplicity conjecture in the case of blow-spherical homeomorphism, we have that blow-spherical regular complex analytic sets are smooth and we obtain a complete classification to complex analytic curves.

The main motivation to study about multiplicity comes from the following problem proposed, in 1971, by O. Zariski (see [69]):

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Question A Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions. If there is a homeomorphism $\varphi : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, then is it $m(V(f), 0) = m(V(g), 0)$?

More than 45 years later, the question above is still an open problem. However, there exist some partial results about it, for example, R. Ephraim in [21] and D. Trotman in [63] showed that the multiplicity is a C^1 invariant. O. Saeki in [57] and Stephen Yau in [65] showed that the multiplicity is an invariant of the embedded topology in the case of isolated quasi-homogeneous surfaces, Greuel in [33] and O'Shea in [53] showed that Question A has a positive answer in the case of quasi-homogeneous hypersurface families with isolated singularities. Several other authors showed partial results about versions of the Zariski's multiplicity conjecture, more recently, we can cite [1], [24], [26] [27], [51], [54], [58], [60], [61], [62], [64] and [66]. In order to know more about this conjecture see the survey [23].

More recently, the author joint with A. Fernandes, in the paper [29], proved that the multiplicity of a complex analytic surface in $\mathbb{C}^3$ is a bi-Lipschitz invariant and W. Neumann and A. Pichon in [52] showed that the multiplicity is a bi-Lipschitz invariant in the case of normal complex analytic surfaces. The bi-Lipschitz invariance of the multiplicity is an advance about a problem that has been extensively studied in recent years, the complex analytic surface classification. The work of L. Birbrair, W. Neumann and A. Pichon in [14] is the most recent significant result on the Lipschitz geometry of singularities about classification of complex analytic surfaces, more specifically: they presented a classification for surfaces with the intrinsic metric module bi-Lipschitz homeomorphisms. More about classification of complex analytic surfaces can be found in [5], [12], [11], [10], [6] and [7]. We have also recent studies about classification of complex analytic curves, see for example [8], [34], [35] and [36].

Another subject of interest for many mathematicians is to know how simple is the topology of complex analytic sets. For example, if topological regularity implies analytic regularity. In general this does not occur, but Mumford in [50] showed a result in this direction, it was stated as follows: *a normal complex algebraic surface with a simply connected link at a point x is smooth in x.* This was a pioneer work in topology of singular algebraic surfaces. From a modern viewpoint this result can be seen as follows: *a topologically regular normal complex algebraic surface is smooth.* Since, in $\mathbb{C}^3$, a surface is normal if and only if it has isolated singularities, the result can be formulated as follows: *a topologically regular complex surface in $\mathbb{C}^3$, with an isolated singularity, is smooth.* The condition of singularity to be isolated is important, since $\{(x, y, z) \in \mathbb{C}^3; y^2 = x^3\}$ is a topologically regular surface, but it is non-smooth. There are also examples of non smooth hypersurfaces in $\mathbb{C}^4$ with topologically regular isolated singularity, for example, E. V. Brieskorn in [15] showed that $\{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4; z_0^3 = z_1^2 + z_2^2 + z_3^2\}$ is a topologically regular hypersurface, but it is non-smooth, as well. However, N. A'Campo in [2] and Lê D. T. in [43] showed that if X is a complex analytic hypersurface in $\mathbb{C}^n$ such that X is a topological submanifold, then X is smooth. Recently, the author in [59] (see also [9]), proved a version of the Mumford's Theorem, showing that Lipschitz regularity in complex analytic sets implies smoothness.

In this paper, we deal with blow-spherical aspects related to the above Zariski's question. More precisely, we consider the questions below:

Question A1. Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets. If X and Y are blow-spherical homeomorphic, then is it $m(X,0) = m(Y,0)$?

Question A2. Let $X, Y \subset \mathbb{C}^n$ be two homogeneous complex algebraic sets. If X and Y are blow-spherical homeomorphic, then is it $m(X,0) = m(Y,0)$?

In Section 3 we define blow-spherical equivalence and we present some properties of this equivalence. In Subsection 3.1, we present some examples of blow-spherical equivalences and in the Subsection 3.2 we show, for example, that the blow-spherical equivalence is different of the topological, intrinsic bi-Lipschitz and bi-Lipschitz equivalences. The others sections are devoted for applications of the results of the Section 3.

In Section 4, we prove a version of the Mumford's Theorem. Namely, we show that if a complex analytic set X is blow-spherical regular, then X is smooth. No restriction on the dimension or co-dimension is needed. No restriction of singularity to be isolated is needed. As an application, we obtain the main result of [9], about Lipschitz regularity of complex analytic sets.

The Section 5 is dedicated for studies about the invariance of the multiplicity. In the Subsection 5.1, we prove Theorem 5.1 that says: The Question A1 has a positive answer if and only if the Question A2 has a positive answer and as a consequence we answer the Questions A1 and A2. In the other subsections, we study those questions in some specific cases. In Subsection 5.2, Theorem 5.5 shows that Question A2 has positive answer for hypersurface singularities whose irreducible components have singular sets with dimension ≤ 1. In particular, in Corollary 5.6 we prove the blow-spherical invariance of the multiplicity of complex analytic surface (not necessarily isolated) singularities in $\mathbb{C}^3$. Moreover, in Subsection 5.3, we prove that the Question A2 has a positive answer in the case of aligned singularities and in the Subsection 5.4, we prove that the Question A2 has a positive answer in the case of families of hypersurfaces.

Finally, in Section 6, we present a complete classification for complex analytic curves in $\mathbb{C}^n$.

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2. Preliminaries

Let $f\colon (\mathbb{C}^n,0)\to(\mathbb{C},0)$ be the germ of a reduced analytic function at the origin with $f\not\equiv 0$. Let $(V(f),0)$ be the germ of the zero set of f at the origin. We recall *the multiplicity of* $V(f)$ at the origin, denoted by $m(V(f),0)$, is defined as following: we write

$$f=f_m+f_{m+1}+\cdots+f_k+\cdots$$

where each f_k is a homogeneous polynomial of degree k and $f_m\neq 0$ and, thus, we denote f_m as $\mathbf{in}(f)$. We define $m(V(f),0):=m$ (see, for example, [18] for a definition of multiplicity in high co-dimension).

Definition 2.1. *Let $A\subset\mathbb{R}^n$ be a subanalytic set such that $x_0\in\overline{A}$. We say that $v\in\mathbb{R}^n$ is a tangent vector of A at $x_0\in\mathbb{R}^n$ if there is a sequence of points $\{x_i\}\subset A\setminus\{x_0\}$ tending to $x_0\in\mathbb{R}^n$ and there is a sequence of positive numbers $\{t_i\}\subset\mathbb{R}^+$ such that*

$$\lim_{i\to\infty}\frac{1}{t_i}(x_i-x_0)=v.$$

Let $C(A,x_0)$ denote the set of all tangent vectors of A at $x_0\in\mathbb{R}^n$. We call $C(A,x_0)$ the **tangent cone** *of A at x_0.*

Notice that $C(A,x_0)$ is the cone $C_3(A,x_0)$ as defined by Whitney (see [68]).

Remark 2.2. Follows from the curve selection lemma for subanalytic sets that, if $A\subset\mathbb{R}^n$ is a subanalytic set and $x_0\in\overline{A}$ is a non-isolated point, then

$$C(A,x_0)=\{v;\ \exists\,\alpha:[0,\varepsilon)\to\mathbb{R}^n \ s.t.\ \alpha(0)=x_0,\ \alpha((0,\varepsilon))\subset A \ and \\ \alpha(t)-x_0=tv+o(t)\}.$$

Remark 2.3. *If $A\subset\mathbb{C}^n$ is a complex analytic set such that $x_0\in A$ then $C(A,x_0)$ is the zero set of a set of homogeneous polynomials (See* [68]*, Chapter 7, Theorem 4D). In particular, $C(A,x_0)$ is a union of complex line passing through at the origin.*

Another way to present the tangent cone of a subset $X\subset\mathbb{R}^n$ at the origin $0\in\mathbb{R}^n$ is via the spherical blow-up of $\mathbb{R}^n$ at the point 0. Let us consider the **spherical blowing-up** (at the origin) of $\mathbb{R}^n$

$$\begin{array}{rccc}\rho_n\colon & \mathbb{S}^{n-1}\times[0,+\infty) & \longrightarrow & \mathbb{R}^n\\ & (x,r) & \longmapsto & rx\end{array}$$

Note that $\rho_n\colon\mathbb{S}^{n-1}\times(0,+\infty)\to\mathbb{R}^n\setminus\{0\}$ is a homeomorphism with inverse mapping $\rho_n^{-1}\colon\mathbb{R}^n\setminus\{0\}\to\mathbb{S}^{n-1}\times(0,+\infty)$ given by $\rho_n^{-1}(x)=(\frac{x}{\|x\|},\|x\|)$. For a subset $X\subset\mathbb{R}^n$, we define the **strict transform** of X (under the spherical blowing-up ρ_n) to be $X':=\rho_n^{-1}(X\setminus\{0\})\cup\partial X'$, where $\partial X':=\overline{\rho_n^{-1}(X\setminus\{0\})}\cap(\mathbb{S}^{n-1}\times\{0\})$.

Remark 2.4. $\partial X' = \mathbb{S}_0 X \times \{0\}$, where $\mathbb{S}_0 X = C(X,0) \cap \mathbb{S}^{n-1}$.

Definition 2.5. *Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in X$. We say that $x \in \partial X'$ is* **simple point of** $\partial X'$, *if there is an open $U \subset \mathbb{R}^{n+1}$ with $x \in U$ such that:*

a) the connected components of $(X' \cap U) \setminus \partial X'$, say $X_1, ..., X_r$, are topological manifolds with $\dim X_i = \dim X$, for all $i = 1, ..., r$;

b) $(X_i \cup \partial X') \cap U$ is a topological manifold with boundary, for all $i = 1, ..., r$;.

Let $Smp(\partial X')$ be the set of all simple points of $\partial X'$.

Since for each connected component C_j of $Smp(\partial X')$ the number of connected components of the germ $(\rho^{-1}(X \setminus \{0\}), x)$ does not depend on $x \in C_j$, the following definition is well posed.

Definition 2.6. *Let $X \subset \mathbb{R}^n$ be a subanalytic set such that $0 \in X$. For each connected component C_j of $Smp(\partial X')$, we define $k_X(C_j)$ to be the number of components of the germ $(\rho^{-1}(X \setminus \{0\}), x)$, for some $x \in C_j \cap Smp(\partial X')$. Moreover, when X is a complex analytic set, there is a complex analytic set σ with $\dim \sigma < \dim X$, such that $X_j \setminus \sigma$ intersect only one connected component C_i (see* [18]*, pp. 132-133), for each irreducible component X_j of tangent cone $C(X,0)$, then we define also $k_X(X_j) := k_X(C_i)$.*

For more about subanalytic sets, see, for example, [4].

Remark 2.7. The number $k_X(C_j)$ equals the n_j defined by Kurdyka e Raby [42], pp. 762 and also equals the k_j defined by Chirka in [18], pp. 132-133, in the case that X is a complex analytic set.

We remind also a very useful result proved by Y.-N. Gau and J. Lipman in the paper [32].

Lemma 2.8 ([32], p. 172, Lemma)**.** *Let $\varphi : A \to B$ be a homeomorphism between two complex analytic sets. If X is an irreducible component of A, then $\varphi(X)$ is an irreducible component of B.*

Remark 2.9. All the sets considered in the paper are supposed to be equipped with the Euclidean metric.

3. Blow-spherical equivalence

Definition 3.1. *Let $(X,0)$ and $(Y,0)$ be subsets germs, respectively at the origin of $\mathbb{R}^n$ and $\mathbb{R}^p$.*

- *A continuous mapping $\varphi : (X,0) \to (Y,0)$, with $0 \notin \varphi(X \setminus \{0\})$, is a* **blow-spherical morphism** *(shortened as* **blow-morphism***), if the mapping*

$$\rho_p^{-1} \circ \varphi \circ \rho_n : X' \setminus \partial X' \to Y' \setminus \partial Y'$$

extends as a continuous mapping $\varphi' : X' \to Y'$.

- *A* **blow-spherical homeomorphism** *(shortened as* **blow-isomorphism***) is a blow-morphism* $\varphi : (X,0) \to (Y,0)$ *such that the extension* φ' *is a homeomorphism. In this case, we say that the germs* $(X,0)$ *and* $(Y,0)$ *are* **blow-spherical equivalent** *or* **blow-spherical homeomorphic** *(or* **blow-isomorphic***).*

The authors in [7] (see also [6]) defined blow-spherical homeomorphism with the additional hypotheses that the blow-spherical homeomorphism needs also to be subanalytic.

Remark 3.2. *If* $\varphi\colon (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ *is a blow-spherical homeomorphism then* $\varphi|_X\colon (X,0) \to (\varphi(X),0)$ *is also a blow-spherical homeomorphism for any subset* $X \subset \mathbb{R}^n$ *with* $0 \in X$*.*

When we say that $\varphi\colon (\mathbb{R}^n,X,0) \to (\mathbb{R}^n,Y,0)$ is a blow-spherical homeomorphism, it means that $\varphi\colon (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ is a blow-spherical homeomorphism such that $\varphi(X) = Y$ and, thus, $\varphi|_X\colon (X,0) \to (Y,0)$ is also a blow-spherical homeomorphism.

Proposition 3.3. *If* X *and* Y *are blow-spherical homeomorphic, then* $C(X,0)$ *and* $C(Y,0)$ *are also blow-spherical homeomorphic.*

Proof. Let $\varphi : X \to Y$ be a blow-isomorphism. Then $\varphi'|_{\partial X'} : \partial X' \to \partial Y'$ is a homeomorphism. We define $d_0\varphi : C(X,0) \to C(Y,0)$ by

$$d_0\varphi(x) = \begin{cases} \|x\| \cdot \nu_\varphi(\frac{x}{\|x\|}), & x \neq 0 \\ 0, & x = 0, \end{cases}$$

where $\varphi'(x,0) = (\nu_\varphi(x),0)$. We have that $d_0\varphi$ is a blow-spherical homeomorphism, since $\rho^{-1} \circ d_0\varphi \circ \rho(x,t) = (\nu_\varphi(x),t)$.

□

Theorem 3.4. *Let* $\varphi : (X,0) \to (Y,0)$ *be a blow-spherical homeomorphism. If* $C(X,0) = \bigcup_{j=1}^r X_j$ *and* $C(Y,0) = \bigcup_{j=1}^s Y_j$ *are, respectively, the decomposition in irreducible components of* $C(X,0)$ *and* $C(Y,0)$*, then there exists a bijection* $\sigma\colon \{1,...,r\} \to \{1,...,s\}$ *such that* $k_X(X_j) = k_Y(Y_{\sigma(j)})$ *for* $j = 1,...,r$*.*

Proof. By Proposition 3.3, we have a homeomorphism $d_0\varphi : C(X,0) \to C(Y,0)$. Thus, by Lemma 2.8, there exists a bijection $\sigma\colon \{1,...,r\} \to \{1,...,s\}$ such that $d_0\varphi(X_j) = Y_{\sigma(j)}$ for $j = 1,...,r$.

Fixed $j \in \{1,...,r\}$, let $p \in \mathbb{S}_0 X_j \times \{0\} \subset \partial X'$ be a generic point. Since $\varphi' : X' \to Y'$ is a homeomorphism, we obtain also a homeomorphism $\varphi'|_{\rho^{-1}(X\setminus\{0\})}\colon (X' \setminus \partial X', p) \to (Y' \setminus \partial Y', \varphi'(p))$, showing that $k_X(X_j) = k_Y(Y_{\sigma(j)})$, since $\varphi'(p) \in \mathbb{S}_0 Y_{\sigma(j)} \times \{0\} \subset \partial Y'$. □

Remark 3.5. Let $X \subset \mathbb{C}^n$ be a complex analytic set and $X_1, ..., X_r$ the irreducible components of $C(X,0)$. Then, $m(X,0) = \sum_{j=1}^{r} k_X(X_j)m(X_j,0)$ (see [18], p. 133, proposition).

3.1. **Examples of blow-spherical equivalences.**

Proposition 3.6. *Let $X, Y \subset \mathbb{R}^m$ be two subsets. If $\varphi\colon (\mathbb{R}^m,0) \to (\mathbb{R}^m,0)$ is a homeomorphism such that φ and φ^{-1} are differentiable at the origin and $\varphi(X) = Y$, then φ and $\varphi|_X\colon X \to Y$ are blow-spherical homeomorphism.*

Proof. Observe that $\nu_\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ given by

$$\nu_\varphi(\mathbf{x}) = \frac{D\varphi_0(\mathbf{x})}{\|D\varphi_0(\mathbf{x})\|}$$

is a homeomorphism with inverse

$$(\nu_\varphi)^{-1}(\mathbf{x}) = \frac{D\varphi_0^{-1}(\mathbf{x})}{\|D\varphi_0^{-1}(\mathbf{x})\|}.$$

Using that $\varphi(t\mathbf{x}) = tD\varphi_0(\mathbf{x}) + o(t)$, we obtain

$$\lim_{t\to 0^+} \frac{\varphi(t\mathbf{x})}{\|\varphi(t\mathbf{x})\|} = \frac{D\varphi_0(\mathbf{x})}{\|D\varphi_0(\mathbf{x})\|} = \nu_\varphi(\mathbf{x}).$$

Then the mapping $\varphi' : \mathbb{R}^{m\prime} \to \mathbb{R}^{m\prime}$ given by

$$\varphi'(\mathbf{x},t) = \begin{cases} \left(\frac{\varphi(t\mathbf{x})}{\|\varphi(t\mathbf{x})\|}, \|\varphi(t\mathbf{x})\|\right), & t \neq 0 \\ (\nu_\varphi(\mathbf{x}),0), & t = 0, \end{cases}$$

is a homeomorphism. Therefore, φ is a blow-spherical homeomorphism and by Remark 3.2, $\varphi|_X\colon X \to Y$ is also a blow-spherical homeomorphism. □

We do a slight digression to remind the notion of inner distance on a path connected Euclidean subset.

Let $Z \subset \mathbb{R}^m$ be a path connected subset. Given two points $q, \tilde{q} \in Z$, we define the *inner distance* in Z between q and $\tilde{q}$ by the number $d_Z(q,\tilde{q})$ below:

$$d_Z(q,\tilde{q}) := \inf\{\text{length}(\gamma) \mid \gamma \text{ is an arc on } Z \text{ connecting } q \text{ to } \tilde{q}\}.$$

Definition 3.7. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A mapping $f\colon X \to Y$ is called* **Lipschitz** *(resp.* **intrinsic Lipschitz***) if there exists $\lambda > 0$ such that is $\|f(x_1) - f(x_2)\| \leq \lambda\|x_1 - x_2\|$ (resp. $d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1,x_2)$) for all $x_1, x_2 \in X$. A Lipschitz (resp. intrinsic Lipschitz) mapping $f\colon X \to Y$ is called* **bi-Lipschitz** *(resp.* **intrinsic bi-Lipschitz***) if its inverse mapping exists and is Lipschitz (resp. intrinsic Lipschitz) and, in this case, we say that X and Y are* **bi-Lipschitz equivalent** *(resp.* **intrinsic bi-Lipschitz equivalent***).*

If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, we remind that a mapping $f\colon X \to Y$ is a subanalytic mapping, when its graph is a subanalytic subset of $\mathbb{R}^n \times \mathbb{R}^m$ (see definition of a subanalytic subset in [4]). So, a subanalytic bi-Lipschitz homeomorphism is a bi-Lipschitz homeomorphism such that its graph is a subanalytic subset.

The next proposition shows another example of blow-spherical homeomorphism and it was already proved in [6, Proposition 4.1] and it is a direct consequence of [41, Theorem 2.25], but we also present a proof here.

Proposition 3.8. *Let $X, Y \subset \mathbb{R}^m$ be two subanalytic sets. If $\varphi : (X,0) \to (Y,0)$ is a subanalytic bi-Lipschitz homeomorphism (with respect to the metric induced). Then, φ is a blow-spherical homeomorphism.*

Proof. Observe that $\nu_\varphi : C(X,0) \cap \mathbb{S}^{m-1} \to C(Y,0) \cap \mathbb{S}^{m-1}$ given by

$$\nu_\varphi(\mathbf{x}) = \frac{D_0\varphi(\mathbf{x})}{\|D_0\varphi(\mathbf{x})\|}$$

is a homeomorphism with inverse

$$(\nu_\varphi)^{-1}(\mathbf{x}) = \frac{D_0\varphi^{-1}(\mathbf{x})}{\|D_0\varphi^{-1}(\mathbf{x})\|},$$

where $D_0\varphi : C(X,0) \to C(Y,0)$ (resp. $D_0\varphi^{-1} : C(Y,0) \to C(X,0)$) is the Lipschitz derivative of φ (resp. φ^{-1}) at the origin, as defined by A. Bernig and A. Lytichak in [3]. We know that $D_0\varphi$ is given by

$$D_0\varphi(v) = \lim_{t\to 0^+} \frac{\varphi(\alpha(t))}{t},$$

where $\alpha : [0,\varepsilon) \to X$ is a subanalytic curve such that $\alpha(t) = tv + o(t)$. By McShane-Whitney-Kirszbraun's Theorem (see [48], [67] and [40]), there exists $\Phi : \mathbb{R}^m \to \mathbb{R}$ a Lipschitz mapping such that $\Phi|_X = \varphi$. Moreover, if $v \in C(X,0)$, we have $D_0\varphi(v) = \lim\limits_{t\to 0^+} \frac{\Phi(tv)}{t} = D_0\Phi(v)$, since $\Phi(\alpha(t)) - \Phi(tv) = o(t)$ whenever $\alpha(t) = tv + o(t)$. In particular, $\nu_\varphi(v) = \frac{D_0\Phi(v)}{\|D_0\Phi(v)\|}$. For a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X \setminus \{0\}$ such that $x_n \to 0$ and $\frac{x_n}{\|x_n\|} \to \mathbf{x} \in C(X,0) \cap \mathbb{S}^{m-1}$, we have

$$\nu_\varphi(\mathbf{x}) = \lim_{n\to\infty} \frac{\varphi(x_n)}{\|\varphi(x_n)\|} = \lim_{n\to\infty} \frac{\Phi(x_n)}{\|\Phi(x_n)\|},$$

since $\lim\limits_{n\to\infty} \frac{\Phi(x_n)}{\|\Phi(x_n)\|} = \frac{D_0\Phi(\mathbf{x})}{\|D_0\Phi(\mathbf{x})\|}$. Then the mapping $\varphi' : X' \to Y'$ given by

$$\varphi'(\mathbf{x},t) = \begin{cases} \left(\frac{\varphi(t\mathbf{x})}{\|\varphi(t\mathbf{x})\|}, \|\varphi(t\mathbf{x})\|\right), & t \neq 0 \\ (\nu_\varphi(\mathbf{x}), 0), & t = 0, \end{cases}$$

is a continuous mapping. Using above φ^{-1} instead of φ, we obtain that φ is a blow-spherical homeomorphism. □

3.2. **Blow-spherical equivalence and other equivalences.** Now we give some examples that separate blow-spherical equivalence of others equivalences.

Example 3.9. *$X = \{(z, x_1, x_2, x_3) \in \mathbb{C}^4;\ z^3 = x_1^2 + x_2^2 + x_3^2\}$ and $Y = \{(z, x_1, x_2, x_3) \in \mathbb{C}^4;\ z = 0\}$ are topological equivalent (see [15]). However, by Theorem 4.3, they are not blow-spherical equivalent.*

Example 3.10. *$X=\{(x,y)\in\mathbb{C}^2;\, y^2=x^3\}$ and $Y=\{(x,y)\in\mathbb{C}^2;\, y^2=x^5\}$ are blow-spherical equivalent (see Theorem 6.1), but they are not bi-Lipschitz equivalent (see* [28, Example 2.1]*).*

Example 3.11. *$X=\{(x,y)\in\mathbb{C}^2;\, y^2=x^3\}$ and $Y=\{(x,y)\in\mathbb{C}^2;\, y=0\}$ are intrinsic bi-Lipschitz equivalent, however by Theorem 4.3, they are not blow-spherical equivalent.*

In [41], the authors presented the following definition.

Definition 3.12. *We say that a homeomorphism $h:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ satisfies condition* semiline-*(SSP), if $h(\ell)$ has a unique direction for all semilines ℓ.*

Here, we say that a such h is a **semiline homeomorphism**.

Remark 3.13. *We have some considerations between the definitions of blow-spherical homeomorphism and semiline homeomorphism:*

(i) A semiline homeomorphism have to be defined in some open neighborhood of 0, but we do not ask that in definition 3.1 (the definition of blow-spherical homeomorphism is intrinsic);

(ii) By Theorem 6.1, $X=\{(x,y)\in\mathbb{C}^2; y^2=x^3\}$ and $Y=\{(x,y)\in\mathbb{C}^2; y^2=x^5\}$ are blow-spherical homeomorphic, however there is no semiline homeomorphism $h\colon(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ such that $h(X)=Y$;

(iii) Because the before items, in order to compare these two notions, we need to consider only those blow-spherical homeomorphism $h\colon(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$;

(iv) If $h\colon(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ is a bi-Lipschitz homeomorphism and a semiline homeomorphism (as in Proposition 2.20 in [41]*), then similarly as Proposition 3.8, we obtain that h is a blow-spherical homeomorphism;*

(v) If $h\colon(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ is a blow-spherical homeomorphism, then h is a semiline homeomorphism;

(vi) The converse of Item (v) is not true. Consider $h\colon(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ given by $h(x,y)=(x,y+x^{\frac{2}{3}})$. For each semiline $\ell\subset\mathbb{R}^2$ there exists $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ such that $\ell=\{(tx,ty);t\geq 0\}$. Therefore,

$$\lim_{t\to 0^+}\frac{h(tx,ty)}{\|h(tx,ty)\|}=\begin{cases}(0,1) & ,\ \text{if } x\neq 0\\ (0,\frac{y}{|y|}) & ,\ \text{if } x=0.\end{cases}$$

This implies that $h(\ell)$ has a unique direction and thus h is a semiline homeomorphism. However, it is clear that h is not a blow-spherical homeomorphism.

Thus, we see that these two notions share important similarities, but they are different and, moreover, this paper and paper [41] *approach different problems.*

4. Regularity of complex analytic sets

Definition 4.1. *A subset $X \subset \mathbb{R}^n$ is called* **blow-spherical regular** *at $0 \in X$ if there is an open neighborhood U of 0 in X which is blow-spherical homeomorphic to an Euclidean ball.*

We remember Prill's Theorem proved in [55].

Lemma 4.2 ([55], Theorem)**.** *Let $C \in \mathbb{C}^n$ be a complex cone which is a topological manifold. Then C is a plane in $\mathbb{C}^n$.*

Theorem 4.3. *Let $X \subset \mathbb{C}^n$ be a complex analytic set. If X is blow-spherical regular at $0 \in X$, then $(X,0)$ is smooth.*

Proof. By Proposition 3.3, $C(X,0)$ is blow-spherical homeomorphic to $\mathbb{C}^k$, where $k = \dim X$, then $C(X,0)$ is irreducible, since $\mathbb{C}^k$ is irreducible. In particular, $C(X,0)$ is a topological manifold. By Prill's Theorem, $C(X,0)$ is a plane. Hence, $m(C(X,0),0) = 1$ and by Theorem 3.4, $k_X(C(X,0)) = 1$ and using the Remark 3.5, $m(X,0) = k_X(C(X,0)) \cdot m(C(X,0),0) = 1$. Then, $(X,0)$ is smooth. □

Definition 4.4. *A subset $X \subset \mathbb{R}^n$ is called* **Lipschitz regular** *(respectively* **subanalytically Lipschitz regular***) at $x_0 \in X$ if there is an open neighborhood U of x_0 in X which is bi-Lipschitz homeomorphic (respectively subanalytic bi-Lipschitz homeomorphic) to an Euclidean ball.*

In the paper [9], the authors defined the notion of Lipschitz regular complex analytic sets as the sets being subanalytically Lipschitz regular as in the definition above.

Now, we give another proof of the main theorem of [9].

Corollary 4.5 ([9])**.** *If $X \subset \mathbb{C}^n$ is a complex analytic set and subanalytically Lipschitz regular at 0, then $(X,0)$ is smooth.*

Proof. By Proposition 3.8, X is blow-spherical regular, then by Theorem 4.3, $(X,0)$ is smooth. □

5. Invariance of the multiplicity

In this Section, we give some partial answers to versions of the Zariski's multiplicity conjecture.

5.1. **Answers to Questions A1 and A2.**

Theorem 5.1 (Reduction for homogeneous sets)**.** *The Question A1 has a positive answer if, and only if, the Question A2 has a positive answer.*

Proof. Obviously, we just need to prove that a positive answer to the Question A2 implies a positive answer to the Question A1. Let $X, Y \subset \mathbb{C}^n$ be two complex analytic set and $\varphi : (X, 0) \to (Y, 0)$ be a blow-spherical homeomorphism. Let us denote by $X_1, \dots, X_r$ and $Y_1, \dots, Y_s$ the irreducible components of the tangent cones $C(X, 0)$ and $C(Y, 0)$ respectively. It comes from Proposition 3.3 and Theorem 3.4 that $r = s$ and the blow-spherical homeomorphism $d_0\varphi : C(X, 0) \to C(Y, 0)$, up to re-ordering of indexes, sends X_i onto Y_i and $k_X(X_i) = k_Y(Y_i)\ \forall\ i$.

We know that X_i and Y_i are irreducible homogeneous algebraic sets. Since the Question A2 has a positive answer, we get $m(X_i, 0) = m(Y_i, 0)\ \forall\ i$. Finally, using the Remark 3.5, we obtain $m(X, 0) = m(Y, 0)$. □

Corollary 5.2. *Question A1 has a positive answer in general if and only if* $\dim X \leq 2$.

Proof. Let us prove that Question A1 has a positive answer when $\dim X \leq 2$. By Theorem 5.1, it is enough to show Corollary 5.2 when X and Y are two irreducible homogeneous complex algebraic sets. Thus, we suppose that X and Y are two irreducible homogeneous complex algebraic sets and they are blow-spherical homeomorphic. The cases $\dim X = 0$ and $\dim X = 1$ are obvious, since in both cases $m(X, 0) = m(Y, 0) = 1$. Thus, we can suppose that $\dim X = \dim Y = 2$. However, since $(X, 0)$ and $(Y, 0)$ are blow-spherical homeomorphic, then they are, in particular, homeomorphic and, therefore, by Proposition 3.5 in [30], we obtain $m(X, 0) = m(Y, 0)$, since degree and multiplicity are equal for homogeneous complex algebraic sets.

Now, let us prove that Question A1 has a negative answer when $\dim X > 2$. In fact, we are going to prove that for each $n \geq 3$, there exists a family of n-dimensional homogeneous complex algebraic sets $\{Y_i\}_{i\in\mathbb{Z}}$ such that for each pair $i \neq j$, $(Y_i, 0)$ and $(Y_j, 0)$ are blow-spherical homeomorphic and $m(Y_i, 0) \neq m(Y_j, 0)$. In order to do that, let $\{p_i\}_{i\in\mathbb{Z}}$ be the family of odd prime numbers and let $n \geq 3$. In the proof of Theorem in [13] was shown that there exists a family $\{X_i\}_{i\in\mathbb{Z}}$ of 2-dimensional projective varieties, such that each $X_i \subset \mathbb{C}P^{m_i}$ is obtained by the embedding of a very ample bundle L_i on $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ of bi-degree $(2, p_i)$. For each $i \in \mathbb{Z}$ we denote by $C(X_i) \subset \mathbb{C}^{m_i+1}$ to be the affine cone of the projective variety X_i. Thus, it is also shown in [13] that for each pair $i \neq j$ the links $S_i := C(X_i) \cap \mathbb{S}^{2m_i+1}$ and $S_j := C(X_j) \cap \mathbb{S}^{2m_j+1}$ are diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^3$ and, in particular, S_i and S_j are bi-Lipschitz homeomorphic. Moreover, $m(C(X_i), 0) = 4p_i$, for all $i \in \mathbb{Z}$. Let $\phi\colon S_i \to S_j$ be a bi-Lipschitz homeomorphism. Then, the mapping $\Phi\colon C(X_i) \to C(X_j)$ given by

$$\Phi(x) = \begin{cases} \|x\| \cdot \phi(\frac{x}{\|x\|}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a blow-spherical homeomorphism. Moreover, for each $k \in \mathbb{Z}$ we define $Y_k := C(X_k) \times \mathbb{C}^{n-3}$. Then the mapping $\Psi\colon Y_i \to Y_j$ given by $\Psi(x, y) = (\Phi(x), y)$ for all $(x, y) \in C(X_i) \times \mathbb{C}^{n-3}$ is also a blow-spherical homeomorphism.

Thus, for each pair $i \neq j$, $(Y_i, 0)$ and $(Y_j, 0)$ are blow-spherical and this finishes the proof, since $m(Y_i, 0) = m(C(X_i), 0) = 4p_i$, for all $i \in \mathbb{Z}$. □

5.2. **Multiplicity of analytic sets with 1-dimensional singular set.** Let $f : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial with $\deg f = d$. We recall the map $\phi : \mathbb{S}^{2n-1} \setminus f^{-1}(0) \to \mathbb{S}^1$ given by $\phi(z) = \frac{f(z)}{|f(z)|}$ is a locally trivial fibration (see [49], §4). Notice that, $\psi : \mathbb{C}^n \setminus f^{-1}(0) \to \mathbb{C} \setminus \{0\}$ defined by $\psi(z) = f(z)$ is a locally trivial fibration such that its fibers are diffeomorphic the fibers of ϕ. Moreover, we can choose as geometric monodromy the homeomorphism $h_f : F_f \to F_f$ given by $h_f(z) = e^{\frac{2\pi i}{d}} \cdot z$, where $F_f := f^{-1}(1)$ is the (global) Milnor fiber of f (see [49], §9).

In the proof of Theorem 2.2 in [29] was proved the following result.

Proposition 5.3. *Let $f, g : \mathbb{C}^{n+1} \to \mathbb{C}$ be two reduced homogeneous complex polynomials. If $\varphi : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ is a homeomorphism and $\chi(F_f) \neq 0$, then $m(V(f), 0) = m(V(g), 0)$.*

Definition 5.4. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a complex polynomial with*

$$\dim \mathrm{Sing}(V(f)) = 1 \text{ and } \mathrm{Sing}(V(f)) = C_1 \cup \ldots \cup C_r.$$

Then $b_i(f)$ denotes the i-th Betti number of the Milnor fiber of f at the origin, $\mu'_j(f)$ is the Milnor number of a generic hyperplane slice of f at $x_j \in C_j \setminus \{0\}$ sufficiently close to the origin we write $\mu'(f) = \sum_{i=1}^{r} \mu'_i(f)$.

Theorem 5.5. *Let $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be two reduced homogeneous complex polynomials such that $\dim \mathrm{Sing}(V(f_i), 0) \leq 1$, where $f = f_1 \cdots f_r$ is the decomposition of f in irreducible polynomials. If $\varphi : (\mathbb{C}^{n+1}, V(f), 0) \to (\mathbb{C}^{n+1}, V(g), 0)$ is a homeomorphism, then $m(V(f), 0) = m(V(g), 0)$.*

Proof. By Proposition 5.3, we can suppose $\chi(F_f) = \chi(F_g) = 0$ and by additivity of the multiplicity, we can suppose that f and g are irreducible homogeneous polynomials with degree d and k, respectively. By A'Campo-Lê's Theorem, we have that $\varphi(\mathrm{Sing}(V(f))) = \mathrm{Sing}(V(g))$ and, in particular, $\dim \mathrm{Sing}(V(f)) = \dim \mathrm{Sing}(V(g))$. Thus, we can suppose that $d, k > 1$.

If $\dim \mathrm{Sing}(V(f)) = 1$, then by Theorem 5.11 in [56], we have

$$(d-1)^n - \mu'(d-1) + (-1)^{n-1} - \mu'(f) = 0$$

and

$$(k-1)^n - \mu'(k-1) + (-1)^{n-1} - \mu'(g) = 0.$$

Thus, we define the polynomial $P : \mathbb{R} \to \mathbb{R}$ by

$$P(t) = t^n - \mu'(f)t + (-1)^{n-1} - \mu'(f), \quad \forall t \in \mathbb{R}.$$

Since $\mu'(f) = \mu'(g)$ (see [44], Proposition and Thérème 2.3), then $d-1$ and $k-1$ are zeros of the polynomial $P(t)$. By Descartes' Rule, the polynomial $P(t)$ has at most one positive zero, since $\mu'(f) = \mu'(g) \geq 1$. Thus, $d = k$.

If $\dim \mathrm{Sing}(V(f)) = 0$, let $\widetilde{f}, \widetilde{g} : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ given by $\widetilde{f}(z, z_{n+1}) = f(z)$ and $\widetilde{g}(z, z_{n+1}) = g(z)$. It is easy to see that $m(V(\widetilde{f}), 0) = m(V(f), 0)$ and $m(V(\widetilde{g}), 0) = m(V(g), 0)$. Moreover, $V(\widetilde{f}) = V(f) \times \mathbb{C}$ and $V(\widetilde{g}) = V(g) \times \mathbb{C}$, then we define $\widetilde{\varphi} : (\mathbb{C}^n \times \mathbb{C}, V(\widetilde{f}), 0) \to (\mathbb{C}^n \times \mathbb{C}, V(\widetilde{g}), 0)$ by $\widetilde{\varphi}(z, z_{n+1}) = (\varphi(z), z_{n+1})$. We have that $\widetilde{\varphi}$ is a homeomorphism. Therefore, by first part of this prove, $m(V(\widetilde{f}), 0) = m(V(\widetilde{g}), 0)$ and this finish the proof. □

Corollary 5.6. *Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two reduced homogeneous complex polynomials. If $\varphi : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ is a homeomorphism, then $m(V(f), 0) = m(V(g), 0)$.*

Theorems 5.1 and 5.5 give also the following consequences.

Corollary 5.7. *Let $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions such that $\dim \mathrm{Sing}(V(f_i), 0) \leq 1$, where $\mathbf{in}(f) = f_1 \cdots f_r$ is the decomposition of $\mathbf{in}(f)$ in irreducible polynomials. If $\varphi : (\mathbb{C}^{n+1}, V(f), 0) \to (\mathbb{C}^{n+1}, V(g), 0)$ is a blow-spherical homeomorphism, then $m(V(f), 0) = m(V(g), 0)$.*

Corollary 5.8. *Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions. If $\varphi : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ is a blow-spherical homeomorphism, then $m(V(f), 0) = m(V(g), 0)$.*

5.3. **Multiplicity of aligned singularities.**

Definition 5.9. *If $h : U \to \mathbb{C}$ is an analytic function, a good stratification for h at a point $p \in V(h)$ is an analytic stratification, $S = \{S_\alpha\}$, of the hypersurface $V(h)$ in a neighborhood, U, of p such that the smooth part of $V(h)$ is a stratum and so that the stratification satisfies Thom's a_h condition with respect to $U \setminus V(h)$. That is, if q_i is a sequence of points in $U \setminus V(h)$ such that $q_i \to q \in S_\alpha$ and $T_{q_i} V(h - h(q_i))$ converges to some hyperplane T, then $T_q S_\alpha \subset T$.*

Definition 5.10. *If $h : (U, 0) \to (\mathbb{C}, 0)$ is an analytic function, then an aligned good stratification for h at the origin is a good stratification for h at the origin in which the closure of each stratum of the singular set is smooth at the origin. If such an aligned good stratification exists, we say that h has an aligned singularity at the origin.*

Theorem 5.11. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced homogeneous complex polynomials. Suppose that f and g have an aligned singularity at the origin. If $\varphi : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ is a homeomorphism, then $m(V(f), 0) = m(V(g), 0)$.*

Proof. We can suppose that $d = m(V(f), 0) > 1$ and $k = m(V(g), 0) > 1$. By Corollary 4.7 in [46], we have

$$(d-1)^n = \sum_{i=1}^{s} \lambda^i_{f,z}(0)(d-1)^i$$

and

$$(k-1)^n = \sum_{i=1}^{s} \lambda^i_{g,z}(0)(k-1)^i,$$

where $s = \dim \mathrm{Sing}(V(f)) = \dim \mathrm{Sing}(V(g))$ and $\lambda^0_{f,z}(0), \cdots, \lambda^s_{f,z}(0)$ (resp. $\lambda^0_{g,z}(0), \cdots$, $\lambda^s_{g,z}(0)$) are the Lê's numbers of f (resp. g) at the origin (see the definition and some properties of the Lê's numbers in [46]). By Corollary 7.8 in [46], we obtain $\lambda^i_{f,z}(0) = \lambda^i_{g,z}(0)$, for $i = 0, ..., s$. Then $d-1$ and $k-1$ are zeros of the following equation

$$t^n - \sum_{i=1}^{s} \lambda^i_z t^i = 0, \tag{1}$$

where $\lambda^i_z := \lambda^i_{f,z}(0)$, for $i = 0, ..., s$. By Descartes' Rule, the equation (1) has only one positive zero, since $\lambda^i_z \geq 0$, for $i = 0, ..., s$. Then $d - 1 = k - 1$, i.e., $m(V(f), 0) = m(V(g), 0)$. □

Corollary 5.12. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic function. Suppose that $f_1, ..., f_s, g_1, ..., g_s$ have an aligned singularity at the origin, where* $\mathbf{in}(f) = f_1 \cdots f_s$ *(resp.* $\mathbf{in}(g) = g_1 \cdots g_s$*) is the decomposition of* $\mathbf{in}(f)$ *(resp.* $\mathbf{in}(g)$*) in irreducible polynomials. If $\varphi : (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$ is a blow-spherical homeomorphism, then $m(V(f), 0) = m(V(g), 0)$.*

Remark 5.13. We finish this Section remarking that the Corollaries 5.7, 5.8, and 5.12 hold true as well when we consider a bi-Lipschitz homeomorphism instead of a blow-spherical homeomorphism, using Theorem 2.1 in [29] instead of Theorem 5.1.

5.4. **The Question A2 in the case of families of hypersurfaces.**

Definition 5.14. *The family of complex analytic functions $\{f_t\}_{t\in[0,1]}$ (resp. the family of complex analytic hypersurfaces $\{V(f_t)\}_{t\in[0,1]}$) is said* **topologically $\mathcal{R}$-equisingular** *(resp.* **topologically V-equisingular***) if there are an open $U \subset \mathbb{C}^n$ and a continuous map $\varphi : U \times [0,1] \to \mathbb{C}^n$ such that $\varphi_t := \varphi(\cdot, t) : U \to \varphi(U \times \{t\})$ is a homeomorphism, $\varphi(0, t) = 0$ and $f_t = f_0 \circ \varphi_t$ (resp. $\varphi_t(V(f_t)) = V(f_0)$) for all $t \in [0, 1]$.*

Remark 5.15. Changing φ_t by $\varphi_0^{-1} \circ \varphi_t$, we can suppose that $\varphi_0 =$id.

Definition 5.16. *We say that an analytic family $\{f_t\}_{t\in[0,1]}$ (resp. $\{V(f_t)\}_{t\in[0,1]}$) is* **equimultiple** *if $\mathrm{ord}_0(f_0) = \mathrm{ord}_0(f_t)$ (resp. $m(V(f_t), 0) = m(V(f_0), 0)$) for all $t \in [0, 1]$.*

Lemma 5.17 ([21], Theorem 2.6)**.** *If the germ of $V = V(f)$ at the origin is irreducible, then there exists $\varepsilon > 0$ such that for any $0 < r < \varepsilon$, $H_1(B_r \setminus V; \mathbb{Z}) \cong \mathbb{Z}$.*

Lemma 5.18 ([21], Theorem 2.7)**.** *Let V be a hypersurface, and suppose that the germ of V at the origin is irreducible. Let f be an analytic function on B_ε which generates the ideal of V at all points of B_ε. Then $f_* : H_1(B_r \setminus V; \mathbb{Z}) \to H_1(\mathbb{C} \setminus \{0\}; \mathbb{Z})$ is a isomorphism for all r, $0 < r < \varepsilon$.*

In the next result, U_1 and U_2 are open of $\mathbb{C}^n$, V_1 and V_2 are hypersurfaces of $\mathbb{C}^n$.

Lemma 5.19 ([21], Theorem 2.8)**.** *Suppose ε is chosen as above to serve for both V_1 and V_2. Assume*

$$\varphi : (U_1, V_1, 0) \to (U_2, V_2, 0)$$

is a homeomorphism. Choose $0 < r < \varepsilon$ and $0 < s < \varepsilon$ such that $B_r \subset \varphi(B_\varepsilon)$ and $\varphi(B_s) \subset B_r$. Then, $\varphi_* : H_1(B_s \setminus V_1; \mathbb{Z}) \to H_1(B_r \setminus V_2; \mathbb{Z})$ is an isomorphism.

Theorem 5.20. *Let $F : \mathbb{C}^n \times [0,1] \to \mathbb{C}$ be a (not necessarily continuous) subanalytic function. Suppose that for each $t \in [0,1]$, $f_t := F(\cdot, t) : \mathbb{C}^n \to \mathbb{C}$ is a (not necessarily reduced) complex homogeneous polynomial. If $\{V(f_t)\}_{t\in[0,1]}$ is a topologically V-equisingular family, then it is equimultiple.*

Proof. Let $\varphi : U \times [0,1] \to \mathbb{C}^n$ be a continuous map such that $\varphi_t := \varphi(\cdot, t) : U \to \varphi(U \times \{t\})$ is a homeomorphism, $\varphi(0,t) = 0$ and $\varphi_t(V(f_t)) = V(f_0)$ for all $t \in [0,1]$.

Note that there is $v \in \mathbb{C}^n \setminus \bigcup_{t\in[0,1]} V(f_t)$. In fact, we denote $V = F^{-1}(0) \subset \mathbb{C}^n \times [0,1]$ and $V_t = V(f_t) \times \{t\}$, then $V = \bigcup_{t\in[0,1]} V_t$. Moreover, $V \setminus \mathrm{Sing}(V)$ is a smooth submanifold of $\mathbb{C}^n \times [0,1] = \mathbb{R}^{2n} \times [0,1]$, then for each $x \in V \setminus \mathrm{Sing}(V)$, there are a neighborhood U_x and a diffeomorphism $\phi_x : B_2(0) \subset \mathbb{R}^m \to U_x$, where m is the dimension of $V \setminus \mathrm{Sing}(V)$. Using the co-area formula, we obtain that

$$\mathcal{H}^{2n+1}(V) = \int_V \|\nabla p(x)\| dx = \int_0^1 \mathcal{H}^{2n}(V \cap p^{-1}(t)) dt = 0,$$

where $p : \mathbb{C}^n \times [0,1] \to [0,1]$ is the canonical projection (here, $\mathcal{H}^k(X)$ denote the Hausdorff measure k-dimensional of the set X). The last equality is why $V(f_t)$ is a complex algebraic set with dimension $n-1$ and, in particular, $\mathcal{H}^{2n-1}(V(f_t)) = \mathcal{H}^{2n}(V(f_t)) = 0$. Therefore, $m < 2n+1$.

Suppose that $m = 2n$. Using the co-area formula once more, we obtain that

$$\int_{\overline{B_1(0)}} \|\nabla h(x)\| dx = \int_{h(\overline{B_1(0)})} \mathcal{H}^{2n-1}(h^{-1}(t)) dt = 0,$$

where $h = p \circ \phi_x : \overline{B_1(0)} \to [0,1]$. However, $\|\nabla h(x)\| \not\equiv 0$, then $\mathcal{H}^{2n}(\overline{B_1(0)}) = 0$, but this is a contradiction. Then, $m \leq 2n-1$ and, therefore, $\mathcal{H}^{2n}(V) = 0$.

Thus, $\mathcal{H}^{2n}(\bigcup_{t\in[0,1]} V(f_t)) = 0$, since the canonical projection $\pi : \mathbb{C}^n \times [0,1] \to \mathbb{C}^n$ is a Lipschitz map and $\pi(V) = \bigcup_{t\in[0,1]} V(f_t)$. In particular, $\bigcup_{t\in[0,1]} V(f_t) \subsetneq \mathbb{C}^n$.

Let $L \subset \mathbb{C}^n$ be a complex line given by $L = \{\lambda v;\ \lambda \in \mathbb{C}\}$. Then $L \cap (\bigcup_{t\in[0,1]} V(f_t)) = \{0\}$. Moreover, by Lemma 2.8, we can suppose that f_t is an irreducible polynomial, for all $t \in [0,1]$.

Fixed $t_0 \in [0,1]$, choose $0 < r, s < \varepsilon$ as in Lemma 5.19 and let γ be a generator of $H_1(\mathbb{D}_L; \mathbb{Z})$, where $\mathbb{D}_L := \{z \in L;\ 0 < \|z\| \leq \delta\} \subset B_r \cap B_s$ and $B_\varepsilon \subset U$.

Then, $(f_{t_0}|_{\mathbb{D}_L})_*(\gamma) = \pm m(V(f_{t_0}),0)$ and $(f_0|_{\mathbb{D}_L})_*(\gamma) = \pm m(V(f_0),0)$. In particular, $i_*(\gamma) = \pm m(V(f_{t_0}),0)$, where $i : \mathbb{D}_L \to B_s \setminus V(f_{t_0})$ is the inclusion map, since $(f_{t_0})_* : H_1(B_s \setminus V(f_{t_0});\mathbb{Z}) \to H_1(\mathbb{C}\setminus\{0\};\mathbb{Z})$ is an isomorphism.

However, $(\varphi_{t_0})_* : H_1(B_s\setminus V(f_{t_0});\mathbb{Z}) \to H_1(B_\varepsilon\setminus V(f_0);\mathbb{Z})$ is also an isomorphism, then $(\varphi_{t_0})_*(i_*(\gamma)) = \pm m(V(f_{t_0}),0)$. Therefore,

$$(f_0\circ\varphi_{t_0}|_{\mathbb{D}_L})_*(\gamma) = \pm m(V(f_{t_0}),0), \tag{2}$$

since $(f_0)_* : H_1(B_\varepsilon \setminus V(f_0);\mathbb{Z}) \to H_1(\mathbb{C}\setminus\{0\});\mathbb{Z})$ is an isomorphism, as well.

Claim. $f_0\circ\varphi_{t_0}|_{\mathbb{D}_L}$ is homotopy equivalent to $f_0|_{\mathbb{D}_L}$.

In fact, $L\cap V(f_t) = \{0\}$ for all $t\in[0,1]$ and, in particular, for each $t\in[0,1]$, $f_t(w)\neq 0$ for all $w\in\mathbb{D}_L$. Thus, the function $H : \mathbb{D}_L\times[0,1]\to\mathbb{C}\setminus\{0\}$ given by $H(z,\lambda) = f_0\circ\varphi(z,\lambda t_0)$ is a homotopy between $f_0|_{\mathbb{D}_L}$ and $f_0\circ\varphi|_{\mathbb{D}_L}$, since $\varphi_0 = id$. In particular, $(f_0\circ\varphi|_{\mathbb{D}_L})_* = (f_0|_{\mathbb{D}_L})_*$. Then

$$\pm m(V(f_{t_0}),0) \overset{(2)}{=} (f_0\circ\varphi_{t_0}|_{\mathbb{D}_L})_*(\gamma) = (f_0|_{\mathbb{D}_L})_*(\gamma) = \pm m(V(f_0),0).$$

Therefore, $m(V(f_{t_0})) = m(V(f_0))$. □

Remark 5.21. After this paper have been finished the author knew that it had already been proved in ([25], Chapter 1, Corollary 5.3) that if there exists a complex lines L such that $L\cap(\bigcup\limits_{t\in[0,1]} V(f_t)) = \{0\}$ then $\{f_t\}$ is equimultiple.

6. Classification of complex analytic curves in the space

In this Section, we prove that the blow-spherical geometry and the multiplicity are essentially the same object, in the case of complex analytic curves. Moreover, we present a classification of the germs of complex analytic curves modulo blow-spherical equivalence.

Theorem 6.1. *Two germs of irreducible complex analytic curves are blow-spherical homeomorphic if, and only if, they have the same multiplicity.*

Proof. Let $X,\widetilde{X}\subset\mathbb{C}^n$ be two irreducible analytic curves. In this case, $C(X,0)$ and $C(\widetilde{X},0)$ are complex lines and, thus, $m(C(X,0)) = m(C(\widetilde{X},0)) = 1$. Therefore, $m(X,0) = k_X(C(X,0))$ and $m(\widetilde{X},0) = k_{\widetilde{X}}(C(\widetilde{X},0))$.

Thus, if X and $\widetilde{X}$ are blow-spherical homeomorphic, then by Theorem 3.4, we obtain that they have the same multiplicity.

Reciprocally, suppose that $k = m(X,0) = m(\widetilde{X},0)$. After changes of linear coordinates, if necessary, we can suppose that the tangent cone of X and of $\widetilde{X}$ is $\{(\xi,0)\in\mathbb{C}^n;\ \xi\in\mathbb{C}\}$. Let $\psi : D_\varepsilon\to X$ and $\widetilde{\psi} : D_\varepsilon\to\widetilde{X}$ be the Puiseux's parametrizations of X and $\widetilde{X}$, resp., given by

$$\psi(t) = (t^k,\phi(t)) = (t^k,\phi_2(t),...,\phi_n(t))$$

and

$$\widetilde{\psi}(t) = (t^k,\widetilde{\phi}(t)) = (t^k,\widetilde{\phi}_2(t),...,\widetilde{\phi}_n(t)),$$

where $ord_0\phi_i > k$ and $ord_0\widetilde{\phi}_i > k$, for $i = 2, ..., n$. Define $\varphi : X \to \widetilde{X}$ given by $\varphi = \widetilde{\psi} \circ \psi^{-1}$.

Claim. φ is a blow-isomorphism such that $\varphi'|_{\partial X'} = id_{\partial X'}$.

In fact, it is enough to show that for each $(\mathbf{x}, 0) \in \partial X'$ and for any sequence $\{z_m\}_{m\in\mathbb{N}} \subset X' \setminus \partial X'$ such that $\lim\limits_{m\to\infty} z_m = (\mathbf{x}, 0)$, then we have $\lim\limits_{m\to\infty} \varphi'(z_m) = (\mathbf{x}, 0)$.

Thus, let $(\mathbf{x}, 0) \in \partial X'$ and $\{z_m\}_{m\in\mathbb{N}} \subset X' \setminus \partial X'$ such that $\lim\limits_{m\to\infty} z_m = (\mathbf{x}, 0)$. For each m, we write $z_m = (\mathbf{x}_m, t_m)$ and $s_m = \psi^{-1}(t_m\mathbf{x}_m)$. Then, we have

$$\begin{aligned}\varphi'(z_m) &= \left(\frac{\varphi(t_m\mathbf{x}_m)}{\|\varphi(t_m\mathbf{x}_m)\|}, \|\varphi(t_m\mathbf{x}_m)\|\right)\\ &= \left(\frac{(s_m^k, \widetilde{\phi}(s_m))}{\|(s_m^k, \widetilde{\phi}(s_m))\|}, \|(s_m^k, \widetilde{\phi}(s_m))\|\right).\end{aligned}$$

But $t_m\mathbf{x}_m = (s_m^k, \phi(s_m))$ and, hence,

$$z_m = \left(\frac{(s_m^k, \phi(s_m))}{\|(s_m^k, \phi(s_m))\|}, \|(s_m^k, \phi(s_m))\|\right).$$

Therefore, $\lim\limits_{m\to\infty} \varphi'(z_m) = \lim\limits_{m\to\infty} z_m = (\mathbf{x}, 0)$, since $ord_0\phi > k$ and $ord_0\widetilde{\phi} > k$. □

Let $X, Y \subset \mathbb{C}^n$ be two complex analytic curves. Let $X_1, ..., X_r \subset \mathbb{C}^n$ be the irreducible components of X and let $Y_1, ..., Y_s \subset \mathbb{C}^n$ be the irreducible components of Y. Then, we have the following

Corollary 6.2. *X and Y are blow-spherical equivalent if and only if there is a bijection $\sigma : \{1, ..., r\} \to \{1, ..., s\}$ such that*

1) $m(X_i, 0) = m(Y_{\sigma(i)}, 0)$, for all $i = 1, ..., r$.
2) there is a homeomorphism $h : (C(X, 0), 0) \to (C(Y, 0), 0)$ satisfying

$$h(C(X_i, 0)) = C(Y_{\sigma(i)}, 0), \text{ for all } i = 1, ..., r.$$

In Proposition 3.14 in [41] was showed that the number of tangent lines at a point of a complex analytic curve is a bi-Lipschitz invariant.

Let $p_1 : \mathbb{Z}_{>0} \times \mathcal{P}(\mathbb{Z}^2_{>0}) \to \mathbb{Z}_{>0}$ and $p_2 : \mathbb{Z}_{>0} \times \mathcal{P}(\mathbb{Z}^2_{>0}) \to \mathcal{P}(\mathbb{Z}^2_{>0})$ be the canonical projections, where $\mathcal{P}(\mathbb{Z}^2_{>0})$ denotes the power set of $\mathbb{Z}^2_{>0}$ and $\mathbb{Z}_{>0} = \{n \in \mathbb{Z}; n > 0\}$; Let $\mathcal{A}$ the subset of $Z_{>0} \times \mathcal{P}(\mathbb{Z}^2_{>0})$ formed by finite and non-empty subsets A satisfying the following:

i) $p_1(A) = \{1, ..., N\}$ for some $N \in \mathbb{Z}_{>0}$;
ii) $p_2(p_1^{-1}(\ell)) = \{(k_{\ell 1}, m_{\ell 1}), ..., (k_{\ell r_\ell}, m_{\ell r_\ell})\} \subset \mathbb{Z}^2_{>0}$ and $k_{\ell j} < k_{\ell(j+1)}$ for all $j \in \{1, ..., r_\ell - 1\}$ and for all $\ell \in \{1, ..., N\}$.
iii) $r_\ell \leq r_{\ell+1}$ for all $\ell \in \{1, ..., N - 1\}$.

For a set $A \in \mathcal{A}$ as above, we define the realization of A to be the curve

$$X_A = \bigcup_{\ell=1}^{N} \{(x, y) \in \mathbb{C}^2; \prod_{j=1}^{r_\ell} \prod_{m=1}^{m_{\ell j}} ((y - \ell x)^{k_{\ell j}} - m(y + \ell x)^{k_{\ell j}+1}) = 0\}.$$

Thus, by Corollary 6.2 and definition of $\mathcal{A}$, it is not hard to verify the following classification:

Theorem 6.3. *For each complex analytic curve $X \subset \mathbb{C}^n$ such that $0 \in X$ there exists a unique set $A \in \mathcal{A}$ such that $(X_A, 0)$ is blow-spherical homeomorphic to $(X, 0)$.*

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