MODELS FOR DAMPED WATER WAVES

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ABSTRACT. In this paper we derive some new weakly nonlinear asymptotic models describing viscous waves in deep water with or without surface tension effects. These asymptotic models take into account several different dissipative effects and are obtained from the free boundary problems formulated in the works of Dias, Dyachenko and Zakharov (*Physics Letters A, 2008*), Jiang, Ting, Perlin and Schultz (*Journal of Fluid Mechanics,1996*) and Wu, Liu and Yue (*Journal of Fluid Mechanics, 2006*).

CONTENTS

1. INTRODUCTION

The motion of a free boundary irrotational and incompressible flow is a classical research topic [35]. In most applications, the flow is also assumed to be inviscid [7, 22].

However, even if in most situations in coastal engineering the assumption of inviscid flow leads to very accurate results, there are other physical scenarios where the viscosity needs to be taken into account. Moreover, there are many situations in which the viscosity is very large and the vorticity is small and its effect negligible. Actually, certain discrepancies between experiments and inviscid theory have been previously reported in the literature. For instance, Wu [39] found that

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From this comparison the theory appears quite satisfactory in predicting the wave phases during the inward focusing and the subsequent reflection within a radial distance as far as r = 20, while the peak amplitudes observed in the experiments are slightly smaller than those predicted by the theory. This discrepancy can be ascribed to the neglect of the viscous effects in the theory and to the approximation that the initial wave generated in the tank was not cylindrical in shape and departed slightly from a perfect solitary wave profile in the experiment.

In addition to this, Zabusky and Galvin [40] wrote

A laboratory-data/numerical- solution comparison of the number of crests and troughs and their phases (or relative locations within a period) shows only negligible difference. As one expects, the crest-to-trough amplitudes differ somewhat more because they are more sensitive to dissipative forces. To quantify some of the details we recommend a study including dissipation.

and, furthermore, Longuet-Higgins [24] stated that

For certain applications, however, viscous damping of the waves is important, and it would be highly convenient to have equations and boundary conditions of comparable simplicity as for undamped waves.

The purpose of this paper is to derive new weakly nonlinear asymptotic models (in the spirit of [6,13,14,25–27,31]) describing damped water waves and, at the same time, keeping the features of potential flows. We observe that, at fist sight, the idea of viscous damping of potential flows is somehow paradoxical since the hypothesis of irrotational velocity implies that the viscous term in the Navier-Stokes equations vanishes.

The equations describing the motion of a irrotational, incompressible, inviscid and homogeneous fluid with a free surface are [41]

$$\Delta\phi = 0 \qquad\qquad\qquad\qquad \text{in } \Omega(t) \times [0,T], \qquad (1a)$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) - \gamma\mathcal{K} = 0 \qquad\qquad \text{on } \Gamma(t) \times [0,T], \qquad (1b)$$

$$h_t = \nabla\phi \cdot \left(1 + (\partial_1 h)^2\right)^{1/2} n \qquad \text{on } \Gamma(t) \times [0,T], \qquad (1c)$$

where

$$\Omega(t) = \left\{(x_1, x_2) \in \mathbb{R}^2 \;\middle|\; -L\pi < x_1 < L\pi\,, -\infty < x_2 < h(x_1,t)\,,\ t \in [0,T]\right\}, \qquad (2)$$

$$\Gamma(t) = \left\{(x_1, h(x_1,t)) \in \mathbb{R}^2 \;\middle|\; x_1 \in \mathbb{S}^1\,,\ t \in [0,T]\right\} \qquad (3)$$

are the the region occupied by the fluid and the surface wave, respectively. We write

$$n = (-\partial_1 h, 1)/\left(1 + (\partial_{x_1} h)^2\right)^{1/2}$$

the unit normal to the surface wave, $2L\pi$ to denote the characteristic wavelength of the surface wave, ϕ for the scalar potential of the flow, *i.e.* the velocity field u satisfies $u = \nabla\phi$, ρ is the

density of the fluid, γ for the surface tension coefficient and

$$\mathcal{K} = \frac{\partial_1^2 h}{\left(1 + (\partial_1 h)^2\right)^{3/2}},$$

is the curvature of the surface wave.

The first attempts to include viscosity effects go back as far as to the works of Boussinesq [5] and Lamb [21]. Later on, Ruvinsky & Freidman [34] formulated a system of equations for weakly damped surfaces waves in deep water and used this system to compute capillary-gravity ripples riding on the forward face of steep gravity waves (see also [33]). Then, these first results were generalized by Ruvinsky, Feldstein & Freidman [32] and the following system is proposed

$$\Delta \phi = 0 \qquad\qquad \text{in } \Omega(t) \times [0, T], \qquad (4a)$$

$$\rho \left(\phi_t + \frac{1}{2} |\nabla \phi|^2 + Gh \right) - \gamma \mathcal{K} = -2\mu \partial_2^2 \phi \qquad\qquad \text{on } \Gamma(t) \times [0, T], \qquad (4b)$$

$$h_t = \nabla \phi \cdot \left(1 + (\partial_1 h)^2 \right)^{1/2} n + v \qquad \text{on } \Gamma(t) \times [0, T], \qquad (4c)$$

$$v_t = \partial_{11}^2 \partial_2 \phi \qquad\qquad \text{on } \Gamma(t) \times [0, T], \qquad (4d)$$

where v and μ denote the vertical component of the vortex part of fluid velocity and the dynamic viscosity. Equation (4) was also studied by Khariff, Skandrani & Poitevin [20].

Using a clever change of variables, Longuet-Higgins [24] simplified the previous system and obtained

$$\Delta \phi = 0 \qquad\qquad \text{in } \Omega(t) \times [0, T], \qquad (5a)$$

$$\rho \left(\phi_t + \frac{1}{2} |\nabla \phi|^2 + Gh \right) - \gamma \mathcal{K} = -4\mu \partial_n^2 \phi \qquad\qquad \text{on } \Gamma(t) \times [0, T], \qquad (5b)$$

$$h_t = \nabla \phi \cdot \left(1 + (\partial_1 h)^2 \right)^{1/2} n \qquad \text{on } \Gamma(t) \times [0, T], \qquad (5c)$$

A similar model was also studied by Jiang, Ting, Perlin & Schultz [16] and Wu, Liu & Yue [38], namely

$$\Delta \phi = 0 \qquad\qquad \text{in } \Omega(t) \times [0, T], \qquad (6a)$$

$$\rho \left(\phi_t + \frac{1}{2} |\nabla \phi|^2 + Gh \right) - \gamma \mathcal{K} = -\delta \mathscr{D}^s \phi \qquad\qquad \text{on } \Gamma(t) \times [0, T], \qquad (6b)$$

$$h_t = \nabla \phi \cdot \left(1 + (\partial_1 h)^2 \right)^{1/2} n \qquad \text{on } \Gamma(t) \times [0, T], \qquad (6c)$$

where the dissipative terms are chosen as

$$\mathscr{D}^2 \phi = \partial_2^2 \phi \text{ or } \mathscr{D}^0 \phi = \phi. \qquad (7)$$

Another similar model where the dissipation acts only on the velocity is the one by Joseph & Wang [17, Equation (6.7) and (6.8)] (see also Wang & Joseph [37]).

We would like to remark that, in the models of damped water waves mentioned so far, there are no dissipative effects acting on the free surface.

In a more recent paper, Dias, Dyachenko & Zakharov [9] proposed a system where the free surface experiments dissipative effects. In particular, based on the linear problem, these authors derived

$$\Delta \phi = 0 \qquad\qquad\qquad \text{in } \Omega(t) \times [0, T], \qquad (8a)$$

$$\rho \left(\phi_t + \frac{1}{2} |\nabla \phi|^2 + Gh \right) = -2\mu \partial_2^2 \phi \qquad\qquad \text{on } \Gamma(t) \times [0, T], \qquad (8b)$$

$$h_t = \nabla \phi \cdot \left(1 + (\partial_1 h)^2 \right)^{1/2} n + 2 \frac{\mu}{\rho} \partial_1^2 h \qquad \text{on } \Gamma(t) \times [0, T], \qquad (8c)$$

as a model of viscid water waves. This model was also considered by several other authors. Dutykh & Dias [12] obtain a new set of viscous potential free-surface flow equations in the spirit of (8) taking into account the effects of the bottom topography. These authors also derived a long wave approximation. This approximate model takes the form of a nonlocal (in time) Boussinesq system (see also [10,11]). Kakleas & Nicholls [18], using the analytic dependence of the Dirichlet-Neumann operator, derived a system of two equations modelling (8). These equations are the viscid analog of the classical Craig-Sulem WW2 model and were mathematically studied by Ambrose, Bona & Nicholls [3]. The well-posedness of the full (8) was studied very recently by Ngom & Nicholls [30]. In particular these authors proved global existence of solutions starting from a small enough initial data for the case of non-vanishing surface tension $\gamma \neq 0$.

Some other related results are those by Kharif, Kraenkel, Manna & Thomas [19] and Hunt & Dutykh [15]. Kharif, Kraenkel, Manna & Thomas studied a similar situation to (8) within the framework of a forced and damped nonlinear Schrödinger equation (see also Touboul & Kharif [36]), while Hunt & Dutykh considered the problem of the interface motion under the capillary-gravity and an external electric force in the case of an incompressible, viscous, perfectly conducting fluid.

1.1. **Plan of the paper.** First we obtain the dimensionless Eulerian formulation in the moving domain and transform it to a dimensionless Arbitrary Lagrangian-Eulerian (ALE) formulation in a fixed domain in section 2. Then we introduce the asymptotic expansion and obtain the cascade of linear equations for the different scales presents in the problem with $s = 0$ corresponding to the models by Jiang, Ting, Perlin & Schultz [16] and Wu, Liu & Yue [38] in Section 3. After neglecting errors of $\mathcal{O}(\varepsilon^2)$ we find the nonlocal wave equation modelling the case $s = 0$. After that we consider the case $s = 2$ and, following a similar approach, find the nonlocal wave equation for the model of Dias, Dyachenko, and Zakharov [9]. Finally, we conclude with a parabolic system of Craig-Sulem flavour in section 5.

1.2. **Notation.** Let A be a matrix, and b be a column vector. Then, we write A_j^i for the component of A, located on row i and column j. We will use the Einstein summation convention for expressions with indexes.

We write

$$\partial_j f = \frac{\partial f}{\partial x_j}, \quad f_t = \frac{\partial f}{\partial t}$$

for the space derivative in the j−th direction and for a time derivative, respectively.

Let $f(x_1)$ denote a L^2 function on $\mathbb{S}^1$ (as usually, identified with the interval $[-\pi, \pi]$ with periodic boundary conditions). We define the Hilbert transform $\mathcal{H}$ and the Dirichlet-to-Neumann operator Λ and its powers, respectively, using Fourier series

$$\widehat{\mathcal{H}f}(k) = -i\,\mathrm{sgn}(k)\hat{f}(k), \quad \widehat{\Lambda f}(k) = |k|\hat{f}(k), \quad \widehat{\Lambda^s f}(k) = |k|^s \hat{f}(k), \tag{9}$$

where

$$\hat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{S}^1} f(x_1)\, e^{-ikx_1} dx_1.$$

In particular, for zero-mean functions, we note that

$$\partial_1 \mathcal{H} = \Lambda, \quad \mathcal{H}^2 = -1, \quad \partial_1 \Lambda^{-1} = -\mathcal{H}.$$

These last equalities will be used extensively through the whole text. Finally, we define the commutator as

$$[\![A, B]\!]f = ABf - BAf.$$

2. DAMPED WATER WAVES

2.1. **The equations in the Eulerian formulation.** We consider system the system

$$\Delta\phi = 0 \qquad\qquad \text{in } \Omega(t) \times [0, T], \tag{10a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) - \gamma\mathcal{K} = -\delta_1 \mathscr{D}^s\phi \qquad\qquad \text{on } \Gamma(t) \times [0, T], \tag{10b}$$

$$h_t = \nabla\phi \cdot \left(1 + (\partial_1 h)^2\right)^{1/2} n + \delta_2 \partial_1^2 h \qquad \text{on } \Gamma(t) \times [0, T], \tag{10c}$$

where $\delta_i \geq 0$ are constant, the dissipative terms are as in (7), ϕ is the scalar potential (units of $length^2/time$), h denotes the surface wave (units of $length$) and G (units of $length/time^2$) is the gravity acceleration. The constant δ_1 has units of $mass/(length^2 \cdot time)$ (when $\mathscr{D}^0\phi = \phi$) and of $mass/time$ (when $\mathscr{D}^2\phi = \partial_2^2\phi$) while δ_2 has units of $length^2/time$. We observe that, for appropriate choice of δ_i and s we recover (exactly) (6) and (8). Indeed, if $\delta_2 = 0$ we obtain the same model by Jiang, Ting, Perlin & Schultz [16] ($s = 0$) and Wu, Liu & Yue [38] ($s = 0$ and $s = 2$), while if $\delta_2 = \delta_1/\rho$ and $s = 2$ we recover the model by Dias, Dyachenko & Zakharov [9]. Following the pioneer work of Zakharov [41], we use the trace of the velocity potential $\xi(t, x) = \phi(t, x, h(t, x))$ (units of $length^2/time$). Now we observe that

$$\xi_t(t, x) = \phi_t(t, x, h(t, x)) + \partial_2\phi(t, x, h(t, x))h_t(t, x)$$
$$= \phi_t(t, x, h(t, x)) + \partial_2\phi(t, x, h(t, x))\left(\nabla\phi \cdot (-\partial_x h, 1) + \delta_2\partial_1^2 h\right).$$

Thus, (10) can be written as

$$\Delta\phi = 0 \qquad\qquad \text{in } \Omega(t) \times [0, T], \tag{11a}$$

$$\phi = \xi \qquad\qquad \text{on } \Gamma(t) \times [0, T], \tag{11b}$$

$$\xi_t = \partial_2\phi\left(\nabla\phi \cdot (-\partial_1 h, 1) + \delta_2\partial_1^2 h\right) - \frac{1}{2}|\nabla\phi|^2 - Gh + \frac{\gamma}{\rho}\mathcal{K} - \frac{\delta_1}{\rho}\mathscr{D}^s\phi \quad \text{on } \Gamma(t) \times [0, T], \tag{11c}$$

$$h_t = \nabla\phi \cdot \left(1 + (\partial_1 h)^2\right)^{1/2} n + \delta_2\partial_1^2 h \qquad\qquad \text{on } \Gamma(t) \times [0, T]. \tag{11d}$$

The system (11) is supplemented with an initial condition for h and ξ:

$$h(x,0) = h_0(x), \tag{12}$$

$$\xi(x,0) = \phi(x, h(0,x), 0) = \xi_0(x). \tag{13}$$

2.2. Nondimensional Eulerian formulation.

We denote by H and L the typical amplitude and wavelength of the water wave. We change to dimensionless variables (denoted with $\tilde{\cdot}$)

$$x = L\,\tilde{x}, \qquad\qquad t = \sqrt{\frac{L}{G}}\,\tilde{t}, \tag{14}$$

and unknowns

$$h(x_1,t) = H\,\tilde{h}(\tilde{x}_1,\tilde{t}), \qquad\qquad \phi(x_1,x_2,t) = H\sqrt{GL}\tilde{\phi}(\tilde{x}_1,\tilde{x}_2,\tilde{t}). \tag{15}$$

with the non-dimensionalized fluid domain

$$\widetilde{\Omega}(t) = \left\{ (\tilde{x}_1,\tilde{x}_2) \ \Big| \ -\pi < \tilde{x}_1 < \pi\,, -\infty < \tilde{x}_2 < \frac{H}{L}\tilde{h}(\tilde{x}_1,t)\,, \ t \in [0,T] \right\},$$

$$\widetilde{\Gamma}(t) = \left\{ \left(\tilde{x}_1, \frac{H}{L}\tilde{h}(\tilde{x}_1,t) \right)\,, \ t \in [0,T] \right\}$$

We find the following dimensionless parameters:

$$\varepsilon = \frac{H}{L}, \qquad \alpha_1^s = \frac{\delta_1}{\rho\sqrt{G}L^{s-\frac{1}{2}}}, \qquad \alpha_2 = \frac{\delta_2}{\sqrt{G}L^{3/2}}. \qquad \beta = \frac{\gamma}{\rho G L^2}, \tag{16}$$

where $s = 0$ if $\mathscr{D}\phi = \phi$ and $s = 2$ if $\mathscr{D} = \partial_2^2\phi$. The first parameter is known as the *steepness parameter* and measures the ratio between the amplitude and the wavelength of the wave. The $\alpha's$ consider the ratio between gravity and viscosity forces. Finally, the fourth one is the Bond number that compares gravity forces with capillary forces. Dropping the tildes for the sake of clarity, we have the following dimensionless form of the damped water waves problem

$$\Delta\phi = 0 \qquad\qquad\qquad\qquad \text{in } \Omega(t) \times [0,T], \tag{17a}$$

$$\phi = \xi \qquad\qquad\qquad\qquad \text{on } \Gamma(t) \times [0,T], \tag{17b}$$

$$\xi_t = -\frac{\varepsilon}{2}|\nabla\phi|^2 - h + \frac{\beta\partial_1^2 h}{\left(1 + (\varepsilon\partial_1 h)^2\right)^{3/2}} - \alpha_1^s\mathscr{D}^s\phi$$

$$\qquad\qquad + \varepsilon\partial_2\phi\left(\nabla\phi\cdot(-\varepsilon\partial_1 h, 1) + \alpha_2\partial_1^2 h\right) \qquad \text{on } \Gamma(t) \times [0,T], \tag{17c}$$

$$h_t = \nabla\phi\cdot(-\varepsilon\partial_1 h, 1) + \alpha_2\partial_1^2 h \qquad\qquad \text{on } \Gamma(t) \times [0,T], \tag{17d}$$

where we have used the nondimensional parameters (16).

2.3. The equations in the Arbitrary Lagrangian-Eulerian formulation.

In the present section we want to express system (17) on the reference domain Ω and reference interface Γ

$$\Omega = \mathbb{S}^1 \times (-\infty, 0)\,, \qquad\qquad \Gamma = \mathbb{S}^1 \times \{0\}\,. \tag{18}$$

The easiest way to do so is, supposing that h is regular, defining the following family (parametrized in $t \in [0,T]$) of diffeomorphisms

$$\begin{aligned} \psi: \quad [0,T] \times \Omega \quad &\to \quad \Omega(t)\,, \\ (x_1, x_2, t) \quad &\mapsto \quad \psi(x_1, x_2, t) = (x_1, x_2 + \varepsilon h(x_1, t))\,. \end{aligned}$$

We compute

$$\nabla\psi = \left(\begin{array}{cc} 1 & 0 \\ \varepsilon\partial_1 h\,(x_1,t) & 1 \end{array} \right), \qquad A = (\nabla\psi)^{-1} = \left(\begin{array}{cc} 1 & 0 \\ -\varepsilon\partial_1 h\,(x_1,t) & 1 \end{array} \right). \qquad (19)$$

With such map we can define the push-back of any application θ defined on $\Omega\,(t)$ simply as $\Theta = \theta \circ \psi$, whence in particular we define

$$\Phi = \phi \circ \psi, \ .$$

We let $N = e_2$ denote the outward unit normal to Ω at Γ. We also recall that, if $\Theta = \theta \circ \psi$, the following formula holds

$$\partial_j \theta \circ \psi = A_j^k \partial_k \Theta,$$

where Einstein convention is used. Then, we can rewrite (17) as the following system of variable coefficients nonlinear PDEs posed on a fixed reference domain

$$A_j^\ell \partial_\ell \left(A_j^k \partial_k \Phi \right) = 0 \qquad\qquad \text{in } \Omega \times [0,T], \quad (20a)$$

$$\Phi = \xi \qquad\qquad \text{on } \Gamma \times [0,T], \quad (20b)$$

$$\xi_t = -\frac{\varepsilon}{2} A_j^k \partial_k \Phi A_j^\ell \partial_\ell \Phi - h + \frac{\beta\partial_1^2 h}{\left(1 + (\varepsilon\partial_1 h)^2\right)^{3/2}} - \alpha_1^s \mathcal{D}^s \Phi$$

$$+ \varepsilon A_2^k \partial_k \Phi \left(A_j^\ell \partial_\ell \Phi A_j^2 + \alpha_2\partial_1^2 h \right) \qquad\qquad \text{on } \Gamma \times [0,T], \quad (20c)$$

$$h_t = A_j^k \partial_k \Phi A_j^2 + \alpha_2\partial_1^2 h \qquad\qquad \text{on } \Gamma \times [0,T], \quad (20d)$$

where the operator $\mathcal{D}^s$ is

$$\mathcal{D}^0 \Phi = \xi, \ \mathcal{D}^2 \Phi = A_2^\ell \partial_\ell \left(A_2^k \partial_k \Phi \right).$$

Next we explicit the values of the A_j^i's in the above system (see (19)) obtaining hence

$$\Delta\Phi = \varepsilon \left(\partial_1^2 h\,\partial_2\Phi + 2\partial_1 h\,\partial_{12}\Phi \right) - \varepsilon^2(\partial_1 h)^2\partial_2^2\Phi, \qquad \text{in } \Omega \times [0,T], \qquad (21a)$$

$$\Phi = \xi \qquad\qquad \text{on } \Gamma \times [0,T], \qquad (21b)$$

$$\xi_t = -\frac{\varepsilon}{2} \left[(\partial_1\Phi)^2 + (\varepsilon\partial_1 h\partial_2\Phi)^2 + (\partial_2\Phi)^2 - 2\varepsilon\partial_1 h\partial_2\Phi\partial_1\Phi \right]$$

$$- h + \frac{\beta\partial_1^2 h}{\left(1 + (\varepsilon\partial_1 h)^2\right)^{3/2}} - \alpha_1^s \mathcal{D}^s \Phi$$

$$+ \varepsilon\partial_2\Phi \left(-\varepsilon\partial_1 h\partial_1\Phi + \varepsilon^2(\partial_1 h)^2\partial_2\Phi + \partial_2\Phi + \alpha_2\partial_1^2 h \right) \qquad \text{on } \Gamma \times [0,T], \qquad (21c)$$

$$h_t = -\varepsilon\partial_1 h\partial_1\Phi + \varepsilon^2(\partial_1 h)^2\partial_2\Phi + \partial_2\Phi + \alpha_2\partial_1^2 h \qquad \text{on } \Gamma \times [0,T], \qquad (21d)$$

where

$$\mathcal{D}^0 \Phi = \Phi, \ \mathcal{D}^2 \Phi = \partial_2^2\Phi.$$

3. The asymptotic model for damped water waves when $s = 0$

In this section we consider the case $s = 0$ (the model by Jiang, Ting, Perlin & Schultz [16] and Wu, Liu & Yue [38]). In this case we have that

$$\mathcal{D}^0 \Phi = \Phi.$$

We introduce the following ansatz:

$$\Phi\left(x_1, x_2, t\right) = \sum_n \varepsilon^n \Phi^{(n)}\left(x_1, x_2, t\right),$$

$$\xi\left(x_1, t\right) = \sum_n \varepsilon^n \xi^{(n)}\left(x_1, t\right), \tag{22}$$

$$h\left(x_1, t\right) = \sum_n \varepsilon^n h^{(n)}\left(x_1, t\right).$$

With this ansatz we can re-profile the nonlinear system (21) in an equivalent sequence of linear systems where the evolution of the n-th profile is determined by the evolution of the preceding $n-1$ profiles.

We are interested in a model approximating (21) with an error $\mathcal{O}(\varepsilon^2)$. Using that

$$\frac{1}{\left(1+x^2\right)^{3/2}} = 1 + \mathcal{O}(x^2),$$

we obtain that

$$\frac{\beta \partial_1^2 h}{\left(1+\left(\varepsilon \partial_1 h\right)^2\right)^{3/2}} = \beta \partial_1^2 h + \mathcal{O}(\varepsilon^2).$$

For the case $n = 0$, we have that

$$\Delta \Phi^{(0)} = 0, \qquad\qquad\qquad \text{in } \Omega \times [0, T], \tag{23a}$$

$$\Phi^{(0)} = \xi^{(0)} \qquad\qquad\qquad \text{on } \Gamma \times [0, T], \tag{23b}$$

$$\xi_t^{(0)} = -h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Phi^{(0)} \qquad\qquad \text{on } \Gamma \times [0, T], \tag{23c}$$

$$h_t^{(0)} = \partial_2 \Phi^{(0)} + \alpha_2 \partial_1^2 h^{(0)} \qquad\qquad \text{on } \Gamma \times [0, T]. \tag{23d}$$

Recalling that

$$\widehat{\Phi^{(0)}}\left(k, x_2, t\right) = \xi^{(0)}(k, t) e^{|k| x_2} \qquad \text{in } \Omega \times [0, T]$$

so

$$\partial_2 \Phi^{(0)} = \Lambda \xi^{(0)} \qquad \text{on } \Gamma,$$

we find that (23d) can be equivalently written as

$$h_{tt}^{(0)} = \Lambda\left(-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \xi^{(0)}\right) + \alpha_2 \partial_1^2 h_t^{(0)} \qquad\qquad \text{on } \Gamma \times [0, T].$$

We note that (23d) can be equivalently wwritten as

$$\xi^{(0)} = \Lambda^{-1}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right], \tag{24}$$

thus,

$$h_{tt}^{(0)} = \Lambda\left(-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Lambda^{-1}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right) + \alpha_2 \partial_1^2 h_t^{(0)} \qquad \text{on } \Gamma \times [0, T]. \tag{25}$$

Similarly, in the case $n = 1$, we find that

$$\Delta \Phi^{(1)} = \partial_1^2 h^{(0)} \, \partial_2 \Phi^{(0)} + 2 \partial_1 h^{(0)} \, \partial_{12} \Phi^{(0)}, \qquad\qquad \text{in } \Omega \times [0, T], \tag{26a}$$

$$\Phi^{(1)} = \xi^{(1)} \qquad\qquad\qquad \text{on } \Gamma \times [0, T], \tag{26b}$$

$$\xi_t^{(1)} = \frac{1}{2}\left[\left(\partial_2 \Phi^{(0)}\right)^2 - \left(\partial_1 \Phi^{(0)}\right)^2\right]$$

$$- h^{(1)} + \beta \partial_1^2 h^{(1)} - \alpha_1^0 \Phi^{(1)} + \alpha_2 \partial_2 \Phi^{(0)} \partial_1^2 h^{(0)} \qquad\qquad \text{on } \Gamma \times [0, T], \tag{26c}$$

$$h_t^{(1)} = -\partial_1 h^{(0)} \partial_1 \Phi^{(0)} + \partial_2 \Phi^{(1)} + \alpha_2 \partial_1^2 h^{(1)} \qquad\qquad \text{on } \Gamma \times [0, T], \qquad (26\text{d})$$

Let us define

$$b = \partial_1^2 h^{(0)} \partial_2 \Phi^{(0)} + 2\partial_1 h^{(0)} \partial_{12} \Phi^{(0)},$$

We now use Lemma A.1 in order to compute

$$\partial_2 \widehat{\Phi^{(1)}}(k, 0, t) = \int_{-\infty}^{0} \hat{b}(k, y_2, t) e^{|k|y_2} \mathrm{d}y_2 + |k| \widehat{\xi^{(1)}}(k, t).$$

We want to provide an explicit expression for the term $\int_{-\infty}^{0} \hat{b}(k, y_2, t) e^{|k|y_2} \mathrm{d}y_2$ considering the form of b. We compute that

$$\int_{-\infty}^{0} \hat{b}(k, y_2, t) e^{|k|y_2} \mathrm{d}y_2 = -\int_{-\infty}^{0} e^{(|k|+|m|)y_2}(k-m)(k+m)|m| \widehat{h^{(0)}}(k-m) \widehat{\xi^{(0)}}(m) \, \mathrm{d}y_2,$$

$$= -\frac{|m|\left[|k|^2 - |m|^2\right]}{|k| + |m|} \widehat{h^{(0)}}(k-m) \widehat{\xi^{(0)}}(m),$$

$$= -|m|\left[|k| - |m|\right] \widehat{h^{(0)}}(k-m) \widehat{\xi^{(0)}}(m).$$

Thus, we find that

$$\partial_2 \Phi^{(1)}\Big|_{x_2=0} = \Lambda \xi^{(1)} - [\![\Lambda, h^{(0)}]\!] \Lambda \xi^{(0)}. \qquad (27)$$

The evolution equations for $h^{(1)}$ and $\xi^{(1)}$ become hence

$$h_t^{(1)} = -\partial_1 h^{(0)} \partial_1 \xi^{(0)} + \Lambda \xi^{(1)} - [\![\Lambda, h^{(0)}]\!] \Lambda \xi^{(0)} + \alpha_2 \partial_1^2 h^{(1)}, \qquad (28)$$

$$\xi_t^{(1)} = \frac{1}{2}\left[\left(\Lambda \xi^{(0)}\right)^2 - \left(\partial_1 \xi^{(0)}\right)^2\right]$$
$$- h^{(1)} + \beta \partial_1^2 h^{(1)} - \alpha_1^0 \Phi^{(1)} + \alpha_2 \Lambda \xi^{(0)} \partial_1^2 h^{(0)}. \qquad (29)$$

Using the above equation for $h_t^{(1)}$ (28) we can express $\xi^{(1)}$ as a function of $h^{(0)}$, $\xi^{(0)}$ and $h^{(1)}$ as follows

$$\xi^{(1)} = \Lambda^{-1}\left[h_t^{(1)} + \partial_1 h^{(0)} \partial_1 \xi^{(0)} + [\![\Lambda, h^{(0)}]\!] \Lambda \xi^{(0)} - \alpha_2 \partial_1^2 h^{(1)}\right]. \qquad (30)$$

Time differentiating (28) and inserting (29), we deduce

$$h_{tt}^{(1)} = -\partial_1 h_t^{(0)} \partial_1 \xi^{(0)} - \partial_1 h^{(0)} \partial_1 \xi_t^{(0)} + \frac{1}{2}\Lambda\left[\left(\Lambda \xi^{(0)}\right)^2 - \left(\partial_1 \xi^{(0)}\right)^2\right]$$
$$- \Lambda h^{(1)} + \beta \Lambda \partial_1^2 h^{(1)} - \alpha_1^0 \Lambda \Phi^{(1)} + \alpha_2 \Lambda\left(\Lambda \xi^{(0)} \partial_1^2 h^{(0)}\right)$$
$$- [\![\Lambda, h_t^{(0)}]\!] \Lambda \xi^{(0)} - [\![\Lambda, h^{(0)}]\!] \Lambda \xi_t^{(0)} + \alpha_2 \partial_1^2 h_t^{(1)}.$$

Recalling the definition of the Riesz potential Λ^{-1} and using (23c) and (24) in order to express $\xi^{(0)}$ and $\xi_t^{(0)}$ in terms of $h^{(0)}$, we find that

$$h_{tt}^{(1)} = \partial_1 h_t^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \partial_1 h^{(0)} \partial_1\left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Phi^{(0)}\right]$$
$$+ \frac{1}{2}\Lambda\left\{\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]^2 - \left(\mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right)^2\right\}$$
$$- \Lambda h^{(1)} + \beta \Lambda \partial_1^2 h^{(1)} - \alpha_1^0 \Lambda \Phi^{(1)} + \alpha_2 \Lambda\left[\left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) \partial_1^2 h^{(0)}\right]$$

$$- [\![\Lambda, h_t^{(0)}]\!] \left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) - [\![\Lambda, h^{(0)}]\!] \Lambda \left(-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Phi^{(0)}\right) + \alpha_2 \partial_1^2 h_t^{(1)}.$$

Using Tricomi identity

$$(\mathcal{H}f)^2 - f^2 = 2\mathcal{H}(f\mathcal{H}f),\tag{31}$$

the previous equation can be further simplified and we find that

$$h_{tt}^{(1)} = \partial_1 h_t^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \partial_1 h^{(0)} \partial_1 \left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Phi^{(0)}\right]$$

$$+ \partial_1 \left\{\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right\}$$

$$- \Lambda h^{(1)} + \beta \Lambda \partial_1^2 h^{(1)} - \alpha_1^0 \Lambda \Phi^{(1)} + \alpha_2 \Lambda \left[\left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) \partial_1^2 h^{(0)}\right]$$

$$- [\![\Lambda, h_t^{(0)}]\!] \left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) - [\![\Lambda, h^{(0)}]\!] \Lambda \left(-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Phi^{(0)}\right) + \alpha_2 \partial_1^2 h_t^{(1)}.$$

We can express $\alpha_1^0 \Phi^{(0)}$ in terms of $h^{(0)}$ as follows

$$\alpha_1^0 \Phi^{(0)} \Big|_{x_2=0} = \alpha_1^0 \xi^{(0)} = \alpha_1^0 \Lambda^{-1} \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right],$$

and, inserting the previous formula into (30), we find that

$$\alpha_1^0 \Phi^{(1)} \Big|_{x_2=0} = \alpha_1^0 \xi^{(1)},$$

$$= \alpha_1^0 \Lambda^{-1} \left[h_t^{(1)} + \partial_1 h^{(0)} \partial_1 \xi^{(0)} + [\![\Lambda, h^{(0)}]\!] \Lambda \xi^{(0)} - \alpha_2 \partial_1^2 h^{(1)}\right],$$

$$= \alpha_1^0 \Lambda^{-1} \left\{h_t^{(1)} - \partial_1 h^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] + [\![\Lambda, h^{(0)}]\!] \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \alpha_2 \partial_1^2 h^{(1)}\right\}.$$

Substituting the previous expressions into the equation for $h_{tt}^{(1)}$, we deduce the following equation:

$$h_{tt}^{(1)} = \partial_1 h_t^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \partial_1 h^{(0)} \partial_1 \left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Lambda^{-1}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right]$$

$$+ \partial_1 \left\{\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right\} - \Lambda h^{(1)} + \beta \Lambda \partial_1^2 h^{(1)}$$

$$- \alpha_1^0 \left\{h_t^{(1)} - \partial_1 h^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] + [\![\Lambda, h^{(0)}]\!] \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \alpha_2 \partial_1^2 h^{(1)}\right\}$$

$$+ \alpha_2 \Lambda \left[\left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) \partial_1^2 h^{(0)}\right] - [\![\Lambda, h_t^{(0)}]\!] \left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right)$$

$$- [\![\Lambda, h^{(0)}]\!] \left(-\Lambda h^{(0)} + \beta \Lambda \partial_1^2 h^{(0)} - \alpha_1^0 \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right) + \alpha_2 \partial_1^2 h_t^{(1)}.$$

We are going now to move the linear part of the above equation on the left hand side to better understand the interactions between different scales $h^{(j)}$. This leads us to

$$h_{tt}^{(1)} + \Lambda h^{(1)} + \beta \Lambda^3 h^{(1)} + \alpha_1^0 h_t^{(1)} - \alpha_1^0 \alpha_2 \partial_1^2 h^{(1)} - \alpha_2 \partial_1^2 h_t^{(1)} = \partial_1 h_t^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]$$

$$- \partial_1 h^{(0)} \left[-\partial_1 h^{(0)} + \beta \partial_1^3 h^{(0)} + \alpha_1^0 \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right]$$

$$+ \partial_1 \left\{\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right\}$$

$$- \alpha_1^0 \left\{-\partial_1 h^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] + [\![\Lambda, h^{(0)}]\!] \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right\}$$

$$+ \alpha_2 \Lambda \left[\left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) \partial_1^2 h^{(0)}\right] - [\![\Lambda, h_t^{(0)}]\!] \left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right)$$

$$- [\![\Lambda, h^{(0)}]\!] \left(-\Lambda h^{(0)} + \beta \Lambda \partial_1^2 h^{(0)} - \alpha_1^0 \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right] \right). \quad (32)$$

We group the nonlinear terms according to the coefficient in front. At $\mathcal{O}(1)$ we find that

$$\partial_1 h_t^{(0)} \mathcal{H} h_t^{(0)} + \left(\partial_1 h^{(0)} \right)^2 + \frac{\Lambda}{2} \left\{ \left[h_t^{(0)} \right]^2 - \left(\mathcal{H} h_t^{(0)} \right)^2 \right\} - [\![\Lambda, h_t^{(0)}]\!] h_t^{(0)} + [\![\Lambda, h^{(0)}]\!] \Lambda h^{(0)}$$

$$= -\Lambda \left(\left(\mathcal{H} h_t^{(0)} \right)^2 \right) + \Lambda (h^{(0)} \Lambda h^{(0)}) + \partial_1 \left(h^{(0)} \partial_1 h^{(0)} \right)$$

$$= -\Lambda \left(\left(\mathcal{H} h_t^{(0)} \right)^2 \right) + \partial_1 [\![\mathcal{H}, h^{(0)}]\!] \Lambda h^{(0)}, \quad (33)$$

where we have used the identity (31). At $\mathcal{O}(\beta)$ we obtain that

$$\beta \left(-\partial_1 h^{(0)} \partial_1^3 h^{(0)} - [\![\Lambda, h^{(0)}]\!] \Lambda \partial_1^2 h^{(0)} \right) = \beta \left(\Lambda \left(h^{(0)} \Lambda^3 h^{(0)} \right) - \partial_1 \left(h^{(0)} \partial_1^3 h^{(0)} \right) \right)$$

$$= \beta \partial_1 [\![\mathcal{H}, h^{(0)}]\!] \Lambda^3 h^{(0)}. \quad (34)$$

At $\mathcal{O}(\alpha_2)$ we find the following contribution

$$- \alpha_2 \left[\partial_1 h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + \partial_1 \left\{ h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right\} + \partial_1 \left\{ \partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)} \right\} \right.$$

$$\left. - \Lambda \left[h_t^{(0)} \partial_1^2 h^{(0)} \right] - [\![\Lambda, h_t^{(0)}]\!] \partial_1^2 h^{(0)} \right]$$

$$= -\alpha_2 \left[\partial_1 h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + \partial_1 \left\{ h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right\} + \partial_1 \left\{ \partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)} \right\} \right.$$

$$\left. - 2\Lambda \left[h_t^{(0)} \partial_1^2 h^{(0)} \right] + h_t^{(0)} \Lambda \partial_1^2 h^{(0)} \right]. \quad (35)$$

Using

$$\mathcal{H} f \mathcal{H} g - \mathcal{H} \left(f \mathcal{H} g + g \mathcal{H} f \right) = fg, \quad (36)$$

and

$$-\Lambda \mathcal{H} = \partial_1,$$

we find that

$$\Lambda \left[h_t^{(0)} \partial_1^2 h^{(0)} \right] = \Lambda \left(\mathcal{H} h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right) + \partial_1 \left(h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + \partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)} \right).$$

Thus, we can group terms in (35) as follows

$$- \alpha_2 \left[\partial_1 \left(h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right) - 2\Lambda \left(\mathcal{H} h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right) - \partial_1 \left(h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + \partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)} \right) \right]$$

$$= -\alpha_2 \left[-2\Lambda \left(\mathcal{H} h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right) - \partial_1 \left(\partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)} \right) \right]$$

$$= \alpha_2 \partial_1 [\![\mathcal{H}, \mathcal{H} h_t^{(0)}]\!] \mathcal{H} \partial_1^2 h^{(0)} + \alpha_2 \Lambda \left(\mathcal{H} h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right). \quad (37)$$

At $\mathcal{O}(\alpha_2 \alpha_2)$, we find that

$$\alpha_2^2 \left[\partial_1 \left\{ \partial_1^2 h^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right\} - \Lambda \left[\left(\partial_1^2 h^{(0)} \right)^2 \right] \right] = -\alpha_2^2 \partial_1 [\![\mathcal{H}, \partial_1^2 h^{(0)}]\!] \partial_1^2 h^{(0)}. \quad (38)$$

We group now the $\mathcal{O}(\alpha_1^0)$ terms:

$$\alpha_1^0 \left[-\partial_1 h^{(0)} \mathcal{H} h_t^{(0)} + \partial_1 h^{(0)} \mathcal{H} h_t^{(0)} - [\![\Lambda, h^{(0)}]\!] h_t^{(0)} + [\![\Lambda, h^{(0)}]\!] h_t^{(0)} \right] = 0. \tag{39}$$

Finally, we are left with the $\mathcal{O}(\alpha_1^0 \alpha_2)$ terms. These terms are

$$\alpha_1^0 \alpha_2 \left[\partial_1 h^{(0)} \mathcal{H} \partial_1^2 h^{(0)} - \partial_1 h^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + [\![\Lambda, h^{(0)}]\!] \partial_1^2 h^{(0)} - [\![\Lambda, h^{(0)}]\!] \partial_1^2 h^{(0)} \right] = 0. \tag{40}$$

Thus, using (33), (34), (37), (38), (39) and (40), we conclude that

$$h_{tt}^{(1)} + \Lambda h^{(1)} + \beta \Lambda^3 h^{(1)} + \alpha_1^0 h_t^{(1)} - \alpha_1^0 \alpha_2 \partial_1^2 h^{(1)} - \alpha_2 \partial_1^2 h_t^{(1)}$$
$$= -\Lambda \left(\left(\mathcal{H} h_t^{(0)} \right)^2 \right) + \partial_1 [\![\mathcal{H}, h^{(0)}]\!] \Lambda h^{(0)} + \beta \partial_1 [\![\mathcal{H}, h^{(0)}]\!] \Lambda^3 h^{(0)} + \alpha_2 \partial_1 [\![\mathcal{H}, \mathcal{H} h_t^{(0)}]\!] \mathcal{H} \partial_1^2 h^{(0)}$$
$$+ \alpha_2 \Lambda \left(\mathcal{H} h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} \right) - \alpha_2^2 \partial_1 [\![\mathcal{H}, \partial_1^2 h^{(0)}]\!] \partial_1^2 h^{(0)}.$$

We define the renormalized variable

$$f = h^{(0)} + \varepsilon h^{(1)}. \tag{41}$$

Using

$$\varepsilon h^{(0)} = \varepsilon f + \mathcal{O}(\varepsilon^2),$$

and neglecting errors $\mathcal{O}(\varepsilon^2)$, we conclude the following model:

$$f_{tt} + \Lambda f + \beta \Lambda^3 f + \alpha_1^0 f_t - \alpha_1^0 \alpha_2 \partial_1^2 f - \alpha_2 \partial_1^2 f_t$$
$$= \varepsilon \left[-\Lambda \left((\mathcal{H} f_t)^2 \right) + \partial_1 [\![\mathcal{H}, f]\!] \Lambda f + \beta \partial_1 [\![\mathcal{H}, f]\!] \Lambda^3 f + \alpha_2 \partial_1 [\![\mathcal{H}, \mathcal{H} f_t]\!] \mathcal{H} \partial_1^2 f \right.$$
$$\left. + \alpha_2 \Lambda \left(\mathcal{H} f_t \mathcal{H} \partial_1^2 f \right) - \alpha_2^2 \partial_1 [\![\mathcal{H}, \partial_1^2 f]\!] \partial_1^2 f \right]. \tag{42}$$

When $\alpha_2 = 0$, equation (42) is an asymptotic model of the damped water waves system proposed by Jiang, Ting, Perlin & Schultz [16] and Wu, Liu & Yue [38]. Also, when $\alpha_2 = \alpha_1^0 = 0$, equation (42) recovers the quadratic $h-$model in [1, 2, 6, 25–27].

4. THE ASYMPTOTIC MODEL FOR DAMPED WATER WAVES WHEN $s = 2$

In this section we focus on the case $s = 2$ (the model by Dias, Dyachenko, and Zakharov [9]). In this case we have that

$$\mathcal{D}^2 \Phi = \partial_2^2 \Phi.$$

We use the ansatz (22) and follow the previous steps. The first term in the series solves

$$\Delta \Phi^{(0)} = 0, \qquad\qquad \text{in } \Omega \times [0, T], \tag{43a}$$

$$\Phi^{(0)} = \xi^{(0)} \qquad\qquad \text{on } \Gamma \times [0, T], \tag{43b}$$

$$\xi_t^{(0)} = -h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^2 \partial_2^2 \Phi^{(0)} \qquad\qquad \text{on } \Gamma \times [0, T], \tag{43c}$$

$$h_t^{(0)} = \partial_2 \Phi^{(0)} + \alpha_2 \partial_1^2 h^{(0)} \qquad\qquad \text{on } \Gamma \times [0, T]. \tag{43d}$$

Taking a time derivative of the equation (43d), using the fact that

$$\partial_2 \Phi^{(0)}\Big|_{x_2=0} = \Lambda \xi^{(0)} = h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}$$

and substituting (43c), we find that

$$h_{tt}^{(0)} = \Lambda \left(-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^2 \partial_2^2 \Phi^{(0)} \right) + \alpha_2 \partial_1^2 h_t^{(0)} \qquad \text{on } \Gamma \times [0, T].$$

Similarly, due to the fact that

$$\partial_2^2 \Phi^{(0)}\Big|_{x_2=0} = \Lambda^2 \xi^{(0)} = \Lambda \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right]$$

we find that the previous equation for h_{tt} can be written as

$$h_{tt}^{(0)} = -\Lambda h^{(0)} - \beta \Lambda^3 h^{(0)} + \alpha_1^2 \partial_1^2 h_t^{(0)} - \alpha_1^2 \alpha_2 \partial_1^4 h^{(0)} + \alpha_2 \partial_1^2 h_t^{(0)} \qquad \text{on } \Gamma \times [0, T]. \qquad (44)$$

Analogously as in (26), for $n = 1$, we find that

$$\Delta \Phi^{(1)} = \partial_1^2 h^{(0)} \, \partial_2 \Phi^{(0)} + 2\partial_1 h^{(0)} \, \partial_{12} \Phi^{(0)}, \qquad \text{in } \Omega \times [0, T], \qquad (45a)$$

$$\Phi^{(1)} = \xi^{(1)} \qquad \text{on } \Gamma \times [0, T], \qquad (45b)$$

$$\xi_t^{(1)} = \frac{1}{2} \left[(\partial_2 \Phi^{(0)})^2 - (\partial_1 \Phi^{(0)})^2 \right]$$
$$- h^{(1)} + \beta \partial_1^2 h^{(1)} - \alpha_1^2 \partial_2^2 \Phi^{(1)} + \alpha_2 \partial_2 \Phi^{(0)} \partial_1^2 h^{(0)} \qquad \text{on } \Gamma \times [0, T], \qquad (45c)$$

$$h_t^{(1)} = -\partial_1 h^{(0)} \partial_1 \Phi^{(0)} + \partial_2 \Phi^{(1)} + \alpha_2 \partial_1^2 h^{(1)} \qquad \text{on } \Gamma \times [0, T], \qquad (45d)$$

We use Lemma A.1 and (27) to find that

$$\partial_2 \Phi^{(1)}\Big|_{x_2=0} = \Lambda \xi^{(1)} - [\![\Lambda, h^{(0)}]\!] \Lambda \xi^{(0)}$$

$$\partial_2^2 \Phi^{(1)}\Big|_{x_2=0} = \Lambda^2 \xi^{(1)} + \partial_1^2 h^{(0)} \Lambda \xi^{(0)} + 2\partial_1 h^{(0)} \partial_1 \Lambda \xi^{(0)}.$$

Then we find the following system of equations

$$h_t^{(1)} = -\partial_1 h^{(0)} \partial_1 \xi^{(0)} + \Lambda \xi^{(1)} - [\![\Lambda, h^{(0)}]\!] \Lambda \xi^{(0)} + \alpha_2 \partial_1^2 h^{(1)}, \qquad (46)$$

$$\xi_t^{(1)} = \frac{1}{2} \left[\left(\Lambda \xi^{(0)} \right)^2 - \left(\partial_1 \xi^{(0)} \right)^2 \right]$$
$$- h^{(1)} + \beta \partial_1^2 h^{(1)} - \alpha_1^2 \left[\Lambda^2 \xi^{(1)} + \partial_1^2 h^{(0)} \Lambda \xi^{(0)} + 2\partial_1 h^{(0)} \partial_1 \Lambda \xi^{(0)} \right] + \alpha_2 \Lambda \xi^{(0)} \partial_1^2 h^{(0)}. \qquad (47)$$

These equations are the analog (when $s = 2$) of the equations (28) and (29).

As before, we want to reduce everything to a single equation for $h^{(1)}$ and $h^{(0)}$. Using (43d), we find that

$$\Lambda \xi^{(1)} = h_t^{(1)} - \partial_1 h^{(0)} \mathcal{H} \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right] - \alpha_2 \partial_1^2 h^{(1)} + [\![\Lambda, h^{(0)}]\!] \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right].$$

As a consequence, we have that

$$\alpha_1^2 \partial_2^2 \Phi^{(1)}\Big|_{x_2=0} = \alpha_1^2 \Lambda \left\{ h_t^{(1)} - \partial_1 h^{(0)} \mathcal{H} \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right] + [\![\Lambda, h^{(0)}]\!] \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right] - \alpha_2 \partial_1^2 h^{(1)} \right\}$$
$$+ \alpha_1^2 \left(\partial_1^2 h^{(0)} \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right] + 2\partial_1 h^{(0)} \partial_1 \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)} \right] \right).$$

Time differentiating (46) and inserting (47), we deduce

$$
\begin{aligned}
h_{tt}^{(1)} =\ & -\partial_1 h_t^{(0)}\partial_1\xi^{(0)} + \Lambda\xi_t^{(1)} - [\![\Lambda, h_t^{(0)}]\!]\Lambda\xi^{(0)} + \alpha_2\partial_1^2 h_t^{(1)} - \partial_1 h^{(0)}\partial_1\xi_t^{(0)} - [\![\Lambda, h^{(0)}]\!]\Lambda\xi_t^{(0)}, \\
=\ & \partial_1 h_t^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] - [\![\Lambda, h_t^{(0)}]\!]\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] \\
& + \alpha_2\partial_1^2 h_t^{(1)} - \partial_1 h^{(0)}\partial_1\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\Lambda\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right] \\
& - [\![\Lambda, h^{(0)}]\!]\Lambda\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\Lambda\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right], \\
& + \frac{1}{2}\Lambda\left[\left(\Lambda\xi^{(0)}\right)^2 - \left(\partial_1\xi^{(0)}\right)^2\right] \\
& + \Lambda\left[-h^{(1)} + \beta\partial_1^2 h^{(1)} - \alpha_1^2\left[\Lambda^2\xi^{(1)} + \partial_1^2 h^{(0)}\Lambda\xi^{(0)} + 2\partial_1 h^{(0)}\partial_1\Lambda\xi^{(0)}\right] + \alpha_2\Lambda\xi^{(0)}\partial_1^2 h^{(0)}\right] \\
=\ & \partial_1 h_t^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] - [\![\Lambda, h_t^{(0)}]\!]\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] \\
& + \alpha_2\partial_1^2 h_t^{(1)} - \partial_1 h^{(0)}\partial_1\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\Lambda\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right] \\
& - [\![\Lambda, h^{(0)}]\!]\Lambda\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\Lambda\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right], \\
& + \frac{1}{2}\Lambda\left[\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right)^2 - \left(\mathcal{H}h_t^{(0)} - \alpha_2\mathcal{H}\partial_1^2 h^{(0)}\right)^2\right] \\
& + \Lambda\Big\{-h^{(1)} + \beta\partial_1^2 h^{(1)} - \alpha_1^2\left[\partial_1^2 h^{(0)}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] + 2\partial_1 h^{(0)}\partial_1\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right] \\
& \qquad + \alpha_2\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\partial_1^2 h^{(0)}\Big\} \\
& - \alpha_1^2\Lambda^2\left\{h_t^{(1)} - \partial_1 h^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] - \alpha_2\partial_1^2 h^{(1)} + [\![\Lambda, h^{(0)}]\!]\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right\},
\end{aligned}
$$

where we have used the previous expression for $\Lambda\xi^{(1)}$. We move the linear terms to the left hand side and use Tricomi identity (31) to obtain

$$
\begin{aligned}
h_{tt}^{(1)} - (\alpha_1^2 + \alpha_2)\partial_1^2 h_t^{(1)} + \Lambda h^{(1)} &+ \beta\Lambda^3 h^{(1)} + \alpha_1^2\alpha_2\partial_1^4 h^{(1)} = \partial_1 h_t^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] \\
& - [\![\Lambda, h_t^{(0)}]\!]\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] - \partial_1 h^{(0)}\partial_1\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\Lambda\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right] \\
& - [\![\Lambda, h^{(0)}]\!]\Lambda\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\Lambda\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right] \\
& + \frac{1}{2}\Lambda\left[\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right)^2 - \left(\mathcal{H}h_t^{(0)} - \alpha_2\mathcal{H}\partial_1^2 h^{(0)}\right)^2\right] \\
& + \Lambda\Big\{-\alpha_1^2\left[\partial_1^2 h^{(0)}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] + 2\partial_1 h^{(0)}\partial_1\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right] \\
& \qquad + \alpha_2\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\partial_1^2 h^{(0)}\Big\} \\
& + \alpha_1^2\partial_1^2\left\{-\partial_1 h^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] + [\![\Lambda, h^{(0)}]\!]\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right\}.
\end{aligned}
$$

We group the different nonlinear contributions according to the coefficient in front: at $\mathcal{O}(1)$ we find (33), while at $\mathcal{O}(\beta)$ we have (34). Using Tricomi identity (31) to obtain

$$
\frac{1}{2}\Lambda\left[\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right)^2 - \left(\mathcal{H}h_t^{(0)} - \alpha_2\mathcal{H}\partial_1^2 h^{(0)}\right)^2\right] = \partial_1\left[\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right)\left(\mathcal{H}h_t^{(0)} - \alpha_2\mathcal{H}\partial_1^2 h^{(0)}\right)\right],
$$

we find that the $\mathcal{O}(\alpha_2)$ contribution is given by (35) and, as a consequence, it can be further simplify to conclude (37). At $O(\alpha_2\alpha_2)$ we have the terms (38). We collect now the $\mathcal{O}(\alpha_1^2)$ terms:

$$\alpha_1^2\Big[\partial_1 h^{(0)}\partial_1\Lambda h_t^{(0)} - \partial_1^2\left(\partial_1 h^{(0)}\mathcal{H}h_t^{(0)}\right) - [\![\Lambda, h^{(0)}]\!]\partial_1^2 h_t^{(0)} + \partial_1^2[\![\Lambda, h^{(0)}]\!]h_t^{(0)}$$

$$- \Lambda\left\{\partial_1^2 h^{(0)}h_t^{(0)} + 2\partial_1 h^{(0)}\partial_1 h_t^{(0)}\right\}\Big] = \alpha_1^2\Big[-\partial_1^3 h^{(0)}\mathcal{H}h_t^{(0)} - 2\partial_1^2 h^{(0)}\Lambda h_t^{(0)}$$

$$+ 2[\![\Lambda, \partial_1 h^{(0)}]\!]\partial_1 h_t^{(0)} + [\![\Lambda, \partial_1^2 h^{(0)}]\!]h_t^{(0)} - \Lambda\left\{\partial_1^2 h^{(0)}h_t^{(0)} + 2\partial_1 h^{(0)}\partial_1 h_t^{(0)}\right\}\Big]$$

$$= \alpha_1^2\Big[-\partial_1^3 h^{(0)}\mathcal{H}h_t^{(0)} - 2\partial_1^2 h^{(0)}\Lambda h_t^{(0)} - \partial_1^2 h^{(0)}\Lambda h_t^{(0)} - 2\partial_1 h^{(0)}\Lambda\partial_1 h_t^{(0)}\Big]$$

$$= -\alpha_1^2\partial_1\Big[\partial_1^2 h^{(0)}\mathcal{H}h_t^{(0)} + 2\partial_1 h^{(0)}\Lambda h_t^{(0)}\Big]$$

$$= -\alpha_1^2\partial_1\Big[\partial_1\left(\partial_1 h^{(0)}\mathcal{H}h_t^{(0)}\right) + \partial_1 h^{(0)}\Lambda h_t^{(0)}\Big]$$

$$= -\alpha_1^2\partial_1\Big[\partial_1^2\left(h^{(0)}\mathcal{H}h_t^{(0)}\right) - h^{(0)}\partial_1\Lambda h_t^{(0)}\Big]$$

$$= -\alpha_1^2\partial_1[\![\partial_1^2, h^{(0)}]\!]\mathcal{H}h_t^{(0)}. \quad (48)$$

Finally, we consider the $\mathcal{O}(\alpha_2\alpha_1^2)$ terms and obtain

$$\alpha_1^2\alpha_2\Big[-\partial_1 h^{(0)}\Lambda\partial_1^3 h^{(0)} + [\![\Lambda, h^{(0)}]\!]\partial_1^4 h^{(0)} - \partial_1^2[\![\Lambda, h^{(0)}]\!]\partial_1^2 h^{(0)}$$

$$+ \Lambda\left[\left(\partial_1^2 h^{(0)}\right)^2 + 2\partial_1 h^{(0)}\partial_1^3 h^{(0)}\right] + \partial_1^2\left(\partial_1 h^{(0)}\partial_1\Lambda h^{(0)}\right)\Big]$$

$$= \alpha_1^2\alpha_2\Big[-[\![\Lambda, \partial_1^2 h^{(0)}]\!]\partial_1^2 h^{(0)} - 2[\![\Lambda, \partial_1 h^{(0)}]\!]\partial_1^3 h^{(0)}$$

$$+ \Lambda\left[\left(\partial_1^2 h^{(0)}\right)^2 + 2\partial_1 h^{(0)}\partial_1^3 h^{(0)}\right] + \partial_1^3 h^{(0)}\partial_1\Lambda h^{(0)} + 2\partial_1^2 h^{(0)}\partial_1^2\Lambda h^{(0)}\Big]$$

$$= \alpha_1^2\alpha_2\Big[\partial_1^2 h^{(0)}\Lambda\partial_1^2 h^{(0)} + 2\partial_1 h^{(0)}\Lambda\partial_1^3 h^{(0)}$$

$$+ \partial_1^3 h^{(0)}\partial_1\Lambda h^{(0)} + 2\partial_1^2 h^{(0)}\partial_1^2\Lambda h^{(0)}\Big]$$

$$= \alpha_1^2\alpha_2\partial_1\Big[\partial_1^2 h^{(0)}\Lambda\partial_1 h^{(0)} + 2\partial_1 h^{(0)}\Lambda\partial_1^2 h^{(0)}\Big]$$

$$= \alpha_1^2\alpha_2\partial_1\Big[\partial_1\left(\partial_1 h^{(0)}\Lambda\partial_1 h^{(0)}\right) + \partial_1 h^{(0)}\Lambda\partial_1^2 h^{(0)}\Big]$$

$$= \alpha_1^2\alpha_2\partial_1\Big[\partial_1^2\left(h^{(0)}\Lambda\partial_1 h^{(0)}\right) - h^{(0)}\Lambda\partial_1^3 h^{(0)}\Big]$$

$$= \alpha_1^2\alpha_2\partial_1[\![\partial_1^2, h^{(0)}]\!]\Lambda\partial_1 h^{(0)}. \quad (49)$$

Collecting (33), (34), (37), (38), (48) and (49), we conclude the following equation for $h^{(1)}$

$$h_{tt}^{(1)} - (\alpha_1^2 + \alpha_2)\partial_1^2 h_t^{(1)} + \Lambda h^{(1)} + \beta\Lambda^3 h^{(1)} + \alpha_1^2\alpha_2\partial_1^4 h^{(1)}$$

$$= -\Lambda\left(\left(\mathcal{H}h_t^{(0)}\right)^2\right) + \partial_1\left[\!\left[\mathcal{H}, h^{(0)}\right]\!\right]\Lambda h^{(0)} + \beta\partial_1\left[\!\left[\mathcal{H}, h^{(0)}\right]\!\right]\Lambda^3 h^{(0)}$$

$$+ \alpha_2\partial_1\left[\!\left[\mathcal{H}, \mathcal{H}h_t^{(0)}\right]\!\right]\mathcal{H}\partial_1^2 h^{(0)} + \alpha_2\Lambda\left(\mathcal{H}h_t^{(0)}\mathcal{H}\partial_1^2 h^{(0)}\right) + \alpha_1^2\alpha_2\partial_1\left[\!\left[\partial_1^2, h^{(0)}\right]\!\right]\Lambda\partial_1 h^{(0)}$$

$$- \alpha_1^2\partial_1\left[\!\left[\partial_1^2, h^{(0)}\right]\!\right]\mathcal{H}h_t^{(0)} - \alpha_2\alpha_2\partial_1\left[\!\left[\mathcal{H}, \partial_1^2 h^{(0)}\right]\!\right]\partial_1^2 h^{(0)}.$$

Thus, neglecting errors of order $\mathcal{O}(\varepsilon^2)$, we conclude the following model for the renormalized variable (41):

$$f_{tt} - (\alpha_1^2 + \alpha_2)\partial_1^2 f_t + \Lambda f + \beta\Lambda^3 f + \alpha_1^2\alpha_2\partial_1^4 f$$

$$= \varepsilon\Bigg\{ -\Lambda\left((\mathcal{H}f_t)^2\right) + \partial_1[\![\mathcal{H}, f]\!]\Lambda f + \beta\partial_1[\![\mathcal{H}, f]\!]\Lambda^3 f$$

$$+ \alpha_2\partial_1[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_1^2 f + \alpha_2\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_1^2 f\right) + \alpha_1^2\alpha_2\partial_1\left[\!\left[\partial_1^2, f\right]\!\right]\Lambda\partial_1 f$$

$$- \alpha_1^2\partial_1\left[\!\left[\partial_1^2, f\right]\!\right]\mathcal{H}f_t - \alpha_2\alpha_2\partial_1\left[\!\left[\mathcal{H}, \partial_1^2 f\right]\!\right]\partial_1^2 f\Bigg\}. \quad (50)$$

When $\alpha_2 = \alpha_1^2$, equation (50) is an asymptotic model of the damped water waves system proposed by Dias, Dyachenko, and Zakharov [9]. Also, when $\alpha_2 = \alpha_1^2 = 0$, equation (50) again recovers the quadratic $h-$model in [1, 2, 6, 25–27].

5. Craig-Sulem models for damped water waves

The pioneer work of Craig & Sulem [8] (see also [28, 29]) lead, among other things, to several asymptotic models obtained by truncating a Taylor series for the Dirichlet-to-Neumann operator present in the Zakharov formulation of the water waves problem [41]. Probably the most famous model of this type is the Craig-Sulem WW2 (see [4, 6, 23]):

$$f_t = -\varepsilon\partial_1 f\partial_1\zeta + \Lambda\zeta - \varepsilon[\![\Lambda, f]\!]\Lambda\zeta, \quad (51)$$

$$\zeta_t = \frac{\varepsilon}{2}\left[(\Lambda\zeta)^2 - (\partial_1\zeta)^2\right] - f + \beta\partial_1^2 f. \quad (52)$$

Using Tricomi identity (31),

$$(\Lambda\zeta)^2 - (\partial_1\zeta)^2 = 2\mathcal{H}\left(\partial_1 f\Lambda f\right),$$

so the previous system can be equivalently written as

$$f_t = -\varepsilon\partial_1 f\partial_1\zeta + \Lambda\zeta - \varepsilon[\![\Lambda, f]\!]\Lambda\zeta, \quad (53)$$

$$\zeta_t = \varepsilon\mathcal{H}\left(\partial_1 f\Lambda f\right) - f + \beta\partial_1^2 f. \quad (54)$$

5.1. **Case $s = 0$.** Using (28) and (29) we find that, up to an error $\mathcal{O}(\varepsilon^2)$ the variables

$$f = h^{(0)} + \varepsilon h^{(1)}, \ \zeta = \xi^{(0)} + \varepsilon\xi^{(1)}, \quad (55)$$

solve the system

$$f_t = -\varepsilon\partial_1 f\partial_1\zeta + \Lambda\zeta - \varepsilon[\![\Lambda, f]\!]\Lambda\zeta + \alpha_2\partial_1^2 f, \quad (56)$$

$$\zeta_t = \varepsilon\mathcal{H}\left(\partial_1 f\Lambda f\right) - f + \beta\partial_1^2 f - \alpha_1^0\zeta + \alpha_2\varepsilon\Lambda\zeta\partial_1^2 f. \quad (57)$$

5.2. **Case** $s = 2$. Using (46) and (47), we also find the viscous analog (called Craig-Sulem WWV2 [3,18]) of the Craig-Sulem WW2 model corresponding for the model of Dias, Dyachenko, and Zakharov [9] of water waves with viscosity

$$f_t = -\varepsilon \partial_1 f \partial_1 \zeta + \Lambda \zeta - \varepsilon [\![\Lambda, f]\!] \Lambda \zeta + \alpha_2 \partial_1^2 f, \tag{58}$$

$$\zeta_t = \varepsilon \mathcal{H} \left(\partial_1 f \Lambda f \right) - f + \beta \partial_1^2 f - \alpha_1^2 \left(\Lambda^2 \zeta + \varepsilon \partial_1^2 f \Lambda \zeta + 2\varepsilon \partial_1 f \partial_1 \Lambda \zeta \right) + \alpha_2 \varepsilon \Lambda \zeta \partial_1^2 f. \tag{59}$$

6. Discussion

In this paper we have obtained a number of new models for damped water waves. In particular, we derived two nonlocal wave equations, namely,

$$
\begin{aligned}
f_{tt} &+ \Lambda f + \beta \Lambda^3 f + \alpha_1^0 f_t - \alpha_1^0 \alpha_2 \partial_1^2 f - \alpha_2 \partial_1^2 f_t \\
&= \varepsilon \Big[-\Lambda \left((\mathcal{H} f_t)^2 \right) + \partial_1 [\![\mathcal{H}, f]\!] \Lambda f + \beta \partial_1 [\![\mathcal{H}, f]\!] \Lambda^3 f + \alpha_2 \partial_1 [\![\mathcal{H}, \mathcal{H} f_t]\!] \mathcal{H} \partial_1^2 f \\
&\qquad\qquad\qquad\qquad + \alpha_2 \Lambda \left(\mathcal{H} f_t \mathcal{H} \partial_1^2 f \right) - \alpha_2^2 \partial_1 [\![\mathcal{H}, \partial_1^2 f]\!] \partial_1^2 f \Big],
\end{aligned}
\tag{60}
$$

and

$$
\begin{aligned}
f_{tt} &- (\alpha_1^2 + \alpha_2) \partial_1^2 f_t + \Lambda f + \beta \Lambda^3 f + \alpha_1^2 \alpha_2 \partial_1^4 f \\
&= \varepsilon \Big\{ -\Lambda \left((\mathcal{H} f_t)^2 \right) + \partial_1 [\![\mathcal{H}, f]\!] \Lambda f + \beta \partial_1 [\![\mathcal{H}, f]\!] \Lambda^3 f \\
&\quad + \alpha_2 \partial_1 [\![\mathcal{H}, \mathcal{H} f_t]\!] \mathcal{H} \partial_1^2 f + \alpha_2 \Lambda \left(\mathcal{H} f_t \mathcal{H} \partial_1^2 f \right) + \alpha_1^2 \alpha_2 \partial_1 [\![\partial_1^2, f]\!] \Lambda \partial_1 f \\
&\qquad\qquad\qquad\qquad - \alpha_1^2 \partial_1 [\![\partial_1^2, f]\!] \mathcal{H} f_t - \alpha_2 \alpha_2 \partial_1 [\![\mathcal{H}, \partial_1^2 f]\!] \partial_1^2 f \Big\}.
\end{aligned}
\tag{61}
$$

Equation (60) is an asymptotic model of the damped water waves system proposed by Jiang, Ting, Perlin & Schultz [16] and Wu, Liu & Yue [38], while equation (61) is an asymptotic model of the water waves with viscosity system proposed by Dias, Dyachenko, and Zakharov [9].

Appendix A. The explicit solution of an elliptic problem

Lemma A.1. *Let us consider the Poisson equation*

$$
\begin{cases}
\Delta u \left(x_1, x_2 \right) & = b \left(x_1, x_2 \right), \quad \left(x_1, x_2 \right) \in \mathbb{S}^1 \times (-\infty, 0), \\
u \left(x_1, 0 \right) & = g \left(x_1 \right), \qquad\quad x_1 \in \mathbb{S}^1, \\
\displaystyle\lim_{x_2 \to -\infty} \partial_2 u \left(x_1, x_2 \right) & = 0, \qquad\qquad\quad x_1 \in \mathbb{S}^1,
\end{cases}
\tag{62}
$$

where we assume that the forcing b and the boundary data g are smooth and decay sufficiently fast at infinity. Then, the unique solution u of (62) *is given by*

$$u\left(x_1, x_2\right) = -\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{|k|} \left[\frac{1}{2} \int_{-\infty}^{0} \hat{b}\left(k, y_2\right) e^{|k||y_2|} \mathrm{d}y_2 - |k|\,\hat{g}\left(k\right) \right] e^{|k|x_2} \right.$$
$$\left. -\frac{1}{2\,|k|} \int_{-\infty}^{0} \hat{b}\left(k, y_2\right) e^{|k||y_2|} \mathrm{d}y_2 \; e^{-|k|x_2} \right. \tag{63}$$
$$\left. +\int_{0}^{x_2} \frac{\hat{b}\left(k, y_2\right)}{2\,|k|} \left[e^{|k|(y_2-x_2)} - e^{|k|(x_2-y_2)} \right] \mathrm{d}y_2 \right\} e^{ikx_1},$$

where the operator $\hat{\cdot}$ *denotes the Fourier transform in the variable* x_1*. In particular*

$$\partial_2 u\left(x_1, 0\right) = \int_{-\infty}^{0} e^{y_2 \Lambda} b\left(x_1, y_2\right) \mathrm{d}y_2 + \Lambda g(x_1), \tag{64}$$

$$\partial_2^2 u\left(x_1, 0\right) = -\partial_1^2 g(x_1) + b(x_1, 0). \tag{65}$$

Proof. Let us apply the Fourier transform to the equation (62), this transforms the PDE (62) in the following series of second-order inhomogeneous costant coefficients ODE's

$$\begin{cases} -k^2 \hat{u}\left(k, x_2\right) + \partial_2^2 \hat{u}\left(k, x_2\right) = \hat{b}\left(k, x_2\right), & \left(k, x_2\right) \in \mathbb{Z} \times \left(-\infty, 0\right), \\ \hat{u}\left(k, 0\right) = \hat{g}\left(k\right), & k \in \mathbb{Z}, \\ \lim_{x_2 \to -\infty} \partial_2 \hat{u}\left(k, x_2\right) = 0, & k \in \mathbb{Z}. \end{cases} \tag{66}$$

The generic solution of (66) can be deduced using the variation of parameters method, whence

$$\hat{u}\left(k, x_2\right) = C_1\left(k\right) e^{|k|x_2} + C_2\left(k\right) e^{-|k|x_2} - \int_{0}^{x_2} \frac{\hat{b}\left(k, y_2\right)}{2\,|k|} \left[e^{|k|(y_2-x_2)} - e^{|k|(x_2-y_2)} \right] \mathrm{d}y_2. \tag{67}$$

The boundary conditions determine the values of the C_i's:

$$C_2\left(k\right) = -\frac{1}{2\,|k|} \int_{-\infty}^{0} \hat{b}\left(k, y_2\right) e^{|k|y_2} \mathrm{d}y_2, \qquad C_1\left(k\right) = -C_2\left(k\right) + \hat{g}\left(k\right). \tag{68}$$

We provide now the detailed computations for the sake of clarity. From the generic solution (67) we easily derive that $C_1 = -C_2 + \hat{g}$ simply setting $x_2 = 0$ and solving the resulting equation in C_1. Next we compute $\partial_2 u$, which gives

$$\partial_2 \hat{u}\left(k, x_2\right) = \left(-C_2\left(k\right) + \hat{g}\left(k\right)\right) |k|\, e^{|k|x_2} - C_2\left(k\right) |k|\, e^{-|k|x_2}$$
$$+\frac{1}{2} \int_{0}^{x_2} \hat{b}\left(k, y_2\right) \left[e^{|k|(y_2-x_2)} + e^{|k|(x_2-y_2)} \right] \mathrm{d}y_2. \tag{69}$$

Due to the negative weight on the exponential, we deduce that

$$\lim_{x_2 \to -\infty} \left[\left(-C_2\left(k\right) + \hat{g}\left(k\right)\right) |k|\, e^{|k|x_2} \right] = 0.$$

Let us now consider the limit

$$\lim_{x_2 \to -\infty} \left[\frac{1}{2} \int_{0}^{x_2} \hat{b}\left(k, y_2\right) e^{|k|(x_2-y_2)} \mathrm{d}y_2 \right].$$

We prove now that such limit is equal to zero by dominated convergence. Let us consider the family of functions

$$\left(f_{k,x_2}\left(y_2\right)\right)_{x_2\in\mathbb{R}_-} = \left(1_{[x_2,0]}\left(y_2\right)\hat{b}\left(k,y_2\right)e^{|k|(x_2-y_2)}\right)_{x_2\in\mathbb{R}_-},$$

Since every element of such family is nonzero only when $y_2 \in [x_2, 0]$ we know that $e^{|k|(x_2-y_2)} \leqslant 1$, hence every f_{k,x_2} can be pointwise bounded by

$$f_{k,x_2}\left(y_2\right) \leqslant \left|\hat{b}\left(k,y_2\right)\right|,$$

uniformly in x_2. Moreover we assumed $b \in L^2\left(\mathbb{S}^1; L^1\left(\mathbb{R}_-\right)\right)$, hence for every k we have that $\hat{b}\left(k,\cdot\right) \in L^1\left(\mathbb{R}_-\right)$ and we can indeed apply the Lebesgue dominated convergence theorem in order to deduce

$$\lim_{x_2\to-\infty}\left[\frac{1}{2}\int_0^{x_2}\hat{b}\left(k,y_2\right)e^{|k|(x_2-y_2)}\mathrm{d}y_2\right] = 0,$$

for every $k \in \mathbb{Z}$. What remains is the following equality

$$\lim_{x_2\to-\infty}\left[-C_2\left(k\right)|k|\,e^{-|k|x_2} + \frac{1}{2}\int_0^{x_2}\hat{b}\left(k,y_2\right)e^{|k|(y_2-x_2)}\mathrm{d}y_2\right] = 0,$$

which in turn gives the required constant

$$C_2\left(k\right) = -\frac{1}{2\,|k|}\int_{-\infty}^0\hat{b}\left(k,y_2\right)e^{|k|y_2}\mathrm{d}y_2.$$

Setting $x_2 = 0$ in (69) we find that

$$\partial_2\hat{u}\left(k,0\right) = -2\,|k|\,C_2\left(k\right) + |k|\,\hat{g}\left(k\right), \tag{70}$$

which reduces to (64). We now differentiate (69) in x_2 obtaining

$$\partial_2^2\hat{u}\left(k,x_2\right) = \left(-C_2\left(k\right) + \hat{g}\left(k\right)\right)|k|^2\,e^{|k|x_2} + C_2\left(k\right)|k|^2\,e^{-|k|x_2}$$
$$+ \hat{b}\left(k,x_2\right) - \frac{|k|}{2}\int_0^{x_2}\hat{b}\left(k,y_2\right)\left[e^{|k|(y_2-x_2)} - e^{|k|(x_2-y_2)}\right]\mathrm{d}y_2. \tag{71}$$

Fixing $x_2 = 0$ in (71) the previous equation simplifies to

$$\partial_2^2\hat{u}\left(k,0\right) = |k|^2\,\hat{g}\left(k\right) + \hat{b}\left(k,x_2\right),$$

which proves (65). □

Acknowledgments

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