Characterising the Rankings Produced by Combinatorial Optimisation Problems and Finding their Intersections

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ABSTRACT

The aim of this paper is to introduce the concept of intersection between combinatorial optimisation problems. We take into account that most algorithms, in their machinery, do not consider the exact objective function values of the solutions, but only a comparison between them. In this sense, if the solutions of an instance of a combinatorial optimisation problem are sorted into their objective function values, we can see the instances as (partial) rankings of the solutions of the search space. Working with specific problems, particularly, the linear ordering problem and the symmetric and asymmetric traveling salesman problem, we show that they can not generate the whole set of (partial) rankings of the solutions of the search space, but just a subset. First, we characterise the set of (partial) rankings each problem can generate. Secondly, we study the intersections between these problems: those rankings which can be generated by both the linear ordering problem and the symmetric/asymmetric traveling salesman problem, respectively. The fact of finding large intersections between problems can be useful in order to transfer heuristics from one problem to another, or to define heuristics that can be useful for more than one problem.

CCS CONCEPTS

• **Mathematics of computing → Combinatorial optimization**;

KEYWORDS

Permutation-based combinatorial optimisation problems, rankings, traveling salesman problem, linear ordering problem.

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1 INTRODUCTION

In the last few decades, the solution of combinatorial optimisation problems (COP) has gained importance because they are ubiquitous in many different fields, such as transportation, industry, economy, telecommunications, logistics, or planning. Many methods have been designed for solving these problems, from general metaheuristics [2, 3, 13, 14, 18] to specific algorithms for particular problems [5, 10, 16, 19, 21, 22]. Among these proposals of algorithms, we find the heuristics which evaluate one or more solutions at each step, compare them with the current solution or solutions, and discard or accept them according to different criteria. These approaches do not make use of the exact value of the objective function of the solutions, but they are only interested in knowing if the objective function value of one solution is higher/lower than the value of another solution. Any evolutionary algorithm that uses tournament or ranking selection operators [3], local search based algorithms such as tabu search [9], variable neighbourhood search [7], iterated local search [12, 13, 21], etc., are some examples of this kind of algorithms. Therefore, all of them will behave equally for two instances of two different COPs that generate the same ranking of solutions, even when they have different objective function values. We will denote these metaheuristics as ranking-based algorithms. Notice that other algorithms - such as simulated annealing [17, 20], or any evolutionary algorithm that uses roulette wheel selection [11], as well as some tabu search techniques that utilize the exact objective function values to determine the tabu tenure [1] - do not fit in this work.

Taking the previous argument into account, for all those ranking-based algorithms, an instance of a COP can be seen as a ranking of the solutions of the search space. Particularly, given an instance, all the solutions of the search space can be sorted into their objective function value, from the best to the worst. Note that this ranking will be a partial ranking when at least two solutions of the search space have the same objective function value. Considering instances of COPs as (partial) rankings of the search space can provide new insights into the analysis of these problems. Specifically, characterising the rankings generated by different COPs is useful when analysing the possible intersections between the ranking spaces, and therefore, intersections between problems. In this sense, no matter whether a problem addresses distances between cities, or if it copes with flows between factories, those rankings (instances) that fall into the intersection of both problems will be solved similarly for ranking-based algorithms.

In this paper, we work with different permutation-based combinatorial optimisation problems: the linear ordering problem (LOP) and the symmetric and asymmetric traveling salesman problem (TSP). First, the necessary conditions for a ranking to be generated by each of the three problems are specified, accompanied with examples. Although we can not prove the sufficiency, we take a step forward in this direction, analysing some restrictions in the rankings by means of theorems. We also determine an upper bound for the number of the different possible rankings produced in each case. Secondly, we analyse the intersection between the ranking space of the LOP and those of the two versions of the TSP. That is, we answer the following question: is there any ranking that could be produced by an instance of the LOP and also by an instance of the symmetric/asymmetric TSP?

The rest of the paper is organised as follows. The permutation-based combinatorial optimisation problems analysed in the paper are formally introduced in Section 2. In Section 3, we carry out the characterisation of the rankings induced by each of the three problems, and the intersections between them are studied in Section 4. Finally, in Section 5, the conclusions and future work are presented.

2 PERMUTATION-BASED COMBINATORIAL OPTIMISATION PROBLEMS

A combinatorial optimisation problem (COP) consists of finding the solution (or solutions) that optimises a function f

$$f: \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R} \\ \sigma & \longmapsto & f(\sigma) \end{array},$$

where the solutions σ are in a finite or countable infinite search space Ω. Specifically, we work with instances of permutation-based COPs. So, from now on, Ω is the set of permutations of size n, $\Omega = S_n$, and a permutation $\sigma \in S_n$ is a bijection of the set of integers {1,2,...,n} onto itself. A permutation is understood as an order of the items {1,2,...,n}, i.e.:

$$\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n)),$$

where $\sigma(i) \in \{1,2,\ldots,n\}$ is the item in the i-th position and $\sigma(i) \neq \sigma(j), \forall i \neq j$.

2.1 Linear Ordering Problem

Given a matrix $A = [a_{ij}]_{n\times n}$ of numerical entries, the Linear Ordering Problem (LOP) consists of finding a simultaneous permutation $\sigma \in S_n$ of the rows and columns of B, such that the sum of the entries above the main diagonal is maximised, or equivalently, the sum of the entries below the main diagonal is minimised [6]. In this paper, we consider the version of minimisation, as we will refer to all minimisation problems. The equation below formalises the LOP function:

$$f_{LOP}(\sigma) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{\sigma(j)\sigma(i)}. \tag{1}$$

The search space is the whole space of permutations of size n, so its size is $|S_n| = n!$.

It is important to note that if a permutation

$$\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n-1)\sigma(n))$$

minimises the objective function f_{LOP}, its reverse,

$$\sigma^r = (\sigma(n)\sigma(n-1)\cdots\sigma(2)\sigma(1)),$$

maximises it.

2.2 Traveling Salesman Problem

Given a list of n cities and their pairwise distances $D = [d_{ij}]_{n\times n}$, the aim of the TSP is to find the shortest tour that visits each city exactly once, returning to the initial city [8]. As the problem has n cities, the search space is specified by the set of permutations of n elements, S_n, and the objective function to minimise is:

$$f_{TSP}(\sigma) = \sum_{i=1}^{n-1} d_{\sigma(i)\sigma(i+1)} + d_{\sigma(n)\sigma(1)}, \tag{2}$$

where $d_{\sigma(i)\sigma(j)}$ represents the distance between the cities $\sigma(i)$ and $\sigma(j)$, $i \neq j$.

2.2.1 Symmetric TSP. In the symmetric version of the traveling salesman problem (STSP), the distance from one city i to another city j is considered the same as from j to i. That is, $d_{ij} = d_{ji}, \forall i \neq j$. In this problem, one solution (tour) can be represented by $2n$ different permutations, and therefore, the search space is of size $n!/2n = (n-1)!/2$. For example, for $n = 4$, the 8 permutations $\sigma_1, \ldots, \sigma_8$ represent the same tour:

$$\sigma_1 = (1234)\ ;\ \sigma_2 = (2341)\ ;\ \sigma_3 = (3412)\ ;\ \sigma_4 = (4123);$$
$$\sigma_5 = (4321)\ ;\ \sigma_6 = (3214)\ ;\ \sigma_7 = (2143)\ ;\ \sigma_8 = (1432).$$

The objective function value for this tour is:

$$f_{STSP}(\sigma_i) = d_{12} + d_{23} + d_{34} + d_{41}\ ,\ \ \forall i = 1,\ldots,8.$$

2.2.2 Asymmetric TSP. The asymmetric traveling salesman problem (ATSP) considers that the distance from one city i to another city j is not necessarily the same as from j to i. In this case, one solution can be represented by n different permutations, and thus, the search space is of size $n!/n = (n-1)!$. For example, for $n = 4$, there are 4 permutations $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ representing the same tour.

$$\sigma_1 = (1234)\ ;\ \sigma_2 = (2341)\ ;\ \sigma_3 = (3412)\ ;\ \sigma_4 = (4123).$$

However, it is different from the tour represented by the 4 permutations $\sigma_5, \sigma_6, \sigma_7, \sigma_8$:

$$\sigma_5 = (4321)\ ;\ \sigma_6 = (3214)\ ;\ \sigma_7 = (2143)\ ;\ \sigma_8 = (1432).$$

The objective function values for these two different tours are:

$$f_{ATSP}(\sigma_i) = d_{12} + d_{23} + d_{34} + d_{41}\ ,\ \ \forall i = 1,\ldots,4,$$
$$f_{ATSP}(\sigma_j) = d_{21} + d_{32} + d_{43} + d_{14}\ ,\ \ \forall j = 5,\ldots,8.$$

3 RANKINGS GENERATED BY COMBINATORIAL OPTIMISATION PROBLEMS

A permutation-based COP has been naturally understood as the pair (f, S_n) where f is the objective function and S_n the search space. If all the solutions of the search space are evaluated by the objective function, we could sort these solutions from the best to the worst one. In this sense, assuming a minimisation problem and considering an injective function f, we can define a ranking $\mathcal{R}_{n!}(f)$ given by f as the permutation in $S_{n!}$ that reorders the elements in S_n so that the first one has the lowest value of the objective function and the last one has the highest value:

$$\mathcal{R}_{n!}(f) = (\sigma_1 \sigma_2 \cdots \sigma_{n!}),$$

with $f(\sigma_1) < f(\sigma_2) < \ldots < f(\sigma_{n!})$.

For the cases where the function f is not injective, that is, more than one solution has the same objective function value, we can define a partial ranking. The partial rankings accept a set of different solutions at each position:

$$\mathcal{PR}_{n!}(f) = (\{\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1k_1}\} \cdots \{\sigma_{m1}, \ldots, \sigma_{mk_m}\}),$$

with

$$f(\sigma_{11}) = f(\sigma_{12}) = \ldots = f(\sigma_{1k_1}) < \ldots < f(\sigma_{m1}) = \ldots = f(\sigma_{mk_m}).$$

Considering this concept of ranking, we can associate each COP with the set of all possible rankings produced by itself [4]. One of the advantages of this point of view is that, while the space composed by all the instances of a COP is infinite, the ranking space induced by a COP is finite. We analyse the properties of the rankings generated by the LOP and the symmetric and asymmetric TSP.

3.1 Rankings of the LOP

It is already known that the rankings generated by the LOP have specific properties [5, 15]. If a solution σ is in the first position of the ranking (it is the best solution, that is, a global optimum) its reverse σ^r is located at the last position of the ranking (it is the worst solution). In general, if a solution σ' is in the k-th position of the ranking, its reverse σ'^r is located at the $(n! - k + 1)$-th position of the ranking. Henceforth, we denote this kind of rankings as reversely symmetric (RS) rankings. Assuming an injective function, the number of all possible RS rankings would be:

$$|\mathcal{R}_{n!}(RS)| = 2^{n!/2} \cdot \left(\frac{n!}{2}\right)!. \tag{3}$$

This is, there is a total of $n!$ solutions in the search space, so a total of $\frac{n!}{2}$ pairs of solutions. As for each pair of solutions (σ, σ^r), one solution will be located in the superior half of the ranking while the other will be located in the inferior half of the ranking, the different possible ways of reordering all the solutions in the superior half of the ranking (and consequently in the inferior half) is $\left(\frac{n!}{2}\right)!$. It comes multiplied by the different possible ways of choosing each permutation, σ or σ^r, from each pair: $2^{n!/2}$.

An example of a ranking of an LOP instance of size 3 is the following, where the 4th, 5th and 6th permutations are the reverses of the 3rd, 2nd and 1st permutations, respectively:

$$\mathcal{R}_{3!}(RS) = \big((123)\ (132)\ (213)\ (312)\ (231)\ (321)\big).$$

So, every instance of an LOP can be seen as a reversely symmetric ranking. The question that arises is: can every reversely symmetric ranking be generated by an instance of the LOP? In order to answer this question, we provide the following example.

Example 3.1. Let us consider a reversely symmetric ranking for permutations of size $n = 4$ of the following form:

$$\mathcal{R}_{4!}(RS) = \big((1234)\ (1243)\ (1423)\ (1342)\ (1324)\ (1432)\ (2134)\ (2143)\ (2314)\ (2413)\ (3124)\ (3214)\ (4123)\ (4213)\ (3142)\ (4132)\ (3412)\ (4312)\ (2341)\ (4231)\ (2431)\ (3241)\ (3421)\ (4321)\big).$$

If an instance of the LOP which generates this ranking exists, it means that it is possible to find a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

such that $\forall \sigma_i, \sigma_j$ elements of $\mathcal{R}_{4!}(RS)$ with $i < j$, $f(\sigma_i) < f(\sigma_j)$.

First, we take, for example, the permutations $\sigma_2 = (1243)$ and $\sigma_3 = (1423)$, so that $f(\sigma_2) < f(\sigma_3)$ has to be fulfilled (see (1) for the calculation of the objective function).

$$\left.\begin{array}{l} f(\sigma_2) = f(1243) = a_{21} + a_{41} + a_{31} + a_{42} + a_{32} + a_{34} \\ f(\sigma_3) = f(1423) = a_{41} + a_{21} + a_{31} + a_{24} + a_{34} + a_{32} \end{array}\right\} \Rightarrow$$

$$\Rightarrow a_{21} + a_{41} + a_{31} + a_{42} + a_{32} + a_{34} < a_{41} + a_{21} + a_{31} + a_{24} + a_{34} + a_{32}.$$

Thus,

$$a_{42} < a_{24}. \tag{4}$$

Secondly, we choose the permutations $\sigma_4 = (1342)$ and $\sigma_5 = (1324)$, so $f(\sigma_4) < f(\sigma_5)$ has to be fulfilled.

$$\left.\begin{array}{l} f(\sigma_4) = f(1342) = a_{31} + a_{41} + a_{21} + a_{43} + a_{23} + a_{24} \\ f(\sigma_5) = f(1324) = a_{31} + a_{21} + a_{41} + a_{23} + a_{43} + a_{42} \end{array}\right\} \Rightarrow$$

$$\Rightarrow a_{31} + a_{41} + a_{21} + a_{43} + a_{23} + a_{24} < a_{31} + a_{21} + a_{41} + a_{23} + a_{43} + a_{42}.$$

Thus,

$$a_{24} < a_{42}. \tag{5}$$

As can be observed, the inequalities given by (4) and (5) are inconsistent.

This is a simple counterexample that shows that not all the RS rankings can be generated by instances of the LOP. So, the number of all possible RS rankings calculated in (3) is an upper bound for the number of all possible rankings that the LOP can generate (assuming an injective function). In Theorem 3.2, we provide sufficient conditions for an RS ranking (or partial ranking) not to correspond with any LOP instance.

THEOREM 3.2. *Given a reversely symmetric ranking $\mathcal{R}_{n!}(RS)$, and $i, j, k, r \in \mathbb{N}$ such that $1 \leq i < j < k < r \leq n!$, if the following three conditions are fulfilled*

(i) *$\exists \sigma_i, \sigma_j$ elements of $\mathcal{R}_{n!}(RS)$ such that $\sigma_i(p) = \sigma_j(p+1)$, $\sigma_i(p+1) = \sigma_j(p)$ and $\sigma_i(s_1) = \sigma_j(s_1), \forall s_1 \neq p, p+1$,*

(ii) *$\exists \sigma_k, \sigma_r$ elements of $\mathcal{R}_{n!}(RS)$ such that $\sigma_k(q) = \sigma_r(q+1)$, $\sigma_k(q+1) = \sigma_r(q)$ and $\sigma_k(s_2) = \sigma_r(s_2), \forall s_2 \neq q, q+1$,*

(iii) *$\sigma_i(p) = \sigma_r(q)$ and $\sigma_i(p+1) = \sigma_r(q+1)$,*

then, $\mathcal{R}_{n!}(RS)$ can not be generated by any instance of the LOP.

PROOF. Let's suppose an RS ranking containing four permutations σ_i, σ_j, σ_k and σ_r that fulfil the three conditions. According to the first condition (i), σ_i and σ_j just differ from one adjacent swap: all their items are equal except the p-th and $(p+1)$-th, which are exchanged. Thus, calculating the difference of their objective function values for the LOP:

$$f(\sigma_i) - f(\sigma_j) = a_{\sigma_i(p+1)\sigma_i(p)} - a_{\sigma_j(p+1)\sigma_j(p)}.$$

As $\sigma_j(p+1) = \sigma_i(p)$ and $\sigma_j(p) = \sigma_i(p+1)$,

$$f(\sigma_i) - f(\sigma_j) = a_{\sigma_i(p+1)\sigma_i(p)} - a_{\sigma_i(p)\sigma_i(p+1)}.$$

It is known that $f(\sigma_i) < f(\sigma_j)$, because $i < j$, therefore

$$a_{\sigma_i(p+1)\sigma_i(p)} - a_{\sigma_i(p)\sigma_i(p+1)} < 0$$
$$\Rightarrow a_{\sigma_i(p+1)\sigma_i(p)} < a_{\sigma_i(p)\sigma_i(p+1)}. \quad (6)$$

Taking into account the second condition (ii), σ_k and σ_r have also all their items equal except the q-th and $(q+1)$-th, which are swapped. Thus, calculating the difference of their objective function values:

$$f(\sigma_k) - f(\sigma_r) = a_{\sigma_k(q+1)\sigma_k(q)} - a_{\sigma_r(q+1)\sigma_r(q)}.$$

As $\sigma_k(q+1) = \sigma_r(q)$ and $\sigma_k(q) = \sigma_r(q+1)$,

$$f(\sigma_k) - f(\sigma_r) = a_{\sigma_r(q)\sigma_r(q+1)} - a_{\sigma_r(q+1)\sigma_r(q)}.$$

It is known that $f(\sigma_k) < f(\sigma_r)$, because $k < r$, therefore

$$a_{\sigma_r(q)\sigma_r(q+1)} < a_{\sigma_r(q+1)\sigma_r(q)}. \quad (7)$$

Because of the third condition (iii), $\sigma_r(q) = \sigma_i(p)$ and $\sigma_r(q+1) = \sigma_i(p+1)$, and therefore (7) can be rewritten as

$$a_{\sigma_i(p)\sigma_i(p+1)} < a_{\sigma_i(p+1)\sigma_i(p)}. \quad (8)$$

Inequalities given by (6) and (8) are inconsistent: an LOP instance that generates a ranking under these conditions does not exist.

□

Theorem 3.2 can be extended to RS partial rankings. Notice, that Theorem 3.2 provides sufficient conditions for a ranking not to be an LOP instance, but we do not state that these conditions are necessary. That is, we know that there is not an instance of the LOP that generates a ranking of these characteristics, but we do not know if these properties are enough to characterise all the impossible rankings for the LOP.

3.2 Rankings of the Symmetric TSP

As seen in Section 2.2.1, in the STSP, there are $2n$ different permutations that represent the same solution. Thus, for this problem, we can refer just to partial rankings, in which at each position there are, at least, $2n$ solutions. Precisely, the rankings generated by the instances of this problem are such that at the i-th position we find the set of permutations $C_r(\sigma_i)$:

$$C_r(\sigma_i) = \Big\{(\sigma_i(1)\cdots\sigma_i(n)),\ (\sigma_i(2)\cdots\sigma_i(n)\sigma_i(1)),\ldots,$$
$$,\ (\sigma_i(n)\sigma_i(1)\cdots\sigma_i(n-1)),\ (\sigma_i(n)\sigma_i(n-1)\cdots\sigma_i(1)),$$
$$,(\sigma_i(n-1)\cdots\sigma_i(1)\sigma_i(n)),\ldots,(\sigma_i(1)\sigma_i(n)\sigma_i(n-1)\cdots\sigma_i(2))\Big\}.$$

From now on, we denote this kind of partial rankings as reverse-cyclic based (RC) rankings. Assuming that at each position of the ranking there is just one set $C_r(\sigma_i)$, the number of all possible RC rankings is

$$|\mathcal{R}_{n!}(RC)| = \left(\frac{(n-1)!}{2}\right)!. \quad (9)$$

An example of a ranking of an STSP of size 4 would be the following:

$$\mathcal{R}_{4!}(RC) = \Big(C_r(1234)\ C_r(1243)\ C_r(1324)\Big),$$

where

$C_r(1234) = \{(1234), (2341), (3412), (4123), (4321), (3214), (2143), (1432)\}$,
$C_r(1243) = \{(1243), (2431), (4312), (3124), (3421), (4213), (2134), (1342)\}$,
$C_r(1324) = \{(1324), (3241), (2413), (4132), (4231), (2314), (3142), (1423)\}$.

Every instance of an STSP generates an RC ranking. Again, the question that arises is: can every RC ranking be generated by an instance of the STSP? In order to answer this question, we provide the following example.

Example 3.3. Let us suppose an RC ranking for permutations of size $n = 7$, where the best four sets of solutions are

$$\mathcal{R}_{7!}(RC) = \Big(C_r(1234567)\ C_r(1235467)\ C_r(1264537)\ C_r(1265437)\cdots\Big).$$

If an instance of the STSP which generates this ranking exists, it means that it is possible to find a matrix

$$D = \begin{pmatrix} 0 & d_{12} & \cdots & d_{17} \\ d_{12} & 0 & \cdots & d_{27} \\ \vdots & \vdots & \ddots & \vdots \\ d_{17} & d_{27} & \cdots & 0 \end{pmatrix}$$

such that $\forall\sigma'_i \in C_r(\sigma_i), \forall\sigma'_j \in C_r(\sigma_j)$ with $i < j$, $f(\sigma'_i) < f(\sigma'_j)$.

First, we take, for example, the permutations $\sigma_1 = (1234567)$ and $\sigma_2 = (1235467)$, so that $f(\sigma_1) < f(\sigma_2)$ has to be fulfilled.

$$\left.\begin{array}{l} f(1234567) = d_{12} + d_{23} + d_{34} + d_{45} + d_{56} + d_{67} + d_{71} \\ f(1235467) = d_{12} + d_{23} + d_{35} + d_{54} + d_{46} + d_{67} + d_{71} \end{array}\right\}$$

$$\Rightarrow d_{12} + d_{23} + d_{34} + d_{45} + d_{56} + d_{67} + d_{71} <$$
$$< d_{12} + d_{23} + d_{35} + d_{54} + d_{46} + d_{67} + d_{71}.$$

Thus,

$$d_{34} + d_{56} < d_{35} + d_{46}. \quad (10)$$

Secondly, we choose the permutations $\sigma_3 = (1264537)$ and $\sigma_4 = (1265437)$, so $f(\sigma_3) < f(\sigma_4)$:

$$\left.\begin{array}{l} f(1264537) = d_{12} + d_{26} + d_{64} + d_{45} + d_{53} + d_{37} + d_{71} \\ f(1265437) = d_{12} + d_{26} + d_{65} + d_{54} + d_{43} + d_{37} + d_{71} \end{array}\right\}$$

$$\Rightarrow d_{12} + d_{26} + d_{64} + d_{45} + d_{53} + d_{37} + d_{71} <$$
$$< d_{12} + d_{26} + d_{65} + d_{54} + d_{43} + d_{37} + d_{71}.$$

Thus,

$$d_{64} + d_{53} < d_{65} + d_{43}. \quad (11)$$

As $d_{ij} = d_{ji}, \forall i, j$, the inequalities given by (10) and (11) are inconsistent.

This is a simple counterexample that shows that not all the RC rankings can be generated by instances of the STSP. So, the number of all possible RC rankings calculated in (9) is an upper bound for the number of all possible rankings that the STSP can generate (when assuming that any two different tours have different objective function values). In Theorem 3.4, we provide sufficient conditions for an RC ranking not to correspond with any STSP instance.

THEOREM 3.4. *Given a reverse-cyclic based ranking $\mathcal{R}_{n!}(RC)$, and $i,j,k,r \in \mathbb{N}$ such that $1 \le i < j < k < r \le n!$, if the following three conditions are fulfilled*

(i) $\exists \sigma_i, \sigma_j$ *elements of* $\mathcal{R}_{n!}(RC)$ *such that* $\sigma_i(p) = \sigma_j(p+1)$, $\sigma_i(p+1) = \sigma_j(p)$ *and* $\sigma_i(s_1) = \sigma_j(s_1), \forall s_1 \neq p, p+1$,

(ii) $\exists \sigma_k, \sigma_r$ *elements of* $\mathcal{R}_{n!}(RC)$ *such that* $\sigma_k(q) = \sigma_r(q+1)$, $\sigma_k(q+1) = \sigma_r(q)$ *and* $\sigma_k(s_2) = \sigma_r(s_2), \forall s_2 \neq q, q+1$,

(iii)
$$\begin{cases} \sigma_i(p) = \sigma_r(q) \\ \sigma_i(p+1) = \sigma_r(q+1) \\ \sigma_i(p-1) = \sigma_r(q-1) \\ \sigma_i(p+2) = \sigma_r(q+2) \end{cases} \quad or \quad \begin{cases} \sigma_i(p) = \sigma_r(q+1) \\ \sigma_i(p+1) = \sigma_r(q) \\ \sigma_i(p-1) = \sigma_r(q+2) \\ \sigma_i(p+2) = \sigma_r(q-1) \end{cases}$$

then, $\mathcal{R}_{n!}(RC)$ can not be generated by any instance of the STSP.

PROOF. Let's suppose an RC ranking containing four permutations σ_i, σ_j, σ_k and σ_r that fulfil the three conditions. According to the first condition (i), σ_i and σ_j just differ from one adjacent swap: all their items are equal except the p-th and $(p+1)$-th, which are swapped. Thus, calculating the difference of their objective function values for the STSP:

$$f(\sigma_i) - f(\sigma_j) = \left(d_{\sigma_i(p-1)\sigma_i(p)} + d_{\sigma_i(p+1)\sigma_i(p+2)}\right) - \\ - \left(d_{\sigma_j(p-1)\sigma_j(p)} + d_{\sigma_j(p+1)\sigma_j(p+2)}\right).$$

As $\sigma_j(p+1) = \sigma_i(p)$ and $\sigma_j(p) = \sigma_i(p+1)$,

$$f(\sigma_i) - f(\sigma_j) = \left(d_{\sigma_i(p-1)\sigma_i(p)} + d_{\sigma_i(p+1)\sigma_i(p+2)}\right) - \\ - \left(d_{\sigma_i(p-1)\sigma_i(p+1)} + d_{\sigma_i(p)\sigma_i(p+2)}\right).$$

It is known that $f(\sigma_i) < f(\sigma_j)$, because $i < j$, therefore

$$\left(d_{\sigma_i(p-1)\sigma_i(p)} + d_{\sigma_i(p+1)\sigma_i(p+2)}\right) - \\ - \left(d_{\sigma_i(p-1)\sigma_i(p+1)} + d_{\sigma_i(p)\sigma_i(p+2)}\right) < 0 \Rightarrow$$

$$d_{\sigma_i(p-1)\sigma_i(p)} + d_{\sigma_i(p+1)\sigma_i(p+2)} < \\ < d_{\sigma_i(p-1)\sigma_i(p+1)} + d_{\sigma_i(p)\sigma_i(p+2)}. \tag{12}$$

Taking into account the second condition (ii), σ_k and σ_r have also all their items equal except the q-th and $(q+1)$-th, which are swapped. Thus, calculating the difference of their objective function values:

$$f(\sigma_k) - f(\sigma_r) = \left(d_{\sigma_k(q-1)\sigma_k(q)} + d_{\sigma_k(q+1)\sigma_k(q+2)}\right) - \\ - \left(d_{\sigma_r(q-1)\sigma_r(q)} + d_{\sigma_r(q+1)\sigma_r(q+2)}\right).$$

As $\sigma_k(q+1) = \sigma_r(q)$ and $\sigma_k(q) = \sigma_r(q+1)$,

$$f(\sigma_k) - f(\sigma_r) = \left(d_{\sigma_r(q-1)\sigma_r(q+1)} + d_{\sigma_r(q)\sigma_r(q+2)}\right) - \\ - \left(d_{\sigma_r(q-1)\sigma_r(q)} + d_{\sigma_r(q+1)\sigma_r(q+2)}\right).$$

It is known that $f(\sigma_k) < f(\sigma_r)$, because $k < r$, therefore

$$d_{\sigma_r(q-1)\sigma_r(q+1)} + d_{\sigma_r(q)\sigma_r(q+2)} < \\ < d_{\sigma_r(q-1)\sigma_r(q)} + d_{\sigma_r(q+1)\sigma_r(q+2)}. \tag{13}$$

Because of the third condition (iii):

(iii-I) if $\sigma_r(q) = \sigma_i(p)$, $\sigma_r(q+1) = \sigma_i(p+1)$, $\sigma_r(q-1) = \sigma_i(p-1)$ and $\sigma_r(q+2) = \sigma_i(p+2)$, then (13) can be rewritten as

$$d_{\sigma_i(p-1)\sigma_i(p+1)} + d_{\sigma_i(p)\sigma_i(p+2)} < \\ < d_{\sigma_i(p-1)\sigma_i(p)} + d_{\sigma_i(p+1)\sigma_i(p+2)}. \tag{14}$$

(iii-II) or if $\sigma_r(q+1) = \sigma_i(p)$, $\sigma_r(q) = \sigma_i(p+1)$, $\sigma_r(q+2) = \sigma_i(p-1)$ and $\sigma_r(q-1) = \sigma_i(p+2)$, then (13) can be rewritten as

$$d_{\sigma_i(p+2)\sigma_i(p)} + d_{\sigma_i(p+1)\sigma_i(p-1)} < \\ < d_{\sigma_i(p+2)\sigma_i(p+1)} + d_{\sigma_i(p)\sigma_i(p-1)}. \tag{15}$$

In both cases (iii-I) and (iii-II), inequalities given by (12) and (14), and (12) and (15), respectively, are inconsistent: an STSP instance that generates a ranking under these conditions does not exist. □

Theorem 3.4 can be extended to RC partial rankings with sets of sizes $k \cdot 2n$ $(1 < k \in \mathbb{N})$ at the different positions of the ranking. Notice, that Theorem 3.4 provides sufficient conditions for a ranking not to be an STSP instance, but we do not state that these conditions are enough to characterise all the impossible rankings for the STSP.

3.3 Rankings of the Asymmetric TSP

In the ATSP, there are n different permutations that represent the same solution. As in the case of the STSP, we can refer to partial rankings with, at least, n solutions having the same objective function value at each position of the ranking. Precisely, the rankings generated by the instances of this problem are such that at the i-th position we find the set of permutations $C(\sigma_i)$:

$$C(\sigma_i) = \Big\{(\sigma_i(1)\cdots\sigma_i(n)), (\sigma_i(2)\cdots\sigma_i(n)\sigma_i(1)), \\ ,\ldots,(\sigma_i(n)\sigma_i(1)\ \cdots\sigma_i(n-1))\Big\}.$$

Henceforth, we denote this kind of rankings as nonreverse-cyclic based (NRC) rankings. Supposing that at each position of the ranking there is just one set $C(\sigma_i)$, the number of all possible NRC rankings is

$$|\mathcal{R}_{n!}(NRC)| = ((n-1)!)!. \tag{16}$$

An example of a ranking of an ATSP of size 4 is the following:

$$\mathcal{R}_{4!}(NRC) = \Big(C(1234)\,C(1243)\,C(1324)\,C(1342)\,C(1423)\,C(1432)\Big),$$

where

$$C(1234) = \{(1234), (2341), (3412), (4123)\},$$
$$C(1243) = \{(1243), (2431), (4312), (3124)\},$$
$$C(1324) = \{(1324), (3241), (2413), (4132)\},$$
$$C(1342) = \{(1342), (3421), (4213), (2134)\},$$
$$C(1423) = \{(1423), (4231), (2314), (3142)\},$$
$$C(1432) = \{(1432), (4321), (3214), (2143)\}.$$

The same question as in the previous two cases is answered by means of a counterexample: can every NRC ranking be generated by an ATSP instance?

Example 3.5. Let us suppose an NRC ranking for permutations of size $n = 7$, where the best four sets of solutions are
$\mathcal{R}_{7!}(NRC) = \big(C_r(1234567)\ C_r(1235467)\ C_r(1735462)\ C_r(1734562)\ \ldots\big)$.

If an instance of the ATSP which generates this ranking exists, it means that it is possible to find a matrix

$$D = \begin{pmatrix} 0 & d_{12} & \cdots & d_{17} \\ d_{21} & \cdots & \cdots & d_{27} \\ \vdots & & & \vdots \\ d_{71} & d_{72} & \cdots & 0 \end{pmatrix}$$

such that $\forall \sigma_i' \in C(\sigma_i), \forall \sigma_j' \in C(\sigma_j)$ with $i < j$, $f(\sigma_i') < f(\sigma_j')$.

In the same way as in Example 3.3, we obtain an inconsistency when analysing the objective function values. On the one hand, choosing the permutations $\sigma_1 = (1234567)$ and $\sigma_2 = (1235467)$:

$$d_{34} + d_{45} + d_{56} < d_{35} + d_{54} + d_{46}. \tag{17}$$

On the other hand, for permutations $\sigma_3 = (1735462)$ and $\sigma_4 = (1734562)$:

$$d_{35} + d_{54} + d_{46} < d_{34} + d_{45} + d_{56}. \tag{18}$$

This counterexample shows that not all the NRC rankings can be generated by instances of the ATSP. So, the number of all possible NRC rankings calculated in (16) is an upper bound for the number of all possible rankings that the ATSP can generate (when assuming that any two different tours have different objective function values). In Theorem 3.6, we provide sufficient conditions for an NRC partial ranking not to correspond with any ATSP instance.

THEOREM 3.6. *Given a nonreverse-cyclic based ranking $\mathcal{R}_{n!}(NRC)$, and $i, j, k, r \in \mathbb{N}$ such that $1 \le i < j < k < r \le n!$, if the following three conditions are fulfilled*

(i) *$\exists \sigma_i, \sigma_j$ elements of $\mathcal{R}_{n!}(NRC)$ such that $\sigma_i(p) = \sigma_j(p+1)$, $\sigma_i(p+1) = \sigma_j(p)$ and $\sigma_i(s_1) = \sigma_j(s_1), \forall s_1 \neq p, p+1$,*

(ii) *$\exists \sigma_k, \sigma_r$ elements of $\mathcal{R}_{n!}(NRC)$ such that $\sigma_k(q) = \sigma_r(q+1)$, $\sigma_k(q+1) = \sigma_r(q)$ and $\sigma_k(s_2) = \sigma_r(s_2), \forall s_2 \neq q, q+1$,*

(iii) *$\sigma_i(p) = \sigma_r(q)$, $\sigma_i(p+1) = \sigma_r(q+1)$, $\sigma_i(p-1) = \sigma_r(q-1)$, $\sigma_i(p+2) = \sigma_r(q+2)$,*

then, $\mathcal{R}_{n!}(NRC)$ can not be generated by any instance of the ATSP.

PROOF. The proof is similar to that of Theorem 3.4. □

Theorem 3.6 can be extended to NRC partial rankings with sets of sizes $k \cdot n$ $(1 < k \in \mathbb{N})$ at the different positions of the ranking. Theorem 3.6 provides sufficient conditions for a ranking not to be an ATSP instance.

4 INTERSECTION BETWEEN RANKINGS

All the COPs have at least one ranking in common: the partial ranking produced by a constant objective function. That is, in all cases we can find a ranking where all the solutions share the same objective function value; thus, all the solutions are located in the first position of the ranking. However, we are interested in knowing if two COPs have more rankings in common. As mentioned in the introduction, if this happens, every algorithm that does not take into account the absolute objective function values, but just takes into account the rank of the solutions, will behave in the same manner in instances that produce the same ranking.

4.1 Intersection between LOP and STSP

If a ranking could be generated by an instance of the LOP and also by an instance of the STSP, on the one hand, it would be an RS ranking, and, on the other hand, it would be an RC ranking.

First, if the ranking is RC, it means that $2n$ solutions have the same objective function value. Particularly, any permutation σ and its reverse σ^r have the same value. Secondly, if the ranking is RS, the global optimum σ^* is in the first position of the ranking, and its reverse σ^{*r} is in the last one. These two conditions imply that all the solutions of the ranking (the whole search space) have the same objective function value. In other words:

$$LOP \cap STSP = \{f_c\},$$

where f_c represents the constant objective functions.

4.2 Intersection between LOP and ATSP

If a ranking is generated by an instance of the LOP and also by an instance of the ATSP, on the one hand, it is an RS ranking, and, on the other hand, it is an NRC ranking. At a first glance, we do not find any contradiction between these two kinds of rankings, as the reverse permutations σ^r do not have any relation with σ in the NRC rankings. In fact, one can find numerous rankings belonging to both LOP and ATSP. Here, an example is provided.

Example 4.1. Given the following partial ranking for permutation size $n = 4$

$$\mathcal{R}_{4!} = \big(C(1234)\ C(1423)\ C(1342)\ C(1243)\ C(1324)\ C(1432)\big),$$

we find an instance of the LOP and an instance of the ATSP that generate it. Notice that it is an RS ranking, because all the permutations in the 1st, 2nd, and 3rd positions have their reverses at 6th, 5th and 4th positions, respectively. It is also an NRC ranking, as at each position a set of $n = 4$ cyclic permutations is found.

For instance, evaluating the LOP instance given by the following matrix

$$A = \begin{pmatrix} 0 & 4 & 1.5 & 0.5 \\ 1 & 0 & 9.5 & 0 \\ 2 & 4 & 0 & 8.5 \\ 3 & 2.5 & 3.5 & 0 \end{pmatrix}$$

the resultant objective function values for each permutation of the search space are:

$$\begin{aligned} f(1234) = f(2341) = f(3412) = f(4123) &= 16.0, \\ f(1423) = f(4231) = f(2314) = f(3142) &= 18.5, \\ f(1342) = f(3421) = f(4213) = f(2134) &= 19.0, \\ f(2431) = f(1243) = f(3124) = f(4312) &= 21.0, \\ f(3241) = f(1324) = f(4132) = f(2413) &= 21.5, \\ f(4321) = f(1432) = f(2143) = f(3214) &= 24.0. \end{aligned}$$

Thus, the LOP instance given by matrix A produces the ranking that was desired.

Evaluating the ATSP instance given by the following distance matrix

$$D = \begin{pmatrix} 0 & 1.7 & 7.1 & 8.5 \\ 8 & 0 & 1.6 & 7 \\ 6 & 9 & 0 & 1.5 \\ 1 & 2 & 8 & 0 \end{pmatrix}$$

the resultant objective function values for each permutation of the search space are:

$$f(1234) = f(2341) = f(3412) = f(4123) = 5.8,$$
$$f(1423) = f(4231) = f(2314) = f(3142) = 18.1,$$
$$f(1342) = f(3421) = f(4213) = f(2134) = 18.6,$$
$$f(2431) = f(1243) = f(3124) = f(4312) = 22.7,$$
$$f(3241) = f(1324) = f(4132) = f(2413) = 24.1,$$
$$f(4321) = f(1432) = f(2143) = f(3214) = 33.5.$$

Thus, the STSP instance given by matrix D produces the same ranking.

It is concluded that the intersection between the LOP and the ATSP is higher than the set composed by just the constant objective functions. The total amount of NRC rankings that, at the same time, are RS rankings, is

$$|\mathcal{R}_{n!}(RS \cap NRC)| = 2^{(n-1)!/2} \cdot \left(\frac{(n-1)!}{2}\right)!. \tag{19}$$

However, as has been explained in Section 3, some RS rankings and NRC rankings are impossible to be produced by LOP and ATSP instances, respectively. Therefore, we conjecture that (19) is an upper bound for the number of rankings that the LOP and the ATSP have in common when assuming that at each position of the ranking there is just one set of cyclic solutions.

5 CONCLUSIONS AND FUTURE WORK

Analysing the combinatorial optimisation problems and understanding the behaviour of the algorithms when dealing with them, have been the main target of the combinatorial optimisation field. Some of the solving techniques proposed in the literature are focused on specific COPs. However, the association between problems and algorithms is still unknown: given a specific problem, which is the most suitable algorithm that solves it?

Based on the fact that most heuristics do not consider the exact objective function values, but just a comparison between them, in this paper, we have treated the COPs as sets of rankings of the solutions. Although the definition of the distinct problems is completely different, the ranking-based algorithms "see" all those instances that generate the same ranking in the same way. Thus, their performance will be exactly the same with these instances. In this sense, we can analyse the intersection between two COPs, that is, the subset of rankings that both COPs have in common. So, if we accept that an algorithm performs well for a specific COP and we find that this COP has a large intersection with another COP, we can also predict that the algorithm will work well when applied to this second COP.

We have worked with the LOP and the symmetric/asymmetric TSP. First, we have shown the properties of the rankings generated by the three problems. However, we have proved that not all the rankings with these characteristics are possible to be generated by instances of these problems. In order to take a step forward in the characterisation of the necessary and sufficient conditions of the generated rankings, we provide theorems with some ranking restrictions. Secondly, we have focused on the intersection between the set of rankings produced by the LOP and those generated by the symmetric TSP. Also, the intersection between those of the LOP and of the asymmetric TSP has been analysed. It has been found that the LOP and the symmetric TSP do not share any ranking, except the one that is common in all the problems: the ranking where all the solutions are in the same position (with the same objective function value). Nevertheless, the LOP and the asymmetric TSP do share a number of rankings. Until now, algorithms have been designed, mainly, taking into account the properties of the problem at hand. However, these results show that it could be more efficient if the algorithms are designed for specific kinds (sets) of rankings.

In order to delve into this analysis, first, it would be interesting to study if the instances belonging to this intersection between the LOP and the ATSP are commonly found in the usual benchmarks or in real life. As a first step to test this, it would be useful to discover the restrictions in the elements of the LOP matrix that make the resultant ranking of the solutions an NRC ranking, or conversely, the restrictions in the elements of an ATSP matrix which provoke an RS ranking. Secondly, providing the upper bounds for the number of RS, RC and NRC rankings that satisfy Theorem 3.2, Theorem 3.4 and Theorem 3.6, respectively, would be useful to know the magnitude of the ranking spaces of each problem, and also, to give an approximation about the magnitude of the intersection between the LOP and the ATSP. Of course, this work can be extended to more different combinatorial optimisation problems, not necessarily based on permutations.

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