

A Bilinear Strategy for Calderón's Problem

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Abstract

Electrical Impedance Imaging would suffer a serious obstruction if for two different conductivities the potential and current measured at the boundary were the same. The Calderón's problem is to decide whether the conductivity is indeed uniquely determined by the data at the boundary. In \mathbb{R}^d , for $d = 5, 6$, we show that uniqueness holds when the conductivity is in $W^{1+\frac{d-5}{2p}, p}(\Omega)$, for $d \leq p < \infty$. This improves on recent results of Haberman, and of Ham, Kwon and Lee. The main novelty of the proof is an extension of Tao's bilinear Theorem.

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1 Introduction

Electrical Impedance Imaging is a technique to reconstruct the inner structure of a body from measurements of potential and current at the boundary. At least since the 30', geophysicists have used this technique to identify different layers of earth underground [22]. In pioneering work, Calderón [7] posed the problem of deciding whether the conductivity is uniquely determined by measurements at the boundary. Calderón went on to show uniqueness, roughly, when the conductivity is close to one.

The electrical potential u in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary satisfies the differential equation

$$\begin{aligned} L_\gamma u &:= \operatorname{div}(\gamma \nabla u) = 0, \\ u|_{\partial\Omega} &= f, \end{aligned} \tag{1}$$

where γ is the conductivity and f the potential at the boundary. We assume that $\gamma \in L^\infty(\Omega)$ and that $\gamma \geq c > 0$. If $f \in H^{1/2}(\partial\Omega)$, then a solution $u \in H^1(\Omega)$ exists. The electrical current at the boundary is $\gamma \partial_\nu u|_{\partial\Omega}$, where ν is the outward-pointing normal, and the operator $\Lambda_\gamma : u|_{\partial\Omega} \mapsto \gamma \partial_\nu u|_{\partial\Omega}$ is called the Dirichlet-to-Neumann map; we can define the map Λ_γ rigorously as

$$\langle \Lambda_\gamma f, g \rangle := \int_\Omega \gamma \nabla u \cdot \nabla \bar{v}, \quad (2)$$

where u solves (1) and $v \in H^1(\Omega)$ is *any* extension of $g \in H^{1/2}(\partial\Omega)$; hence $\Lambda_\gamma : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$. If we choose v such that $L_\gamma v = 0$, then we see that Λ_γ is symmetric. Uniqueness fails if two different conductivities γ_1 and γ_2 satisfy $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$; this were the case, for every $f_1, f_2 \in H^{\frac{1}{2}}(\partial\Omega)$ we would have

$$0 = \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2})f_1, f_2 \rangle = \int_\Omega (\gamma_1 - \gamma_2) \nabla u_1 \cdot \nabla \bar{u}_2, \quad (3)$$

where $L_{\gamma_1} u_1 = 0$ and $L_{\gamma_2} u_2 = 0$ are extensions of f_1 and f_2 respectively. Most of the proofs of uniqueness show that the collection of functions $\{\nabla u_1 \cdot \nabla \bar{u}_2\}$ is dense, so γ_1 and γ_2 cannot be different.

Kohn and Vogelius [16] showed that for smooth conductivities γ_1 and γ_2 , uniqueness holds at the boundary to all orders, so $\partial_\nu^N \gamma_1 = \partial_\nu^N \gamma_2$ at $\partial\Omega$ for every integer N . In particular, if the conductivities are analytic, then $\gamma_1 = \gamma_2$ in Ω .

In [24], Sylvester and Uhlmann introduced the method that most of the proofs follow nowadays. If u_j solve the equation (1) for γ_j , then the function $w_j := \gamma_j^{\frac{1}{2}} u_j$ solves the equation $(-\Delta + q_j)w_j = 0$ with $q_j = \gamma_j^{-\frac{1}{2}} \Delta \gamma_j^{\frac{1}{2}}$, and the relationship (3) is replaced by

$$\int_{\mathbb{R}^d} (q_1 - q_2) w_1 w_2 = 0; \quad (4)$$

then, they had to prove that the collection of function $\{w_1 w_2\}$ is dense. The integral is evaluated over \mathbb{R}^d because the functions γ_1 and γ_2 are extended to the whole space, and are arranged so that $\gamma_1 = \gamma_2 = 1$ outside a ball containing Ω . Since $e^{\zeta \cdot x}$ is harmonic when $\zeta \in \mathbb{C}^d$ satisfies $\zeta \cdot \zeta = 0$, then they used the ansatz $w_j = e^{\zeta_j \cdot x} (1 + \psi_j)$, expecting that ψ_j is somehow negligible for $|\zeta_1|, |\zeta_2| \rightarrow \infty$. These solutions w_j are called Complex Geometrical Optics (CGO) solutions. Sylvester and Uhlmann selected ζ_1 and ζ_2 such that $\zeta_1 + \zeta_2 = i\xi$ for $\xi \in \mathbb{R}^d$; then, on the assumed smallness of ψ_j for $|\zeta_1|, |\zeta_2| \rightarrow \infty$,

equation (4) means that $\hat{q}_1 = \hat{q}_2$, and this implies that $\gamma_1 = \gamma_2$. Their argument works well for conductivities in $C^2(\Omega)$.

In \mathbb{R}^2 , Astala and Päivärinta [1] proved that uniqueness holds in $L^\infty(\Omega)$, the best possible result. In higher dimensions, Brown [4] proved uniqueness for conductivities in $C^{\frac{3}{2}+}(\Omega)$, and this was improved to $W^{\frac{3}{2}, 2d+}(\Omega)$ by Brown and Torres [6]. By analogy with unique continuation, it is conjectured that the lowest possible regularity is $W^{1,d}(\Omega)$.

The function ψ in the CGO solution $w = e^{\zeta \cdot x}(1 + \psi)$ satisfies the equation

$$\Delta_\zeta \psi := \Delta \psi + 2\zeta \cdot \nabla \psi = q(1 + \psi). \quad (5)$$

Then, it is necessary to prove that a solution exists and is small. In [12], Haberman and Tataru introduced a Bourgain-type space adapted to $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$, the symbol of Δ_ζ . The space is defined as

$$\dot{X}_\zeta^b := \{u \mid \|u\|_{\dot{X}_\zeta^b}^2 := \int_{\mathbb{R}^d} |p_\zeta(\xi)|^{2b} |\hat{u}|^2 d\xi < \infty\},$$

and it follows immediately that $\|\Delta_\zeta^{-1}\|_{\dot{X}_\zeta^{-\frac{1}{2}} \rightarrow \dot{X}_\zeta^{\frac{1}{2}}} = 1$. The dual of \dot{X}_ζ^b is \dot{X}_ζ^{-b} . If we define the multiplication operator $M_q : u \mapsto qu$, then the existence of ψ follows from $\|\Delta_\zeta^{-1} M_q\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{\frac{1}{2}}} \leq \|M_q\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}} \leq c < 1$, and the smallness of ψ follows from the smallness of $\|q\|_{\dot{X}_\zeta^{-\frac{1}{2}}}$. Using these spaces Haberman and Tataru proved uniqueness for Lipschitz conductivities close to one.

Caro and Rogers [8] proved uniqueness for Lipschitz conductivities without further restriction. They used Carleman estimates, in the spirit of [15] and [9].

After an observation in [21], Haberman refined in [11] the method of Bourgain spaces, and proved uniqueness for conductivities in $W^{1,3}(\Omega)$ for $d = 3$, and $W^{1+\frac{d-4}{2p}, p}(\Omega)$ for $p \geq d$ and $d = 4, 5, 6$. He argued as follows: for γ_1 and γ_2 he wanted to show that $\|M_{q_j}\|_{\dot{X}_{\zeta_j}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta_j}^{-\frac{1}{2}}}$ and $\|q_j\|_{\dot{X}_{\zeta_j}^{-\frac{1}{2}}}$ are small for some ζ_1 and ζ_2 that satisfy $\zeta_1 + \zeta_2 = i\xi$, so Haberman proved that there exist sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$ for which $\|M_{q_j}\|_{\dot{X}_{\zeta_{j,k}}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta_{j,k}}^{-\frac{1}{2}}}$ and $\|q_j\|_{\dot{X}_{\zeta_{j,k}}^{-\frac{1}{2}}}$ tend to zero as $|\zeta_{1,k}|, |\zeta_{2,k}| \rightarrow \infty$. To find the sequences, he proved that the expected value of both norms goes to zero as $|\zeta_1|, |\zeta_2| \rightarrow \infty$.

Theorem 1 (Haberman [11]). *Let us write $\zeta(U, \tau) := \tau(Ue_1 - iUe_2)$ for $\tau \geq 1$ and $U \in O_d$ a rotation. If $\nabla \log \gamma_1$ and $\nabla \log \gamma_2$ are in $W^{\frac{d-4}{2p}, p}(\mathbb{R}^d)$ for $d \leq p < \infty$, or in $L^3(\mathbb{R}^d)$ for $d = 3$, then*

$$\frac{1}{M} \int_M \int_{O_d} \|M_{q_j}\|_{\dot{X}_{\zeta(U, \tau)}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta(U, \tau)}^{-\frac{1}{2}}}^p dU d\tau \text{ and } \frac{1}{M} \int_M \int_{O_d} \|q_j\|_{\dot{X}_{\zeta(U, \tau)}^{-\frac{1}{2}}}^2 dU d\tau \xrightarrow{M \rightarrow \infty} 0.$$

The idea is that, when $|\zeta_j|$ is large, the set of bad pairs (ζ_1, ζ_2) for which $\|M_{q_j}\|_{\dot{X}_{\zeta_j, k}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta_j, k}^{-\frac{1}{2}}}$ or $\|q_j\|_{\dot{X}_{\zeta_j, k}^{-\frac{1}{2}}}$ is large has measure close to zero, then it is possible to extract sequences such that these norms are small and such that $\zeta_1 + \zeta_2 = i\xi$.

The estimates of Haberman are very good, and most of the argument works well just for $\gamma \in W^{1, d}(\Omega)$. The bottle-neck is to get a strong upper bound of $\|M_{\partial_i f}\|_{\dot{X}_{\zeta(U, \tau)}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta(U, \tau)}^{-\frac{1}{2}}}$, where $f \in W^{s, p}$ for some $s \geq 0$.

In Section 2 we proof the next theorem.

Theorem 2 (Vanishing of the Expected Value). *Let us write $\zeta(U, \tau) := \tau(Ue_1 - iUe_2)$ for $\tau \geq 1$ and $U \in O_d$ a rotation. Suppose that f is a function supported in the unit ball. If $f \in W^{\frac{d-5}{2p} +, p}(\mathbb{R}^d)$ for $d \leq p < \infty$, then*

$$\frac{1}{M} \int_M \int_{O_d} \|M_{\partial_i f}\|_{\dot{X}_{\zeta(U, \tau)}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta(U, \tau)}^{-\frac{1}{2}}} dU d\tau \xrightarrow{M \rightarrow \infty} 0. \quad (6)$$

The main consequence of this theorem is the next improvement on Calderón's problem.

Theorem 3. *For $d = 5, 6$ suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. If γ_1 and γ_2 are in $W^{1 + \frac{d-5}{2p} +, p}(\Omega) \cap L^\infty$ for $d \leq p < \infty$, and if $\gamma_1, \gamma_2 \geq c > 0$, then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \text{ implies } \gamma_1 = \gamma_2.$$

We write $\gamma \in W^{1 + \frac{d-5}{2p} +, p}(\Omega) \cap L^\infty$ to emphasize that $\gamma \in L^\infty$, but it follows from Sobolev embedding for domains with Lipschitz boundaries. We note that Theorem 2 holds for $d \geq 3$, and the restriction $d = 5, 6$ in Theorem 3 seems technical; in fact, we can state the following consequence of the vanishing of the expected value.

Theorem 4. For $d \geq 7$ suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. If γ_1 and γ_2 are in $W^{1+\frac{d-5}{2p}+p}(\Omega) \cap L^\infty$ for $d \leq p < \infty$, if $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ at $\partial\Omega$, and if $\gamma_1, \gamma_2 \geq c > 0$, then

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \text{implies} \quad \gamma_1 = \gamma_2.$$

By the trace theorem the normal derivative $\partial_\nu \gamma$ is well-defined. The proof of Theorem 3 and Theorem 4 has been already summarized in this introduction, and we provide some more details in Section 2. We refer the reader to the literature to reconstruct the whole argument, in particular to Haberman [11] and to Ham, Kwon and Lee [13].

We have added an appendix with an example that shows that averaging is necessary.

1.1 Restriction Theory

Ham, Kwon and Lee [13] applied deep estimates from restriction theory to improve on Harberman results, and we will follow most of their arguments. We give here a brief introduction to restriction theory and the way it comes in Calderón's problem; a detailed exposition of restriction theory can be found in [20, part IV].

We control the norm $\|M_{\partial_i f}\|_{\dot{X}_{\zeta(U,\tau)}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta(U,\tau)}^{-\frac{1}{2}}}$ by duality, so we need an upper bound of

$$\langle (\partial_i f)u, v \rangle = \int_{\mathbb{R}^d} (\partial_i f)u\bar{v} \, dx \quad \text{for } u, v \in \dot{X}_{\zeta(U,\tau)}^{\frac{1}{2}}. \quad (7)$$

The contribution coming from frequencies close the null set of $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$, which we call the characteristic set Σ_ζ , is the hardest part we have to deal with.

The characteristic set Σ_ζ is a $(d-2)$ -sphere, and we have to control the duality pairing when the Fourier transform of u and v is concentrated close to Σ_ζ . This is just the setting for which restriction theory has been developed; a few classical examples of applications are [10, 14, 2, 3].

In restriction theory, we seek to prove the best possible bounds $\|\widehat{f}|_S\|_p \leq C\|f\|_q$, where S is a manifold or just a set. One of earliest and most important result is due to Tomas [27] and Stein (unpublished); for the proof see *e.g.* [23, chp. 9].

Theorem 5. (*Tomas-Stein Inequality*) Suppose that $S \subset \mathbb{R}^n$ is a compact surface with non-vanishing curvature. If $f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2\frac{n+1}{n+3}$, then

$$\|\widehat{f}|_S\|_2 \leq C\|f\|_p. \quad (8)$$

The dual operator is called the extension operator, and it is the Fourier transform of a measure $f dS$ supported on the set S . The function $(f dS)^\vee$ is the prototype of a function with frequencies highly concentrated close to S . In the dual side, the Tomas-Stein inequality is

$$\|(f dS)^\vee\|_{L^{p'}(\mathbb{R}^n)} \leq C\|f\|_{L^2(S)} \quad \text{for } 2\frac{n+1}{n-1} \leq p' \leq \infty. \quad (9)$$

Since the earliest days of restriction theory, a kind of stability of bilinear estimates was exploited; for example, the bound $\|(f dS)^\vee\|_{L^4(\mathbb{R}^2)} \leq C\|f\|_2$ is false, but the bound $\|(f_1 dS_1)^\vee (f_2 dS_2)^\vee\|_{L^2(\mathbb{R}^2)} \leq C\|f_1\|_2\|f_2\|_2$ is true, whenever the lines S_1 and S_2 are transversal; curvature is not required. This stability of bilinear estimates was clarified and refined by Tao, Vargas and Vega [26].

If we are to expect some improvement of a bilinear estimate, we have to require a separation condition on the surfaces S_1 and S_2 involved. For example, if $\|(f_1 dS_1)^\vee (f_2 dS_2)^\vee\|_{L^2(\mathbb{R}^2)} \leq C\|f_1\|_2\|f_2\|_2$ were true in any case, then just setting $S_1 = S_2$ would provide a linear estimate, a false one in this case. One of the key outcomes of [26] is a general strategy to get linear bounds from bilinear bounds, and we will follow this strategy in Section 3.1.

If we are to use the bilinear strategy, we need strong bilinear upper bounds. For some time, the bilinear analogue of the Tomas-Stein inequality in \mathbb{R}^n , for $n \geq 3$, was known as Klainerman-Machedon conjecture. Wolff made the first big progress, proving the conjecture when the surfaces are subsets of the cone [30]. Subsequently, Tao refined the method and proved the conjecture when the surfaces are subsets of a surface with positive curvature [25]. Vargas [29] and Lee [18] proved the conjecture when the surfaces are subsets of the hyperboloid, dealing with unusual obstructions.

Since we are interested in the sphere, we need to prove the bilinear theorem for this case. To avoid antipodal points in the bilinear inequality, we restrict ourselves to the surface

$$S := \{(\xi', \xi_n) \mid \xi_n = 1 - \sqrt{1 - |\xi'|^2} \text{ and } |\xi'| < \frac{1}{\sqrt{2}} + \frac{1}{10}\} \quad (10)$$

Following [26], we define also surfaces of elliptic type.

Definition 6. (Surfaces of Elliptic Type) A surface S is of ε -elliptic type if:

- The surface is the graph of a C^∞ function $\Phi : B_1 \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$.
- $\Phi(0) = 0$ and $\nabla\Phi(0) = 0$.
- The eigenvalues of $D^2\Phi(x)$ lie in $[1 - \varepsilon, 1 + \varepsilon]$ for every $x \in B_1$.

For every $\varepsilon > 0$ and for every point in a surface with positive curvature, we can find a sufficiently small neighborhood U so that U is of ε -elliptic type, up to a linear transformation.

We prove in Section 4 the next extension of Tao's bilinear theorem.

Theorem 7 (Bilinear Theorem). *Suppose that $S_1, S_2 \subset \mathbb{R}^n$ are two open subsets of a surface of elliptic type or the hemisphere in (10), and suppose that their diameter is $\lesssim 1$ and they lie at distance ~ 1 of each other. If f_μ and g_ν are functions with Fourier transforms supported in a μ -neighborhood of S_1 and a ν -neighborhood of S_2 respectively, for $\mu \leq \nu < \mu^{\frac{1}{2}} < 1$, then for every $\delta > 0$ it holds that*

$$\|f_\mu g_\nu\|_{p'} \leq C_\delta \mu^{\frac{n}{2p} - \delta} \nu^{\frac{1}{p} - \delta} \|f_\mu\|_2 \|g_\nu\|_2, \quad \text{for } 1 \leq p' \leq \frac{n}{n-1}. \quad (11)$$

For surfaces of ε -elliptic type, the constant C_δ may depend on ε and on the semi-norms $\|\partial^N \Phi\|_\infty$. The inequalities are best possible in μ and ν , up to δ -losses.

Unexpected phenomena appear: when μ is much smaller than ν , *i.e.* when $\mu^{\frac{1}{2}} \leq \nu$, then bilinearity does not play any role; moreover, the curvature of the support of g_ν is of no importance, and the bounds that Tomas-Stein yield cannot be improved. If we try to get bilinear bounds for f_μ and g_ν by averaging over translations of the surface and then applying Tao's bilinear theorem, we do not reach the optimal result (11), except when $\mu = \nu$.

The reader can consult the symbols and notations we use at the end of the article.

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2 Outline of the Proof

The proof that Theorem 2 implies Theorem 3 is long, and many steps are already well described in the literature. We refer the reader to [11, 13] for details.

First, we extend carefully γ_1 and γ_2 to the whole space. By the definition of $W^{s,p}(\Omega)$, we can extend γ_1 to a function in $W^{s,p}(\mathbb{R}^d)$. Since $\gamma_j \in W^{1+\frac{d-5}{2p}+p}(\Omega)$, then by a theorem of Brown in [5] we have that $\gamma_1 = \gamma_2$ at $\partial\Omega$ if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Now we define the function

$$\eta := \begin{cases} \gamma_2 - \gamma_1 & \text{if } \Omega \\ 0 & \text{if } \Omega^c. \end{cases}$$

Since η is zero at $\partial\Omega$ and $\frac{d-5}{2p}+ \leq \frac{1}{p}$, then $\eta \in W^{1+\frac{d-5}{2p}+p}(\mathbb{R}^d)$ (see [19, Theorem 1]); this explains the condition $d \leq 6$ in Theorem 3. We can thus define the extension $\gamma_2 := \gamma_1 + \eta \in W^{1+\frac{d-5}{2p}+p}(\mathbb{R}^d)$. Finally, we arrange the extensions so that $\gamma_1 = \gamma_2 = 1$ outside a ball containing Ω . For $d \geq 7$ we are in the case $\frac{d-5}{2p}+ > \frac{1}{p}$, and we need additionally the condition $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ at $\partial\Omega$ to be able to extend the conductivities. This is the condition that we included in Theorem 4.

For all $w_1, w_2 \in H_{\text{loc}}^1(\mathbb{R}^d)$ that solve $(-\Delta + q_j)w_j = 0$ with $q_j = \gamma_j^{-\frac{1}{2}}\Delta\gamma_j^{\frac{1}{2}}$, we want to show that the collection of functions $\{w_1 w_2\}$ is dense, which implies that $\gamma_1 = \gamma_2$; see [6] for a rigorous justification. Notice that q_j is compactly supported.

For $\zeta_j \cdot \zeta_j = 0$, the function $w_j = e^{\zeta_j \cdot x}(1 + \psi_j)$ is a CGO solution. The function $\psi_j \in H_{\text{loc}}^1(\mathbb{R}^d)$ has to satisfy the equation

$$(-\Delta_\zeta + q_j)\psi_j = -q_j. \quad (12)$$

If we choose ζ_1 and ζ_2 such that $\zeta_1 + \zeta_2 = i\xi$ and replace in (4), then we get

$$\begin{aligned} \int_{\mathbb{R}^d} (q_1 - q_2)e^{i\xi \cdot x} &= \int e^{i\xi \cdot x} \psi_2 q_2 - \int e^{i\xi \cdot x} \psi_1 q_1 + \\ &+ \int e^{i\xi \cdot x} \psi_1 \Delta_\zeta \psi_2 - \int e^{i\xi \cdot x} \psi_2 \Delta_\zeta \psi_1. \end{aligned} \quad (13)$$

We expect that the functions ψ_j are negligible, so if we ignore them, we would get that $\widehat{q}_1(\xi) = \widehat{q}_2(\xi)$ for every $\xi \in \mathbb{R}^d$, which implies $\gamma_1 = \gamma_2$.

The space $H_{\text{loc}}^1(\mathbb{R}^d)$ does not seem to be the best suited space to solve (12). Following Haberman and Tataru [12], we use the spaces \dot{X}_ζ^b and X_ζ^b . Since the inclusion $\dot{X}_\zeta^{\frac{1}{2}} \subset H_{\text{loc}}^1(\mathbb{R}^d)$ holds true, then we have

$$(-\Delta_\zeta + q) : \dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}.$$

The goal is to find a pair of sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$ that satisfy the following conditions:

- $\zeta_{1,k} + \zeta_{2,k} = i\xi$ and $|\zeta_{j,k}| \rightarrow \infty$ as $k \rightarrow \infty$.
- There exist solutions $\psi_{j,k} \in \dot{X}_{\zeta_{j,k}}^{\frac{1}{2}}$ of the equation (12).
- $\|\psi_{j,k}\|_{\dot{X}_{\zeta_{j,k}}^{\frac{1}{2}}} \rightarrow 0$ as $k \rightarrow \infty$.

To solve (12) we write $(I - \Delta_\zeta^{-1}q)\psi = \Delta_\zeta^{-1}q$. To invert the operator $(I - \Delta_\zeta^{-1}M_q)$, where $M_q : u \mapsto qu$, it suffices to prove that $\|M_q\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}} \leq c < 1$. We also have the upper bound

$$\|\psi\|_{\dot{X}_\zeta^{\frac{1}{2}}} \leq \|(I - \Delta_\zeta^{-1}M_q)^{-1}\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{\frac{1}{2}}} \|q\|_{\dot{X}_\zeta^{-\frac{1}{2}}} \leq \frac{1}{1-c} \|q\|_{\dot{X}_\zeta^{-\frac{1}{2}}}.$$

Then, we can rewrite the goal as: to find a pair of sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$ that satisfy the following conditions:

- $\zeta_{1,k} + \zeta_{2,k} = i\xi$ and $|\zeta_{j,k}| \rightarrow \infty$ as $k \rightarrow \infty$.
- $\|M_{q_j}\|_{\dot{X}_{\zeta_{j,k}}^{\frac{1}{2}} \rightarrow \dot{X}_{\zeta_{j,k}}^{-\frac{1}{2}}} \leq c < 1$ for sufficiently large k .
- $\|q_j\|_{\dot{X}_{\zeta_{j,k}}^{-\frac{1}{2}}} \rightarrow 0$ as $k \rightarrow \infty$.

To find the sequences $\{\zeta_{1,k}\}$ and $\{\zeta_{2,k}\}$, Haberman proved that the expected value of $\|M_{q_j}\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}}$ and $\|q_j\|_{\dot{X}_\zeta^{-\frac{1}{2}}}$ over $|\zeta| \sim M \geq 1$ is small; see Theorem 1. The reader can see in [11, sec. 7] how to find the sequences from the vanishing of the expected value.

To prove the vanishing of the expected value of $\|q_j\|_{\dot{X}_\zeta^{-\frac{1}{2}}}$, it suffices to assume that $\nabla \log \gamma_j \in L^d(\mathbb{R}^d)$, so we will not turn our attention to it.

To control $\|M_{q_j}\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}}$ we write $q = \frac{1}{2}\Delta \log \gamma + \frac{1}{4}|\nabla \log \gamma|^2 = \frac{1}{2}\operatorname{div}(\mathbf{f}) + \frac{1}{4}|\mathbf{f}|^2$, where the components of $\mathbf{f} = (f^1, \dots, f^n)$ belong to $W^{s-1,p}(\mathbb{R}^d)$. We can divide M_q into the terms $M_{\partial_i f}$ and $M_{|\mathbf{f}|^2}$. Haberman proved that the expected value of $\|M_{|\mathbf{f}|^2}\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}}$ goes to zero if $f \in L^d(\mathbb{R}^d)$, so we are left with $\|M_{\partial_i f}\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}}$.

The estimates for $\|M_{\partial_i f}\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}}$ are not strong enough to get the vanishing in the limit for $f \in L^d$. To prove Theorem 2, we assume the following theorem, which we will prove in the next section.

Theorem 8. *Suppose that f is supported in the unit ball. If $f \in W^{\frac{d-5}{2p}+p}(\mathbb{R}^d)$ for $d \leq p < \infty$, then*

$$\int_M \int_{O_d} \|M_{\partial_i f}\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}} \rightarrow X_{\zeta(U,\tau)}^{-\frac{1}{2}}} dU d\tau \leq C \|f\|_{\frac{d-5}{2p}+p}. \quad (14)$$

Proof of Theorem 2. Since f is compactly supported, then $\|M_{\partial_i f}\|_{\dot{X}_\zeta^{\frac{1}{2}} \rightarrow \dot{X}_\zeta^{-\frac{1}{2}}} \lesssim \|M_{\partial_i f}\|_{X_\zeta^{\frac{1}{2}} \rightarrow X_\zeta^{-\frac{1}{2}}}$; see [12, Lemma 2.2(3-4)]. We estimate M_g by duality as

$$|\langle gu, v \rangle| \leq \|g\|_\infty \|u\|_2 \|v\|_2 \leq \frac{1}{|\zeta|} \|g\|_\infty \|u\|_{X_\zeta^{\frac{1}{2}}} \|v\|_{X_\zeta^{-\frac{1}{2}}}.$$

For some $A \leq 1$ to be fixed later, we define $g = P_{\leq A} \partial_i f$, where $P_{\leq A}$ is the projection to frequencies $\lesssim A$. By Young inequality for convolutions we get

$$\|M_g\|_{X_\zeta^{\frac{1}{2}} \rightarrow X_\zeta^{-\frac{1}{2}}} \leq \frac{1}{|\zeta|} \|g\|_\infty \lesssim \frac{A^2}{|\zeta|} \|f\|_d.$$

The expected value is thus bounded as

$$\begin{aligned} \int_M \int_{O_d} \|M_{\partial_i f}\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}} \rightarrow X_{\zeta(U,\tau)}^{-\frac{1}{2}}} dU d\tau &\lesssim \frac{A^2}{M} \|f\|_d + \int_M \int_{O_d} \|M_{P_{>A} \partial_i f}\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}} \rightarrow X_{\zeta(U,\tau)}^{-\frac{1}{2}}} dU d\tau \\ &\lesssim \frac{A^2}{M} \|f\|_d + \|P_{>A} f\|_{\frac{d-5}{2p}+p}. \end{aligned}$$

If we choose $A = M^{\frac{1}{4}}$ and let $M \rightarrow \infty$, then we get the vanishing. \square

3 Upper Bounds for the Expected Value

In this and the next section, we use duality to get an upper bound of $\|M_{\partial_j f}\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}} \rightarrow X_{\zeta(U,\tau)}^{-\frac{1}{2}}}$ in terms of f , U and τ . We want to get an upper bound

$$|\langle (\partial_j f)u, v \rangle| = \left| \int_{\mathbb{R}^d} (\partial_j f)u\bar{v} dx \right| \leq A(U, \tau, f) \|u\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}}} \|v\|_{X_{\zeta(U,\tau)}^{-\frac{1}{2}}}, \quad (15)$$

with a constant $A(U, \tau, f)$ depending on some quantity related to $\|f\|_{W^{s,p}}$ for $s = \frac{d-5}{2p} +$ and $d \leq p < \infty$.

The characteristic set Σ_{ζ} of $p_{\zeta}(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$, the symbol of Δ_{ζ} , is a $(d-2)$ -sphere in the hyperplane $\{\xi \mid \langle Ue_1, \xi \rangle = 0\}$, with center τUe_2 and radius $\tau \geq 1$. If $d(\xi, \Sigma)$ denotes the distance from ξ to Σ_{ζ} , then

$$|p_{\zeta}(\xi)| \sim \begin{cases} \tau d(\xi, \Sigma_{\zeta}), & \text{for } d(\xi, \Sigma_{\zeta}) \leq \frac{1}{10}\tau, \\ \tau^2 + |\xi|^2, & \text{for } d(\xi, \Sigma_{\zeta}) > \frac{1}{10}\tau \end{cases}$$

We break up the frequencies accordingly into characteristics and non-characteristics, and define the corresponding projections as

$$\begin{aligned} (Q_l f)^{\wedge}(\xi) &:= \zeta(\tau^{-1}d(\xi, \Sigma_{\zeta}))\hat{f}(\xi) \\ (Q_h f)^{\wedge}(\xi) &:= (1 - \zeta(\tau^{-1}d(\xi, \Sigma_{\zeta})))\hat{f}(\xi), \end{aligned}$$

where $\zeta \in C_c^{\infty}(\mathbb{R})$ is supported inside $(-\frac{1}{10}, \frac{1}{10})$. It follows that

$$\|Q_h u\|_2 \leq \tau^{-1} \|u\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}}} \quad (16)$$

$$\|\partial_j Q_h u\|_2 \leq \|u\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}}}. \quad (17)$$

In Lemma 3.3 of [11] Haberman proved, using Tomas-Stein inequality, that

$$\|u\|_{\frac{2d}{d-2}} \lesssim \|u\|_{X_{\zeta(U,\tau)}^{\frac{1}{2}}}. \quad (18)$$

With the help of inequalities (16), (17) and (18), we can control in (15) all the terms involving non-characteristic frequencies. In fact,

$$\begin{aligned} \langle (\partial_j f)u, v \rangle &= \langle (\partial_j f)Q_h u, Q_h v \rangle + \langle (\partial_j f)Q_h u, Q_l v \rangle + \\ &\quad + \langle (\partial_j f)Q_l u, Q_h v \rangle + \langle (\partial_j f)Q_l u, Q_l v \rangle. \end{aligned}$$

For the first term at the right, after integration by parts, we have

$$\begin{aligned}
|\langle (\partial_j f) Q_h u, Q_h v \rangle| &\leq \|f\|_d (\|\partial_j Q_h u\|_2 \|Q_h v\|_{\frac{2d}{d-2}} + \\
&\quad + \|Q_h u\|_{\frac{2d}{d-2}} \|\partial_j Q_h v\|_2) \\
&\lesssim \|f\|_d \|u\|_{X_{\zeta(U, \tau)}^{\frac{1}{2}}} \|v\|_{X_{\zeta(U, \tau)}^{\frac{1}{2}}}. \tag{19}
\end{aligned}$$

For the mixed terms we have

$$\begin{aligned}
|\langle (\partial_j f) Q_h u, Q_l v \rangle| &\leq \|f\|_d (\|\partial_j Q_h u\|_2 \|Q_l v\|_{\frac{2d}{d-2}} + \\
&\quad + \|Q_h u\|_2 \|\partial_j Q_l v\|_{\frac{2d}{d-2}}) \\
&\lesssim \|f\|_d \|u\|_{X_{\zeta(U, \tau)}^{\frac{1}{2}}} \|v\|_{X_{\zeta(U, \tau)}^{\frac{1}{2}}}, \tag{20}
\end{aligned}$$

where we used the localization of $Q_l v$ to frequencies $\leq 5\tau$, so that $\|\partial_j Q_l v\|_{\frac{2d}{d-2}} \lesssim \tau \|Q_l v\|_{\frac{2d}{d-2}}$; this follows from Young inequality. We are left then with the characteristic frequencies.

We assume that the support of the Fourier transform of u and v lie in a $\frac{1}{10}$ -neighborhood of Σ_ζ . We define the transformation

$$u_{\tau U}(x) := \tau^{-d} u(\tau^{-1} U x), \tag{21}$$

so that the frequencies of $u_{\tau U}$ are supported in a $\frac{1}{10}$ -neighborhood of the S^{d-2} sphere centered at e_2 in the hyperplane normal to e_1 . The Fourier transform of $u_{\tau U}$ is $\widehat{u}_{\tau U}(\xi) = \widehat{u}(\tau U \xi)$, and the $X_{\zeta(U, \tau)}^b$ -norm scales as

$$\|u\|_{X_{\zeta(U, \tau)}^b} = \tau^{\frac{d}{2} + 2b} \|u_{\tau U}\|_{X_{\zeta(1), 1/\tau}^b}. \tag{22}$$

We change variables in the pairing (15) to get

$$\begin{aligned}
\langle (\partial_j f) u, v \rangle &= \tau^{-d} \int (\partial_j f)(\tau^{-1} U x) u(\tau^{-1} U x) \bar{v}(\tau^{-1} U x) dx \\
&= \tau^{2d+1} \int_{B_\tau} (\partial_j f_{\tau U}) u_{\tau U} \bar{v}_{\tau U} dx \\
&= \tau^{2d+1} \langle (\partial_{U e_j} f_{\tau U}) u_{\tau U}, v_{\tau U} \rangle, \tag{23}
\end{aligned}$$

where we used the identity

$$(\partial_j f)(\tau^{-1} U x) = \int \xi_j \widehat{f}(\xi) e^{i(\tau^{-1} U x) \cdot \xi} d\xi = \tau^{d+1} (\partial_{U e_j} f_{\tau U})(x).$$

Therefore, we assume that the characteristic sphere S^{d-2} lies in the normal plane to e_1 , has radius 1 and is centered at e_2 . We assume also that the function f is supported in $B_\tau(0)$.

We apply the Hardy-Littlewood decomposition to $f = \sum_{\tau^{-1} \leq \lambda} P_\lambda f$, and decompose u and v into dyadic projections u_μ and v_ν , where $(u_\mu)^\wedge = \zeta(\mu^{-1}d(\xi, \Sigma_\zeta))\hat{u}$ and $\zeta \in C_c^\infty(\mathbb{R})$ is supported in $(\frac{1}{2}, 2)$. Then, the pairing (15) gets into

$$\begin{aligned} \langle (\partial_w f)u, v \rangle &= \sum_{\tau^{-1} \leq \lambda, \mu, \nu \lesssim 1} \langle (\partial_w P_{\lambda, \sup(\mu, \nu)} f)u_\mu, v_\nu \rangle \\ &= \sum_{\substack{\tau^{-1} \leq \lambda \lesssim 1 \\ \tau^{-1} \leq \mu \leq \nu \lesssim 1}} \langle (\partial_w P_{\lambda, \nu} f)u_\mu, v_\nu \rangle + \sum_{\substack{\tau^{-1} \leq \lambda \lesssim 1 \\ \tau^{-1} \leq \mu > \nu \lesssim 1}} \dots, \end{aligned} \quad (24)$$

where ∂_w is the derivative in some direction w , and $P_{\lambda, \sup(\mu, \nu)}$ is the projection to frequencies $|\xi| \sim \lambda$ and $|\xi_1| \lesssim \sup(\mu, \nu)$. By symmetry, we can assume that $\mu \leq \nu$.

We use Toma-Stein to control the low frequency terms, $\lambda \lesssim \nu^{\frac{1}{2}}$, and the terms with very different characteristic regions, $\mu^{\frac{1}{2}} \leq \nu$.

Theorem 9. *If f_μ and g_ν are functions in \mathbb{R}^n , and their Fourier transform are supported in a μ - and ν -neighborhood of S^{n-1} respectively, where $\mu \leq \nu$, then*

$$\|f_\mu g_\nu\|_{p'} \lesssim \mu^{\frac{n+1}{2p'}} \|f_\mu\|_2 \|g_\nu\|_2, \quad \text{for } 1 \leq p' \leq \frac{n+1}{n}. \quad (25)$$

Proof. We use Hölder to get

$$\|f_\mu g_\nu\|_{p'} \leq \|f_\mu\|_{2p'/(2-p')} \|g_\nu\|_2. \quad (26)$$

Since $1 \leq p' \leq \frac{n+1}{n}$, then $2 \leq 2p'/(2-p') \leq 2\frac{n+1}{n-1}$, and the latter is the Tomas-Stein exponent. To bound the term $\|f_\mu\|_r$, for $r = \frac{2p'}{2-p'}$, we interpolate between $p' = 2$ and $p' = 2\frac{n+1}{n-1}$.

The point $p' = 2$ is immediate. For $p' = 2\frac{n+1}{n-1}$, we write \hat{f}_μ as an average over spheres

$$f_\mu(x) = \int r^{n-1} \int_{S^{n-1}} \hat{f}_\mu(r\theta) e(\langle rx, \theta \rangle) d\theta dr := \int r^{n-1} (f_\mu^r dS)^\vee(rx) dr$$

We apply Minkowski, Tomas-Stein and Cauchy-Schwarz to find $\|f_\mu\|_{2\frac{n+1}{n-1}} \leq C\mu^{\frac{1}{2}} \|f_\mu\|_2$; this leads to

$$\|f_\mu\|_r \lesssim \mu^{\frac{n+1}{2}(\frac{1}{2}-\frac{1}{r})} \|f_\mu\|_2, \quad \text{for } 2 \leq r \leq 2\frac{n+1}{n-1}.$$

We replace it in (26) to get

$$\|f_\mu g_\nu\|_{p'} \lesssim \mu^{\frac{n+1}{2p}} \|f_\mu\|_2 \|g_\nu\|_2,$$

which is what we wanted. \square

By Hölder, we can bound each term in (24) as

$$|\langle (\partial_w P_{\lambda,\nu} f) u_\mu, v_\nu \rangle| \leq \lambda \|P_{\lambda,\nu} f\|_p \|u_\mu v_\nu\|_{p'}. \quad (27)$$

To bound the bilinear term, we begin by writing it as

$$\int |u_\mu v_\nu|^{p'} dx = \iint |u_\mu(x_1, \tilde{x}) v_\nu(x_1, \tilde{x})|^{p'} d\tilde{x} dx_1. \quad (28)$$

We fix x_1 as a parameter and define the function $u_\mu^{x_1}(\tilde{x}) = u_\mu(x_1, \tilde{x})$; its Fourier transform is the term in parentheses in the formula

$$u_\mu(x_1, \tilde{x}) = \int \left(\int \hat{u}_\mu(x) e^{ix_1 \cdot \xi_1} d\xi_1 \right) e^{i\tilde{x} \cdot \tilde{\xi}} d\tilde{\xi} = \int \hat{u}_\mu^{x_1}(\tilde{\xi}) e^{i\tilde{x} \cdot \tilde{\xi}} d\tilde{\xi}.$$

The support of $\hat{u}_\mu^{x_1}$ lies in a μ -neighborhood of the sphere $S^{d-2} \subset \mathbb{R}^{d-1}$. Hence, we can apply Theorem 9 with $n = d - 1$ to the inner integral at the right of (28) to get

$$\int |u_\mu v_\nu|^{p'} dx \leq \mu^{p' \frac{d}{2p}} \int \|u_\mu(x_1, \cdot)\|_2^{p'} \|v_\nu(x_1, \cdot)\|_2^{p'} dx_1. \quad (29)$$

Since \hat{u}_μ is supported in the μ -neighborhood of the hyperplane normal to e_1 , then we can use the formula $u_\mu = u_\mu * \phi_\mu$, where $\phi_\mu(x) = \mu \phi(\mu x)$ and $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ is a smooth function whose Fourier transform equals one in a μ -neighborhood of the origin. Hence, by Minkowski we have

$$\begin{aligned} \|u_\mu(x_1, \cdot)\|_2 &= \left(\int \left| \int u_\mu(x_1 - y_1, \tilde{x}) \phi_\mu(y_1) dy_1 \right|^2 d\tilde{x} \right)^{1/2} \\ &\leq \int \|u_\mu(x_1 - y_1, \cdot)\|_2 \phi_\mu(y_1) dy_1 \\ &= (\|u_\mu^{z_1}\|_{L^2_{\frac{x}{\mu}}} * \phi_\mu)(x_1). \end{aligned}$$

This fact and the next lemma allow us to bound the integral at the right of (29).

Lemma 10. *Let a and b be two functions in the real line, then*

$$\|(a * \phi_\mu)b\|_{p'} \leq C\mu^{\frac{1}{p}}\|a\|_2\|b\|_2, \quad \text{for } 1 \leq p' \leq 2. \quad (30)$$

The inequality is best possible in μ .

Proof. We use Hölder and Young inequalities to get

$$\|(a * \phi_\mu)b\|_{p'} \leq \|a * \phi_\mu\|_{2p'/(2-p')} \|b\|_2 \leq \|\phi_\mu\|_{p'} \|a\|_2 \|b\|_2,$$

where $\|\phi_\mu\|_{p'} = \mu^{\frac{1}{p}}\|\phi_1\|_{p'}$. The example $a = b = \mathbf{1}_{(-\mu^{-1}, \mu^{-1})}$ shows that the constant $\mu^{\frac{1}{p}}$ is best possible. \square

With the aid of Lemma 10 and $\|u_\mu\|_2 \lesssim \mu^{-\frac{1}{2}}\|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}}$, we continue (29) as

$$\|u_\mu v_\nu\|_{p'} \leq \mu^{\frac{d+2}{2p}}\|u_\mu\|_2\|v_\nu\|_2 \lesssim \mu^{\frac{d+2}{2p}-\frac{1}{2}}\nu^{-\frac{1}{2}}\|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}}\|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}}. \quad (31)$$

Furthermore, when we are restricted to low frequencies $\lambda \lesssim \nu^{\frac{1}{2}}$, we can use this bound and (27) in the pairing (24) to get for $p = d$

$$\begin{aligned} |\langle (\partial_w f)u, v \rangle| &\lesssim \left(\sum_{\substack{\tau^{-1} \leq \lambda \lesssim \nu^{\frac{1}{2}} \\ \tau^{-1} \leq \mu \leq \nu \lesssim 1}} \lambda \mu^{\frac{1}{d}} \nu^{-\frac{1}{2}} \|P_\lambda f\|_d \right) \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} + \dots \\ &\lesssim \left(\sum_{\tau^{-1} \leq \lambda \lesssim \nu^{\frac{1}{2}}} \lambda \nu^{\frac{1}{d}-\frac{1}{2}} \|P_\lambda f\|_d \right) \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \\ &\lesssim \left(\sum_{\tau^{-1} \leq \lambda \lesssim 1} \lambda^{\frac{2}{d}} \|P_\lambda f\|_d \right) \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \\ &\lesssim \left(\sum_{\tau^{-1} \leq \lambda \lesssim 1} \|P_\lambda f\|_d^d \right)^{\frac{1}{d}} \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \\ &\lesssim \|f\|_d \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} + \dots \end{aligned} \quad (32)$$

On the other hand, when the characteristic frequencies are very different, *i.e.* $\mu^{\frac{1}{2}} \leq \nu$, again by (31) and (27) in the pairing (24), we get

$$|\langle (\partial_w f)u, v \rangle| \leq \left(\sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \lesssim 1 \\ \tau^{-1} \leq \mu \leq \nu^2 \lesssim 1}} \lambda \mu^{\frac{d+2}{2p}-\frac{1}{2}} \nu^{-\frac{1}{2}} \|P_{\lambda,\nu} f\|_p \right) \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} + \dots \quad (33)$$

We are left thus with the case of high frequencies $\lambda \gtrsim \nu^{\frac{1}{2}}$, and similar characteristic frequencies $\mu \leq \nu \leq \mu^{\frac{1}{2}}$.

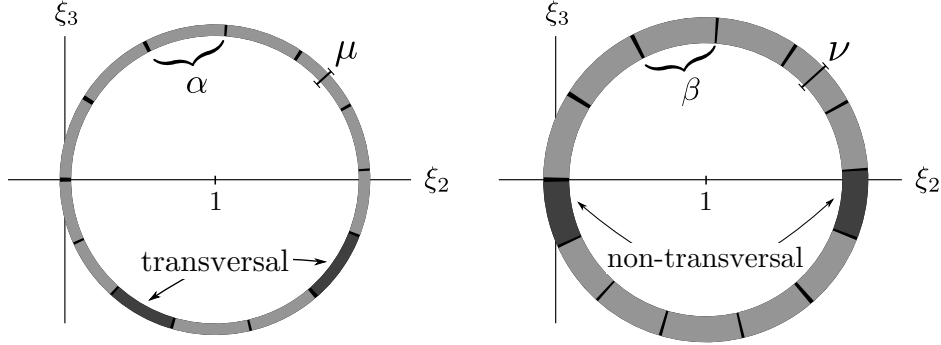


Figure 1: The decomposition of the μ - and the ν -neighborhoods of the sphere $S^{d-2} + e_2$ into caps α and β .

3.1 Bilinear Strategy

In this section we assume that $\mu \leq \nu < \mu^{\frac{1}{2}}$, so that the bilinear inequality in Theorem 7 give us a small improvement over Tomas-Stein inequality. To pass from bilinear to linear inequalities, we follow the strategy in [26].

Using smooth partitions of unity $\{\varphi_\alpha\}_\alpha$ and $\{\varphi_\beta\}_\beta$ we decompose, respectively, the μ - and the ν -neighborhoods of the sphere $S^{d-2} + e_2$ into caps α and β of radius $\rho_0 \ll 1$; see Figure 1. If the angle between the normal vectors to two caps α and β , respectively, is $\gtrsim \rho_0$, then we call them transversal and denote it by $\alpha \sim \beta$; otherwise the caps are not transversal, $\alpha \not\sim \beta$. For transversal caps we can use the Bilinear Theorem 7 for the sphere. We define $\hat{u}_{\mu,\alpha} := \varphi_\alpha \hat{u}_\mu$ and $\hat{v}_{\nu,\beta} := \varphi_\beta \hat{v}_\nu$, and write so the bilinear term as

$$u_\mu \bar{v}_\nu = \sum_{\alpha,\beta} u_{\mu,\alpha} \bar{v}_{\nu,\beta} = \sum_{\alpha \sim \beta} u_{\mu,\alpha} \bar{v}_{\nu,\beta} + \sum_{\alpha \not\sim \beta} u_{\mu,\alpha} \bar{v}_{\nu,\beta}.$$

Since we cannot apply the Bilinear Theorem to non-transversal caps, we decompose them again into caps of radius $\rho_1 = \frac{1}{2}\rho_0$, and we still denote the smaller caps as α and β . If the angle between the normal vectors to two caps α and β , respectively, is $\sim \rho_1$, then we call them transversal and denote it again by $\alpha \sim \beta$; otherwise the caps are not transversal, $\alpha \not\sim \beta$. For transversal caps we will use a rescaled version of the Bilinear Theorem 7 for surfaces of elliptic type, choosing ρ_0 sufficiently small. We continue the process of subdivision of non-transversal caps until the radius of the caps is

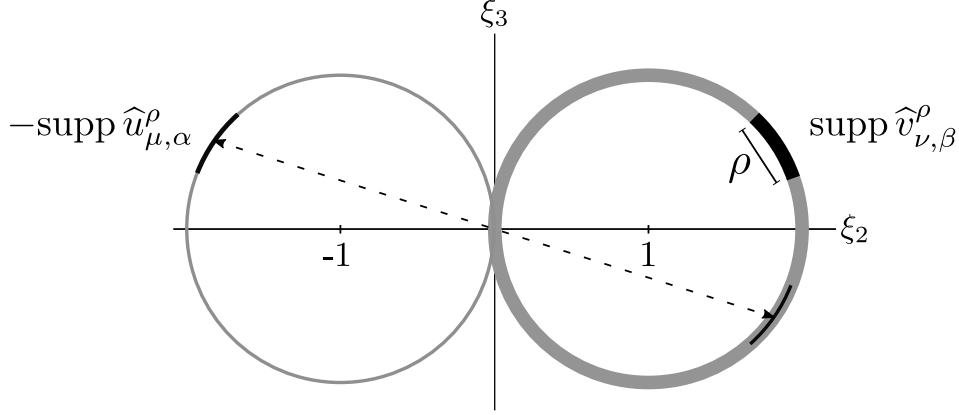


Figure 2: Two neighboring, transversal caps.

$\rho \sim \nu^{\frac{1}{2}}$, and write

$$\begin{aligned} \langle (\partial_w f)u, v \rangle = & \sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \lesssim 1 \\ \mu \leq \nu < \mu^{\frac{1}{2}}} } \left[\sum_{\substack{\nu^{1/2} < \rho \lesssim 1 \\ \alpha \sim \beta}} \langle (\partial_w P_{\lambda,\nu} f)u_{\mu,\alpha}^\rho, v_{\nu,\beta}^\rho \rangle + \right. \\ & \left. + \sum_{\alpha \not\sim \beta} \langle (\partial_w P_{\lambda,\nu} f)u_{\mu,\alpha}^{\rho^*}, v_{\nu,\beta}^{\rho^*} \rangle \right], \quad (34) \end{aligned}$$

where the sum over non-transversal terms is at scale $\rho^* \sim \nu^{\frac{1}{2}}$. The superscript in $u_{\mu,\alpha}^\rho$ is to keep track of the radius of the caps α .

The support of the inverse Fourier transform of $u_{\mu,\alpha}^\rho \bar{v}_{\nu,\beta}^\rho$ has some special properties, and they determine when the pairing $\langle (\partial_w P_{\lambda,\nu} f)u_{\mu,\alpha}^\rho, v_{\nu,\beta}^\rho \rangle$ either vanishes or not. Recall that the support of the convolution $\check{u}_{\mu,\alpha}^\rho * \widehat{v}_{\nu,\beta}^\rho$ lies in the Minkowski sum of the sets $-\alpha \supset \text{supp } \check{u}_{\mu,\alpha}^\rho$ and $\beta \supset \text{supp } \widehat{v}_{\nu,\beta}^\rho$; see Figure 2. The reader will find easier to evaluate the Minkowski sum of $-\alpha + e_2$ and $\beta - e_2$.

When the caps α and β have radius ρ_0 and are transversal, then we have that

$$-\alpha + \beta \subset \{(\xi_1, \tilde{\xi}) \mid \frac{\rho_0}{2} \leq |\tilde{\xi}| \leq 2 - \frac{\rho_0^2}{2}, \quad |\xi_1| \leq 2\nu\};$$

Hence, all the terms $\langle (\partial_w P_{\lambda,\nu} f)u_{\mu,\alpha}^{\rho_0}, v_{\nu,\beta}^{\rho_0} \rangle$ vanish for $\lambda \leq c\rho_0$.

When the caps have radius $\rho < \rho_0$, we have to distinguish between neighboring and antipodal caps. Two caps are neighboring if there exists a ball

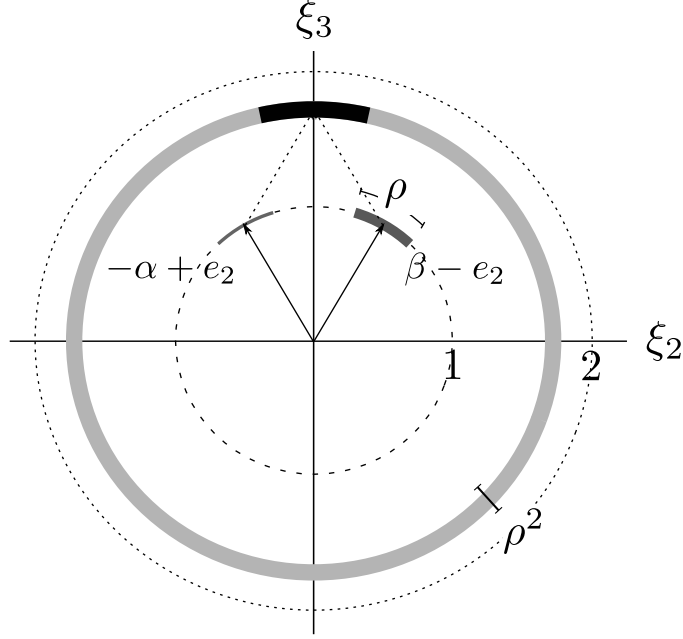


Figure 3: The Minkowski sum of two antipodal, transversal caps at scale ρ .

of radius $2\rho_0$ that contains both of them, and two caps are antipodal if they lie in different and opposite balls of radius $2\rho_0$. We refer to neighboring and antipodal, transversal caps as $\alpha \sim_n \beta$ and $\alpha \sim_a \beta$ respectively.

If two caps of radius $\nu^{\frac{1}{2}} \leq \rho < \rho_0$ are neighboring and transversal, then for the Minkowski sum we get

$$-\alpha + \beta \subset \{(\xi_1, \tilde{\xi}) \mid |\tilde{\xi}| \sim \rho, \quad |\xi_1| \leq 2\nu\}.$$

Hence, only the terms $\langle (\partial_w P_{\lambda, \nu} f) u_{\mu, \alpha}^\rho, v_{\nu, \beta}^\rho \rangle$ for which $\lambda \sim \rho$ survive. When the caps are non-transversal, the Minkowski sum lies in $\{|\tilde{\xi}| \leq c\nu^{\frac{1}{2}}\}$, but we already considered the low frequency terms $\lambda \lesssim \nu^{\frac{1}{2}}$ in the previous section, so $\langle (\partial_w P_{\lambda, \nu} f) u_{\mu, \alpha}^{\rho*}, v_{\nu, \beta}^{\rho*} \rangle$ always vanishes.

If two caps of radius $\nu^{\frac{1}{2}} \leq \rho < \rho_0$ are antipodal and transversal, then for the Minkowski sum we get

$$-\alpha + \beta \subset S_{\nu, \rho} := \{(\xi_1, \tilde{\xi}) \mid 2 - |\tilde{\xi}| \sim \rho^2, \quad |\xi_1| \leq 2\nu\}. \quad (35)$$

Only the terms $\langle (\partial_w P_{\lambda, \nu} f) u_{\mu, \alpha}^\rho, v_{\nu, \beta}^\rho \rangle$ for which $\lambda \sim 1$ survive, but now we need more detailed information about $-\alpha + \beta$; see Figure 3. We can see

that $-\alpha + \beta$ forms a cap of radius $\sim \rho$ lying in the ρ^2 -neighborhood of the sphere with radius $2 - \rho^2$ centered at zero, which we called $S_{\nu,\rho}$. Fixing ρ , the collection of all the the caps $\{-\alpha + \beta\}$, where $\alpha \sim_a \beta$, is an almost disjoint covering of $S_{\nu,\rho}$. In fact, let x be a point in $S_{\nu,\rho}$, c_α be the center of α and c_β be the center of β ; if x and $-c_\alpha + e_2$ make an angle $\gtrsim \rho$, since $\alpha \sim_a \beta$ then the sum $-\alpha + \beta$ necessarily lies away from x . Hence, only the caps $-\alpha + \beta$ for which α and β make an angle $\lesssim \rho$ with x can cover it. For future reference let us write it down as a lemma.

Lemma 11. *For fixed μ, ν and $\nu^{\frac{1}{2}} < \rho < \rho_0$, let α and β denote caps at scale ρ , then*

$$\sum_{\alpha \sim_a \beta} \mathbf{1}_{-\alpha + \beta} \leq C_d \mathbf{1}_{S_{\nu,\rho}}, \quad (36)$$

where $S_{\nu,\rho}$ is defined in (35), and C_d does not depend either on μ , on ν or on ρ .

A similar statement holds for non-transversal caps at scale $\nu^{\frac{1}{2}}$, but the caps $-\alpha + \beta$ lie now in a ν -neighborhood of $2S^{d-2}$.

We will follow the argument of the previous section to bound the terms $\langle (\partial_w P_{\lambda,\nu} f) u_{\mu,\alpha}^\rho, v_{\nu,\beta}^\rho \rangle$; however, the Bilinear Theorem is only stated for transversal caps at scale $\sim 1 \sim \rho_0$. To remedy this situation, we use parabolic rescaling.

Theorem 12. *Let $f_{\mu,\alpha}$ and $g_{\nu,\beta}$ be two functions with Fourier transform supported in a μ - and ν -neighborhood of S^{n-1} . If the caps α and β are transversal at scale $\rho \leq \rho_0$, then for $1 \leq p' \leq \frac{n+1}{n}$ it holds that*

$$\begin{aligned} \|f_{\mu,\alpha} g_{\nu,\beta}\|_{p'} &\leq C_\varepsilon \rho^{-\frac{1}{p}} \mu^{\frac{n}{2p} - \varepsilon} \nu^{\frac{1}{p} - \varepsilon} \|f_{\mu,\alpha}\|_2 \|g_{\nu,\beta}\|_2 \quad \text{for } \rho > \nu \mu^{-\frac{1}{2}}, \\ \|f_{\mu,\alpha} g_{\nu,\beta}\|_{p'} &\leq C \mu^{\frac{n+1}{2}} \|f_{\mu,\alpha}\|_2 \|g_{\nu,\beta}\|_2 \quad \text{for } \nu^{\frac{1}{2}} \leq \rho \leq \nu \mu^{-\frac{1}{2}}. \end{aligned} \quad (37)$$

Proof. The case $\rho_0 \sim 1 \gtrsim \nu \mu^{-\frac{1}{2}}$ is Theorem 7 for the sphere, so we assume that $\rho < \rho_0$. By conjugation and modulation, we assume further that both caps lie in the hypersurface given by the graph of

$$\varphi(\xi') = 1 - \sqrt{1 - |\xi'|^2} = \frac{1}{2} |\xi'|^2 + O(|\xi'|^4),$$

where $\xi = (\xi', \xi_n) \in \mathbb{R}^n$; we assume also that the center of the caps are symmetrically placed in the axis ξ_1 . Since the caps are at distance $\sim \rho$

of each other, after applying the scaling $\xi \mapsto (\rho^{-1}\bar{\xi}, \rho^{-2}\xi_d)$, the support of the new functions $\widehat{F}(\xi) := \widehat{f}_{\mu,\alpha}(\rho\xi', \rho^2\xi_n)$ and $\widehat{G}_{\nu,\beta} := \widehat{g}_{\nu,\beta}(\rho\xi', \rho^2\xi_n)$ lie at distance ~ 1 of each other, and the hypersurface transforms accordingly to the graph of

$$\varphi_\rho(\xi') := \rho^{-2}\varphi(\rho\xi') = \rho^{-2} - \sqrt{\rho^{-4} - |\rho^{-1}\bar{\xi}|^2} = \frac{1}{2}|\xi'|^2 + O(\rho_0^2|\xi'|^4).$$

If $\rho < \rho_0$ is sufficiently small, then the semi-norms $\|\partial^N \varphi_\rho\|_\infty$ are uniformly bounded, and the bilinear theorem holds uniformly. The rescaled functions F and G are

$$\begin{aligned} F(x) &= \rho^{-n-1} f_{\mu,\alpha}(\rho^{-1}x', \rho^{-2}x_n) \\ G(x) &= \rho^{-n-1} g_{\nu,\beta}(\rho^{-1}x', \rho^{-2}x_n). \end{aligned}$$

Since the Fourier transforms of F and G are supported now in sets of width $\rho^{-2}\mu$ and $\rho^{-2}\nu$ respectively, then we should apply the Bilinear Theorem 7 whenever $\rho^{-2}\nu < (\rho^{-2}\mu)^{\frac{1}{2}}$, and Tomas-Stein otherwise.

If $\rho > \nu\mu^{-\frac{1}{2}}$, then we apply the Bilinear Theorem to F and G to find

$$\begin{aligned} \|f_{\mu,\alpha}g_{\nu,\beta}\|_{p'} &= \rho^{2(n+1)-\frac{n+1}{p'}} \|FG\|_{p'} \\ &\leq C_\varepsilon \rho^{2(n+1)-\frac{n+1}{p'}-\frac{n+2}{p}} \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|F\|_2 \|G\|_2 \\ &= C_\varepsilon \rho^{-\frac{1}{p}} \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f_{\mu,\alpha}\|_2 \|g_{\nu,\beta}\|_2; \end{aligned}$$

if we use Tomas-Stein instead, then we get the result for $\rho \leq \nu\mu^{-\frac{1}{2}}$ □

If we define the quantity

$$K_{\mu,\nu}^\rho(p') := \sup_{\substack{\|f_{\mu,\alpha}\|_2=1 \\ \|g_{\nu,\beta}\|_2=1}} \|f_{\mu,\alpha}g_{\nu,\beta}\|_{p'}, \quad (38)$$

where the supremum runs over functions $f_{\mu,\alpha}$ and $g_{\nu,\beta}$ with Fourier transform supported in caps at scale ρ , then we can restate Theorem 12 as

$$K_{\mu,\nu}^\rho(p') \leq \begin{cases} C_\varepsilon \rho^{-\frac{1}{p}} \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} & \text{for } \rho > \nu\mu^{-\frac{1}{2}} \\ C\mu^{\frac{n+1}{2p}} & \text{for } \nu^{\frac{1}{2}} \leq \rho \leq \nu\mu^{-\frac{1}{2}} \end{cases}$$

By Lemma 10 and Theorem 12, for $n = d - 1$, we get

$$\begin{aligned}
\sum_{\alpha \sim \beta} \|u_{\mu,\alpha}^\rho v_{\nu,\beta}^\rho\|_{p'} &\lesssim \mu^{\frac{1}{p}} K_{\mu,\nu}^\rho \sum_{\alpha \sim \beta} \|u_{\mu,\alpha}\|_2 \|v_{\nu,\beta}\|_2 \\
&\lesssim \mu^{\frac{1}{p}} K_{\mu,\nu}^\rho \|u_\mu\|_2 \|v_\nu\|_2 \\
&\lesssim \mu^{\frac{1}{p} - \frac{1}{2}} \nu^{-\frac{1}{2}} K_{\mu,\nu}^\rho \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}}.
\end{aligned}$$

Now let us consider only transversal, neighboring caps at scale ρ . By the decomposition (34) we get

$$\begin{aligned}
|\langle (\partial_w f)u, v \rangle| &\leq \sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \sum_{\substack{\nu^{1/2} \leq \rho \sim \lambda \\ \alpha \sim_n \beta}} |\langle (\partial_w P_{\lambda,\nu} f) u_{\mu,\alpha}^\rho, v_{\nu,\beta}^\rho \rangle| + \dots \\
&\lesssim \sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \lambda \mu^{\frac{1}{p} - \frac{1}{2}} \nu^{-\frac{1}{2}} \|P_{\lambda,\nu} f\|_p \sum_{\nu^{1/2} \leq \rho \sim \lambda} K_{\mu,\nu}^\rho \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \\
&\lesssim_\varepsilon \left(\sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \leq \nu \mu^{-\frac{1}{2}} \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \lambda \mu^{\frac{d+2}{2p} - \frac{1}{2}} \nu^{-\frac{1}{2}} \|P_{\lambda,\nu} f\|_p + \right. \\
&\quad \left. \sum_{\substack{\nu \mu^{-\frac{1}{2}} \leq \lambda \lesssim 1 \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \lambda^{1 - \frac{1}{p}} \mu^{\frac{d+1}{2p} - \frac{1}{2}} \nu^{\frac{1}{p} - \frac{1}{2} - \varepsilon} \|P_{\lambda,\nu} f\|_p \right) \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}}.
\end{aligned} \tag{39}$$

The operator $P_{\lambda,\nu}$ is the projection to frequencies $|\xi| \sim \lambda$ and $|\xi_1| \lesssim \nu$.

When the caps α and β are antipodal, we have to refine the projection $P_{\lambda,\nu}$, so we project also to the cap $-\alpha + \beta$ and denote this projection as $P_{\lambda,\nu,\alpha,\beta}$. We argue as above to get

$$\begin{aligned}
|\langle (\partial_w f)u, v \rangle| &\leq \sum_{\substack{\lambda \sim 1 \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \left(\sum_{\substack{\nu^{1/2} < \rho \\ \alpha \sim_a \beta}} |\langle (\partial_w P_{\lambda,\nu} f) u_{\mu,\alpha}^\rho, v_{\nu,\beta}^\rho \rangle| + |\langle (\partial_w P_{\lambda,\nu} f) u_{\mu,\alpha}^{\rho*}, v_{\nu,\beta}^{\rho*} \rangle| \right) \\
&\lesssim \sum_{\substack{\lambda \sim 1 \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \lambda \mu^{\frac{1}{p} - \frac{1}{2}} \nu^{-\frac{1}{2}} \sum_{\nu^{1/2} \leq \rho} K_{\mu,\nu}^\rho \sup_{\alpha \sim_a \beta} \|P_{\lambda,\nu,\alpha,\beta} f\|_p \|u\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1),1/\tau}^{\frac{1}{2}}} + \dots.
\end{aligned} \tag{40}$$

We have already bounded all the contributions, and we can say that for some functional $A'(f)$ we got an upper bound

$$|\langle (\partial_w f)u, v \rangle| \leq (\|f\|_d + A'(f)) \|u\|_{X_{\zeta(1), 1/\tau}^{\frac{1}{2}}} \|v\|_{X_{\zeta(1), 1/\tau}^{\frac{1}{2}}}$$

If we return to the original variables, and replace u and v by $u_{\tau U}$ and $v_{\tau U}$, and w by Ue_j , then by (21), (22) and (23) we get

$$\begin{aligned} \langle (\partial_j f)u, v \rangle &= \tau^{2d+1} \langle (\partial_w f_{\tau U})u_{\tau U}, v_{\tau U} \rangle \\ &\lesssim \tau^{2d+1} (\|f_{\tau U}\|_d + A'(f_{\tau U})) \|u_{\tau U}\|_{X_{\zeta(1), 1/\tau}^{1/2}} \|v_{\tau U}\|_{X_{\zeta(1), 1/\tau}^{1/2}} \\ &= (\|f\|_d + \tau^{d-1} A'(f_{\tau U})) \|u\|_{X_{\zeta(\tau, U)}^{1/2}} \|v\|_{X_{\zeta(\tau, U)}^{1/2}}. \end{aligned}$$

If $m_{\lambda, \nu, \alpha, \beta}$ is the multiplier of $P_{\lambda, \nu, \alpha, \beta}$, then

$$\begin{aligned} (P_{\lambda, \nu, \alpha, \beta} f_{\tau U})(x) &= (m_{\lambda, \nu, \alpha, \beta}(\cdot) \widehat{f}(\tau U \cdot)) \check{\gamma}(x) \\ &= \tau^{-d} (P_{\tau\lambda, \tau\nu, \alpha, \beta}^U f)(\tau^{-1} Ux), \end{aligned}$$

where the multiplier of P^U is $m(U^{-1}\xi)$. Hence,

$$\|P_{\lambda, \nu, \alpha, \beta} f_{\tau U}\|_p = \tau^{-\frac{d}{p'}} \|P_{\tau\lambda, \tau\nu, \alpha, \beta}^U f\|_p$$

We collect all the estimates (19), (20), (32), (33), (39) and (40) to conclude this section with the following theorem.

Theorem 13. *For $d \leq p \leq \infty$, the norm of the operator $M_{\partial_j f} : u \in X_{\zeta(U, \tau)}^{\frac{1}{2}} \mapsto (\partial_j f)u \in X_{\zeta(U, \tau)}^{-\frac{1}{2}}$ has the upper bound*

$$\|M_{\partial_j f}\|_{X_{\zeta(\tau, U)}^{1/2} \mapsto X_{\zeta(\tau, U)}^{-1/2}} \lesssim_\varepsilon \|f\|_d + \tau^{\frac{d}{p}-1} A(\tau, U, f), \quad (41)$$

where

$$\begin{aligned} A(\tau, U, f) &:= \sum_{\substack{\nu^{\frac{1}{2}} \leq \lambda \leq 1 \\ \tau^{-1} \leq \mu \leq \nu}} Q(\lambda, \mu, \nu) \|P_{\tau\lambda, \tau\nu}^U f\|_p + \\ &\quad + \sum_{\substack{\lambda \sim 1 \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \lambda \mu^{\frac{1}{p}-\frac{1}{2}} \nu^{-\frac{1}{2}} \sum_{\nu^{1/2} \leq \rho} K_{\mu, \nu}^\rho \sup_{\alpha \sim_a \beta} \|P_{\tau\lambda, \tau\nu, \alpha, \beta}^U f\|_p. \end{aligned} \quad (42)$$

The constant $K_{\mu, \nu}^\rho$ is defined in (38), and

$$Q(\lambda, \mu, \nu) := \begin{cases} \lambda^{1-\frac{1}{p}} \mu^{\frac{d+1}{2p}-\frac{1}{2}} \nu^{\frac{1}{p}-\frac{1}{2}-\varepsilon} & \text{for } \lambda > \nu \mu^{-\frac{1}{2}} \text{ and } \nu \leq \mu^{\frac{1}{2}} \\ \lambda \mu^{\frac{d+2}{2p}-\frac{1}{2}} \nu^{-\frac{1}{2}} & \text{otherwise.} \end{cases}$$

3.2 End of the Proof

In this section we average the norm $\|M_{\partial_j f}\|_{\dot{X}_{\zeta(\tau,U)}^{1/2} \mapsto \dot{X}_{\zeta(\tau,U)}^{-1/2}}$ over τ and U . We follow the method of Haberman [11] and of Ham, Kwon and Lee [13].

By Theorem 13 we have

$$\begin{aligned}
& \int_M \int_{O_d} \|M_{\partial_i f}\|_{X_{\zeta(\tau,U)}^{1/2} \mapsto X_{\zeta(\tau,U)}^{-1/2}} dU d\tau \lesssim_\varepsilon \|f\|_d + \\
& \quad + M^{\frac{d}{p}-1} \sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \lesssim 1 \\ M^{-1} \leq \mu \leq \nu}} Q(\lambda, \mu, \nu) \int_M \int_{O_d} \|P_{\tau\lambda, \tau\nu}^U f\|_p dU d\tau + \\
& \quad + M^{\frac{d}{p}-1} \sum_{\substack{\lambda \sim 1 \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \lambda \mu^{\frac{1}{p}-\frac{1}{2}} \nu^{-\frac{1}{2}} \sum_{\nu^{1/2} \leq \rho} K_{\mu, \nu}^\rho \int_M \int_{O_d} \sup_{\alpha, \beta} \|P_{\tau\lambda, \tau\nu, \alpha, \beta}^U f\|_p dU d\tau. \quad (43)
\end{aligned}$$

The first average at the right has been already bounded by Haberman.

Lemma 14. (Haberman, Lemma 5.1 in [11]) Let $P_{\tau\lambda, \tau\nu}^U$ be the projection to frequencies $|\xi| \sim \tau\lambda$ and to frequencies $|\langle Ue_1, \xi \rangle| \leq 2\tau\nu$. If $f \in L^p(\mathbb{R}^d)$, then

$$\left(\int_{O_d} \|P_{\tau\lambda, \tau\nu}^U f\|_p^p dU \right)^{\frac{1}{p}} \leq C \left(\frac{\nu}{\lambda} \right)^{\frac{1}{p}} \|f\|_p \quad \text{for } 2 \leq p \leq \infty. \quad (44)$$

The second average at the right of (43) has been already bounded by Ham, Kwon and Lee.

Lemma 15. (Ham, Kwon and Lee, Lemma 4.3 in [13]) For fixed $\tau^{-\frac{1}{2}} \leq \nu^{\frac{1}{2}} < \rho < \rho_0$ and $\tau^{-1} \leq \lambda \leq 1$, let α and β denote all the transversal, antipodal caps at scale ρ , or all the non-transversal, antipodal caps at scale $\sim \nu^{\frac{1}{2}}$, as described in Section 3.1. If $P_{\tau\lambda, \tau\nu, \alpha, \beta}^U$ is the projection to frequencies $|\xi| \sim \tau\lambda$, $|\langle Ue_1, \xi \rangle| \leq 2\tau\nu$ and $\{\xi \mid \xi \in \tau U(-\alpha + \beta)\}$, then

$$\left(\int_M \int_{O_d} \sup_{\alpha \sim \beta} \|P_{\tau\lambda, \tau\nu, \alpha, \beta}^U f\|_p^p dU d\tau \right)^{\frac{1}{p}} \leq C \left(\frac{\nu}{\lambda} \right)^{\frac{1}{p}} \rho^{\frac{2}{p}} \|f\|_p \quad \text{for } 2 \leq p \leq \infty. \quad (45)$$

Sketch of the proof. The proof is by interpolation. Since $-\alpha + \beta$ forms a cap of dimensions $\rho \times \cdots \times \rho \times \rho^2$, then for the point $p = \infty$ we get

$$\sup_{\alpha, \beta, U, \tau} \|P_{\tau\lambda, \tau\nu, \alpha, \beta}^U f\|_\infty \lesssim \|f\|_\infty.$$

Let us denote by $m_{\tau\lambda,\tau\nu,\alpha,\beta}^U$ the multiplier of $P_{\tau\lambda,\tau\nu,\alpha,\beta}^U$. For $p = 2$ we get

$$\begin{aligned} \int_M \int_{O_d} \sum_{\alpha,\beta} \|m_{\tau\lambda,\tau\nu,\alpha,\beta}^U \hat{f}\|_2^2 dU d\tau = \\ \int |\hat{f}(\xi)|^2 \int_M \int_{S^{d-1}} \sum_{\alpha,\beta} |m_{\tau\lambda,\tau\nu,\alpha,\beta}|^2 (|\xi|\omega) d\omega d\tau d\xi. \end{aligned}$$

By Lemma 11 we have that $\sum_{\alpha,\beta} |m_{\tau\lambda,\tau\nu,\alpha,\beta}|^2 \lesssim \mathbf{1}_{\tau S_{\nu,\rho}}$, where $S_{\nu,\rho}$ is defined in (35). The set $\tau S_{\nu,\rho}$ is a $(d-2)$ -sphere of radius $\tau(2-\rho^2)$, width $2\tau\nu$ in the direction e_1 , and width $\tau\rho^2$ in $\{\xi_1 = 0\}$. For fixed ξ we get

$$\int_M \int_{S^{d-1}} \sum_{\alpha \sim_a \beta} |m_{\tau\lambda,\tau\nu,\alpha,\beta}|^2 (|\xi|\omega) d\omega d\tau \lesssim \mathbf{1}_{\{|\xi| \sim M\}} \frac{\nu}{\lambda} \rho^2 |\xi| M^{-1},$$

which leads to

$$\begin{aligned} \int_M \int_{O_d} \sum_{\alpha,\beta} \|m_{\tau\lambda,\tau\nu,\alpha,\beta}^U \hat{f}\|_2^2 dU d\tau &\lesssim \frac{\nu}{\lambda} \rho^2 \int_{\{|\xi| \sim M\}} |\hat{f}|^2 d\xi \\ &\leq \frac{\nu}{\lambda} \rho^2 \|f\|_2^2, \end{aligned}$$

and then (45) follows. \square

We use Lemma 14, Lemma 15 and Hölder in (43) to get

$$\begin{aligned} \int_M \int_{O_d} \|M_{\partial_j} f\|_{X_{\zeta(\tau,U)}^{1/2} \mapsto X_{\zeta(\tau,U)}^{-1/2}} dU d\tau &\lesssim_\varepsilon \|f\|_d + \\ &+ M^{\frac{d}{p}-1} \sum_{\substack{\nu^{\frac{1}{2}} \lesssim \lambda \lesssim 1 \\ M^{-1} \leq \mu \leq \nu}} Q(\lambda, \mu, \nu) \nu^{\frac{1}{p}} \lambda^{-\frac{1}{p}} \|P_{M\lambda} f\|_p + \\ &+ M^{\frac{d}{p}-1} \sum_{\substack{\lambda \sim 1 \\ \mu \leq \nu < \mu^{\frac{1}{2}}}} \lambda^{1-\frac{1}{p}} \mu^{\frac{1}{p}-\frac{1}{2}} \nu^{\frac{1}{p}-\frac{1}{2}} \sum_{\nu^{1/2} \leq \rho} K_{\mu,\nu}^\rho \rho^{\frac{2}{p}} \|P_{M\lambda} f\|_p \\ &\sim_\varepsilon \|f\|_d + A_1 + A_2. \end{aligned}$$

To bound A_1 we use the definition of $Q(\lambda, \mu, \nu)$ in Theorem 13:

$$Q(\lambda, \mu, \nu) := \begin{cases} \lambda^{1-\frac{1}{p}} \mu^{\frac{d+1}{2p}-\frac{1}{2}} \nu^{\frac{1}{p}-\frac{1}{2}-\varepsilon} & \text{for } \lambda > \nu \mu^{-\frac{1}{2}} \text{ and } \nu \leq \mu^{\frac{1}{2}} \\ \lambda \mu^{\frac{d+2}{2p}-\frac{1}{2}} \nu^{-\frac{1}{2}} & \text{otherwise.} \end{cases}$$

We sum first in ν , then in μ and finally in λ to get

$$\begin{aligned}
A_1 &= \left(\sum_{\substack{\nu^{\frac{1}{2}} \leq \lambda \leq 1 \\ \mu^{\frac{1}{2}} \leq \nu}} + \sum_{\substack{\nu^{\frac{1}{2}} \leq \lambda \leq \nu \mu^{-\frac{1}{2}} \\ \mu \leq \nu \leq \mu^{\frac{1}{2}}}} + \sum_{\nu \mu^{-\frac{1}{2}} \leq \lambda \leq 1 \\ \mu \leq \nu \leq \mu^{\frac{1}{2}}} \right) Q(\lambda, \mu, \nu) \nu^{\frac{1}{p}} \lambda^{-\frac{1}{p}} M^{\frac{d}{p}-1} \|P_{M\lambda} f\|_p \\
&= \left(\sum_{\mu^{\frac{1}{4}} \leq \lambda} \lambda^{1-\frac{1}{p}} \mu^{\frac{d+2}{2p}-\frac{1}{2}} \sum_{\mu^{\frac{1}{2}} \leq \nu \leq \lambda^2} \nu^{\frac{1}{p}-\frac{1}{2}} + \right. \\
&\quad + \sum_{\mu^{\frac{1}{2}} \leq \lambda} \lambda^{1-\frac{1}{p}} \mu^{\frac{d+2}{2p}-\frac{1}{2}} \sum_{\lambda \mu^{\frac{1}{2}} \leq \nu \leq \min(\lambda^2, \mu^{\frac{1}{2}})} \nu^{\frac{1}{p}-\frac{1}{2}} + \\
&\quad \left. + \sum_{\mu^{\frac{1}{2}} \leq \lambda} \lambda^{1-\frac{2}{p}} \mu^{\frac{d+1}{2p}-\frac{1}{2}} \sum_{\mu \leq \nu \leq \lambda \mu^{\frac{1}{2}}} \nu^{\frac{2}{p}-\frac{1}{2}} \right) M^{\frac{d}{p}-1} \|P_{M\lambda} f\|_p \\
&\lesssim \left(\sum_{M^{-\frac{1}{4}} \leq \lambda} \lambda^{1-\frac{1}{p}} \left[\sum_{\mu \leq \lambda^4} \mu^{\frac{d+3}{2p}-\frac{3}{4}} \right] + \sum_{M^{-\frac{1}{2}} \leq \lambda} \lambda^{\frac{1}{2}} \left[\sum_{\mu \leq \lambda^2} \mu^{\frac{d+3}{2p}-\frac{3}{4}} \right] + \right. \\
&\quad \left. + M^{1-\frac{d+5}{2p}} + \sum_{\lambda} \lambda^{1-\frac{2}{p}} \right) M^{\frac{d}{p}-1} \|P_{M\lambda} f\|_p
\end{aligned}$$

During the summation we used the condition $p \geq d \geq 5$. At the end we get

$$A_1 \leq C M^{\frac{d-5}{2p}+} \sum_{M^{-\frac{1}{2}} \leq \lambda \leq 1} \lambda^{\frac{1}{2}} \|P_{M\lambda} f\|_p \leq C_\varepsilon \|f\|_{W^{\frac{d-5}{2p}+, p}}.$$

We bound now A_2 , recalling that:

$$K_{\mu, \nu}^\rho(p') \leq \begin{cases} C_\varepsilon \rho^{-\frac{1}{p}} \mu^{\frac{d-1}{2p}} \nu^{\frac{1}{p}-\varepsilon} & \text{for } \rho > \nu \mu^{-\frac{1}{2}} \\ C \mu^{\frac{d}{2p}} & \text{for } \nu^{\frac{1}{2}} \leq \rho \leq \nu \mu^{-\frac{1}{2}}, \end{cases}$$

We sum first in ρ , then in ν , in μ and finally in λ to get

$$\begin{aligned}
A_2 &\lesssim_\varepsilon M^{\frac{d}{p}-1} \sum_{\substack{\lambda \sim 1 \\ \mu \leq \nu \leq \mu^{\frac{1}{2}}}} \lambda^{1-\frac{1}{p}} \mu^{\frac{d+1}{2p}-\frac{1}{2}} \nu^{\frac{1}{p}-\frac{1}{2}} \left(\mu^{\frac{1}{2p}} \sum_{\nu^{\frac{1}{2}} \leq \rho \leq \nu \mu^{-\frac{1}{2}}} \rho^{\frac{2}{p}} + \nu^{\frac{1}{p}-} \sum_{\nu \mu^{-\frac{1}{2}} \leq \rho \leq 1} \rho^{\frac{1}{p}} \right) \|P_{M\lambda} f\|_p \\
&\lesssim_\varepsilon \sum_{\lambda \sim 1} \lambda^{1-\frac{1}{p}} \mu^{\frac{d+1}{2p}-\frac{1}{2}} \sum_{\mu \leq \nu \leq \mu^{\frac{1}{2}}} \nu^{\frac{2}{p}-\frac{1}{2}-} \|P_{M\lambda} f\|_p \\
&\lesssim_\varepsilon M^{\frac{d-5}{2p}+} \sum_{\lambda \sim 1} \|P_{M\lambda} f\|_p \\
&\lesssim_\varepsilon \|f\|_{W^{\frac{d-5}{2p}+, p}}.
\end{aligned}$$

The statement of Theorem 8 follows.

4 The Bilinear Theorem

In this section we prove the bilinear theorem for two open subsets of the paraboloid. The paraboloid is technically simpler, so the exposition runs more smoothly. After concluding the proof, we explain how we should modify the proof to get Theorem 7. The proof follows closely the ideas presented by Tao in [25], and we include here the argument for the sake of completeness.

Theorem 7'. *Suppose that S_1 and S_2 are two open subsets of the paraboloid in \mathbb{R}^n with diameter $\lesssim 1$ and at distance ~ 1 of each other. If f_μ and g_ν are functions with Fourier transforms supported in a μ -neighborhood of S_1 and a ν -neighborhood of S_2 respectively, for $\mu \leq \nu < \mu^{\frac{1}{2}} < 1$, then for every $\varepsilon > 0$ it holds that*

$$\|f_\mu g_\nu\|_{p'} \leq C_\varepsilon \mu^{\frac{n}{2p} - \varepsilon} \nu^{\frac{1}{p} - \varepsilon} \|f_\mu\|_2 \|g_\nu\|_2, \quad \text{for } 1 \leq p' \leq \frac{n}{n-1}. \quad (46)$$

The inequalities are best possible, up to ε -losses, in μ and ν .

We can restate the theorem in terms of the quantity

$$K_{\mu,\nu}(p') := \sup_{\|f_\mu\|_2 = \|g_\nu\|_2 = 1} \|f_\mu g_\nu\|_{p'}.$$

We get the upper bound of $K_{\mu,\nu}(p')$ by an argument of induction in scales. With some examples, we show that the upper bound $K_{\mu,\nu}(p')$ is the best possible, up to ε -losses.

When $\mu^{\frac{1}{2}} \leq \nu$, the separation between supports does not yield any improvement over Theorem 9, at least in the range $1 \leq p' \leq \frac{n+1}{n}$.

Example 16 (Case $\mu^{\frac{1}{2}} \leq \nu$). Let $N_\mu(S_1)$ and $N_\nu(S_2)$ be neighborhoods of two open subsets of the paraboloid with diameter ~ 1 and at distance ~ 1 of each other. In $N_\mu(S_1)$ let C_1 be a cap of radius $\mu^{\frac{1}{2}}$ and width μ . In $N_\nu(S_2)$ let $C_2 := C_1 + a \subset N_\nu(S_2)$ for some vector a ; this is possible owing to the hypothesis $\mu^{\frac{1}{2}} \leq \nu$. After replacing for $\hat{u}_\mu = \mathbf{1}_{C_1}$ and $\hat{v}_\nu = \mathbf{1}_{C_2}$ in the bilinear inequality, we get $K_{\mu,\nu}(p') \geq c\mu^{\frac{n+1}{2p}}$.

Theorem 7 holds in \mathbb{R}^2 without ε -losses. The proof is by averaging over translations of the parabola; see for example Lemma 2.4 in [17].

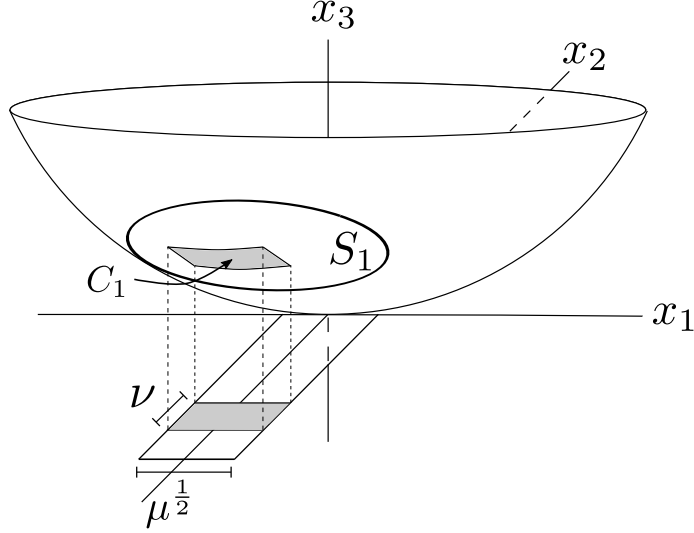


Figure 4: The construction of the cap C_1 .

Example 17 (Case \mathbb{R}^2 and $\mu \leq \nu \leq \mu^{\frac{1}{2}}$). Let $N_\mu(S_1)$ and $N_\nu(S_2)$ be separated in the parabola as in Theorem 7'. In $N_\mu(S_1)$ let C_1 be a cap of diameter ν and width μ . In $N_\nu(S_2)$ let $C_2 := C_1 + a \subset N_\nu(S_2)$ for some vector a . After replacing for $\hat{u}_\mu = \mathbb{1}_{C_1}$ and $\hat{v}_\nu = \mathbb{1}_{C_2}$ in the bilinear inequality, we get $K_{\mu,\nu}(p') \geq c\mu^{\frac{1}{p}}\nu^{\frac{1}{p}}$.

In higher dimensions we consider as example a modification of the squashed caps in Section 2.7 of [26].

Example 18 (Case $n \geq 3$ and $\mu \leq \nu \leq \mu^{\frac{1}{2}}$). Let $N_\mu(S_1)$ and $N_\nu(S_2)$ be separated in the paraboloid as in Theorem 7'. Let $L_\mu \subset \mathbb{R}^{n-1}$ be a $\mu^{\frac{1}{2}}$ -neighborhood of the plane $\{x_1 = \dots = x_{n-2} = 0\}$. In L_μ choose a box \tilde{C}_1 of dimensions $\nu \times \mu^{\frac{1}{2}} \times \dots \times \mu^{\frac{1}{2}}$, so that its lift to the paraboloid lies in S_1 , and thicken it in $N_\mu(S_1)$ creating so a cap C_1 of dimensions $\nu \times \mu^{\frac{1}{2}} \times \dots \times \mu^{\frac{1}{2}} \times \mu$; see Figure 4. Now, let $C_2 := C_1 + a \subset N_\nu(S_2)$ for some vector a . After replacing for $\hat{u}_\mu = \mathbb{1}_{C_1}$ and $\hat{v}_\nu = \mathbb{1}_{C_2}$ in the bilinear inequality, we get $K_{\mu,\nu}(p') \geq c\mu^{\frac{n}{2p}}\nu^{\frac{1}{p}}$.

The rest of this section is devoted to the proof of the inequality (46) in Theorem 7'. We do first some reductions.

By Galilean and rotational symmetry, we can assume that

$$\begin{aligned} S_1 &= \left\{ \left(\xi', \frac{1}{2} |\xi'|^2 \right) \mid |\xi' - c_1 e_1| \leq c_2 \right\} \\ S_2 &= \left\{ \left(\xi', \frac{1}{2} |\xi'|^2 \right) \mid |\xi' + c_1 e_1| \leq c_2 \right\}; \end{aligned}$$

the constant C_ε in (46) depends on c_1 and c_2 .

It suffices to prove the local inequality

$$\|f_\mu g_\nu\|_{L^{p'}(B_{\mu^{-1}})} \leq C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f_\mu\|_2 \|g_\nu\|_2. \quad (47)$$

In fact, cover \mathbb{R}^n with balls $B_{\mu^{-1}}$ and choose a bump function $\zeta_{B_\mu^{-1}} \sim 1$ in $B_{\mu^{-1}}$ so that $\text{supp } \widehat{\zeta}_{B_\mu^{-1}} \subset B_\mu(0)$. Then,

$$\begin{aligned} \|f_\mu g_\nu\|_{p'} &\leq \sum_{B_{\mu^{-1}}} \|f_\mu g_\nu\|_{L^{p'}(B_{\mu^{-1}})} \\ &\lesssim \sum_{B_{\mu^{-1}}} \|(\widehat{f}_\mu * \widehat{\zeta}_{B_\mu^{-1}})^\vee (\widehat{g}_\nu * \widehat{\zeta}_{B_\mu^{-1}})^\vee\|_{L^{p'}(B_{\mu^{-1}})} \end{aligned}$$

The width of the supports of $\widehat{f}_\mu * \widehat{\zeta}_{B_\mu^{-1}}$ and $\widehat{g}_\nu * \widehat{\zeta}_{B_\mu^{-1}}$ are essentially μ and ν respectively. Hence, we can apply the local bilinear inequality (47) to get

$$\begin{aligned} \|f_\mu g_\nu\|_{p'} &\leq C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \sum_{B_{\mu^{-1}}} \|f_\mu \zeta_{B_{\mu^{-1}}}\|_2 \|g_\nu \zeta_{B_{\mu^{-1}}}\|_2 \\ &\leq C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \left(\sum_{B_{\mu^{-1}}} \|f_\mu \zeta_{B_{\mu^{-1}}}\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{B_{\mu^{-1}}} \|g_\nu \zeta_{B_{\mu^{-1}}}\|_2^2 \right)^{\frac{1}{2}} \\ &\leq C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f_\mu\|_2 \|g_\nu\|_2, \end{aligned}$$

which is what we wanted to prove.

At scale μ^{-1} the function f_μ looks like $(fdS)^\vee$ for some function f in the paraboloid, so it suffices to prove the next theorem.

Theorem 19. *Suppose that S_1 and S_2 are two open subsets of the paraboloid in \mathbb{R}^n with diameter ~ 1 and at distance ~ 1 of each other. If fdS is a measure supported in S_1 and g_ν a function with Fourier transform supported in a ν -neighborhood of S_2 , then for $1 < R^{\frac{1}{2}} \leq \nu^{-1} \leq R$ and for every $\varepsilon > 0$ it holds*

$$\|(fdS)^\vee g_\nu\|_{L^{p'}(B_R)} \leq C_\varepsilon R^{\frac{1}{2}(1-\frac{n}{p})+\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f\|_{L^2(S)} \|g_\nu\|_2, \quad (48)$$

where $1 \leq p' \leq \frac{n}{n-1}$.

In fact, after a change of variables $\xi \mapsto (\xi', \frac{1}{2}|\xi'|^2 + t)$ we can write f_μ as

$$\begin{aligned} f_\mu(x) &= \int_{-\mu}^{\mu} \left(\int \widehat{f}_\mu(\xi', \frac{1}{2}|\xi'|^2 + t) e(\langle x', \xi' \rangle + x_n \frac{1}{2}|\xi'|^2) d\xi \right) e(x_n t) dt \\ &= \int_{-\mu}^{\mu} (\widehat{f}_{\mu,t} dS)^\vee e(x_n t) dt, \end{aligned} \quad (49)$$

where $\widehat{f}_{\mu,t}$ is a parabolic slice of \widehat{f}_μ . To bound the local bilinear inequality (47) we use Minkowski to get

$$\|f_\mu g_\nu\|_{L^{p'}(B_{\mu^{-1}})} \leq \int_{-\mu}^{\mu} \|(\widehat{f}_{\mu,t} dS)^\vee g_\nu\|_{L^{p'}(B_{\mu^{-1}})} dt.$$

Then, writing $\mu^{-1} = R$, we can use Theorem 19 and Cauchy-Schwarz inequality to get

$$\begin{aligned} \|f_\mu g_\nu\|_{p'} &\leq C_\varepsilon \mu^{\frac{1}{2}(\frac{n}{p}-1)-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \int_{-\mu}^{\mu} \|f_{\mu,t}\|_2 dt \|g_\nu\|_2 \\ &\leq C_\varepsilon \mu^{\frac{n}{2p}-\varepsilon} \nu^{\frac{1}{p}-\varepsilon} \|f_\mu\|_2 \|g_\nu\|_2. \end{aligned}$$

Therefore, we must prove now Theorem 19.

The point $p' = 1$ of Theorem 19 can be proven readily. By Cauchy-Schwarz and by the trace inequality $\|(f dS)^\vee\|_2 \leq CR^{\frac{1}{2}} \|f\|_{L^2(S)}$ we get

$$\|(f dS)^\vee g_\nu\|_{L^1(B_R)} \leq CR^{\frac{1}{2}} \|f\|_2 \|g_\nu\|_2.$$

Hence, it suffices to prove the inequality (48) at the point $p' = \frac{n}{n-1}$.

We begin the proof in the next section with the wave packet decomposition. This decomposition is nowadays a classical change of basis, so we only outline it.

4.1 Wave Packet Decomposition

Let f be a function in \mathbb{R}^{n-1} , and decompose the space into caps α of radius $R^{-\frac{1}{2}}$ and center $c_\alpha \in \mathbb{R}^{n-1}$. Choose a smooth partition of unity $\{\zeta_\alpha\}$ adapted to the caps α so that $\sum_\alpha \zeta_\alpha^2 = 1$. Use Fourier series adapted to each α to expand $f\zeta_\alpha$ into frequencies ω , and develop f as

$$f(\xi) = |\alpha|^{-\frac{1}{2}} \sum_{\alpha, \omega} a(\alpha, \omega) \zeta_\alpha(\xi) e(\langle \omega, \xi - c_\alpha \rangle),$$

where $\omega = R^{\frac{1}{2}}\mathbb{Z}^{n-1}$. The coefficients a satisfy the next properties:

$$a(\alpha, \omega) = \frac{1}{|\alpha|^{\frac{1}{2}}} \int f \zeta_{\alpha} e(-\langle \omega, \xi - c_{\alpha} \rangle) d\xi, \quad (50)$$

$$\sum_{\alpha, \omega} |a(\alpha, \omega)|^2 = \|f\|_2^2. \quad (51)$$

By the linearity of the extension operator, we can write $(fdS)^{\vee}$ as

$$(fdS)^{\vee}(x) = \sum_{\alpha, \omega} a(\alpha, \omega) \phi_{T(\alpha, \omega)},$$

where ϕ_T is a function essentially supported in a tube T of dimensions $R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}} \times R$; the angle and position of T are determined by α and ω respectively. Furthermore,

$$|\phi_T(x)| \leq C_M R^{-\frac{n-1}{2}} \frac{1}{\langle R^{-\frac{1}{2}}(x' + \omega + x_n c_{\alpha}) \rangle^M}, \quad \text{for } |x_n| \leq R;$$

so ϕ_T is concentrated in a tube T of direction $(-c_{\alpha}, 1)$ whose main axis passes through $(-\omega, 0)$. We deduce also that for $\delta > 0$, for $x \notin R^{\delta}T$, and for $|x_n| \leq R$ it holds

$$|\phi_T(x)| \leq C_{\delta} R^{-100n}, \quad (52)$$

where possibly $C_{\delta} \rightarrow \infty$ as $\delta \rightarrow 0$.

The function g_{ν} can be written similarly. We decompose $N_{\nu}(S_2)$ into rectangles β of dimensions $\nu \times R^{-\frac{1}{2}} \times \cdots \times R^{-\frac{1}{2}}$ and center $c_{\beta} \in \mathbb{R}^n$, where c_{β} is now a point in S_2 . Arguing as before we have

$$\hat{g}_{\nu}(\xi) = |\beta|^{-\frac{1}{2}} \sum_{\beta, \omega} b(\alpha, \omega) \zeta_{\beta}(\xi) e(\langle \omega, \xi - c_{\beta} \rangle),$$

where ω belongs to some rotation of the grid $\nu^{-1}\mathbb{Z} \times R^{\frac{1}{2}}\mathbb{Z}^{n-1}$. Again, we get

$$b(\beta, \omega) = \frac{1}{|\beta|^{\frac{1}{2}}} \int \hat{g}_{\nu} \zeta_{\beta} e(-\langle \omega, \xi - c_{\beta} \rangle) d\xi \quad (53)$$

$$\sum_{\beta, \omega} |b(\beta, \omega)|^2 = \|g_{\nu}\|_2^2. \quad (54)$$

By the linearity of the Fourier transform, we can write g_{ν} as

$$g_{\nu} = \sum_{\beta, \omega} b(\beta, \omega) \phi_{T(\beta, \omega)},$$

where T are now tubes of dimensions $\nu^{-1} \times R^{\frac{1}{2}} \times \cdots \times R^{\frac{1}{2}}$. Again, we get

$$\begin{aligned} |\phi_T(x)| &\leq C_M \nu R^{-\frac{n-1}{2}} \frac{1}{\langle R^{-\frac{1}{2}} |x' + \omega' + x_n c'_\beta| + \nu |x_n + \omega_n| \rangle^M}, \\ |\phi_T(x)| &\leq C_\delta \nu R^{-100n}, \quad \text{for } x \notin R^\delta T \text{ and for } \delta > 0. \end{aligned} \quad (55)$$

We replace the wave packet decomposition into the bilinear inequality (48), so we must prove that for $\|a\|_2 = 1$ and $\|b\|_2 = 1$ we have

$$\left\| \sum_{T_1, T_2} a_{T_1} b_{T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{n}{n-1}}(B_R)} \leq C_\varepsilon R^\varepsilon \nu^{\frac{1}{n} - \varepsilon}.$$

Since $|\phi_{T_1}|$ and $|\phi_{T_2}|$ decay strongly outside the tubes, then we can ignore all the tubes that do not intersect the ball $10B_R$, so the number of tubes in each group is $\lesssim R^{Cn}$; recall that $\nu^{-1} \geq R^{\frac{1}{2}}$.

Now, for all the terms that satisfy $|a_{T_1}|$ or $|b_{T_2}| \lesssim R^{-Cn}$ the contribution to the bilinear inequality is negligible, so we can ignore all these terms and do pigeonholing in $|a_{T_1}|$ and $|b_{T_2}|$; here, we introduce logarithmic losses. Hence, for two collections of tubes \mathbb{T}_1 and \mathbb{T}_2 that intersect the ball $10B_R$ we must prove that

$$\left\| \sum_{T_1 \in \mathbb{T}_1, T_2 \in \mathbb{T}_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{n}{n-1}}(B_R)} \leq C_\varepsilon R^\varepsilon \nu^{\frac{1}{n} - \varepsilon} |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}. \quad (56)$$

The proof of this inequality begins with an induction on scales in the next section.

4.2 Induction on Scales

We want to control the quantity

$$K_\nu(R) := \sup_{\|f\|_2 = \|g_\nu\|_2 = 1} \|(fdS)^\vee g_\nu\|_{L^{p'}(B_R)}.$$

Rough estimates show that $K_{\mu, \nu}(R)$ is finite, thus well defined, and we want to prove that $K_{\mu, \nu}(R) \leq C_\varepsilon R^\varepsilon \nu^{\frac{1}{n} - \varepsilon}$.

The induction on scales seeks to control $K_{\mu, \nu}(R)$ in terms of $K_{\mu, \nu}(R^{1-\delta})$ for some $\delta > 0$, which we keep fixed in what follows, so we lower scales and stop at scale $\sim \nu^{-1}$, when Tao's bilinear theorem provides the best possible upper bound, up to ε -losses. From now on, we write R' for $R^{1-\delta}$.

We begin the induction by breaking up the ball B_R into balls $B_{R'}$. Now, we define a relationship between balls and tubes, so that a tube is related to a ball if the contribution of ϕ_T to the bilinear term is large in that ball. We need first decompose B_R into balls q of radius $R^{\frac{1}{2}}$, and now we introduce the following group of definitions for a dyadic number μ_2 :

$$\mathbb{T}_2(q) := \{T_2 \in \mathbb{T}_2 \mid R^\delta T_2 \cap q \neq \emptyset\} \quad (57)$$

$$q(\mu_2) := \{q \subset B_R \mid \mu_2 \leq |\mathbb{T}_2(q)| < 2\mu_2\} \quad (58)$$

$$\lambda(T_1, \mu_2, B_{R'}) := |\{q \in q(\mu_2) \mid q \subset B_{R'} \text{ and } R^\delta T_1 \cap q \neq \emptyset\}|. \quad (59)$$

Definition 20 (Relation between tubes and balls). For every number μ_2 and every tube $T_1 \in \mathbb{T}_1$ choose a ball $B_{R'}^*(\mu_2, T_1)$, if it exists, that satisfies

$$\lambda(T_1, \mu_2, B_{R'}^*) = \max_{B_{R'}} \lambda(T_1, \mu_2, B_{R'}) > 0.$$

We say that a tube $T_1 \in \mathbb{T}_1$ is related to a ball $B_{R'} \subset B_R$, or $T_1 \sim B_{R'}$, if $B_{R'} \subset 10B_{R'}^*(\mu_2, T_1)$ for some μ_2 . The negation of $T_1 \sim B_{R'}$ is $T_1 \not\sim B_{R'}$. Symmetrically, we can define a relation between tubes $T_2 \in \mathbb{T}_2$ and balls $B_{R'}$.

Every tube in \mathbb{T}_j intersects a number $\lesssim R^\delta$ of balls $B_{R'} \subset B_R$, but each tube is related only to $\lesssim \log R$ balls. This follows from the condition $1 \leq \mu_2 \lesssim R^{\frac{n-1}{2} + C\delta}$.

Now, we bound the bilinear term as

$$\begin{aligned} \left\| \sum_{\substack{T_1 \in \mathbb{T}_1 \\ T_2 \in \mathbb{T}_2}} \phi_{T_1} \phi_{T_2} \right\|_{L^{p'}(B_R)} &\leq \sum_{B_{R'} \subset B_R} \left\| \sum_{T_1, T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{p'}(B_{R'})} \\ &\leq \sum_{B_{R'} \subset B_R} \left(\left\| \sum_{T_1 \sim B_{R'}, T_2 \sim B_{R'}} \phi_{T_1} \phi_{T_2} \right\|_{L^{p'}(B_{R'})} + \right. \\ &\quad \left. + \left\| \sum_{T_1 \not\sim B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{p'}(B_{R'})} + \left\| \sum_{T_1 \sim B_{R'}, T_2 \not\sim B_{R'}} \phi_{T_1} \phi_{T_2} \right\|_{L^{p'}(B_{R'})} \right). \\ &= \text{I} + \text{II} + \text{III} \end{aligned} \quad (60)$$

For the first term I at the right we use the inductive hypothesis, Cauchy-

Schwarz, and the bound $|\{B_{R'} \mid T_j \sim B_{R'}\}| \lesssim \log R$ to get

$$\begin{aligned}
\sum_{B_{R'} \subset B_R} \left\| \sum_{\substack{T_1 \sim B_{R'} \\ T_2 \sim B_{R'}}} \phi_{T_1} \phi_{T_2} \right\|_{L^{p'}(B_{R'})} &\leq K(R') \sum_{B_{R'} \subset B_R} |\{T_1 \sim B_{R'}\}|^{\frac{1}{2}} |\{T_2 \sim B_{R'}\}|^{\frac{1}{2}} \\
&\leq K(R') \left(\sum_{B_{R'}, T_1} \mathbb{1}_{\{T_1 \sim B_{R'}\}} \right)^{\frac{1}{2}} \left(\sum_{B_{R'}, T_2} \mathbb{1}_{\{T_2 \sim B_{R'}\}} \right)^{\frac{1}{2}} \\
&\leq C(\log R) K(R') |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}. \tag{61}
\end{aligned}$$

We have bounded so the main contribution with an acceptable logarithmic loss.

We turn now to II in (60); the term III can be similarly controlled, so we will not describe it. We bound the $L^{\frac{n}{n-1}}$ -norm by interpolation between the points $p' = 1$ and $p' = 2$. For $p' = 1$ we use Cauchy-Schwarz and the trace inequality to get

$$\left\| \sum_{T_1 \not\sim B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^1(B_{R'})} \lesssim R^{\frac{1}{2}} |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}; \tag{62}$$

recall that $\sum_{T_1 \not\sim B_{R'}} \phi_{T_1} = (fdS)^\vee$ for some function f in S , and $\sum_{T_2} \phi_{T_2} = g_\nu$ for some function g_ν , so we only applied the trace theorem to $(fdS)^\vee$, and used (51) and (54). We are left with the point $p' = 2$.

If we are to prove (56) by interpolation, we must get the upper bound

$$\left\| \sum_{T_1 \not\sim B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^2(B_{R'})} \lesssim_\delta R^{\frac{1}{2}(1-\frac{n}{2})+C\delta} \nu^{\frac{1}{2}} |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}.$$

This inequality is in general false, if we do not put constrains over the tubes. The simple example $f = 1$ and $g_\nu = 1$ in $N_\nu(S_2)$ is enough, and worst examples can be given. Hence, we have to exploit the special structure of the tubes $T_1 \not\sim B_{R'}$.

We use the decomposition of B_R into cubes q of radius $R^{\frac{1}{2}}$ and the definition (58) to write the L^2 -norm as

$$\left\| \sum_{T_1 \not\sim B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^2(B_{R'})}^2 = \sum_{\mu_2} \sum_{q \in q(\mu_2)} \left\| \sum_{T_1 \not\sim B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2.$$

By pigeonholing, it suffices to control the norm for a fixed μ_2 . We introduce now the definitions

$$\lambda(T_1, \mu_2) := |\{q \in q(\mu_2) \mid R^\delta T_1 \cap q \neq \emptyset\}| \tag{63}$$

$$\mathbb{T}_1[\mu_2, \lambda_1] := \{T_1 \in \mathbb{T}_1 \mid \lambda_1 \leq \lambda(T_1, \mu_2) < 2\lambda_1\}. \tag{64}$$

Since $1 \leq \lambda_1 \lesssim R^{\frac{1}{2}+C\delta}$, by pigeonholing again it suffices to prove

$$\sum_{q \in q(\mu_2)} \left\| \sum_{\substack{T_1 \not\sim B_{R'}, T_1 \in \mathbb{T}_1[\mu_2, \lambda_1] \\ T_2}} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \lesssim_\delta R^{1-\frac{n}{2}+C\delta} \nu |\mathbb{T}_1| |\mathbb{T}_2|. \quad (65)$$

The case $\lambda(T_1, \mu_2) = 0$ is handled with (52). In the next section, we use the special nature of the L^2 -norm to decouple the frequencies.

4.3 Decoupling at Scale $R^{\frac{1}{2}}$

We need first a L^2 upper bound of the bilinear operator. Recall that the extension operator is defined as

$$(fdS)^\vee(x) = \int_{\mathbb{R}^{n-1}} f(\xi) e(\langle x', \xi' \rangle + x_n \varphi(\xi')) d\xi',$$

where $\varphi(\xi') = \frac{1}{2}|\xi'|^2$ and $\xi = (\xi', \xi_n)$. For an open subset S_1 of the paraboloid, we denote by $\pi(S_1)$ its projection to \mathbb{R}^{n-1} .

We need also the Radon transform of a function, and we define it as

$$Rf(\xi', \theta) := \int_{\mathbb{R}^{n-1}} f(\xi' + \eta) \delta(\langle \eta, \theta \rangle) d\eta;$$

the Radon transform $Rf(\xi', \theta)$ is the integral over the hyperplane with normal θ that passes through ξ' .

Lemma 21. *Let S_1 and S_2 be two open subsets of the paraboloid with radius ~ 1 and at distance ~ 1 of each other. Suppose that fdS and gdS are measures with support in S_1 and S_2 respectively. Then, it holds that*

$$\|(fdS)^\vee (gdS)^\vee\|_2^2 \leq C \|f\|_1 \sup_{\substack{\xi' \in \pi(S_1) \\ \xi'' \in \pi(S_2)}} R|f|(\xi', \frac{\xi' - \xi''}{|\xi' - \xi''|}) \|g\|_1 \|g\|_\infty \quad (66)$$

Proof. We compute the square of the extension operator as

$$\begin{aligned} |(fdS)^\vee(x)|^2 &= \int_{\mathbb{R}^{2(n-1)}} f(\xi'_1 + \xi'_2) \bar{f}(\xi'_2) \\ &\quad e(\langle x', \xi'_1 \rangle + x_n(\varphi(\xi'_1 + \xi'_2) - \varphi(\xi'_2))) d\xi'_1 d\xi'_2 \\ &= \int \left(\int f(\xi'_1 + \xi'_2) \bar{f}(\xi'_2) \delta(\varphi(\xi'_1 + \xi'_2) - \varphi(\xi'_2) - t) d\xi'_2 \right) e(\langle x, \xi_1 \rangle) d\xi_1 \\ &:= \tilde{F}(x), \end{aligned}$$

where F is the function in parentheses. Thus, we get

$$\|(fdS)^\vee(gdS)^\vee\|_2^2 = \int (F * G)^\vee(x) dx = (F * G)^\vee(0).$$

We develop the convolution and change variables, so that

$$\begin{aligned} \|(fdS)^\vee(gdS)^\vee\|_2^2 &= \int f(\xi'_2) \bar{g}(\xi''_2) \\ &\int \bar{f}(\xi'_2 + \xi'_1) g(\xi''_2 + \xi'_1) \delta(\varphi(\xi'_2) - \varphi(\xi'_1 + \xi'_2) + \xi_{1,n}) \delta(\varphi(\xi'_1 + \xi''_2) - \varphi(\xi''_2) - \xi_{1,n}) d\xi_1 \\ &\quad d\xi'_2 d\xi''_2. \end{aligned}$$

We can use Fubini to put inside the integral with respect to $\xi_{1,n}$, so that after the change of variables $\xi_{1,n} \mapsto \xi_{1,n} + \varphi(\xi'_1 + \xi''_2) - \varphi(\xi''_2)$ we get

$$\begin{aligned} I &:= \int \delta(\varphi(\xi'_2) - \varphi(\xi'_1 + \xi'_2) + \xi_{1,n}) \delta(\varphi(\xi'_1 + \xi''_2) - \varphi(\xi''_2) - \xi_{1,n}) d\xi_{1,n} \\ &= \delta(\langle \xi'_1, \xi'_2 - \xi''_2 \rangle). \end{aligned} \quad (67)$$

Then, the L^2 norm gets into

$$\begin{aligned} \|(fdS)^\vee(gdS)^\vee\|_2^2 &\leq \int |f|(\xi'_2) |g|(\xi''_2) \int |f|(\xi'_2 + \xi'_1) |g|(\xi''_2 + \xi'_1) \delta(\langle \xi'_1, \xi'_2 - \xi''_2 \rangle) d\xi'_1 d\xi'_2 d\xi''_2 \\ &\leq \|f\|_1 \|g\|_1 \|g\|_\infty \sup_{\xi'_2, \xi''_2} \int |f|(\xi'_2 + \xi'_1) \delta(\langle \xi'_1, \xi'_2 - \xi''_2 \rangle) d\xi'_1. \end{aligned}$$

Finally, by the identity $\delta(at) = a^{-1} \delta(t)$, and the condition of separation between S_1 and S_2 , we get

$$\int |f|(\xi'_2 + \xi'_1) \delta(\langle \xi'_1, \xi'_2 - \xi''_2 \rangle) d\xi'_1 \leq CR |f|(\xi'_2, \frac{\xi'_2 - \xi''_2}{|\xi'_2 - \xi''_2|}),$$

which concludes the proof. \square

We use now Lemma 21 to bound each term at the left side of the inequality (65). To simplify, let us define $\mathbb{T}'_1 := \{T_1 \not\sim B_{R'}\} \cap \mathbb{T}_1[\mu_2, \lambda_1]$. By (50) and (53) we can neglect the contribution from tubes such that $R^\delta T \cap q = \emptyset$. We define so the functions

$$\begin{aligned} f_q(\xi) &:= |\alpha|^{-\frac{1}{2}} \sum_{T_1 \in \mathbb{T}'_1(q)} \zeta_\alpha(\xi) e(\langle \omega, \xi - c_\alpha \rangle) \\ \hat{g}_{\nu, q}(\xi) &:= |\beta|^{-\frac{1}{2}} \sum_{T_2 \in \mathbb{T}_2(q)} \zeta_\beta(\xi) e(\langle \omega, \xi - c_\beta \rangle). \end{aligned}$$

We write $g_{\nu,q}$ as an average over paraboloids as in (49), and by Minkowski and Cauchy-Schwarz we get

$$\begin{aligned} \left\| \sum_{T_1 \in \mathbb{T}'_1(q), T_2 \in \mathbb{T}_2(q)} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 &\leq \| (f_q dS)^\vee g_{\nu,q} \|_2^2 \\ &\leq \| (f_q dS)^\vee \int (\widehat{g}_{\nu,q}^t dS)^\vee e(x_n t) dt \|_2^2 \\ &\leq \nu \int \| (f_q dS)^\vee (\widehat{g}_{\nu,q}^t dS)^\vee \|_2^2 dt \end{aligned}$$

We apply Lemma 21 to the integrand, using the inequalities

$$\begin{aligned} \|f_q\|_1 &\leq R^{-\frac{n-1}{4}} |\mathbb{T}'_1(q)| \\ \|\widehat{g}_{\nu,q}^t\|_1 &\leq \nu^{-\frac{1}{2}} R^{-\frac{n-1}{4}} |\mathbb{T}_2(q)|, \quad \|\widehat{g}_{\nu,q}^t\|_\infty \leq \nu^{-\frac{1}{2}} R^{\frac{n-1}{4} + C\delta}, \end{aligned}$$

to get

$$\left\| \sum_{\substack{T_1 \in \mathbb{T}'_1(q) \\ T_2 \in \mathbb{T}_2(q)}} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \leq C\nu R^{-\frac{n-1}{4} + C\delta} |\mathbb{T}'_1(q)| |\mathbb{T}_2(q)| \sup_{\substack{\xi' \in \pi(S_1) \\ \xi'' \in \pi(S_2)}} R |f_q| \left(\xi', \frac{\xi' - \xi''}{|\xi' - \xi''|} \right). \quad (68)$$

Let $\mathbb{T}'_1(q)(\xi', \xi' - \xi'')$ denote the collection of tubes in $\mathbb{T}'_1(q)$ such that the corresponding cap α intersects the hyperplane with normal $(\xi' - \xi'')/|\xi' - \xi''|$ that passes through ξ' . Then,

$$\begin{aligned} \sup_{\substack{\xi' \in \pi(S_1) \\ \xi'' \in \pi(S_2)}} R |f_q| \left(\xi', \frac{\xi' - \xi''}{|\xi' - \xi''|} \right) &\leq R^{-\frac{n-1}{4} + \frac{1}{2}} \sup_{\substack{\xi' \in \pi(S_1) \\ \xi'' \in \pi(S_2)}} |\mathbb{T}'_1(q)(\xi', \xi' - \xi'')| \\ &:= R^{-\frac{n-1}{4} + \frac{1}{2}} \nu(q, \mu_2, \lambda_1); \end{aligned}$$

we choose the last definition with the same notation as Tao in [25]. We replace in (68) to find

$$\left\| \sum_{T_1 \in \mathbb{T}'_1(q), T_2 \in \mathbb{T}_2(q)} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \leq C\nu R^{1 - \frac{n}{2} + C\delta} \nu(q, \mu_2, \lambda_1) |\mathbb{T}'_1(q)| |\mathbb{T}_2(q)|,$$

where $\mathbb{T}'_1 := \{T_1 \not\sim B_{R'}\} \cap \mathbb{T}_1[\mu_2, \lambda_1]$. Summing over all the cubes $q \in q(\mu_2)$ we get

$$\sum_{q \in q(\mu_2)} \left\| \sum_{T_1 \in \mathbb{T}'_1(q), T_2 \in \mathbb{T}_2(q)} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 \leq C\nu R^{1 - \frac{n}{2} + C\delta} \sum_{q \in q(\mu_2)} \nu(q, \mu_2, \lambda_1) |\mathbb{T}'_1(q)| |\mathbb{T}_2(q)|. \quad (69)$$

The term at the right does not involve oscillations, so we achieved a decoupling of the oscillating tubes at the left. To conclude the proof of (65), we must get an upper bound of $\nu(q, \mu_2, \lambda_1)$, which we do in the next section.

4.4 A Kakeya-type Estimate

In this section we aim to prove the inequality

$$\nu(q_0, \mu_2, \lambda_1) \lesssim R^{C\delta} \frac{|\mathbb{T}_2|}{\mu_2 \lambda_1}, \quad (70)$$

for some fixed $q_0 \in q(\mu_2)$, μ_2 and λ_1 . For any $\xi' \in \pi(S_1)$ and $\xi'' \in \pi(S_2)$ we consider then the following bilinear expression

$$B := \int_{\substack{q \in q(\mu_2) \\ B_R \setminus 10B_{R'}}} \sum_{T_1 \in \mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1} \sum_{T_2 \in \mathbb{T}_2} \mathbb{1}_{2R^\delta T_2}.$$

By the definition of $q(\mu_2)$ we get

$$B \gtrsim \mu_2 \sum_{T_1 \in \mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')} \int_{B_R \setminus 10B_{R'}} \mathbb{1}_{2R^\delta T_1}.$$

Since for $T_1 \in \{T_1 \not\subset B_{R'}\} \cap \mathbb{T}_1[\mu_2, \lambda_1]$ it holds that $|\{q \in q(\mu_2) \mid R^\delta T_1 \cap q \neq \emptyset\}| \sim \lambda_1$, we see that

$$|\{q \in q(\mu_2) \mid q \subset B_R \setminus 10B_{R'} \text{ and } R^\delta T_1 \cap q \neq \emptyset\}| \gtrsim R^{-\delta} \lambda_1.$$

Then,

$$B \gtrsim R^{\frac{n}{2} - C\delta} \lambda_1 \mu_2 |\mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')| \quad (71)$$

To get an upper bound of B , we re-order the summations so that

$$B \leq \sum_{T_2 \in \mathbb{T}_2} \int_{B_R \setminus 10B_{R'}} \mathbb{1}_{2R^\delta T_2} \sum_{T_1 \in \mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1}.$$

Since all the tubes intersect $q_0 \subset B_{R'}$, we see that

$$\sum_{T_1 \in \mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1}(x) \lesssim R^{C\delta} \quad \text{for } x \in B_R \setminus 10B_{R'}.$$

The tubes in $\mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')$ have directions $(-c_\alpha, 1)$, where c_α lies at distance $< R^{-\frac{1}{2}}$ from a hyperplane with normal direction $\xi' - \xi''$ that passes through ξ' . Then, the main axis of all the tubes in $\mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')$ make an angle $< R^{-\frac{1}{2}}$ with a hyperplane with normal direction $(\xi' - \xi'', \langle \xi', \xi' - \xi'' \rangle)$ that passes through q_0 . It amounts to saying that the support of $\sum_{T_1 \in \mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')} \mathbb{1}_{2R^\delta T_1}$ lies inside the $R^{\frac{1}{2} + \delta}$ -neighborhood of a hyperplane that passes through q_0 . Furthermore, every tube from \mathbb{T}_2 intersects the hyperplane transversally, making an angle $> c$ uniformly. Then,

$$B \lesssim R^{\frac{n}{2} + C\delta} |\mathbb{T}_2|. \quad (72)$$

We use (71) and (72) to conclude that

$$|\mathbb{T}'_1(q_0)(\xi', \xi' - \xi'')| \lesssim R^{C\delta} \frac{|\mathbb{T}_2|}{\lambda_1 \mu_2},$$

which is what we wanted to prove.

4.5 End of the Proof

In this section we reap all the bounds we have obtained. We plug (70) into (69) to get

$$\begin{aligned} \sum_{q \in q(\mu_2)} \left\| \sum_{\substack{T_1 \in \mathbb{T}'_1(q) \\ T_2 \in \mathbb{T}_2(q)}} \phi_{T_1} \phi_{T_2} \right\|_{L^2(q)}^2 &\leq \nu R^{1 - \frac{n}{2} + C\delta} |\mathbb{T}_2| \sum_{q \in q(\mu_2)} \lambda_1^{-1} |\mathbb{T}_1[\mu_2, \lambda_1](q)| \\ &\lesssim \nu R^{1 - \frac{n}{2} + C\delta} |\mathbb{T}_2| \sum_{T_1 \in \mathbb{T}_1[\lambda_1, \mu_2]} \lambda_1^{-1} \sum_{q \in q(\mu_2)} \mathbb{1}_{\{T_1 \cap R^\delta q \neq \emptyset\}} \\ &\lesssim \nu R^{1 - \frac{n}{2} + C\delta} |\mathbb{T}_1| |\mathbb{T}_2|, \end{aligned}$$

This concludes the proof of (65).

We interpolate the bilinear norm between the points $p' = 1$ in (62) and $p' = 2$ in (65) to get

$$\left\| \sum_{T_1 \neq B_{R'}, T_2} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{n}{n-1}}(B_{R'})} \leq C_\delta (\log R)^C R^{C\delta} \nu^{\frac{1}{n}} |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}}.$$

This bound joins the inequalities (60) and (61) to yield

$$\left\| \sum_{\substack{T_1 \in \mathbb{T}_1 \\ T_2 \in \mathbb{T}_2}} \phi_{T_1} \phi_{T_2} \right\|_{L^{\frac{n}{n-1}}(B_R)} \leq C_\delta (\log R)^C (K_\nu(R') + R^{C\delta} \nu^{\frac{1}{n}}) |\mathbb{T}_1|^{\frac{1}{2}} |\mathbb{T}_2|^{\frac{1}{2}};$$

in other words,

$$K_\nu(R) \leq C_\delta (\log R)^C (K_\nu(R^{1-\delta}) + R^{C\delta} \nu^{\frac{1}{n}}).$$

When we iterate, we get at the N -th step

$$K_\nu(R) \leq C_\delta^N (\log R)^{NC} (K_\nu(R^{(1-\delta)^N}) + NR^{C\delta} \nu^{\frac{1}{n}}).$$

We stop when $R^{(1-\delta)^N} \leq \nu^{-1} < R^{(1-\delta)^{N-1}}$; the number of steps is

$$N \leq -\frac{1}{\log(1-\delta)} + 1 \leq 2\delta^{-1}.$$

If $r \leq \nu^{-1}$, then we can average over translations of the paraboloid and apply Tao's bilinear to get $K_\nu(r) \leq C_\varepsilon r^{1-\frac{n+2}{2p}+\varepsilon} \nu^{\frac{1}{2}}$. We have thus that

$$K_\nu(R) \leq C_\delta R^{C\delta} (\nu^{-1+\frac{n+2}{2n}+\frac{1}{2}} + \nu^{\frac{1}{n}}) \leq C_\delta R^{C\delta} \nu^{\frac{1}{n}}.$$

This concludes the proof of Theorem 19, which implies Theorem 7'.

4.5.1 Additional Remarks

We indicate here the changes we need to do for surfaces of elliptic type or the hemisphere. The argument is sufficiently robust to admit perturbations.

For surfaces of ε -elliptic type, the semi-norms $\|\partial^N \Phi\|_\infty$ enter in the constants C_δ of (52) and (55). Since the eigenvalues of $D^2 \Phi$ are close to one, then the tubes have approximately the same length.

The delta function in (67) gets into

$$\delta(\Phi(\xi_2') - \Phi(\xi_1' + \xi_2') + \Phi(\xi_1' + \xi_2'') - \Phi(\xi_2'')) = \delta(\langle A(\xi_2' - \xi_2''), \xi_1' \rangle)$$

for some matrix A with eigenvalues in $[1 - \varepsilon, 1 + \varepsilon]$. Then $|\langle A(\xi_2' - \xi_2''), \xi_1' \rangle - \langle \xi_2' - \xi_2'', \xi_1' \rangle| \leq C\varepsilon$, and instead of an integral over the hyperplane H with normal direction $\xi_2' - \xi_2''$ that passes through ξ_2' , we integrate over a $(n-2)$ -surface \tilde{H} that lies in a ε -neighborhood of H and passes through ξ_2' .

A tube associated with a cap with center c_α has velocity $(-\nabla \Phi(c_\alpha), 1)$. If $\tilde{P} \subset \mathbb{R}^n$ is a $(n-1)$ -cone with center in a cube q generated by all the lines with directions $(-\nabla \Phi(\eta), 1)$ for $\eta \in \tilde{H}$, then we must verify that all the tubes coming from the separated set S_2 are transversal to \tilde{P} . In fact, notice that for any point $\xi_2' + \xi_1' \in \tilde{H}$, a vector v tangent to \tilde{H} satisfies the equation

$$\langle \nabla \Phi(\xi_1' + \xi_2'') - \nabla \Phi(\xi_1' + \xi_2'), v \rangle = 0;$$

hence, $\langle A(\xi_2'' - \xi_2'), v \rangle = 0$ for some matrix A close to I . Then, the vectors normal to \tilde{P} have the form $(A(\xi_2'' - \xi_2'), \langle \nabla \Phi(\xi_2' + \xi_1'), A(\xi_2'' - \xi_2') \rangle)$. If we take the inner product of these vectors with $(-\nabla \Phi(\eta_2), 1)$ for $\eta_2 \in \pi(S_2)$, then we get

$$\langle A(\xi_2'' - \xi_2'), \nabla \Phi(\xi_2' + \xi_1') - \nabla \Phi(\eta_2) \rangle = \langle A(\xi_2'' - \xi_2'), A'(\xi_2' + \xi_1' - \eta_2) \rangle;$$

hence, the inner product is basically equal to $\langle \eta_1 - \eta_2, \eta_1' - \eta_2' \rangle$ for all the pairs $\eta_1, \eta_1' \in \pi(S_1)$ and $\eta_2, \eta_2' \in \pi(S_2)$, and $|\langle \eta_1 - \eta_2, \eta_1' - \eta_2' \rangle| \geq c > 0$, then \tilde{P} is uniformly transversal to all the tubes coming from S_2 . The estimates hold uniformly in $\varepsilon \ll 1$.

The case of the hemi-sphere is similar. The term (67) is almost as simple as for the paraboloid. By symmetry, we can assume that $\xi_2' = -ae_1$ and $\xi_2'' = ae_1$ for some $0 < a \leq \frac{1}{\sqrt{2}} + \frac{1}{10}$. Then, the $(n-2)$ -surface \tilde{H} is again a hyperplane H with normal direction e_1 that passes through ξ_2' . The cone \tilde{P} is a translation of a portion of the quadratic cone $\{\xi \mid \xi_1^2 = a^2|\xi|^2\}$. It is intuitively clear that the portion of the cone generated by direction from S_1 is uniformly transversal to tubes from S_2 .

Appendix: Non-averaged Upper Bounds

We may wonder whether it is really necessary to average, or we just have not pushed as much as possible the estimates for $\|M_{\partial_i f}\|_{X_\zeta^{\frac{1}{2}} \rightarrow X_\zeta^{-\frac{1}{2}}}$. We show that averaging is indeed necessary.

Theorem 22. *If $f \in W^{\frac{d-2}{2p}, p}(\mathbb{R}^n)$, for $d \leq p < \infty$, is a function with support in B_1 , then*

$$\|M_{\partial_i f}\|_{X_\zeta^{\frac{1}{2}} \rightarrow X_\zeta^{-\frac{1}{2}}} \leq C \|f\|_{\frac{d-2}{2p}, p}, \quad (73)$$

where C does not depend on ζ . The inequality is best possible, in the sense that it is not possible to lower the regularity of f .

Proof. It is not necessary to use bilinear theory to get (73), the computations of Haberman in Section 4 of [11] are enough. To see that the result is best possible, we fix $\zeta = \tau(e_1 - ie_2)$ and consider the $\tau^{-\frac{1}{2}}$ -neighborhood of a 2-plane of side-length 1 lying in the plane (x_1, x_2) , and denote this set by F . We define $f(x) := e^{2\pi i(2\tau)x_2} \varphi_F(x)$, where φ_F is a smooth cut-off function of F —see Figure 5(a), and we have that $\|f\|_{s,p} \sim \tau^{s - \frac{d-2}{2p}}$. To estimate the

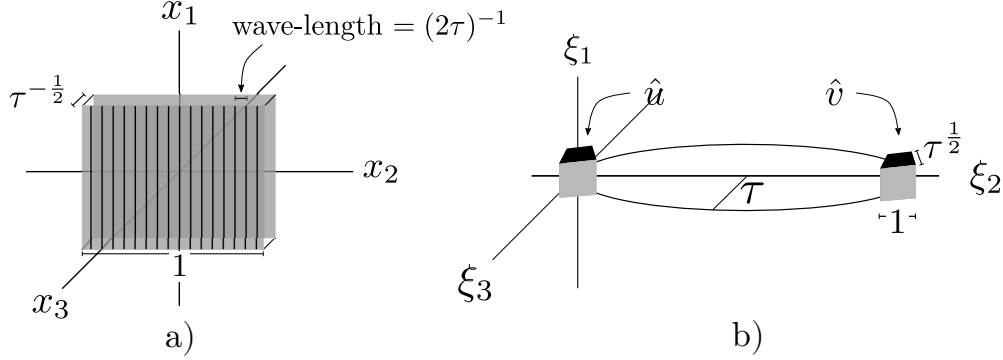


Figure 5: (a) Representation of the function f , and (b) of the Fourier transforms of u and v .

operator norm of $\partial_2 f$ we consider the box B of dimensions $1 \times 1 \times \tau^{\frac{1}{2}} \times \dots \times \tau^{\frac{1}{2}}$ centered at zero, and take $\hat{u} = \varphi_B$ and $\hat{v} = \varphi_B(\cdot - 2\tau e_2)$, for which $\|u\|_{X_\zeta^{\frac{1}{2}}} = \|v\|_{X_\zeta^{\frac{1}{2}}} = \tau^{\frac{d}{4}}$; see Figure 5(b). The duality pairing gives $|\langle \partial_2 f u, v \rangle| \gtrsim \tau^{\frac{d}{2}}$. If K is the best constant in (73), then we get

$$\tau^{\frac{d}{2}} \lesssim K \tau^{s - \frac{d-2}{2p}} \tau^{\frac{d}{2}};$$

if K is to be uniformly bounded in ζ , then necessarily $s \geq \frac{d-2}{2p}$. \square

If we did not need bilinearity to get the sharp upper bound (73), where does the gain of bilinearity come from? It comes mainly from Lemma 15. Where does bilinearity fail to improve over (73)? At the second term at the right of (42).

Notations

- Relations: $A \lesssim_\epsilon B$ if $A \leq C_\epsilon B$; $A \sim B$ if $A \lesssim B \lesssim A$; $A \ll 1$ if $A \leq c$, where c is chosen sufficiently small.
- Various: $e(z) := e^{2\pi iz}$. $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. $B_r(x)$ a ball of radius r with center at x . $\int_M d\tau := \frac{1}{M} \int_M d\tau$. $a+ := a + \epsilon$ for $\epsilon \ll 1$. If E is a set, then $\mathbb{1}_E$ is the characteristic function of the set, and $|E|$ is its measure, where the measure can be deduced from the context. If T is a tube with main axis l , then AT is a dilation of T by a factor $A > 0$ and same main axis l .

- Multipliers: $m(D)f = (\widehat{mf})^\vee$, where m stands for *multiplier*; $Pf = m(D)f$, where m is a smooth cut-off for a set of frequencies where we want to project to.
- The operator $\Delta_\zeta := \Delta + \zeta \cdot \nabla$ has symbol $p_\zeta(\xi) := -|\xi|^2 + 2i\zeta \cdot \xi$ and characteristic $\Sigma_\zeta := \{\xi \mid p_\zeta(\xi) = 0\}$.
- $\zeta(U, \tau) := \tau(Ue_1 - iUe_2)$, where $\{e_i\}$ is the canonical basis, $\tau \geq 1$ and $U \in O_d$ is a rotation.
- $\|u\|_{X_\zeta^b}^2 := \int |p_\zeta(\xi)|^{2b} |\widehat{u}(\xi)|^2 d\xi$.
- $\|u\|_{X_{\zeta,\sigma}^b}^2 := \int (|p_\zeta(\xi)| + \sigma)^{2b} |\widehat{u}(\xi)|^2 d\xi$ for $\sigma > 0$; $\|u\|_{X_\zeta^b} = \|u\|_{X_{\zeta,|\zeta|}^b}$.
- Sobolev-Slobodeckij spaces: For $1 < p < \infty$, $W^{s,p}(\mathbb{R}^d)$ is the space of distributions f such that

$$\|f\|_{s,p} := \sum_{|\alpha| \leq s} \|D^\alpha f\|_p < \infty \quad \text{for } s \text{ integer.}$$

$$\|f\|_{s,p} := \|P_{\leq 1} f\|_p + \left(\sum_{k>0} 2^{skp} \|P_k f\|_p^p \right)^{\frac{1}{p}} < \infty \quad \text{for } 0 < s \neq \text{integer.}$$

For a domain $\Omega \subset \mathbb{R}^d$, we define $W^{s,p}(\Omega) := \{f|_\Omega \mid f \in W^{s,p}(\mathbb{R}^d)\}$. The space $\dot{W}^{s,p}(\Omega)$ is the completion in $W^{s,p}(\mathbb{R}^d)$ of test functions $D(\Omega) := \{\varphi \in C^\infty(\Omega) \mid \text{supp } \varphi \Subset \Omega\}$. For further details, see *e.g.* [28, 19].

- $(fdS)^\vee(x) := \int_{\mathbb{R}^{n-1}} f(\xi) e(\langle x', \xi \rangle + x_n \varphi(\xi)) d\xi$, where S is the graph of φ and $(x', x_n) \in \mathbb{R}^n$.

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