BILINEAR IDENTITIES INVOLVING THE k-PLANE TRANSFORM AND FOURIER EXTENSION OPERATORS

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ABSTRACT. We prove certain $L^2(\mathbb{R}^n)$ bilinear estimates for Fourier extension operators associated to spheres and hyperboloids under the action of the k-plane transform. As the estimates are L^2-based, they follow from bilinear identities: in particular, these are the analogues of a known identity for paraboloids, and may be seen as higher-dimensional versions of the classical $L^2(\mathbb{R}^2)$-bilinear identity for Fourier extension operators associated to curves in $\mathbb{R}^2$.

1. Introduction

For $n \geqslant 2$, let U be an open subset in $\mathbb{R}^{n-1}$ and $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function parametrising a hypersurface $S = \{(\xi, \phi(\xi)) : \xi \in U\}$. Associated to S, define the Fourier extension operator

$$Ef(z) := \int_U e^{i(x\cdot\xi + t\phi(\xi))} f(\xi)\, \mathrm{d}\xi,$$

where $z = (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $f \in L^1(U)$. The terminology *extension* comes from the fact that E is the adjoint operator to the restriction of the Fourier transform to S, that is $E^*h(\xi) = \widehat{h}(\xi, \phi(\xi))$. Stein observed in the late 1960s that under certain curvature hypothesis on S it is possible to obtain $L^p(U) - L^q(\mathbb{R}^n)$ estimates for E besides the trivial $L^1(U) - L^\infty(\mathbb{R}^n)$ ones implied by Minkowski's inequality. In particular, the *Fourier restriction conjecture* asserts that if S is compact and has everywhere non-vanishing Gaussian curvature

$$\|Ef\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(U)}$$

should hold for all $q > \frac{2n}{n-1}$ and $\frac{1}{q} \leqslant \frac{n-1}{n+1}\frac{1}{p'}$. This conjecture is fully solved for $n = 2$ [16, 33], but is still open for $n \geqslant 3$ and constitutes one of the main open problems in Euclidean Harmonic Analysis. The first fundamental result in this direction was the Stein–Tomas [31, 28] restriction estimate

$$\|Ef\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \leqslant C\|f\|_{L^2(U)}; \tag{1.1}$$

note that this estimate is best possible in terms of the exponent q for $f \in L^2(U)$. Over the last few years, there has been a great interest in establishing the sharp value of C and the existence and characterisation of extremisers in (1.1) depending on the underlying surface S: see for instance [17] or the most recent survey [20].

Substantial improvements on (1.1) have been achieved over the last few decades. An important ingredient for this has been the bilinear and multilinear approach. These estimates generally adopt the form

$$\|\prod_{j=1}^k E_j f_j\|_{L^{q/k}(\mathbb{R}^n)} \leqslant C \prod_{j=1}^k \|f_j\|_{L^p(U_j)}, \tag{1.2}$$

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where the E_j are associated to hypersurfaces S_j satisfying certain *transversality* hypotheses. A key feature of these inequalities is that, under such additional hypotheses, it is possible to obtain estimates for $p = 2$ and $\frac{2n}{n-1} < q < \frac{2(n+1)}{n-1}$. The interested reader is referred, for instance, to [32, 29] for the theory of bilinear restriction estimates and to [4] for the multilinear approach; see also the survey papers [30, 1].

An elementary instance of a bilinear estimate is in fact the identity

$$\|E_1 f_1 E_2 f_2\|_{L^2(\mathbb{R}^2)}^2 = (2\pi)^2 \int_{U_1 \times U_2} \frac{|f_1(\xi_1)|^2 |f_2(\xi_2)|^2}{|\phi_1'(\xi_1) - \phi_2'(\xi_2)|} \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2, \tag{1.3}$$

which follows from an application of Plancherel's theorem and a change of variables; note that under the *transversality* hypothesis $|\phi_1'(\xi_1) - \phi_2'(\xi_2)| > c > 0$ for $\xi_1 \in U_1$, $\xi_2 \in U_2$, one may interpret the identity (1.3) in the framework of (1.2). Of course the presence of L^2 on the left-hand-side in (1.3) is key for the use of Plancherel's theorem. This bilinear approach has its roots in the work of Fefferman [16] and may also be extended to higher dimensions. Identifying $E_j f_j = \widehat{g_j \mathrm{d}\mu_j}$, where $g_j : \mathbb{R}^n \to \mathbb{R}$ is the lift of f_j to S_j, i.e., $g_j(\xi, \phi_j(\xi)) = f_j(\xi)$ and $\mathrm{d}\mu_j$ is the parametrised measure in S_j defined via

$$\int_{\mathbb{R}^n} g(\eta) \,\mathrm{d}\mu(\eta) = \int_{U_j} g(\xi, \phi_j(\xi)) \,\mathrm{d}\xi,$$

one may obtain the $L^2(\mathbb{R}^n)$ bilinear estimate

$$\|E_1 f_1 E_2 f_2\|_{L^2(\mathbb{R}^n)}^2 \leqslant \||g_1|^2 \mathrm{d}\mu_1 * |g_2|^2 \mathrm{d}\mu_2\|_{L^1(\mathbb{R}^n)} \|\mathrm{d}\mu_1 * \mathrm{d}\mu_2\|_{L^\infty(\mathbb{R}^n)} \leqslant C \|f_1\|_{L^2(U_1)}^2 \|f_2\|_{L^2(U_2)}^2, \tag{1.4}$$

after an application of Plancherel's theorem and the Cauchy–Schwarz inequality, provided one assumes the transversality condition $\|\mathrm{d}\mu_1 * \mathrm{d}\mu_2\|_{L^\infty(\mathbb{R}^n)} \leqslant C < \infty$. It should be remarked that the exponent here is $q = 4 \geqslant \frac{2(n+1)}{n-1}$ if $n \geqslant 3$. This is very much in contrast to the setting (1.2), in which the main goal is to obtain estimates when $q < \frac{2(n+1)}{n-1}$; bilinear and multilinear estimates of that type are deep and difficult and will not be explored in this paper.

It is interesting to compare (1.3) and (1.4). The first observation is that (1.3) is an identity, whilst (1.4) is an inequality. The second one is the presence of the weight factor $|\phi_1'(\xi_1) - \phi_2'(\xi_2)|^{-1}$ in (1.3); the transversality weight $|\mathrm{d}\mu_1 * \mathrm{d}\mu_2|$ in (1.4) does not necessarily have a closed form in terms of the variables of integration of f_1 and f_2.

The main purpose of this paper is to further exploit the elementary 2-dimensional analysis in (1.3) into higher dimensions. More precisely, we wish to obtain a bilinear identity in higher dimensions which incorporates an explicit weight factor amounting to some *transversality* condition; we note that an alternative higher dimensional version of (1.3) has recently been obtained by Bennett and Iliopoulou [5] in a n-linear level. To this end, we shall replace the $L^2(\mathbb{R}^n)$ in (1.4) by a mixed-norm $L^1(\mathbb{R}^{n-2}) \times L^2(\mathbb{R}^2)$. Given $x = (\bar{x}, x'') \in \mathbb{R}^{n-2} \times \mathbb{R}^2$, taking the L^1-norm in the $\bar{x}$ variables will essentially reduce matters to a 2-dimensional analysis in the $x'' = (x_{n-1}, x_n)$ variables, where the resulting extension operators E_1 and E_2 will correspond to sections of the original surfaces by 2-dimensional planes parallell to $\xi_1 = \cdots = \xi_{n-2} = 0$. This question has already been addressed by Planchon and the second author [26] if the underlying hypersurfaces are paraboloids. The motivation in their work came from the relevant role played by this type of inequalities in the global behaviour of large solutions of non-linear Schrödinger equations; more concretely we refer to the so called interaction Morawetz inequality introduced by Colliander, Keel, Staffillani, Takaoka and Tao in [15]. Here we further explore the existence of those bilinear identities for two other fundamental surfaces, such as the sphere and the hyperboloid.

Before describing our results in detail we shall first review the known results in the case of paraboloids, as they will provide the framework and context to understand our results.

1.1. Estimates for paraboloids.

In recent years, starting with the work of Ozawa and Tsutsumi [25] for the paraboloid $S_1 = S_2 = \{(\xi, |\xi|^2) : \xi \in \mathbb{R}^{n-1}\}$, there has been an increasing interest in understanding the weight $|\mathrm{d}\mu_1 * \mathrm{d}\mu_2|$ in (1.4) so that a L^2-bilinear estimate

$$\|E_1 f_1 E_2 f_2\|_{L^2(\mathbb{R}^n)}^2 \leqslant C \int_{U_1\times U_2} K_{S_1,S_2}(\xi_1,\xi_2)|f_1(\xi_1)|^2|f_2(\xi_2)|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \tag{1.5}$$

holds for some kernel K_{S_1,S_2} and such that the constant C is best possible; in many cases, extremisers for the above kind of inequalities have also been characterised. This has been mostly studied for paraboloids [9], cones [7], spheres [19, 10] and hyperboloids [24, 22], with the corresponding natural interpretations in PDE.

It should be noted that the bilinear estimates (1.3) and (1.4) also hold when $E_2 f_2$ is replaced by its complex conjugate $\overline{E_2 f_2}$. This is of course of interest when $S_1 = S_2$ and $f_1 = f_2$, as then the bilinear estimates can be reinterpreted as L^2 estimates of $|Ef|^2$. In particular, in the case of paraboloids, the identity (1.3) may be reinterpreted as

$$\int_{\mathbb{R}\times\mathbb{R}} |D_x|u|^2|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} |\xi-\eta||\widehat{u_0}(\xi)|^2|\widehat{u_0}(\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{1.6}$$

or simply

$$\int_{\mathbb{R}\times\mathbb{R}} |D_x^{1/2}|u|^2|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\|u_0\|_{L^2(\mathbb{R})}^2\|u_0\|_{L^2(\mathbb{R})}^2$$

in order to avoid the singularity of the resulting weight $|\phi'(\xi)-\phi'(\eta)| = 2|\xi-\eta|$; here we interpret the extension operator $u(x,t) = E\widetilde{u_0}(x,t)$ as the solution of the free Schrödinger equation $i\partial_t u - \Delta u = 0$ in $\mathbb{R}^d$ associated to the initial data $u(x,0) = u_0(x)$, with the normalisation of the Fourier transform considered in §2.1. Note that, for this specific case, it is crucial that the multiplier associated to D_x coincides precisely with $|\phi'(\xi)-\phi'(\eta)|$. Moreover, Ozawa and Tsutsumi [25] made use of the Radon transform to obtain the higher dimensional version

$$\|(-\Delta)^{(2-d)/4}|u|^2\|_{L^2_{x,t}(\mathbb{R}^d\times\mathbb{R})}^2 \leqslant \mathbf{OT}(d)\|u_0\|_{L^2(\mathbb{R}^d)}^2\|u_0\|_{L^2(\mathbb{R}^d)}^2 \tag{1.7}$$

where the constant $\mathbf{OT}(d) = \frac{2^{-d}\pi^{(2-d)/2}}{\Gamma(d/2)}$ is sharp after verifying that for $u_0(x) = e^{-|x|^2}$ the inequality becomes an identity; see also [9, 3]. An interesting feature from the identity (1.6) is that it may be used to prove local well-posedness of cubic non-linear Schrödinger equations.

Motivated by the applications to non-linear PDE in [15], Planchon and the second author [26] established certain higher dimensional analogues of the $\mathbb{R}^{1+1}$ identity (1.6). Up to that point all higher dimensional versions of (1.6), or more generally (1.3), were inequalities rather than *identities*. Their estimates also involved the Radon transform in the spatial variables[1], which in fact features in the statement of the identity. Recall that given a linear k-dimensional subspace $\pi \in \mathcal{G}_{k,n}$ and $y \in \pi^\perp$, the k-plane transform of a function f is defined as

$$T_{k,n}f(\pi,y) := \int_\pi f(x+y)\,\mathrm{d}\lambda_\pi(x),$$

where $\mathcal{G}_{k,n}$ denotes the Grassmanian manifold of all k-dimensional subspaces in $\mathbb{R}^n$ and $\mathrm{d}\lambda_\pi$ is the induced Lebesgue measure on π. The cases $k=1$ and $k=n-1$ correspond to the X-ray transform X and the Radon transform[2] $\mathcal{R}$ respectively. With this, it was shown in [26] that given $\omega \in \mathbb{S}^{n-1}$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_s \mathcal{R}(|u(\cdot,t)|^2)(\omega,s)|^2\,\mathrm{d}s\,\mathrm{d}t + J_\omega(u) = \frac{\pi}{(2\pi)^{2d+1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |(\xi-\eta)\cdot\omega||\widehat{u_0}(\xi)|^2|\widehat{u_0}(\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{1.8}$$

where

$$J_\omega(u) := \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{(\langle\omega\rangle^\perp)^2} \left|u(x+s\omega,t)\partial_s u(y+s\omega,t) - u(y+s\omega,t)\partial_s u(x+s\omega,t)\right|^2 \mathrm{d}\lambda_{(\langle\omega\rangle^\perp)^2}(x,y)\,\mathrm{d}s\,\mathrm{d}t.$$

[1]Note that the Radon transform in the spatial variables in $u(x,t)$ amounts to a $(n-2)$-plane transform in the context of the extension operators $Ef(z)$.

[2]The Radon transform $\mathcal{R}f$ is identified with a function in $\mathbb{S}^{n-1}_+ \times \mathbb{R}$ setting $\mathcal{R}f(\omega,s) \equiv \mathcal{R}f(\langle\omega\rangle^\perp, s\omega)$.

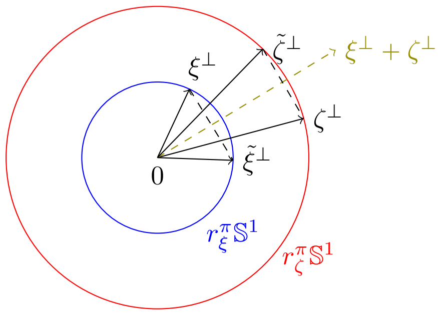

FIGURE 1. The new points $\tilde{\xi}^\perp \in r_\xi^\pi \mathbb{S}^1$ and $\tilde{\zeta}^\perp \in r_\zeta^\pi \mathbb{S}^1$ in $\pi^\perp$ are the reflected points of $\xi^\perp$ and $\zeta^\perp$ with respect to $\xi^\perp + \zeta^\perp$.

Of course fixing $\omega = e_d$ (or any other coordinate vector) the first term on the left-hand-side amounts to $\|\partial_s \| |u|^2 \|_{L^1(\mathbb{R}^{d-1})} \|^2_{L^2_{x_d,t}(\mathbb{R}^2)}$, which in the absence of the derivative ∂_s becomes $\|u\|^4_{L^4_{x_d,t}(\mathbb{R}^2; L^2(\mathbb{R}^{d-1}))}$; note the contrast with the L^4-nature of (1.3) and (1.4).

The approach in [26] to establish (1.8) uses integration-by-parts arguments as in [15] and extends to versions of (1.8) for nonlinear Schrödinger equations with nonlinearity of the type $\pm|u|^{p-1}u$, where $p \geqslant 1$. Estimates like (1.8) were used, for instance, to show scattering for solutions of those non-linear equations (previously obtained by Nakanishi [23]; see also [14]) or to deduce Bourgain's [8] bilinear refinement of the Strichartz estimate.

However, it is possible to obtain (1.8) via Fourier analysis taking advantage of applications of Plancherel's theorem, in the spirit of (1.3) and (1.4). In this paper we will see how to exploit this approach to obtain variants of (1.8) when the underlying extension operator is associated to spheres or hyperboloids. This should be understood as an exploration of the interaction of the k-plane transform and $|Ef|^2$; see also the recent paper [2] or the upcoming preprint [6] for further examples of this phenomenon.

1.2. **Estimates for the sphere.** In the case of the sphere $\mathbb{S}_r^{n-1} \equiv r\mathbb{S}^{n-1}$ of radius r in $\mathbb{R}^n$, it is considered the more classical form of the extension operator

$$g \mapsto \widehat{g \mathrm{d}\sigma_r^n},$$

where $\mathrm{d}\sigma_r^n$ denotes the induced normalised Lebesgue measure on $\mathbb{S}_r^{n-1}$ and $g \in L^1(\mathbb{S}_r^{n-1})$. The following L^2-identities for $T_{n-2,n}(\widehat{g_1 \mathrm{d}\sigma_r^n}\overline{\widehat{g_2 \mathrm{d}\sigma_r^n}})$ are obtained.

Theorem 1.1. *Let $n \geqslant 3$. Let $\pi \in \mathcal{G}_{n-2,n}$ and let $\pi^\perp$ denote the orthogonal subspace to π. For each $z \in \mathbb{R}^n$, write $z = z^\pi + z^\perp$, where z^π is the orthogonal projection of z into π. Then*

$$\int_{\pi^\perp} \Big| (-\Delta_y)^{1/4} T_{n-2,n}(\widehat{g_1 \mathrm{d}\sigma_r^n}\overline{\widehat{g_2 \mathrm{d}\sigma_r^n}})(\pi, y) \Big|^2 \mathrm{d}\lambda_{\pi^\perp}(y) \tag{1.9}$$
$$= \mathbf{C}_{\mathbb{S}^{n-1}} \int_{(\mathbb{S}_r^{n-1})^2} K_{\pi,\mathbb{S}_r^{n-1}}(\xi,\zeta) g_1(\xi) \bar{g}_2(\xi^\pi + \tilde{\xi}^\perp) g_2(\zeta) \bar{g}_1(\zeta^\pi + \tilde{\zeta}^\perp) \, \mathrm{d}\sigma_r^n(\xi) \, \mathrm{d}\sigma_r^n(\zeta)$$

where

$$K_{\pi,\mathbb{S}_r^{n-1}}(\xi,\zeta) := \frac{2}{|\xi^\perp + \zeta^\perp|}, \qquad \mathbf{C}_{\mathbb{S}^{n-1}} := (2\pi)^{2(n-1)},$$

$r_\xi = \sqrt{r^2 - |\xi^\pi|^2}$ and $\tilde{\xi}^\perp, \tilde{\zeta}^\perp \in \pi^\perp$ are the reflected points of $\xi^\perp$ and $\zeta^\perp$ in $\pi^\perp$ with respect to the line passing through the origin and $\xi^\perp + \zeta^\perp$, that is $\xi^\perp + \zeta^\perp = \tilde{\xi}^\perp + \tilde{\zeta}^\perp$ (see Figure 1).

Of course the L^2-nature of the inequality on its left-hand-side allows one to take advantage of Plancherel's theorem. As briefly described before §1.1, the key presence of the $(n-2)$-plane transform reduces the problem to a 2-dimensional analysis, and one is left to understand the convolution of two weighted measures associated to concentric circles of different radii in the subspace $\pi^\perp \simeq \mathbb{R}^2$. The main advantage with respect to (1.4) is that in this setting it is possible to express $h_1 d\sigma^2_{r_1} * h_2 d\sigma^2_{r_2}(\xi^\perp + \zeta^\perp)$ as the weight $d\sigma^2_{r_1} * d\sigma^2_{r_2}(\xi^\perp + \zeta^\perp)$ times an evaluation of the functions h_1 and h_2 at points depending on $\xi^\perp$ and $\zeta^\perp$. Note that the scenario (1.4) is *overdetermined* and prevents one from obtaining an identity when L^2-norm with respect to all variables is taken to the bilinear term $E_1 f_1 E_2 f_2$.

Given complex numbers $a, b, c, d \in \mathbb{C}$, the well known identity

$$a\bar{b}\bar{c}d = \frac{1}{2}\big(|ac|^2 + |bd|^2 - |a\bar{c} - b\bar{d}|^2\big) + i\mathrm{Im}(a\bar{b}\bar{c}d) \tag{1.10}$$

may be used in Theorem 1.1 to replace the 4-linear *wave interaction*

$$g_1(\xi)\bar{g}_2(\xi^\pi + \tilde{\xi}^\perp) g_2(\zeta)\bar{g}_1(\zeta^\pi + \tilde{\zeta}^\perp)$$

in (1.9) by an alternative expression involving $|g_1(\xi)|^2 |g_2(\zeta)|^2$ and which is closer in spirit to (1.8).

Corollary 1.2. *Let $n \geqslant 3$ and $\pi \in \mathcal{G}_{n-2,n}$. Then*

$$\int_{\pi^\perp} \Big|(-\Delta_y)^{1/4} T_{n-2,n}(\widehat{g_1 d\sigma^n_r}\overline{\widehat{g_2 d\sigma^n_r}})(\pi, y)\Big|^2 \, dy \tag{1.11}$$

$$= \mathbf{C}_{\mathbb{S}^{n-1}} \int_{(\mathbb{S}^{n-1}_r)^2} K_{\pi, \mathbb{S}^{n-1}_r}(\xi, \zeta) |g_1(\xi)|^2 |g_2(\zeta)|^2 \, d\sigma^n_r(\xi) \, d\sigma^n_r(\zeta) - I_{\pi, \mathbb{S}^{n-1}_r}(g_1, g_2),$$

where

$$I_{\pi, \mathbb{S}^{n-1}_r}(g_1, g_2) := \frac{\mathbf{C}_{\mathbb{S}^{n-1}}}{2} \int_{(\mathbb{S}^{n-1}_r)^2} K_{\pi, \mathbb{S}^{n-1}_r}(\xi, \zeta) |g_1(\xi) g_2(\zeta) - g_2(\xi^\pi + \tilde{\xi}^\perp) g_1(\zeta^\pi + \tilde{\zeta}^\perp)|^2 \, d\sigma^n_r(\xi) \, d\sigma^n_r(\zeta).$$

Of course the term $I_{\pi, \mathbb{S}^{n-1}_r}(g_1, g_2) \geqslant 0$ and is identically zero if g_1 and g_2 are constant functions, so it may be dropped from (1.11) at the expense of losing the identity, leading to a sharp inequality which fits in the context of (1.5). Thus, the term $I_{\pi, \mathbb{S}^{n-1}_r}(g_1, g_2)$ may be interpreted as the *distance* of such a resulting inequality to become an identity.[3]

As the k-plane transform satisfies the Fourier transform relation

$$\mathcal{F}_y T_{k,n} f(\pi, \xi) = \widehat{f}(\xi) \qquad \text{for } \xi \in \pi^\perp, \tag{1.12}$$

one may easily obtain by means of Plancherel's theorem the relation

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \frac{(2\pi)^{-k}}{|\mathcal{G}_{n-k-1,n-1}|} \|(-\Delta_y)^{k/4} T_{k,n} f\|^2_{L^2(\mathcal{G}_{k,n}, L^2(\pi^\perp))}. \tag{1.13}$$

Thus, on averaging Theorem 1.1 over all $\pi \in \mathcal{G}_{n-2,n}$ one has the following.

Corollary 1.3. *Let $n \geqslant 3$. Then*

$$\|(-\Delta)^{\frac{3-n}{4}} (\widehat{g_1 d\sigma^n_r}\overline{\widehat{g_2 d\sigma^n_r}})\|^2_{L^2(\mathbb{R}^n)} \leqslant (2\pi)^{2-n} \mathbf{C}_{\mathbb{S}^{n-1}} \int_{(\mathbb{S}^{n-1}_r)^2} K_{\mathbb{S}^{n-1}_r}(\xi, \zeta) |g_1(\xi)|^2 |g_2(\zeta)|^2 \, d\sigma^n_r(\xi) \, d\sigma^n_r(\zeta)$$

where

$$K_{\mathbb{S}^{n-1}_r}(\xi, \zeta) := \frac{1}{|\mathcal{G}_{1,n-1}|} \int_{\mathcal{G}_{n-2,n}} K_{\pi, \mathbb{S}^{n-1}_r}(\xi, \zeta) \, d\mu_{\mathcal{G}}(\pi).$$

In the particular case $n = 3$ and after setting $g_1 = g_2$, the right-hand-side in Corollary 1.3 amounts to a bilinear quantity appearing in the work of Foschi [18] on the sharp constant in the Stein–Tomas inequality (1.1) for $\mathbb{S}^2$. Thus, appealing to his work, one can deduce the following.

Corollary 1.4 (Stein–Tomas [31], Foschi [18])**.**

$$\|\widehat{g d\sigma^3}\|_{L^4(\mathbb{R}^3)} \leqslant 2\pi \|g\|_{L^2(\mathbb{S}^2)}. \tag{1.14}$$

Besides the value for the sharp constant, Foschi [18] also showed that the only real valued extremisers are constant functions; the existence of extremisers was previously verified in [12, 13].

[3] The inequality resulting from dropping $I_{\pi, \mathbb{S}^{n-1}_r}(g_1, g_2)$ in (1.11) may be obtained more directly by an application of the Cauchy–Schwarz inequality: see §5.1

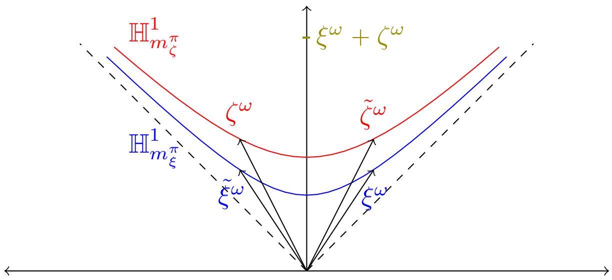

FIGURE 2. If $\xi^\omega + \zeta^\omega$ lies in the vertical axis, the new points $\tilde\xi^\omega$ and $\tilde\zeta^\omega$ are the reflected points of ξ^ω and ζ^ω with respect to that axis. For ease of notation, ξ^ω is identified with the point $(\xi^\omega, \phi_{m^\pi_\xi}(\xi^\omega)) \in \mathbb{H}^1_{m^\pi_\xi}$, and similarly for the other points.

Solution to the Helmholtz equation. Consider the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^n$. If $\sup_{R>0} \frac{1}{R}\int_{B_R} |u|^2 < \infty$, then there exists $g \in L^2(\mathbb{S}^{n-1}_k)$ such that $u = \widehat{g\mathrm{d}\sigma_{\mathbb{S}^{n-1}_k}}$. Theorem 1.1 and the subsequent corollaries may be then interpreted in that context.

1.3. **Estimates for the hyperboloid.** A similar analysis to the one described for $\mathbb{S}^{n-1}$ may be carried for one of the components of the elliptic hyperboloid in $\mathbb{R}^{d+1}$, defined by

$$\mathbb{H}^d_m := \{(\xi, \xi_{d+1}) \in \mathbb{R}^d \times \mathbb{R} : \xi_{d+1} = \phi_m(\xi) := \sqrt{m^2 + |\xi|^2}\}$$

and equipped with the Lorentz invariance measure $\mathrm{d}\sigma_{\mathbb{H}^d_m}$, defined by

$$\int_{\mathbb{H}^d_m} g(\xi, \xi_{d+1})\, \mathrm{d}\sigma_{\mathbb{H}^d_m}(\xi, \xi_{d+1}) = \int_{\mathbb{R}^d} g(\xi, \phi_m(\xi)) \frac{\mathrm{d}\xi}{\phi_m(\xi)}. \tag{1.15}$$

A function $f \in L^1(\mathbb{R}^d)$ is identified with its lift g to $\mathbb{H}^d_m$, given by $g(\xi, \phi_m(\xi)) = f(\xi)$, and note

$$\widehat{g\mathrm{d}\sigma_{\mathbb{H}^d_m}}(x,t) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it\sqrt{m^2+|\xi|^2}} f(\xi) \frac{\mathrm{d}\xi}{\sqrt{m^2+|\xi|^2}}$$

where $(x,t) \in \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}$. A natural reason to split into a space-time domain is in view of the connection of $\widehat{g\mathrm{d}\sigma_{\mathbb{H}^d_m}}$ with the Klein–Gordon propagator $e^{it\sqrt{m^2-\Delta}}f$; this will be further discussed below. Thus, considering the Radon transform in the space variables - as in (1.8) and as opposed to Theorem 1.1, where no time role is given and therefore $(n-2)$-plane transform is taken - one obtains the following.

Theorem 1.5. *Let $d \geqslant 2$. Let $\omega \in \mathbb{S}^{d-1}_+$ and let $\pi := \langle\omega\rangle^\perp \in \mathcal{G}_{d-1,d}$ be the orthogonal subspace to $\langle\omega\rangle$. For each $x \in \mathbb{R}^d$, write $x = x^\pi + x^\omega\omega$, where $x^\omega = x\cdot\omega$. Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Big|\partial_s^{1/2}\mathcal{R}(\widehat{g_1\mathrm{d}\sigma_{\mathbb{H}^d_m}}(\cdot,t)\overline{\widehat{g_2\mathrm{d}\sigma_{\mathbb{H}^d_m}}(\cdot,t)})(\omega,s)\Big|^2 \mathrm{d}s\,\mathrm{d}t$$
$$= \mathbf{C}_{\mathbb{H}^d} \int_{(\mathbb{R}^d)^2} K_{\omega,\mathbb{H}^d_m}(\xi,\zeta) f_1(\xi)\bar f_2(\xi^\pi + \tilde\xi^\omega\omega) f_2(\zeta)\bar f_1(\zeta^\pi + \tilde\zeta^\omega\omega)\, \frac{\mathrm{d}\xi}{\phi_m(\xi)}\,\frac{\mathrm{d}\zeta}{\phi_m(\zeta)}$$

where

$$K_{\omega,\mathbb{H}^d_m}(\xi,\zeta) := \frac{|\xi^\omega - \tilde\xi^\omega|^{1/2}|\zeta^\omega - \tilde\zeta^\omega|^{1/2}}{|\xi^\omega\phi_m(\zeta) - \zeta^\omega\phi_m(\xi)|} \qquad \text{and} \qquad \mathbf{C}_{\mathbb{H}^d} = (2\pi)^{2d}.$$

Above, the points $(\tilde\xi^\omega, \phi_{m^\pi_\xi}(\tilde\xi^\omega)) \in \mathbb{H}^1_{m^\pi_\xi}$ and $(\tilde\zeta^\omega, \phi_{m^\pi_\zeta}(\tilde\zeta^\omega)) \in \mathbb{H}^1_{m^\pi_\zeta}$ are the image under L^{-1} of the reflected points of $L((\xi^\omega, \phi_{m^\pi_\xi}(\xi^\omega)))$ and $L((\zeta^\omega, \phi_{m^\pi_\zeta}(\zeta^\omega)))$ in $\mathbb{R}^2$ with respect to the vertical axis respectively, where L is the unique Lorentz transformation mapping $(\xi^\omega + \zeta^\omega, \phi_{m^\pi_\xi}(\xi^\omega) + \phi_{m^\pi_\zeta}(\zeta^\omega))$ to the vertical axis and $m^\pi_\xi := \sqrt{m^2 + |\xi^\pi|^2}$ (see Figure 2).

As in the case of the sphere, the use of the Radon transform in $\mathbb{R}^d$ and Plancherel's theorem reduces the above estimate to explicitly understand $h_1 \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * h_2 \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(\xi^\omega + \zeta^\omega, \phi_{m_1}(\xi^\omega) + \phi_{m_2}(\zeta^\omega))$. In fact, note that the value of $K_{\omega,\mathbb{H}^d_m}$ amounts to the expression $\left|\begin{pmatrix} 1 & 1 \\ \phi'_{m^\pi_\xi}(\xi^\omega) & \phi'_{m^\pi_\zeta}(\zeta^\omega)\end{pmatrix}\right|$ corrected with the natural weight $\frac{1}{\phi_{m^\pi_\xi}(\xi^\omega)\phi_{m^\pi_\zeta}(\zeta^\omega)}$ coming from the definition of $\mathrm{d}\sigma_{\mathbb{H}^1_m}$. This should be compared with the elementary two-dimensional identity (1.3). The presence of the numerator $|\xi^\omega - \tilde{\xi}^\omega|^{1/2}|\zeta^\omega - \tilde{\zeta}^\omega|^{1/2}$ is due to the action of $\partial_s^{1/2}$ on $\mathcal{R}(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{H}^d_m}}(\cdot,t)\overline{\widehat{g_2 \mathrm{d}\sigma_{\mathbb{H}^d_m}}}(\cdot,t))$. Moreover, one can explicitly write $\tilde{\xi}^\omega$ and $\tilde{\zeta}^\omega$ in terms of ξ, ζ and ω, leading to the more compact expression

$$K_{\omega,\mathbb{H}^d_m}(\xi,\zeta) = \frac{2(\phi_m(\xi)+\phi_m(\zeta))}{(\phi_m(\xi)+\phi_m(\zeta))^2 - ((\xi+\zeta)\cdot\omega)^2}.$$

As before, one may use (1.10) to rewrite Theorem 1.5 in the spirit of (1.8).

Corollary 1.6. *Let $d \geqslant 2$ and $\omega \in \mathbb{S}^{d-1}_+$. Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\partial_s^{1/2}\mathcal{R}(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{H}^d_m}}(\cdot,t)\overline{\widehat{g_2 \mathrm{d}\sigma_{\mathbb{H}^d_m}}}(\cdot,t))(\omega,s)\right|^2 \mathrm{d}s\,\mathrm{d}t$$
$$= \mathbf{C}_{\mathbb{H}^d}\int_{(\mathbb{R}^d)^2} K_{\omega,\mathbb{H}^d_m}(\xi,\zeta)|f_1(\xi)|^2|f_2(\zeta)|^2 \frac{\mathrm{d}\xi}{\phi_m(\xi)}\frac{\mathrm{d}\zeta}{\phi_m(\zeta)} - I_{\omega,\mathbb{H}^d_m}(f_1,f_2)$$

where

$$I_{\omega,\mathbb{H}^d_m}(f_1,f_2) := \frac{\mathbf{C}_{\mathbb{H}^d}}{2}\int_{(\mathbb{R}^d)^2} K_{\omega,\mathbb{H}^d_m}(\xi,\zeta)|f_1(\xi)f_2(\zeta) - f_2(\xi^\pi + \tilde{\xi}^\omega\omega)f_1(\zeta^\pi + \tilde{\zeta}^\omega\omega)|^2 \frac{\mathrm{d}\xi}{\phi_m(\xi)}\frac{\mathrm{d}\zeta}{\phi_m(\zeta)}.$$

As $\mathcal{R} = T_{d-1,d}$, the use of the Plancherel's relation (1.13) after averaging over $\omega \in \mathbb{S}^{d-1}_+$ yields the following.

Corollary 1.7. *Let $d \geqslant 2$. Then*

$$\|(-\Delta_x)^{\frac{2-d}{4}}(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{H}^d_m}}\overline{\widehat{g_2 \mathrm{d}\sigma_{\mathbb{H}^d_m}}})\|^2_{L^2_{x,t}(\mathbb{R}^d\times\mathbb{R})} \leqslant (2\pi)^{1-d}\mathbf{C}_{\mathbb{H}^d}\int_{(\mathbb{R}^d)^2} K_{\mathbb{H}^d_m}(\xi,\zeta)|f_1(\xi)|^2|f_2(\zeta)|^2\frac{\mathrm{d}\xi}{\phi_m(\xi)}\frac{\mathrm{d}\zeta}{\phi_m(\zeta)}$$

where

$$K_{\mathbb{H}^d_m}(\xi,\zeta) := \frac{1}{2}\int_{\mathbb{S}^{d-1}} K_{\omega,\mathbb{H}^d_m}(\xi,\zeta)\,\mathrm{d}\sigma^d(\omega).$$

The Klein–Gordon propagator. The solution to the Klein–Gordon equation $-\partial_t^2 u + \Delta u = m^2 u$ in $\mathbb{R}^d \times \mathbb{R}$, with initial data $u(x,0) = f_0(x)$, $\partial_t u(x,0) = f_1(x)$ is given by

$$u(x,t) = e^{it\sqrt{m^2-\Delta}}f_-(x) + e^{-it\sqrt{-\Delta}}f_+(x)$$

where $f_+ = \frac{1}{2}(f_0 + i(\sqrt{m^2-\Delta})^{-1}f_1)$ and $f_- = \frac{1}{2}(f_0 - i(\sqrt{m^2-\Delta})^{-1}f_1)$ and

$$e^{\pm it\sqrt{m^2-\Delta}}f(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{\pm it\sqrt{m^2+|\xi|^2}}\widehat{f}(\xi)\,\mathrm{d}\xi.$$

Note that $e^{\pm it\sqrt{m^2-\Delta}}f(x) = (2\pi)^{-d}(\widehat{g}\mathrm{d}\sigma_{\mathbb{H}^d_m})\,\widehat{}\,(x,t)$; where $\widehat{g}$ is the lift of $\widehat{f}\sqrt{m^2+|\cdot|^2}$ to $\mathbb{H}^d_m$. Thus, Theorem 1.5 and Corollaries 1.6 and 1.7 may be re-interpreted in terms of $e^{it\sqrt{m^2-\Delta}}$; in particular, setting $\mathbf{KG(d)} = (2\pi)^{-4d}\mathbf{C}_{\mathbb{H}^d}$, the estimate in Corollary 1.7 reads as

$$\|(-\Delta_x)^{\frac{2-d}{4}}(e^{it\sqrt{m^2-\Delta}}f_1\overline{e^{it\sqrt{m^2-\Delta}}f_2})\|^2_{L^2_{x,t}(\mathbb{R}^{d+1})} \leqslant \mathbf{KG(d)}\int_{(\mathbb{R}^d)^2} K_{\mathbb{H}^d_m}(\xi,\zeta)|\widehat{f}_1(\xi)|^2|\widehat{f}_2(\zeta)|^2\phi_m(\xi)\phi_m(\zeta)\mathrm{d}\xi\mathrm{d}\zeta.$$

Structure of the paper. Section 2 contains some notation and standard observations which will be useful throughout the paper. In Section 3 we revisit the convolution of weighted measures of circles and hyperbolas. Section 4 contains the proofs of Theorems 1.1 and 1.5 whilst Section 5 is concerned with the derivation of the several corollaries. Finally, we provide a Fourier analytic proof of the identity (1.8), together with a further discussion on Fourier bilinear identities associated to paraboloids.

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2. Notation and preliminaries

2.1. **Fourier transform.** We work with the normalisation of the Fourier transform

$$\mathcal{F}(f)(\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^n}e^{iz\cdot\xi}f(z)\,\mathrm{d}z \qquad \text{and} \qquad \mathcal{F}^{-1}(f)(z)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-iz\cdot\xi}f(\xi)\,\mathrm{d}\xi.$$

With this normalisation,

$$\widehat{f*g}=\widehat{f}\cdot\widehat{g}, \qquad \widehat{fg}(\xi)=(2\pi)^{-n}\widehat{f}*\widehat{g}(\xi), \qquad \widehat{\widehat{f}}(z)=(2\pi)^n\tilde{f}(z), \qquad \widehat{\widehat{\bar{f}}}(z)=(2\pi)^n\bar{f}(z),$$

where $\tilde{f}(z):=f(-z)$, Plancherel's theorem adopts the form

$$\|\widehat{f}\|_{L^2(\mathbb{R}^n)}=(2\pi)^{n/2}\|f\|_{L^2(\mathbb{R}^n)}$$

and the n-dimensional Dirac delta is

$$\delta_n(a)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ia\cdot z}\,\mathrm{d}z.$$

2.2. **k-plane transform.** The Grassmannian manifold $\mathcal{G}_{k,n}$ of all k-dimensional subspaces of $\mathbb{R}^n$ is equipped with an invariant measure $\mathrm{d}\mu_{\mathcal{G}}$ under the action of the orthogonal group. This measure is unique up to a constant, and is chosen to be normalised as

$$|\mathcal{G}_{k,n}|=\int_{\mathcal{G}_{k,n}}\mathrm{d}\mu_{\mathcal{G}}(\pi)=\frac{|\mathbb{S}^{n-1}|\cdots|\mathbb{S}^{n-k}|}{|\mathbb{S}^{k-1}|\cdots|\mathbb{S}^{0}|}.$$

Given $\pi\in\mathcal{G}_{k,n}$ and $\xi\in\pi^{\perp}$, the relation (1.12) between the k-plane transform $T_{k,n}$ and the Fourier transform easily follows from the definition

$$\mathcal{F}_yT_{k,n}f(\pi,\xi)=\int_{\pi^\perp}e^{iy\cdot\xi}T_{k,n}f(\pi,y)\,\mathrm{d}\lambda_{\pi^\perp}(y)=\int_{\pi^\perp}e^{iy\cdot\xi}\int_{\pi}f(x+y)\,\mathrm{d}\lambda_{\pi}(x)\,\mathrm{d}\lambda_{\pi^\perp}(y)=\widehat{f}(\xi) \tag{2.1}$$

after changing variables $z=x+y$ and noting that $\xi\cdot x=0$ for $\xi\in\pi^\perp$. This and the known identity (see for instance [21, Chapter 2])

$$\int_{\mathbb{S}^{n-1}}f(\omega)\,\mathrm{d}\sigma^n(\omega)=\frac{1}{|\mathcal{G}_{n-k-1,n-1}|}\int_{\mathcal{G}_{k,n}}\int_{\mathbb{S}^{n-1}\cap\pi^\perp}f(\omega)\,\mathrm{d}\sigma^n_{\pi^\perp}(\omega)\,\mathrm{d}\mu_{\mathcal{G}}(\pi), \tag{2.2}$$

yield via Plancherel's theorem and a change to polar coordinates the Plancherel-type identity (1.13) for the k-plane transform.

2.3. **Lorentz transformations.** The Lorentz group $\mathcal{L}$ is defined as the group of invertible linear transformations in $\mathbb{R}^{d+1}$ preserving the bilinear form

$$(z,u)\mapsto z_{d+1}u_{d+1}-z_du_d-\cdots-z_1u_1.$$

It is well known that the measure $\mathrm{d}\sigma_{\mathbb{H}^d_m}$ is invariant under the action of the subgroup of $\mathcal{L}$ that preserves the hyperboloid $\mathbb{H}^d_m$, denoted by $\mathcal{L}^+$. More precisely,

$$\int_{\mathbb{H}^d_m}f\circ L\,\mathrm{d}\sigma_{\mathbb{H}^d_m}=\int_{\mathbb{H}^d_m}f\,\mathrm{d}\sigma_{\mathbb{H}^d_m}$$

for all $L\in\mathcal{L}^+$. It is also a well known fact that given $P=(\xi,\tau)\in\mathbb{R}^{d+1}$ with $\tau>|\xi|$, there exists a Lorentz transformation $L\in\mathcal{L}^+$ such that $L(\xi,\tau)=(0,\sqrt{\tau^2-|\xi|^2})$; see for instance [27]. For $d=1$, this transformation is given by

$$L\equiv L_{\gamma_P}:=\begin{pmatrix}\cosh\gamma_P & -\sinh\gamma_P\\ -\sinh\gamma_P & \cosh\gamma_P\end{pmatrix}, \qquad \text{where} \quad \gamma_P:=\ln\sqrt{\frac{\tau+\xi}{\tau-\xi}}; \tag{2.3}$$

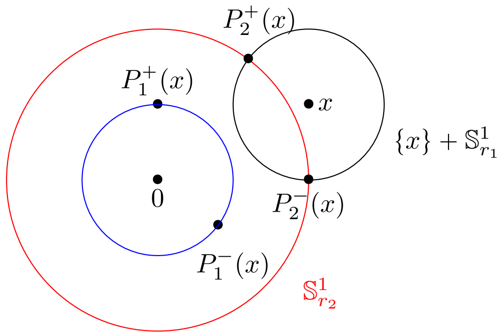

FIGURE 3. The points $P_2^+(x), P_2^-(x) \in \mathbb{S}^1_{r_2} \cap (\{x\} + \mathbb{S}^1_{r_1})$ and the points $P_1^+(x) := x - P_2^-(x), P_1^-(x) := x - P_2^+(x) \in \mathbb{S}^1_{r_1}$.

recall that P may be expressed in hyperbolic coordinates as $P = (\xi, \tau) = (r_P \sinh \gamma_P, r_P \cosh \gamma_P)$, where $r_P := \sqrt{\tau^2 - \xi^2}$. The inverse Lorentz transformation that maps $(0, r_P)$ back to (ξ, τ) is given by $L_{-\gamma_P}$.

3. Convolution of weighted measures

As is discussed in the introduction, a key ingredient in the proof of Theorems 1.1 and 1.5 is to understand convolutions of two weighted measures associated to concentric circles of different radii in $\mathbb{R}^2$ and to rectangular hyperbolas in $\mathbb{R}^2$ with foci lying on the same line but with different major axis. The computation of such convolutions is standard; see, for instance [18, 10] for the circular case or [27, 11] for the hyperbolic case. The main key feature here is that the convolution is carried with respect to weighted measures, and thanks to the key fact of being in $\mathbb{R}^2$, one can give a precise evaluation of such weights at certain points.

3.1. **Circles.** Given $r \in \mathbb{R}_+$, let $\mathrm{d}\sigma^2_r$ denote the normalised Lebesgue measure of $\mathbb{S}^1_r \equiv r\mathbb{S}^1$, that is

$$\int_{\mathbb{S}^1_r} g(\omega)\, \mathrm{d}\sigma^2_r(\omega) = \int_{\mathbb{S}^1} g(r\omega)\, \mathrm{d}\sigma^2(\omega),$$

and recall that $\mathrm{d}\sigma^2(\omega) = \delta_1(1-|\omega|)\,\mathrm{d}\omega = 2\delta_1(1-|\omega|^2)\,\mathrm{d}\omega$, where $\mathrm{d}\omega$ denotes the Lebesgue measure on $\mathbb{R}^2$.

Given $0 < r_1 \leqslant r_2$, the domain of integration in $\mathrm{d}\sigma^2_{r_1} * \mathrm{d}\sigma^2_{r_2}(x)$ is $\mathbb{S}^1_{r_2} \cap (\{x\} + \mathbb{S}_{r_1})$. This set is non-empty if and only if $|x| \in [r_2 - r_1, r_2 + r_1]$ and consists of one point in the *tangent* case $|x| = r_2 - r_1$ or $|x| = r_2 + r_1$ and of two points otherwise. In the non-empty case, fix $v_x \in \mathbb{S}^1$ such that $v_x \cdot x = 0$ and is the $\pi/2$ degrees rotation of x in the anti-clockwise direction, and let $P_2^+(x)$ and $P_2^-(x)$ denote the points in $\mathbb{S}^1_{r_2} \cap (\{x\} + \mathbb{S}_{r_1})$ such that $P_2^+(x) \cdot v_x \geqslant 0$ and $P_2^-(x) \cdot v_x \leqslant 0$ respectively; note that $P_2^+(x) = P_2^-(x)$ in the tangent case. Define $P_1^-(x) := x - P_2^+(x) \in \mathbb{S}^1_{r_1}$ and $P_1^+(x) := x - P_2^-(x) \in \mathbb{S}^1_{r_1}$. Observe that $P_j^+(x)$ and $P_j^-(x)$ are reflected points one another with respect to the line passing through the origin containing x: see Figure 3.

Lemma 3.1. *Let $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 \leqslant r_2$. Then*

$$g_1 \mathrm{d}\sigma^2_{r_1} * g_2 \mathrm{d}\sigma^2_{r_2}(x) = \frac{2g_1(P_1^+(x))g_2(P_2^-(x)) + 2g_1(P_1^-(x))g_2(P_2^+(x))}{\sqrt{-(|x|^2 - (r_2+r_1)^2)(|x|^2 - (r_2 - r_1)^2)}}$$

if $|x| \in [r_2 - r_1, r_2 + r_1]$.

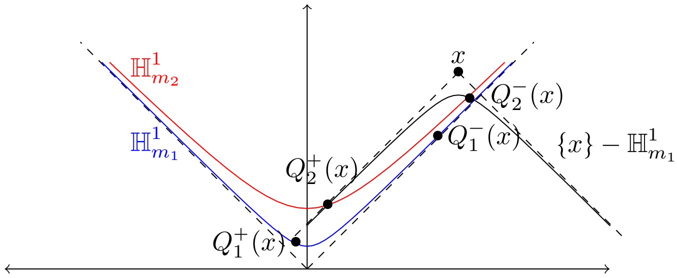

FIGURE 4. The points $Q_2^+(x), Q_2^-(x) \in \mathbb{H}^1_{m_2} \cap (\{x\} - \mathbb{H}^1_{m_1})$ and the points $Q_1^+(x) := x - Q_2^-(x), Q_1^-(x) := x - Q_1^+(x) \in \mathbb{H}^1_{m_1}$.

Proof. A computation shows

$$\begin{aligned} g_1\, \mathrm{d}\sigma^2_{r_1} * g_2\, \mathrm{d}\sigma^2_{r_2}(x) &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} g_1(r_1\omega_1)g_2(r_2\omega_2)\delta_2(x - r_1\omega_1 - r_2\omega_2)\, \mathrm{d}\sigma^2(\omega_1)\, \mathrm{d}\sigma^2(\omega_2) \\ &= \frac{2}{r_2^2}\int_{\mathbb{S}^1}\int_{\mathbb{R}^2} g_1(r_1\omega_1)g_2(r_2\omega_2)\delta_2\Big(\frac{x}{r_2} - \frac{r_1}{r_2}\omega_1 - \omega_2\Big)\delta_1(1 - |\omega_2|^2)\, \mathrm{d}\sigma^2(\omega_1)\, \mathrm{d}\omega_2 \\ &= \frac{1}{r_1|x_1|}\int_{\mathbb{S}^1} g_1(r_1\omega_1)g_2(x - r_1\omega_1)\delta_1\Big(\frac{r_2^2}{2r_1|x|} - \frac{|x|}{2r_1} - \frac{r_1}{2|x|} + \frac{x}{|x|}\cdot\omega_1\Big)\, \mathrm{d}\sigma^2(\omega_1) \\ &=: \mathrm{I}^+(x) + \mathrm{I}^-(x), \end{aligned}$$

where $\mathrm{I}^+(x)$ corresponds to the integration over $\mathbb{S}^1_+(x) := \{\omega \in \mathbb{S}^1 : x\cdot\omega \geqslant 0\}$ and $\mathrm{I}^-(x)$ to the integration over $\mathbb{S}^1_-(x) := \mathbb{S}^1 \backslash \mathbb{S}^1_+(x) = \{\omega \in \mathbb{S}^1 : x \cdot \omega \leqslant 0\}$.

Denoting by α_x the clockwise angle between e_1 and x and $P_x(u) = (\cos(\alpha_x + \arccos(u)), \sin(\alpha_x + \arccos(u)))$, the expression for $\mathrm{I}^+(x)$ becomes, after a change of variable,

$$\begin{aligned} \mathrm{I}^+(x) &= \frac{1}{r_1|x_1|}\int_{-1}^{1}\delta_1\Big(\frac{r_2^2}{2r_1|x|} - \frac{|x|}{2r_1} - \frac{r_1}{2|x|} + u\Big)(1-u^2)^{-1/2} g_1(r_1P_x(u))g_2(x - r_1P_x(u))\, \mathrm{d}u \\ &= \frac{1}{r_1|x|}\Big(1 - \Big(\frac{r_2^2}{2r_1|x|} - \frac{|x|}{2r_1} - \frac{r_1}{2|x|}\Big)^2\Big)^{-1/2} g_1(P_1^+(x))g_2(P_2^-(x))\chi_{\{r_2 - r_1 \leqslant |x| \leqslant r_2 + r_1\}}(x) \\ &= \frac{2g_1(P_1^+(x))g_2(P_2^-(x))}{\sqrt{-(|x|^2 - (r_2 + r_1)^2)(|x|^2 - (r_2 - r_1)^2)}}\chi_{\{r_2 - r_1 \leqslant |x| \leqslant r_2 + r_1\}}(x) \end{aligned}$$

noting that $r_1P_x(u) = P_1^+(x)$ after integrating in u. Arguing similarly for $\mathrm{I}^-(x)$ concludes the proof. □

3.2. **Hyperbolas.** Consider the Lorentz invariant measure $\mathrm{d}\sigma_{\mathbb{H}^1_m}$ defined in (1.15). Given $0 < m_1 \leqslant m_2$, the domain of integration in $\mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(x)$ is $\mathbb{H}^1_{m_2} \cap (\{x\} - \mathbb{H}^1_{m_1})$. Reasoning as in the previous case, this set is non-empty if and only if $\sqrt{x_2^2 - x_1^2} \geqslant m_1 + m_2$ and consists of one single point in the *tangent* case $\sqrt{x_2^2 - x_1^2} = m_1 + m_2$ and of two points otherwise; here $x = (x_1, x_2) \in \mathbb{R}^2$. In the non-empty case, let $Q_2^+(x)$ and $Q_2^-(x)$ denote the points in $\mathbb{H}^1_{m_2} \cap (\{x\} - \mathbb{H}^1_{m_1})$ such that $\big(Q_2^+(x) - x\big)\cdot e_1 \geqslant 0$ and $\big(Q_2^-(x) - x\big)\cdot e_1 \leqslant 0$ respectively; of course $Q_2^+(x) = Q_2^-(x)$ in the tangent case. Define $Q_1^+(x) = x - Q_2^-(x) \in \mathbb{H}^1_{m_1}$ and $Q_1^-(x) = x - Q_2^+(x) \in \mathbb{H}^1_{m_1}$ (see Figure 4).

Lemma 3.2. *Let $m_1, m_2 \in \mathbb{R}$ such that $0 < m_1 \leqslant m_2$. For each $x = (x_1, x_2) \in \mathbb{R}^2$ such that $x_2^2 \geqslant x_1^2$ one has*

$$g_1 \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * g_2 \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(x) = \frac{2g_1(Q_1^+(x))g_2(Q_2^-(x)) + 2g_1(Q_1^-(x))g_2(Q_2^+(x))}{\sqrt{(x_2^2 - x_1^2)^2 - 2(x_2^2 - x_1^2)(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}}\chi_{\{\sqrt{x_2^2 - x_1^2} \geqslant m_1 + m_2\}}(x).$$

Proof. By invariance of the measure $\mathrm{d}\sigma_{\mathbb{H}^1_m}$ under Lorentz transformations, it suffices to prove the above identity for $x = (0, z)$. Indeed, note that if $L_x \in \mathcal{L}^+$ is the Lorentz transformation satisfying

$L_x(x) = (0,z) = (0, \sqrt{x_2^2 - x_1^2})$, then

$$\begin{aligned} g_1 \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * g_2 \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(x) &= \int_{\mathbb{H}^1_{m_1}} \int_{\mathbb{H}^1_{m_2}} g_1(\omega) g_2(\nu) \delta_2(x - \omega - \nu)\, \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}}(\omega)\, \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(\nu) \\ &= \int_{\mathbb{H}^1_{m_1}} \int_{\mathbb{H}^1_{m_2}} g_1(L_x^{-1}(\omega)) g_2(L_x^{-1}(\nu)) \delta_2((0,z) - \omega - \nu)\, \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}}(\omega)\, \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(\nu) \\ &= h_1 \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * h_2 \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(0,z) \end{aligned}$$

where $h_j = g_j \circ L_x^{-1}$; the reduction to the vertical axis then follows from noting that $h_j(Q_\ell^\pm(0,z)) = g_j(Q_\ell^\pm(x))$ for $j, \ell = 1, 2$. Next,

$$\begin{aligned} h_1 \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * h_2 \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(0,z) &= \int_{\mathbb{H}^1_{m_1}} \int_{\mathbb{H}^1_{m_2}} h_1(\omega) h_2(\nu) \delta_2((0,z) - (\omega_1,\omega_2) - (\nu_1,\nu_2))\, \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}}(\omega)\, \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(\nu) \\ &= \int_{\mathbb{R}} h_1(\omega_1, \phi_{m_1}(\omega_1)) h_2(-\omega_1, \phi_{m_2}(\omega_1)) \frac{\delta_1(z - \phi_{m_1}(\omega_1) - \phi_{m_2}(\omega_1))}{\phi_{m_1}(\omega_1)\phi_{m_2}(\omega_1)}\, \mathrm{d}\omega_1. \end{aligned}$$

Splitting $\mathbb{R} = \mathbb{R}_- \cup \mathbb{R}_+$ and doing the change of variables

$$v = \phi_{m_1}(\omega_1) + \phi_{m_2}(\omega_1), \qquad \text{with} \quad \frac{\mathrm{d}\omega_1}{\phi_{m_1}(\omega_1)\phi_{m_2}(\omega_1)} = \frac{\mathrm{d}v}{\omega_1 v},$$

on each half-line one has

$$\begin{aligned} h_1 \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * h_2 \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(0,z) = \int_{m_1+m_2}^\infty \Big(& h_1(\omega_1, \phi_{m_1}(\omega_1)) h_2(-\omega_1, \phi_{m_2}(\omega_1)) \\ & + h_1(-\omega_1, \phi_{m_1}(\omega_1)) h_2(\omega_1, \phi_{m_2}(\omega_1)) \Big) \delta_1(z - v) \frac{\mathrm{d}v}{\omega_1 v} \end{aligned}$$

where ω_1 above is the function of v

$$\omega_1 = \omega_1(v) := \frac{\sqrt{v^4 - 2v^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}}{2v}.$$

Noting that $Q_1^\pm(0,v) = (\pm\omega_1(v), \phi_{m_1}(\omega_1(v)))$ and $Q_2^\pm(0,v) = (\mp\omega_1(v), \phi_{m_2}(\omega_1(v)))$, one has

$$h_1 \mathrm{d}\sigma_{\mathbb{H}^1_{m_1}} * h_2 \mathrm{d}\sigma_{\mathbb{H}^1_{m_2}}(0,z) = \frac{2h_1(Q_1^+(0,z)) h_2(Q_2^-(0,z)) + 2h_1(Q_1^-(0,z)) h_2(Q_2^+(0,z))}{\sqrt{z^4 - 2z^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}} 1_{\{z \geqslant m_1 + m_2\}},$$

completing the proof. □

4. The proof of Theorems 1.1 and 1.5

4.1. **Proof of Theorem 1.1.** By simplicity we work on the unit sphere $r = 1$; the result for $\mathbb{S}_r^{n-1}$ follows analogously. Given $\pi \in \mathcal{G}_{n-2,n}$, let $\pi^\perp$ denote its orthogonal subspace. For each $\xi \in \mathbb{R}^n$, write $\xi = \xi^\pi + \xi^\perp$, where $\xi^\pi \in \pi$ and $\xi^\perp \in \pi^\perp$, and let $r_\xi^\pi := \sqrt{1 - |\xi^\pi|^2}$. Given $x \in \pi$ and $y \in \pi^\perp$,

$$\begin{aligned} \widehat{g_j \mathrm{d}\sigma^n}(x+y) &= \int_{\mathbb{S}^{n-1}} e^{i(x+y)\cdot\xi} g_j(\xi)\, \mathrm{d}\sigma^n(\xi) \\ &= \int_{|\xi^\pi| \leqslant 1} e^{ix\cdot\xi^\pi} \int_{r_\xi^\pi \mathbb{S}^1} e^{iy\cdot\xi^\perp} g_j(\xi^\pi + \xi^\perp)\, \mathrm{d}\sigma^\perp_{r_\xi^\pi}(\xi^\perp)\, \mathrm{d}\lambda_\pi(\xi^\pi) \\ &= \int_{|\xi^\pi| \leqslant 1} e^{ix\cdot\xi^\pi} \mathcal{F}^\perp(g_{j,\xi^\pi} \mathrm{d}\sigma^\perp_{r_\xi^\pi})(y)\, \mathrm{d}\lambda_\pi(\xi^\pi) \end{aligned}$$

where $g_{j,\xi^\pi}(\omega) := g_j(\xi^\pi + \omega)$, $\mathcal{F}^\perp$ denotes the Fourier transform in $\pi^\perp$ and $\mathrm{d}\sigma^\perp_{r_\xi^\pi}$ denotes the induced normalised Lebesgue measure of $r_\xi^\pi \mathbb{S}^1$ in $\pi^\perp$, which can be of course identified with $\mathrm{d}\sigma^2_{r_\xi^\pi}$. Then, by Plancherel's theorem in π,

$$T_{n-2,n}(\widehat{g_1 \mathrm{d}\sigma^n}\overline{\widehat{g_2\mathrm{d}\sigma^n}})(\pi,y) = \int_\pi \widehat{g_1\mathrm{d}\sigma^n}(x+y)\overline{\widehat{g_2\mathrm{d}\sigma^n}}(x+y)\,\mathrm{d}\lambda_\pi(x)$$
$$= (2\pi)^{n-2}\int_{|\xi^\pi|\leqslant 1} \mathcal{F}^\perp(g_{1,\xi^\pi}\mathrm{d}\sigma^\perp_{r^\pi_\xi})(y)\overline{\mathcal{F}^\perp(g_{2,\xi^\pi}\mathrm{d}\sigma^\perp_{r^\pi_\xi})}(y)\,\mathrm{d}\lambda_\pi(\xi^\pi).$$

A further application of Plancherel's theorem in $\pi^\perp$ yields

$$\int_{\pi^\perp} |(-\Delta_y)^{1/4}T_{n-2,n}(\widehat{g_1\mathrm{d}\sigma^n}\overline{\widehat{g_2\mathrm{d}\sigma^n}})(\pi,y)|^2\,\mathrm{d}\lambda_{\pi^\perp}(y)$$
$$= (2\pi)^{2(n-1)}\int_{|\xi^\pi|\leqslant 1}\int_{|\zeta^\pi|\leqslant 1}\int_{\pi^\perp}|v|\big(\tilde{g}_{1,\xi^\pi}\mathrm{d}\sigma^\perp_{r^\pi_\xi} *^\perp \bar{g}_{2,\xi^\pi}\mathrm{d}\sigma^\perp_{r^\pi_\xi}\big)(v)\overline{\big(\tilde{g}_{1,\zeta^\pi}\mathrm{d}\sigma^\perp_{r^\pi_\zeta} *^\perp \bar{g}_{2,\zeta^\pi}\mathrm{d}\sigma^\perp_{r^\pi_\zeta}\big)}(v)$$

where the right-hand-side is integrated with respect to the measure $\mathrm{d}\lambda_{\pi^\perp}(v)\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi)$. For fixed ξ^π, ζ^π with $|\xi^\pi|\leqslant 1$ and $|\zeta^\pi|\leqslant 1$, the innermost integral above equals

$$\int_{(r^\pi_\xi\mathbb{S}^1)^2\times(r^\pi_\zeta\mathbb{S}^1)^2} |\xi^\perp-\eta^\perp|^{1/2}|\zeta^\perp-\mu^\perp|^{1/2} g_{1,\xi^\pi}(\xi^\perp)\bar{g}_{2,\xi^\pi}(\eta^\perp)g_{2,\zeta^\pi}(\zeta^\perp)\bar{g}_{1,\zeta^\pi}(\mu^\perp)\,\mathrm{d}\Sigma^\perp_{\xi^\pi,\zeta^\pi}(\xi^\perp,\eta^\perp,\zeta^\perp,\mu^\perp) \tag{4.1}$$

where

$$\mathrm{d}\Sigma^\perp_{\xi^\pi,\zeta^\pi}(\xi^\perp,\eta^\perp,\zeta^\perp,\mu^\perp) := \delta(\xi^\perp+\zeta^\perp-\eta^\perp-\mu^\perp)\,\mathrm{d}\sigma^\perp_{r^\pi_\xi}(\xi^\perp)\,\mathrm{d}\sigma^\perp_{r^\pi_\xi}(\eta^\perp)\,\mathrm{d}\sigma^\perp_{r^\pi_\zeta}(\zeta^\perp)\,\mathrm{d}\sigma^\perp_{r^\pi_\zeta}(\mu^\perp).$$

Observe that one may rewrite the above integral as

$$\int_{r^\pi_\xi\mathbb{S}^1\times r^\pi_\zeta\mathbb{S}^1} g_{1,\xi^\pi}(\xi^\perp)g_{2,\zeta^\pi}(\zeta^\perp)\big(h_{2,\xi}\mathrm{d}\sigma^\perp_{r^\pi_\xi} *^\perp h_{1,\zeta}\mathrm{d}\sigma^\perp_{r^\pi_\zeta}\big)(\xi^\perp+\zeta^\perp)\,\mathrm{d}\sigma^\perp_{r^\pi_\xi}(\xi^\perp)\,\mathrm{d}\sigma^\perp_{r^\pi_\zeta}(\zeta^\perp)$$

where $h_{2,\xi}(\eta^\perp) := \bar{g}_{2,\xi^\pi}(\eta^\perp)|\xi^\perp-\eta^\perp|^{1/2}$ and similarly for $h_{1,\zeta}$. As $\pi^\perp\cong\mathbb{R}^2$, assuming without loss of generality that $r^\pi_\xi\leqslant r^\pi_\zeta$, one can appeal to Lemma 3.1 to evaluate

$$\big(h_{2,\xi}\mathrm{d}\sigma^\perp_{r^\pi_\xi} *^\perp h_{1,\zeta}\mathrm{d}\sigma^\perp_{r^\pi_\zeta}\big)(\xi^\perp+\zeta^\perp) = \frac{2h_{2,\xi}(\xi^\perp)h_{1,\zeta}(\zeta^\perp)+2h_{2,\xi}(\tilde{\xi}^\perp)h_{1,\zeta}(\tilde{\zeta}^\perp)}{\sqrt{-(|\xi^\perp+\zeta^\perp|^2-(r^\pi_\zeta+r^\pi_\xi)^2)(|\xi^\perp+\zeta^\perp|^2-(r^\pi_\zeta-r^\pi_\xi)^2)}} \tag{4.2}$$

after noting that if $x=\xi^\perp+\zeta^\perp$ then $(P_1^+(x),P_1^-(x),P_2^+(x),P_2^-(x)) = (\xi^\perp,\tilde{\xi}^\perp,\zeta^\perp,\tilde{\zeta}^\perp)$, where $\tilde{\xi}^\perp,\tilde{\zeta}^\perp\in\pi^\perp$ are the reflected points of $\xi^\perp$ and $\zeta^\perp$ with respect to $\xi^\perp+\zeta^\perp$. Note that the implicit support condition $r^\pi_\zeta - r^\pi_\xi \leqslant |\xi^\perp+\zeta^\perp|\leqslant r^\pi_\zeta+r^\pi_\xi$ in (4.2) always holds under the assumption $r^\pi_\xi\leqslant r^\pi_\zeta$. Observe that $h_{2,\xi}(\xi^\perp)=h_{1,\zeta}(\zeta^\perp)=0$, so manipulating the denominator one has

$$\big(h_\xi\mathrm{d}\sigma^\perp_{r^\pi_\xi} *^\perp h_\zeta\mathrm{d}\sigma^\perp_{r^\pi_\zeta}\big)(\xi^\perp+\zeta^\perp) = \left(\frac{|\xi^\perp-\tilde{\xi}^\perp||\zeta^\perp-\tilde{\zeta}^\perp|}{(r^\pi_\xi r^\pi_\zeta)^2-(\xi^\perp\cdot\zeta^\perp)^2}\right)^{1/2}\bar{g}_{2,\xi^\pi}(\tilde{\xi}^\perp)\bar{g}_{1,\zeta^\pi}(\tilde{\zeta}^\perp) \tag{4.3}$$

for all $\xi^\perp\in r^\pi_\xi\mathbb{S}^1$ and $\zeta^\perp\in r^\pi_\zeta\mathbb{S}^1$. Next note that $|\xi^\perp\wedge\zeta^\perp|^2 = (r^\pi_\xi r^\pi_\zeta)^2-(\xi^\perp\cdot\zeta^\perp)^2$, but also $|\xi^\perp\wedge\zeta^\perp|^2 = \frac{1}{4}|\xi^\perp+\zeta^\perp|^2|\xi^\perp-\tilde{\xi}^\perp||\zeta^\perp-\tilde{\zeta}^\perp|$, as the points satisfy the relation $\xi^\perp+\zeta^\perp=\tilde{\xi}^\perp+\tilde{\zeta}^\perp$. Then

$$\left(\frac{|\xi^\perp-\tilde{\xi}^\perp||\zeta^\perp-\tilde{\zeta}^\perp|}{(r^\pi_\xi r^\pi_\zeta)^2-(\xi^\perp\cdot\zeta^\perp)^2}\right)^{1/2} = \frac{2}{|\xi^\perp+\zeta^\perp|},$$

and combining the above estimates one obtains

$$\int_{\pi^\perp}|(-\Delta_y)^{1/4}T_{n-2,n}(\widehat{g_1\mathrm{d}\sigma^n}\overline{\widehat{g_2\mathrm{d}\sigma^n}})(\pi,y)|^2\,\mathrm{d}\lambda_{\pi^\perp}(y)$$
$$= (2\pi)^{2(n-1)}\int_{|\xi^\pi|\leqslant 1}\int_{|\zeta^\pi|\leqslant 1}\int_{r^\pi_\xi\mathbb{S}^1\times r^\pi_\zeta\mathbb{S}^1} K_{\pi,\mathbb{S}^{n-1}}(\xi,\zeta)g_{1,\xi^\pi}(\xi^\perp)g_{2,\zeta^\pi}(\zeta^\perp)\bar{g}_{2,\xi^\pi}(\tilde{\xi}^\perp)\bar{g}_{1,\zeta^\pi}(\tilde{\zeta}^\perp)\,\mathrm{d}\Sigma_\pi(\xi,\zeta)$$
$$= (2\pi)^{2(n-1)}\int_{(\mathbb{S}^{n-1})^2} K_{\pi,\mathbb{S}^{n-1}}(\xi,\zeta)g_1(\xi)g_2(\zeta)\bar{g}_2(\xi^\pi+\tilde{\xi}^\perp)\bar{g}_1(\zeta^\pi+\tilde{\zeta}^\perp)\,\mathrm{d}\sigma^n(\xi)\,\mathrm{d}\sigma^n(\zeta),$$

completing the proof of Theorem 1.1; above $\mathrm{d}\Sigma_\pi(\xi,\zeta) := \mathrm{d}\sigma^\perp_{r^\pi_\xi}(\xi^\perp)\,\mathrm{d}\sigma^\perp_{r^\pi_\zeta}(\zeta^\perp)\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi)$.

4.2. **Proof of Theorem 1.5.** Given $\omega \in \mathbb{S}^{d-1}_+$ and $\pi = \langle\omega\rangle^\perp \in \mathcal{G}_{d-1,d}$ write, for each $\xi \in \mathbb{R}^d$, $\xi = \xi^\pi + \xi^\omega\omega$, where $\xi^\omega = \xi\cdot\omega$ and let $m^\pi_\xi := \sqrt{m^2+|\xi^\pi|^2}$. Given $s \in \mathbb{R}$ and $x \in \pi$,

$$\begin{aligned}\widehat{g_j \mathrm{d}\sigma_{\mathbb{H}^d_m}}(x+s\omega,t) &= \int_{\mathbb{R}^d} e^{i(x+s\omega)\cdot\xi + it\sqrt{m^2+|\xi|^2}} f_j(\xi)\frac{\mathrm{d}\xi}{\sqrt{m^2+|\xi|^2}}\\ &= \int_\pi e^{ix\cdot\xi^\pi}\int_{\mathbb{R}} e^{is\xi^\omega + it\sqrt{m^2+|\xi^\pi|^2+|\xi^\omega|^2}} f_j(\xi^\pi+\xi^\omega\omega)\frac{\mathrm{d}\xi^\omega}{\sqrt{m^2+|\xi^\pi|^2+|\xi^\omega|^2}}\,\mathrm{d}\lambda_\pi(\xi^\pi)\\ &= \int_\pi e^{ix\cdot\xi^\pi}\mathcal{F}^2(g_{j,\xi^\pi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\xi}})(s,t)\,\mathrm{d}\lambda_\pi(\xi^\pi),\end{aligned}$$

where $f_{j,\xi^\pi}(\nu) := f_j(\xi^\pi+\nu\omega)$ for all $\nu \in \mathbb{R}$ and g_{j,ξ^π} denotes the lift of f_{j,ξ^π} to $\mathbb{H}^1_{m^\pi_\xi}$, and $\mathcal{F}^2$ denotes the 2-dimensional Fourier transform. Reasoning as in the proof of Theorem 1.1,

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_s^{1/2}\mathcal{R}\big(\widehat{g_1\mathrm{d}\sigma_{\mathbb{H}^d_m}}(\cdot,t)\overline{\widehat{g_2\mathrm{d}\sigma_{\mathbb{H}^d_m}}(\cdot,t)}\big)(\omega,s)|^2\,\mathrm{d}s\,\mathrm{d}t\\ &= (2\pi)^{2d}\int_\pi\int_\pi\int_{\mathbb{R}}\int_{\mathbb{R}} |v|(\tilde g_{1,\xi^\pi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\xi}} *^2 \bar g_{2,\xi^\pi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\xi}})(v,\tau)\overline{(\tilde g_{1,\zeta^\pi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\zeta}} *^2 \bar g_{2,\zeta^\pi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\zeta}})}(v,\tau),\end{aligned}$$

where the right-hand-side is integrated with respect to the measure $\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi)$. The innermost integral above in $\mathrm{d}v\,\mathrm{d}\tau$ equals

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_{1,\xi^\pi}(\xi^\omega)f_{2,\zeta^\pi}(\zeta^\omega)\big(H_{2,\xi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\xi}} * H_{1,\zeta}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\zeta}}\big)(P_{\xi,\zeta,\omega})\,\frac{\mathrm{d}\xi^\omega}{\phi_{m^\pi_\xi}(\xi^\omega)}\,\frac{\mathrm{d}\zeta^\omega}{\phi_{m^\pi_\zeta}(\zeta^\omega)},$$

where $H_{2,\xi}$ is the lift of $h_{2,\xi}(\eta) := \bar f_{2,\xi^\pi}(\eta)|\xi^\omega-\eta|^{1/2}$ to $\mathbb{H}^1_{m^\pi_\xi}$ (similarly for $H_{1,\zeta}$) and $P_{\xi,\zeta,\omega}$ denotes the point

$$P_{\xi,\zeta,\omega} := \big(\xi^\omega+\zeta^\omega, \phi_{m^\pi_\xi}(\xi^\omega)+\phi_{m^\pi_\zeta}(\zeta^\omega)\big).$$

Denoting by r_P the hyperbolic radius of $P_{\xi,\zeta,\omega}$, that is, $r_P^2 = (\phi_{m^\pi_\xi}(\xi^\omega)+\phi_{m^\pi_\zeta}(\zeta^\omega))^2 - (\xi^\omega+\zeta^\omega)^2$, Lemma 3.2 yields

$$\big(H_{2,\xi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\xi}} * H_{1,\zeta}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\zeta}}\big)(P_{\xi,\zeta,\omega}) = \frac{2h_{2,\xi}(\xi^\omega)h_{1,\zeta}(\zeta^\omega)+2h_{2,\xi}(\tilde\xi^\omega)h_{1,\zeta}(\tilde\zeta^\omega)}{\sqrt{r_P^4-2r_P^2\big((m^\pi_\xi)^2+(m^\pi_\zeta)^2\big)+\big((m^\pi_\xi)^2-(m^\pi_\zeta)^2\big)^2}} \tag{4.4}$$

where $(\tilde\xi^\omega,\phi_{m^\pi_\xi}(\tilde\xi^\omega)) = Q_1^-(P_{\xi,\zeta,\omega}) \in \mathbb{H}^1_{m^\pi_\xi}$ and $(\tilde\zeta^\omega,\phi_{m^\pi_\zeta}(\tilde\zeta^\omega)) = Q_2^+(P_{\xi,\zeta,\omega}) \in \mathbb{H}^1_{m^\pi_\zeta}$. After an algebraic manipulation and noting that $h_{1,\zeta}(\zeta^\omega) = h_{2,\xi}(\xi^\omega) = 0$, (4.4) becomes

$$\big(H_{2,\xi}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\xi}} * H_{1,\zeta}\mathrm{d}\sigma_{\mathbb{H}^1_{m^\pi_\zeta}}\big)(P_{\xi,\zeta,\omega}) = \frac{|\xi^\omega-\tilde\xi^\omega|^{1/2}\bar f_{2,\xi^\pi}(\tilde\xi^\omega)|\zeta^\omega-\tilde\zeta^\omega|^{1/2}\bar f_{1,\zeta^\pi}(\tilde\zeta^\omega)}{|\xi^\omega\phi_{m^\pi_\zeta}(\zeta^\omega)-\zeta^\omega\phi_{m^\pi_\xi}(\xi^\omega)|}.$$

Putting all the estimates together as in the proof of Theorem 1.1 concludes now the proof.

Remark 4.1. As the points in the pairs $(Q_1^+(0,z),Q_1^-(0,z))$ and $(Q_2^+(0,z),Q_2^-(0,z))$ are symmetric with respect to the vertical axis, it is a simple exercise to obtain an expression for $\tilde\xi^\omega$ and $\tilde\zeta^\omega$ via Lorentz transformations. Indeed, let γ_P denote the hyperbolic angle of $P_{\xi,\zeta,\omega}$ and let L_{γ_P} denote, as in (2.3), the Lorentz transformation such that $L_{\gamma_P}(P_{\xi,\zeta,\omega}) = (0,r_P)$. Then

$$\begin{aligned}Q_1^+(0,r_P) &= L_{\gamma_P}(\xi^\omega,\phi_{m^\pi_\xi}(\xi^\omega)) = (m^\pi_\xi\sinh(\gamma_\xi-\gamma_P), m^\pi_\xi\cosh(\gamma_\xi-\gamma_P))\\ Q_2^-(0,r_P) &= L_{\gamma_P}(\zeta^\omega,\phi_{m^\pi_\zeta}(\zeta^\omega)) = (m^\pi_\zeta\sinh(\gamma_\zeta-\gamma_P), m^\pi_\zeta\cosh(\gamma_\zeta-\gamma_P)).\end{aligned}$$

Clearly,

$$\begin{aligned}Q_1^-(0,r_P) &= (-m^\pi_\xi\sinh(\gamma_\xi-\gamma_P), m^\pi_\xi\cosh(\gamma_\xi-\gamma_P))\\ Q_2^+(0,r_P) &= (-m^\pi_\zeta\sinh(\gamma_\zeta-\gamma_P), m^\pi_\zeta\cosh(\gamma_\zeta-\gamma_P))\end{aligned}$$

and

$$\begin{aligned}Q_1^-(P_{\xi,\zeta,\omega}) &= L_{-\gamma_P}(Q_1^-(0,r_P)) = (m^\pi_\xi\sinh(2\gamma_P-\gamma_\xi), m^\pi_\xi\cosh(2\gamma_P-\gamma_\xi))\\ Q_2^+(P_{\xi,\zeta,\omega}) &= L_{-\gamma_P}(Q_2^+(0,r_P)) = (m^\pi_\zeta\sinh(2\gamma_P-\gamma_\zeta), m^\pi_\zeta\cosh(2\gamma_P-\gamma_\zeta)),\end{aligned}$$

so $\tilde{\xi}^\omega = m_\xi^\pi \sinh(2\gamma_P - \gamma_\xi)$ and $\tilde{\zeta}^\omega = m_\zeta^\pi \sinh(2\gamma_P - \gamma_\zeta)$. In particular, this allows one to rewrite the kernel as

$$K_{\omega,\mathbb{H}^d_m}(\xi,\zeta) = \frac{\big(m_\xi^\omega m_\zeta^\omega |\sinh(\gamma_\xi - \gamma_P)||\cosh(\gamma_\xi + \gamma_P)||\sinh(\gamma_\zeta - \gamma_P)||\cosh(\gamma_\zeta + \gamma_P)|\big)^{1/2}}{m_\xi^\omega m_\zeta^\omega |\sinh(\gamma_\xi - \gamma_\zeta)|}.$$

Remark 4.2. Note that

$$|\xi^\omega - \tilde{\xi}^\omega| = |(L_{\gamma_P}^{-1}(L_{\gamma_P}(\xi^\omega, \phi_{m_\xi^\pi}(\xi^\omega)) - L_{\gamma_P}((\tilde{\xi}^\omega, \phi_{m_\xi^\pi}(\tilde{\xi}^\omega))))_1| = |(L_{\gamma_P}^{-1}(2a,0))_1| = 2|a|\cosh(\gamma_P),$$

where $a := m_\xi^\pi \sinh(\gamma_\xi - \gamma_P)$. As $|\xi^\omega - \tilde{\xi}^\omega| = |\zeta^\omega - \tilde{\zeta}^\omega|$ and the denominator in (4.4) is easily seen to be equal to $|a|r_P$ (see the proof of Lemma 3.2), the kernel $K_{\omega,\mathbb{H}^d_m}$ may then be expressed as

$$K_{\omega,\mathbb{H}^d_m} = \frac{2(\phi_{m_\xi^\pi}(\xi^\omega) + \phi_{m_\zeta^\pi}(\zeta^\omega))}{(\phi_{m_\xi^\pi}(\xi^\omega) + \phi_{m_\zeta^\pi}(\zeta^\omega))^2 - (\xi^\omega + \zeta^\omega)^2}$$

after noting that $\cosh(\gamma_P) = (\phi_{m_\xi^\pi}(\xi^\omega) + \phi_{m_\zeta^\pi}(\zeta^\omega))/r_P$.

5. Corollaries

5.1. **Proof of Corollary 1.2.** By (1.9) it is clear that the expression

$$g_1(\xi)\bar{g}_2(\xi^\pi + \tilde{\xi}^\perp)g_2(\zeta)\bar{g}_1(\zeta^\pi + \tilde{\zeta}^\perp) \tag{5.1}$$

on its right-hand-side is real and positive. The identity (1.10) then yields that (5.1) equals to

$$\frac{1}{2}\big(|g_1(\xi)g_2(\zeta)|^2 + |g_2(\xi^\pi + \tilde{\xi}^\perp)g_1(\zeta^\pi + \tilde{\zeta}^\perp)|^2 - |g_1(\xi)g_2(\zeta) - g_2(\xi^\pi + \tilde{\xi}^\perp)g_1(\zeta^\pi + \tilde{\zeta}^\perp)|^2\big).$$

The negative term above immediately gives raise to the expression $I_{\pi,\mathbb{S}^{n-1}}(g_1,g_2)$, whilst the positive terms amount to the same expression over the integral sign, finishing the proof.

Observe that the resulting sharp inequality

$$\int_{\pi^\perp}\Big|(-\Delta_y)^{1/4}T_{n-2,n}(\widehat{g_1\mathrm{d}\sigma^n}\overline{\widehat{g_2\mathrm{d}\sigma^n}})(\pi,y)\Big|^2\mathrm{d}y \leqslant \mathbf{C}_{\mathbb{S}^{n-1}}\int_{(\mathbb{S}^{n-1})^2} K_{\pi,\mathbb{S}^{n-1}}(\xi,\zeta)|g_1(\xi)|^2|g_2(\zeta)|^2\mathrm{d}\sigma^n(\xi)\,\mathrm{d}\sigma^n(\zeta) \tag{5.2}$$

obtained from dropping the negative term in (1.11) may be deduced more directly via a simple application of the Cauchy–Schwarz inequality. Note that (4.1) is a positive quantity, so in particular equals to its modulus. By the triangle inequality, the left-hand-side of (1.9) is controlled by

$$\int_{|\xi^\pi|\leqslant 1}\int_{|\zeta^\pi|\leqslant 1}\int_{(r_\xi^\pi\mathbb{S}^1)^2\times(r_\zeta^\pi\mathbb{S}^1)^2} |\xi^\perp - \eta^\perp|^{1/2}|\zeta^\perp - \mu^\perp|^{1/2}|g_{1,\xi^\pi}(\xi^\perp)||g_{2,\xi^\pi}(\eta^\perp)||g_{2,\zeta^\pi}(\zeta^\perp)||g_{1,\zeta^\pi}(\mu^\perp)| \\ \mathrm{d}\Sigma^\perp_{\xi^\pi,\zeta^\pi}(\xi^\perp,\eta^\perp,\zeta^\perp,\mu^\perp)\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi). \tag{5.3}$$

Applying the Cauchy–Schwarz inequality with respect to the measure $\mathrm{d}\Sigma^\perp_{\xi^\pi,\zeta^\pi}\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi)$, the above is further controlled by

$$\int_{|\xi^\pi|\leqslant 1}\int_{|\zeta^\pi|\leqslant 1}\int_{r_\xi^\pi\mathbb{S}^1\times r_\zeta^\pi\mathbb{S}^1} |g_{1,\xi^\pi}(\xi^\perp)|^2|g_{2,\zeta^\pi}(\zeta^\perp)|^2\big(h_\xi\mathrm{d}\sigma^\perp_{r_\xi^\pi} *^\perp h_\zeta\mathrm{d}\sigma^\perp_{r_\zeta^\pi}\big)(\xi^\perp + \zeta^\perp)\,\mathrm{d}\sigma^\perp_{r_\xi^\pi}(\xi^\perp)\,\mathrm{d}\Sigma_\pi(\xi,\zeta)$$

where $h_\xi(\eta^\perp) := |\xi^\perp - \eta^\perp|^{1/2}$ and similarly for h_ζ; above $\mathrm{d}\Sigma_\pi(\xi,\zeta) := \mathrm{d}\sigma^\perp_{r_\zeta^\pi}(\zeta^\perp)\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi)$. Evaluation of the innermost convolution as in (4.3) yields then the desired inequality (5.2).

5.2. **Proof of Corollary 1.3.** Given $\pi \in \mathcal{G}_{n-2,n}$, Plancherel's theorem and the relation (2.1) yields

$$\int_{\pi^\perp}\Big|(-\Delta_y)^{1/4}T_{n-2,n}h(\pi,y)\Big|^2\mathrm{d}\lambda_{\pi^\perp}(y) = (2\pi)^{-2}\int_{\pi^\perp}|\xi^\perp||\widehat{h}(\xi^\perp)|^2\mathrm{d}\lambda_{\pi^\perp}(\xi^\perp).$$

Averaging over all $\pi \in \mathcal{G}_{n-2,n}$, and using (2.2) and polar coordinates

$$\begin{aligned}\int_{\mathcal{G}_{n-2,n}} \int_{\pi^\perp} &|\xi^\perp|^{3-n}|\widehat{h}(\xi^\perp)|^2|\xi^\perp|^{n-2}\mathrm{d}\lambda_{\pi^\perp}(\xi^\perp)\,\mathrm{d}\mu_{\mathcal{G}}(\pi)\\ &= \int_{\mathcal{G}_{n-2,n}}\int_0^\infty\int_{\mathbb{S}^{n-1}\cap\pi^\perp} r^{3-n}|\widehat{h}(r\omega)|^2 r^{n-2} r\,\mathrm{d}r\,\mathrm{d}\sigma^{n,\perp}(\omega)\,\mathrm{d}\mu_{\mathcal{G}}(\pi)\\ &= |\mathcal{G}_{1,n-1}|\int_0^\infty\int_{\mathbb{S}^{n-1}} r^{3-n}|\widehat{h}(r\omega)|^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma^n(\omega)\\ &= |\mathcal{G}_{1,n-1}|(2\pi)^n\int_{\mathbb{R}^n}||\nabla|^{\frac{3-n}{2}}h(x)|^2\,\mathrm{d}x,\end{aligned}$$

which completes the proof on taking $h = \widehat{g_1\mathrm{d}\sigma^n}\,\overline{\widehat{g_2\mathrm{d}\sigma^n}}$.

5.3. **Proof of Corollary 1.4.** Recall $\xi = \xi^\pi + \xi^\perp$. For $n = 3$, $\pi = \langle\omega\rangle$, where $\omega \in \mathcal{G}_{1,3} \simeq \mathbb{S}^2_+$. Then $\xi^\pi = (\xi\cdot\omega)\,\omega$ and $\xi^\perp = \xi - (\xi\cdot\omega)\,\omega$, so

$$|\xi^\perp + \zeta^\perp|^2 = |\xi+\zeta|^2 + |(\xi+\zeta)\cdot\omega|^2 - 2((\xi+\zeta)\cdot\omega)^2 = |\xi+\zeta|^2\Big(1 - \Big(\frac{(\xi+\zeta)}{|\xi+\zeta|}\cdot\omega\Big)^2\Big).$$

Noting that $|\mathcal{G}_{1,2}| = \pi$,

$$K_{\mathbb{S}^{n-1}}(\xi,\zeta) = \frac{2}{|\mathcal{G}_{1,2}|}\int_{\mathbb{S}^2_+}\frac{\mathrm{d}\sigma^3_+(\omega)}{|\xi^\perp+\zeta^\perp|} = \frac{2\pi}{\pi|\xi+\zeta|}\int_{-1}^1\frac{\mathrm{d}u}{\sqrt{1-u^2}} = \frac{2\pi}{|\xi+\zeta|}.$$

Thus

$$\|\widehat{g\mathrm{d}\sigma^3}\|^4_{L^4(\mathbb{R}^3)} \leqslant (2\pi)^4\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\frac{1}{|\xi+\zeta|}|g(\xi)|^2|g(\zeta)|^2\,\mathrm{d}\sigma^3(\xi)\,\mathrm{d}\sigma^3(\zeta)$$

and the desired sharp Stein–Tomas inequality for the sphere follows from the following fact due to Foschi [18],

$$\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\frac{1}{|\xi+\zeta|}|g(\xi)|^2|g(\zeta)|^2\,\mathrm{d}\sigma^3(\xi)\,\mathrm{d}\sigma^3(\zeta) = \|g\|^4_{L^2(\mathbb{S}^2)}, \tag{5.4}$$

which holds for g antipodally symmetric. The reduction to the antipodally symmetric may be done as in [18], using the Cauchy–Schwarz inequality for real numbers

$$ac + bd \leqslant \sqrt{a^2+b^2}\sqrt{c^2+d^2}. \tag{5.5}$$

Indeed, note that in the proof of (5.2) via the Cauchy–Schwarz inequality given in Section 5.1, one may replace $|g_{\xi^\pi}(\xi^\perp)||g_{\xi^\pi}(\eta^\perp)|$ in the innermost integral in (5.3) by

$$\frac{|g_{\xi^\pi}(\xi^\perp)||g_{\xi^\pi}(\eta^\perp)| + |g_{\xi^\pi}(-\xi^\perp)||g_{\xi^\pi}(-\eta^\perp)|}{2},$$

and using (5.5) this is bounded by $|g^\#_{\xi^\pi}(\xi^\perp)||g^\#_{\xi^\pi}(\eta^\perp)|$, where for any function h, the function $h^\#$ denotes $h^\#(\xi) := \sqrt{(h(\xi)+h(-\xi))/2}$, which is antipodally symmetric. One can argue similarly to replace $|g_{\zeta^\pi}(\zeta^\perp)||g_{\zeta^\pi}(\mu^\perp)|$ by $|g^\#_{\zeta^\pi}(\zeta^\perp)||g^\#_{\zeta^\pi}(\mu^\perp)|$. Thus, the right hand side in (5.2) is replaced by

$$\mathbf{C}_{\mathbb{S}^{n-1}}\int_{|\xi^\pi|\leqslant 1}\int_{|\zeta^\pi|\leqslant 1}\int_{r^\pi_\xi\mathbb{S}^1\times r^\pi_\zeta\mathbb{S}^1}\frac{2}{|\xi^\perp+\zeta^\perp|}|g^\#_{\xi^\pi}(\xi^\perp)|^2|g^\#_{\zeta^\pi}(\zeta^\perp)|^2\,\mathrm{d}\sigma^1_{r_{\xi^\pi}}(\xi^\perp)\,\mathrm{d}\sigma^1_{r_{\zeta^\pi}}(\zeta^\perp)\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi). \tag{5.6}$$

One desires, however, to have $g^\#$ rather than $g^\#_{\xi^\pi}$ and $g^\#_{\zeta^\pi}$. By a change of variables, the integrand $4|g^\#_{\xi^\pi}(\xi^\perp)|^2|g^\#_{\zeta^\pi}(\zeta^\perp)|^2$ may be further replaced by

$$\big(|g^\#_{\xi^\pi}(\xi^\perp)|^2 + |g^\#_{-\xi^\pi}(\xi^\perp)|^2\big)\big(|g^\#_{\zeta^\pi}(\zeta^\perp)|^2 + |g^\#_{-\zeta^\pi}(\zeta^\perp)|^2\big),$$

which equals

$$|g^\#(\xi)|^2|g^\#(\zeta)|^2 + |g^\#(\xi)|^2|g^\#(\zeta^\perp-\zeta^\pi)|^2 + |g^\#(\xi^\perp-\xi^\pi)|^2|g^\#(\zeta)|^2 + |g^\#(\xi^\perp-\xi^\pi)|^2|g^\#(\zeta^\perp-\zeta^\pi)|^2.$$

A further change of variables in each of the terms allows to see that (5.6) equals

$$\mathbf{C}_{\mathbb{S}^{n-1}} \int_{(\mathbb{S}^{n-1})^2} K_{\pi,\mathbb{S}^{n-1}}(\xi,\zeta)|g^{\#}(\xi)|^2|g^{\#}(\zeta)|^2 \,\mathrm{d}\sigma^n(\xi)\,\mathrm{d}\sigma^n(\zeta),$$

as desired for the later application of Foschi's identity (5.4) on antipodally symmetric functions.

5.4. **Proof of Corollary 1.6.** This follows the same argument as that of Corollary 1.2.

5.5. **Proof of Corollary 1.7.** The proof follows from the same argument as in §5.2. Indeed, the elementary argument therein yields the relation

$$\|(-\Delta)^{\ell/2} f\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^{-(d-1)} \|\partial_s^{\frac{d-1}{2}+\ell} \mathcal{R} f\|^2_{L^2_{\omega,s}(\mathbb{S}^{d-1}_+,\mathbb{R})},$$

from which Corollary 1.7 follows from taking $\ell = (2-d)/2$ after averaging over $\omega \in \mathbb{S}^{d-1}_+$; note that ω in the Radon transform $\mathcal{R}$ only runs over $\mathbb{S}^{d-1}_+ \simeq \mathcal{G}_{d-1,d}$.

6. The bilinear identity (1.8) for paraboloids revisited

The purpose of this final section is to provide an alternative proof for the identity (1.8) via Fourier analysis. The proof follows the same scheme as those of Theorems 1.1 and 1.5 with a little twist, which is available when taking one full derivative in the s-variable in the case of paraboloids.

To see this, let $\mathbb{P}^d_a := \{(\xi, |\xi|^2 + a) : \xi \in \mathbb{R}^d\}$ denote the paraboloid in $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ with tangent plane $t = a$ at its vertex; if $a = 0$ we simply denote it by $\mathbb{P}^d$. Let $\mathrm{d}\sigma_{\mathbb{P}^d_a}$ denote the parametrised measure on $\mathbb{P}^d_a$, which satisfies $\widehat{g\mathrm{d}\sigma_{\mathbb{P}^d_a}}(x,t) = Ef(x,t)$ where E is the extension operator associated to $\phi(\xi) := |\xi|^2 + a$ and g is the lift of the function $f : \mathbb{R}^d \to \mathbb{C}$ to $\mathbb{P}^d_a$.

Given $\omega \in \mathbb{S}^{d-1}_+$ and $\pi = \langle\omega\rangle^\perp \in \mathcal{G}_{d-1,d}$ write, for each $\xi \in \mathbb{R}^d$, $\xi = \xi^\pi + \xi^\omega\omega$, where $\xi^\omega = \xi\cdot\omega$. Given $s \in \mathbb{R}$ and $x \in \pi$,

$$\begin{aligned}
\widehat{g_j\mathrm{d}\sigma_{\mathbb{P}^d}}(x+s\omega,t) &= \int_{\mathbb{R}^d} e^{i(x+s\omega)\cdot\xi + it|\xi|^2} f_j(\xi)\,\mathrm{d}\xi \\
&= \int_\pi e^{ix\cdot\xi^\pi} \int_{\mathbb{R}} e^{is\xi^\omega + it|\xi^\pi|^2 + it|\xi^\omega|^2} f_j(\xi^\pi + \xi^\omega\omega)\,\mathrm{d}\xi^\omega\,\mathrm{d}\lambda_\pi(\xi^\pi) \\
&= \int_\pi e^{ix\cdot\xi^\pi} \mathcal{F}^2(g_{j,\xi^\pi}\mathrm{d}\sigma_{\mathbb{P}^2_{|\xi^\pi|^2}})(s,t)\,\mathrm{d}\lambda_\pi(\xi^\pi),
\end{aligned}$$

where $f_{j,\xi^\pi}(\nu) := f_j(\xi^\pi + \nu\omega)$, $\mathcal{F}^2$ denotes the 2-dimensional Fourier transform and g_{j,ξ^π} is the lift of f_{j,ξ^π} to $\mathbb{P}^2_{|\xi^\pi|^2}$. Reasoning as in the proof of Theorem 1.1,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_s \mathcal{R}\big(\widehat{g_1\mathrm{d}\sigma_{\mathbb{P}^d}}(\cdot,t)\overline{\widehat{g_2\mathrm{d}\sigma_{\mathbb{P}^d}}}(\cdot,t)\big)(\omega,s)|^2\,\mathrm{d}s\,\mathrm{d}t \\
&\quad= (2\pi)^{2d}\int_\pi\int_\pi\int_{\mathbb{R}^4} |\xi^\omega - \eta^\omega||\zeta^\omega - \mu^\omega| f_{1,\xi^\pi}(\xi^\omega)\bar{f}_{2,\xi^\pi}(\eta^\omega)\bar{f}_{1,\zeta^\pi}(\mu^\omega) f_{2,\zeta^\pi}(\zeta^\omega)\,\mathrm{d}\Sigma_{\xi^\pi,\zeta^\pi}(\xi^\omega,\eta^\omega,\mu^\omega,\zeta^\omega)
\end{aligned}$$

where

$$\mathrm{d}\Sigma_{\xi^\pi,\zeta^\pi}(\xi^\omega,\eta^\omega,\mu^\omega,\zeta^\omega) := \delta(\xi^\omega - \eta^\omega + \zeta^\omega - \mu^\omega)\delta((\xi^\omega)^2 - (\eta^\omega)^2 + (\zeta^\omega)^2 - (\mu^\omega)^2)\mathrm{d}\xi^\omega\mathrm{d}\eta^\omega\mathrm{d}\mu^\omega\,\mathrm{d}\zeta^\omega\mathrm{d}\lambda_\pi(\xi^\pi)\mathrm{d}\lambda_\pi(\zeta^\pi).$$

Arguing similarly,

$$J_\omega(\widehat{g_1\mathrm{d}\sigma_{\mathbb{P}^d}}, \widehat{g_2\mathrm{d}\sigma_{\mathbb{P}^d}}) = (2\pi)^{2d}\int_\pi\int_\pi\int_{\mathbb{R}^4} (\zeta^\omega\mu^\omega - \zeta^\omega\eta^\omega - \xi^\omega\mu^\omega + \xi^\omega\eta^\omega) f_{1,\xi^\pi}(\xi^\omega)\bar{f}_{2,\xi^\pi}(\eta^\omega)\bar{f}_{1,\zeta^\pi}(\mu^\omega) f_{2,\zeta^\pi}(\zeta^\omega)$$

with respect to the measure $\mathrm{d}\Sigma_{\xi^\pi,\zeta^\pi}(\xi^\omega,\eta^\omega,\mu^\omega,\zeta^\omega)$, where $J_\omega(G_1,G_2)$ is the bilinearisation of $J_\omega(u)$; namely the integrand is replaced by

$$\begin{aligned}
&G_1(x+s\omega,t)\partial_s G_2(y+s\omega,t)\big(\partial_s\bar{G}_1(y+s\omega,t)\bar{G}_2(x+s\omega,t) - \bar{G}_1(y+s\omega,t)\partial_s\bar{G}_2(x+s\omega,t)\big) \\
&\quad - G_2(y+s\omega,t)\partial_s G_1(x+s\omega,t)\big(\bar{G}_2(x+s\omega,t)\partial_s\bar{G}_1(y+s\omega,t) - \partial_s\bar{G}_2(x+s\omega,t)\bar{G}_1(y+s\omega,t)\big).
\end{aligned}$$

Noting that

$$|\xi^\omega - \eta^\omega||\zeta^\omega - \mu^\omega| + (\zeta^\omega\mu^\omega - \zeta^\omega\eta^\omega - \xi^\omega\mu^\omega + \xi^\omega\eta^\omega) = |\xi^\omega - \mu^\omega|^2 \tag{6.1}$$

if $(\xi^\omega, \eta^\omega, \mu^\omega, \zeta^\omega) \in \text{supp}\,(\mathrm{d}\Sigma_{\xi^\pi,\zeta^\pi})$, one can combine the two terms above to obtain

$$\begin{aligned}(6.2)\quad &\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_s \mathcal{R}\big(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{P}^d}}(\cdot,t)\overline{\widehat{g_2 \mathrm{d}\sigma_{\mathbb{P}^d}}(\cdot,t)}\big)(s,\omega)|^2 \,\mathrm{d}s\,\mathrm{d}t + J_\omega(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{P}^d}}, \widehat{g_2 \mathrm{d}\sigma_{\mathbb{P}^d}}) \\ &= (2\pi)^{2d}\int_\pi\int_\pi\int_{\mathbb{R}^4} |\xi^\omega - \mu^\omega|^2 f_{1,\xi^\pi}(\xi^\omega)\bar{f}_{2,\xi^\pi}(\eta^\omega)\bar{f}_{1,\zeta^\pi}(\mu^\omega) f_{2,\zeta^\perp}(\zeta^\omega)\,\mathrm{d}\Sigma_{\xi^\pi,\zeta^\pi}(\xi^\omega,\eta^\omega,\mu^\omega,\zeta^\omega).\end{aligned}$$

For fixed ξ^ω and μ^ω, the only solution for the equations in the δ function is $\eta^\omega = \xi^\omega$ and $\zeta^\omega = \mu^\omega$. Thus, the right-hand-side above equals

$$\frac{(2\pi)^{2d}}{2}\int_{\xi^\pi}\int_{\zeta^\pi}\int_{\mathbb{R}^2} |\xi^\omega - \mu^\omega| f_{1,\xi^\pi}(\xi^\omega)\bar{f}_{2,\xi^\pi}(\xi^\omega)\bar{f}_{1,\zeta^\pi}(\mu^\omega) f_{2,\zeta^\pi}(\mu^\omega)\,\mathrm{d}\xi^\omega\,\mathrm{d}\mu^\omega\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi)$$

and if $f_1 = f_2$,

$$\frac{(2\pi)^{2d}}{2}\int_\pi\int_\pi\int_{\mathbb{R}^2} |\xi^\omega - \mu^\omega||f_{\xi^\pi}(\xi^\omega)|^2|f_{\zeta^\pi}(\mu^\omega)|^2\,\mathrm{d}\xi^\omega\,\mathrm{d}\mu^\omega\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi)$$

which of course is

$$\frac{(2\pi)^{2d}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |(\xi-\eta)\cdot\omega||f(\xi)|^2|f(\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

In the language of the Schrödinger equation, $u = E\widecheck{u_0}$, so the right hand side is

$$\frac{\pi}{(2\pi)^{d+1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |(\xi-\eta)\cdot\omega||\widehat{u_0}(\xi)|^2|\widehat{u_0}(\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and one obtains the desired identity (1.8).

Averaging over all $\omega \in \mathbb{S}^{d-1}_+$ after dropping the term $J_\omega(u)$ from the obtained identity and noting that

$$\int_{\mathbb{S}^{d-1}} |(\xi-\eta)\cdot\omega|\,\mathrm{d}\sigma^n(\omega) = 2|\xi-\eta|\int_0^1 u(1-u^2)^{\frac{d-3}{2}}\,\mathrm{d}u = \frac{2|\xi-\eta|\pi^{\frac{d-1}{2}}}{\Gamma((d+1)/2)},$$

one has

$$(6.3)\quad \|(-\Delta_x)^{\frac{3-d}{4}}(|u|^2)\|^2_{L^2_{x,t}(\mathbb{R}^d\times\mathbb{R})} \leqslant (2\pi)^{1-d}\frac{\pi}{(2\pi)^{d+1}}\frac{\pi^{\frac{d-1}{2}}}{\Gamma((d+1)/2)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\xi-\eta||\widehat{u_0}(\xi)|^2|\widehat{u_0}(\eta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$

and the constant simplifies as $\mathbf{PV}(d) := \frac{2^{-3d}\pi^{\frac{1-5d}{2}}}{\Gamma(\frac{d+1}{2})}$; this inequality was also obtained in [3] in a more direct way.

Remark 6.1. The honest analogue of Theorems 1.1 and 1.5 in the context of paraboloids is given by the bilinear identity

$$\begin{aligned}(6.4)\quad &\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_s^{1/2}\mathcal{R}\big(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{P}^d}}(\cdot,t)\overline{\widehat{g_2 \mathrm{d}\sigma_{\mathbb{P}^d}}(\cdot,t)}\big)(s,\omega)|^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{(2\pi)^{2d}}{2}\int_\pi\int_\pi\int_{\mathbb{R}^2} f_{1,\xi^\pi}(\xi^\omega)\bar{f}_{2,\xi^\pi}(\zeta^\omega)\bar{f}_{1,\zeta^\pi}(\xi^\omega) f_{2,\zeta^\pi}(\zeta^\omega)\,\mathrm{d}\xi^\omega\,\mathrm{d}\zeta^\omega\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi).\end{aligned}$$

In contrast to the previous case, one solves here the equations in the δ functions in terms of ξ^ω and ζ^ω; the solution in terms of ξ^ω and μ^ω is now degenerate in terms of the weight $|\xi^\omega - \eta^\omega|^{1/2}|\zeta^\omega - \mu^\omega|^{1/2}$, which vanishes in this case. Note that, in (6.2), the fact of taking one full derivative with respect to s and adding the term $J_\omega(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{P}^d}}, \widehat{g_2 \mathrm{d}\sigma_{\mathbb{P}^d}})$ had the effect of replacing the weight $|\xi^\omega - \eta^\omega||\zeta^\omega - \mu^\omega|$ by $|\xi^\omega - \mu^\omega|^2$ thanks to the algebraic identity (6.1), allowing to solve in those variables.

As in the case of spheres and hyperboloids, the identity for complex numbers (1.10) allows to rewrite (6.4) as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\partial_s^{1/2}\mathcal{R}\big(\widehat{g_1 \mathrm{d}\sigma_{\mathbb{P}^d}}(\cdot,t)\overline{\widehat{g_2 \mathrm{d}\sigma_{\mathbb{P}^d}}(\cdot,t)}\big)(s,\omega)|^2 \,\mathrm{d}s\,\mathrm{d}t$$
$$= \frac{(2\pi)^{2d}}{2}\int_\pi\int_\pi\int_{\mathbb{R}^2} |f_{1,\xi^\pi}(\xi^\omega)|^2|f_{2,\zeta^\pi}(\zeta^\omega)|^2 \,\mathrm{d}\xi^\omega\,\mathrm{d}\zeta^\omega\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi) - I_\omega(f_1,f_2)$$

where

$$I_\omega(f_1,f_2) := \frac{(2\pi)^{2d}}{4}\int_\pi\int_\pi\int_{\mathbb{R}^2} |f_{1,\xi^\pi}(\xi^\omega)f_{2,\zeta^\pi}(\zeta^\omega) - f_{1,\zeta^\pi}(\xi^\omega)f_{2,\xi^\pi}(\zeta^\omega)|^2 \,\mathrm{d}\xi^\omega\,\mathrm{d}\zeta^\omega\,\mathrm{d}\lambda_\pi(\xi^\pi)\,\mathrm{d}\lambda_\pi(\zeta^\pi).$$

Unlike $J_\omega(f)$, the term $I_\omega(f,f)$ does not have an obvious closed expression in terms of physical variables. Setting $f_1 = f_2$ and averaging over all $\omega \in \mathbb{S}^{d-1}_+$ after dropping $I_\omega(f,f)$ one obtains

$$\|(-\Delta_x)^{\frac{2-d}{4}}(|u|^2)\|^2_{L^2_{x,t}(\mathbb{R}^d\times\mathbb{R})} \leqslant (2\pi)^{1-d}\frac{(2\pi)^{2d}}{2}\frac{|\mathbb{S}^{d-1}|}{2}\|\widetilde{u_0}\|^4_{L^2(\mathbb{R}^d)} = \frac{2^{-d}\pi^{\frac{2-d}{2}}}{\Gamma(d/2)}\|u_0\|^4_{L^2(\mathbb{R}^d)},$$

which is the Ozawa–Tsutsumi estimate (1.7); note that for $d = 2$ this amounts to the $L^4(\mathbb{R}^{2+1})$ Strichartz estimate. The interested reader should look at the work of Bennett, Bez, Jeavons and Pattakos [3] for a unified treatment of the Ozawa–Tsutsumi estimates (1.7), the inequalities deduced from (6.3), and a more general case with an arbitrary number of derivatives on the left-hand-side of such inequalities.

References

[1] J. Bennett. Aspects of multilinear harmonic analysis related to transversality. In *Harmonic analysis and partial differential equations*, volume 612 of *Contemp. Math.*, pages 1–28. Amer. Math. Soc., Providence, RI, 2014.

[2] J. Bennett, N. Bez, T. C. Flock, S. Gutiérrez, and M. Iliopoulou. A sharp k-plane Strichartz inequality for the Schrödinger equation. *Trans. Amer. Math. Soc.*, 370(8):5617–5633, 2018.

[3] J. Bennett, N. Bez, C. Jeavons, and N. Pattakos. On sharp bilinear Strichartz estimates of Ozawa-Tsutsumi type. *J. Math. Soc. Japan*, 69(2):459–476, 2017.

[4] J. Bennett, A. Carbery, and T. Tao. On the multilinear restriction and Kakeya conjectures. *Acta Math.*, 196(2):261–302, 2006.

[5] J. Bennett and M. Iliopoulou. A multilinear Fourier extension identity on $\mathbb{R}^n$. *Math. Res. Lett.*, 25(4):1089–1108, 2018.

[6] J. Bennett and S. Nakamura. Tomography bounds for the Fourier extension operator. *In preparation.*

[7] N. Bez and K. M. Rogers. A sharp Strichartz estimate for the wave equation with data in the energy space. *J. Eur. Math. Soc. (JEMS)*, 15(3):805–823, 2013.

[8] J. Bourgain. Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity. *Internat. Math. Res. Notices*, (5):253–283, 1998.

[9] E. Carneiro. A sharp inequality for the Strichartz norm. *Int. Math. Res. Not. IMRN*, (16):3127–3145, 2009.

[10] E. Carneiro and D. Oliveira e Silva. Some sharp restriction inequalities on the sphere. *Int. Math. Res. Not. IMRN*, (17):8233–8267, 2015.

[11] E. Carneiro, D. Oliveira e Silva, and M. Sousa. Extremizers for Fourier restriction on hyperboloids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(2):389–415, 2019.

[12] M. Christ and S. Shao. Existence of extremals for a Fourier restriction inequality. *Anal. PDE*, 5(2):261–312, 2012.

[13] M. Christ and S. Shao. On the extremizers of an adjoint Fourier restriction inequality. *Adv. Math.*, 230(3):957–977, 2012.

[14] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$. *Comm. Pure Appl. Math.*, 57(8):987–1014, 2004.

[15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math. (2)*, 167(3):767–865, 2008.

[16] C. Fefferman. Inequalities for strongly singular convolution operators. *Acta Math.*, 124:9–36, 1970.

[17] D. Foschi. Maximizers for the Strichartz inequality. *J. Eur. Math. Soc. (JEMS)*, 9(4):739–774, 2007.

[18] D. Foschi. Global maximizers for the sphere adjoint Fourier restriction inequality. *J. Funct. Anal.*, 268(3):690–702, 2015.

[19] D. Foschi and S. Klainerman. Bilinear space-time estimates for homogeneous wave equations. *Ann. Sci. École Norm. Sup. (4)*, 33(2):211–274, 2000.

[20] D. Foschi and D. Oliveira e Silva. Some recent progress on sharp Fourier restriction theory. *Anal. Math.*, 43(2):241–265, 2017.

[21] S. Helgason. *The Radon transform*, volume 5 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, second edition, 1999.
[22] C. Jeavons. A sharp bilinear estimate for the Klein-Gordon equation in arbitrary space-time dimensions. *Differential Integral Equations*, 27(1-2):137–156, 2014.
[23] K. Nakanishi. Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2. *J. Funct. Anal.*, 169(1):201–225, 1999.
[24] T. Ozawa and K. M. Rogers. A sharp bilinear estimate for the Klein-Gordon equation in $\mathbb{R}^{1+1}$. *Int. Math. Res. Not. IMRN*, (5):1367–1378, 2014.
[25] T. Ozawa and Y. Tsutsumi. Space-time estimates for null gauge forms and nonlinear Schrödinger equations. *Differential Integral Equations*, 11(2):201–222, 1998.
[26] F. Planchon and L. Vega. Bilinear virial identities and applications. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(2):261–290, 2009.
[27] R. Quilodrán. Nonexistence of extremals for the adjoint restriction inequality on the hyperboloid. *J. Anal. Math.*, 125:37–70, 2015.
[28] R. S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.
[29] T. Tao. A sharp bilinear restrictions estimate for paraboloids. *Geom. Funct. Anal.*, 13(6):1359–1384, 2003.
[30] T. Tao. Some recent progress on the restriction conjecture. In *Fourier analysis and convexity*, Appl. Numer. Harmon. Anal., pages 217–243. Birkhäuser Boston, Boston, MA, 2004.
[31] P. A. Tomas. A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.*, 81:477–478, 1975.
[32] T. Wolff. A sharp bilinear cone restriction estimate. *Ann. of Math. (2)*, 153(3):661–698, 2001.
[33] A. Zygmund. On Fourier coefficients and transforms of functions of two variables. *Studia Math.*, 50:189–201, 1974.

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