ON LIPSCHITZ RIGIDITY OF COMPLEX ANALYTIC SETS

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Abstract. We prove that any complex analytic set in $\mathbb{C}^n$ which is Lipschitz normally embedded at infinity and has tangent cone at infinity that is a linear subspace of $\mathbb{C}^n$ must be an affine linear subspace of $\mathbb{C}^n$ itself. No restrictions on the singular set, dimension nor codimension are required. In particular, any complex algebraic set in $\mathbb{C}^n$ which is Lipschitz regular at infinity is an affine linear subspace.

1. Introduction

Local Lipschitz geometry of complex algebraic sets has been intensively studied in the last years. In the recent works on this subject [1] and [11], it was showed an important rigidity result of such a local geometric structure of algebraic sets. Indeed, it was proved that Lipschitz regular complex algebraic germs of sets in $\mathbb{C}^n$, that is, germs of complex algebraic sets in $\mathbb{C}^n$ which are bi-Lipschitz homeomorphic to the germ of $\mathbb{R}^d$ at some point, are analytically smooth (see Theorem 3.1 in [1] and Theorem 4.2 in [11]). From another way, looking to scrutinize global Lipschitz geometry of such sets in some sense, we arrived on subsets of $\mathbb{C}^n$ that, outside a compact subset, are bi-Lipschitz homeomorphic to the complement of a Euclidean ball in some $\mathbb{R}^d$; they are called *Lipschitz regular at infinity* (see Definition 2.3).

A path connected subset X of $\mathbb{C}^n$ is called *Lipschitz normally embedded* if there exists a positive real number λ such that

$$d_X(x,y) \leq \lambda \|x-y\|$$

for all $x,y \in X$, where $d_X(x,y)$ (inner distance on X between x and y) is the infimum of the $length(\gamma)$; γ varies on the set of paths on X connecting x to y. We say that X is *Lipschitz normally embedded at infinity* if there exists a compact subset $K \subset X$ such that $X \setminus K$ is Lipschitz normally embedded.

The main result of this paper is the following.

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Theorem 1.1. *Let $X \subset \mathbb{C}^n$ be a closed and pure d-dimensional analytic subset. Suppose X has a unique tangent cone at infinity and this cone is a d-dimensional complex linear subspace of $\mathbb{C}^n$. If X is Lipschitz normally embedded at infinity, then X is an affine linear subspace of $\mathbb{C}^n$.*

We are going to address the notion of tangent cone at infinity and Lipschitz normal embedding in the subsections 2.2 and 2.1 respectively. As a consequence of the above theorem, we also prove that complex algebraic subsets of $\mathbb{C}^n$ which are Lipschitz regular at infinity are affine linear subspaces of $\mathbb{C}^n$. Finally, as a consequence of the above theorem as well, we prove that pure dimension complex algebraic subsets of $\mathbb{C}^n$ which are Lipschitz regular at infinity must be affine linear subspaces of $\mathbb{C}^n$. Let us compare this last result with a celebrated theorem due to Bombieri, De Giorgi and Miranda which says that entire positive minimal graph of functions in Euclidean spaces must be horizontal affine hyperplane; notice that, in our result, we deal with complex analytic sets in $\mathbb{C}^n$ which are not necessarily graph of smooth functions, a priori, they are not supposed even smooth.

We organized the paper in the following way. In Section 2 we present definitions and basic properties on Lipschitz geometry at infinity, more precisely, in Subsection 2.1 we repeat the definition Lipschitz normal embedding at infinity with some examples and we set some basic results; in Subsection 2.2 we present the notion of tangent cones at infinity and prove some basic results as well. Finally, Section 3 is devoted to prove Theorem 1.1.

2. Preliminaries

Let us start this section by reminding the definition of Lipschitz functions, where all the subsets of $\mathbb{R}^n$ (or $\mathbb{C}^n$) are considered equipped with the induced Euclidean metric.

Definition 2.1. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A mapping $f\colon X \to Y$ is called* **Lipschitz** *if there exists $\lambda > 0$ such that is*

$$\|f(x_1) - f(x_2)\| \leq \lambda \|x_1 - x_2\|$$

for all $x_1, x_2 \in X$. A Lipschitz mapping $f\colon X \to Y$ is called **bi-Lipschitz** *if its inverse mapping exists and is Lipschitz.*

Next, we are going to establish the notion of bi-Lipschitz homeomorphims at infinity.

Definition 2.2. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two subsets. We say that X and Y are* **bi-Lipschitz homeomorphic at infinity***, if there exist compact subsets $K \subset \mathbb{R}^n$ and $\widetilde{K} \subset \mathbb{R}^m$ and a bi-Lipschitz homeomorphism $\phi\colon X \setminus K \to Y \setminus \widetilde{K}$.*

Definition 2.3. *A subset $X \subset \mathbb{R}^n$ is called* **Lipschitz regular at infinity** *if X and $\mathbb{R}^k$ are bi-Lipschitz homeomorphic at infinity, for some $k \in \mathbb{N}$.*

Example 2.4. *Let $X \subset \mathbb{R}^3$ be defined by $X = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2 = z^3\}$. We see that X is an algebraic subset of $\mathbb{R}^3$ with an isolated singularity at $0 \in \mathbb{R}^3$. By using the mapping $\pi\colon X \to \mathbb{R}^2$; $\pi(x,y,z) = (x,y)$, it is easy to see that X is Lipschitz regular at infinity.*

Example 2.5. *Let $Y \subset \mathbb{R}^3$ be defined by $Y = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2 = z\}$. We see that Y is a smooth algebraic subset of $\mathbb{R}^3$. From another way, Y is not Lipschitz regular at infinity (this can be seen using e.g. Theorem 2.19).*

2.1. **Lipschitz normal embedding at infinity.** Let us remind the definition of Inner Distance already presented in Section 1. Given a path connected subset $X \subset \mathbb{R}^m$ the *inner distance* on X is defined as follows: given two points $x_1, x_2 \in X$, $d_X(x_1,x_2)$ is the infimum of the lengths of paths on X connecting x_1 to x_2. As we said in the beginning of Section 2 all the sets considered in this paper are supposed to be equipped with the Euclidean induced metric. Whenever we consider the inner distance, we emphasize it clearly.

Definition 2.6 (See [2]). *A subset $X \subset \mathbb{R}^n$ is called* **Lipschitz normally embedded** *if there exists $\lambda > 0$ such that*

$$d_X(x_1,x_2) \leq \lambda \|x_1 - x_2\|$$

for all $x_1, x_2 \in X$.

Proposition 2.7. *If a closed unbounded subset $X \subset \mathbb{R}^n$ is Lipschitz regular at infinity, then there exists a compact $K \subset \mathbb{R}^n$ such that each connected component of $X \setminus K$ is Lipschitz normally embedded.*

Proof. Let $X \subset \mathbb{R}^n$ be a closed and unbounded subset. Let us suppose that X is Lipschitz regular at infinity, that is, there exist compact subsets $K_1 \subset \mathbb{R}^k$ and $K_2 \subset \mathbb{R}^n$ and a bi-Lipschitz homeomorphism $\psi\colon \mathbb{R}^k \setminus K_1 \to X \setminus K_2$. Without loss of generality, one can suppose that K_1 is a Euclidean closed ball. Let us denote $Y = \mathbb{R}^k \setminus K_1$, $Z = X \setminus K_2$.

First, let us suppose that $k > 1$. Since there are positive constant $\lambda_1 < \lambda_2$ such that:

$$\lambda_1 \|p - q\| \leq \|\psi(p) - \psi(q)\| \leq \lambda_2 \|p - q\|, \ \forall p, q \in Y,$$

it follows that

$$\lambda_1 d_Y(p,q) \leq d_Z(\psi(p), \psi(q)) \leq \lambda_2 d_Y(p,q), \ \forall p, q \in Y.$$

On the other hand, $d_Y(p,q) \leq \pi\|p-q\|$, for all $p,q \in Y$ and, by those inequalities above, it follows that

$$d_Z(\psi(p),\psi(q)) \leq \frac{\lambda_2 \pi}{\lambda_1}\|\psi(p)-\psi(q)\|,\ \forall\ p,q \in Y.$$

Therefore,

$$d_Z(x,y) \leq \frac{\lambda_2 \pi}{\lambda_1}\|x-y\|,\ \forall\ x,y \in Z.$$

In other words, Z is Lipschitz normally embedded.

In the case where $k=1$, $\mathbb{R}\setminus K_1$ has two connected components which we denote by Y_1 and Y_2. Actually, Y_1 and Y_2 are half-lines and, therefore, for each $i=1,2$, $d_{Y_i}(p,q) \leq |p-q|$, for all $p,q \in Y_i$. Likewise as it was done above, we have

$$d_{Z_i}(x,y) \leq \frac{\lambda_2}{\lambda_1}\|x-y\|,\ \forall\ x,y \in Z_i,$$

where $Z_1=\psi(Y_1)$ and $Z_2=\psi(Y_2)$ are the connected components of Z. Hence, the proposition is proved. □

Definition 2.8. *A subset $X \subset \mathbb{R}^n$ is* **Lipschitz normally embedded at infinity** *if there exists a compact subset $K \subset \mathbb{R}^n$ such that $X \setminus K$ is Lipschitz normally embedded.*

Let us finish this section pointing out the following result which we are going to use in the proof of Theorem 1.1

Corollary 2.9. *Let $X \subset \mathbb{C}^n$ be a complex algebraic subset. If X is Lipschitz regular at infinity, then X is Lipschitz normally embedded at infinity.*

2.2. **Tangent cone at infinity.** Let us start this section recalling two well known results about semialgebraic sets, namely, the Monotonicity Theorem and the Curve Selection Lemma.

Lemma 2.10 (Theorem 1.8 in [5])**.** *Let $f\colon (a,b) \to \mathbb{R}$ be a semialgebraic function. Then, there are $a=a_0<a_1<...<a_k=b$ such that, for each $i=0,...,k-1$, the restriction $f|_{(a_i,a_{i+1})}$ is analytic and either constant, strictly increasing or strictly decreasing.*

Lemma 2.11 (Theorem 2.5.5 in [3])**.** *Let X be a semialgebraic subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ being a non-isolated point of $\overline{X}$. Then, there exists a continuous semialgebraic mapping $\gamma\colon [0,1] \to \mathbb{R}^n$ such that $\gamma(0)=x$ and $\gamma((0,1]) \subset X$.*

Definition 2.12. *Let $X \subset \mathbb{R}^m$ be an unbounded subset. Given a sequence of real positive numbers $\{t_j\}_{j\in\mathbb{N}}$ such that $t_j \to +\infty$, we say that $v \in \mathbb{R}^m$ is* **tangent to X at infinity with respect to $\{t_j\}_{j\in\mathbb{N}}$** *if there is a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset X$ such that $\lim\limits_{j\to+\infty} \frac{1}{t_j}x_j = v$.*

Definition 2.13. *Let $X \subset \mathbb{R}^m$ be a unbounded subset and let $T = \{t_j\}_{j\in\mathbb{N}}$ be a sequence of real positive numbers such that $t_j \to +\infty$. Denote by $E_T(X)$ the set of $v \in \mathbb{R}^m$ which are tangent to X at infinity with respect to T. We call $E_T(X)$* **a tangent cone of X at infinity**. *When X has a unique tangent cone at infinity, we denote it by $C_\infty(X)$ and we call $C_\infty(X)$* **the tangent cone of X at infinity**.

Let us remark that a tangent cone of a set X at infinity can be non-unique as we can see in the following example..

Example 2.14. *Let $X = \{(x,y) \in \mathbb{R}^2;\ y \cdot \sin(\log(x^2+y^2+1)) = 0\}$. For each $j \in \mathbb{N}$, we define $t_j = (e^{j\pi}-1)^{\frac{1}{2}}$ and $s_j = (e^{j\pi+\pi/2}-1)^{\frac{1}{2}}$. Thus, for $T = \{t_j\}_{j\in\mathbb{N}}$ and $S = \{s_j\}_{j\in\mathbb{N}}$, we have $E_T(X) = \mathbb{R}^2$ and $E_S(X) \subset \{(x,y) \in \mathbb{R}^2;\ y \neq 0\} \cup \{(0,0)\}$. However, $E_T(X) \neq E_S(X)$. In fact, it is clear that $E_T(X)$ and $E_S(X)$ are not even homeomorphic.*

In general, it is not an easy task to verify whether unbounded subsets have a unique tangent cone at infinity, even in the case of some classes of analytic subsets, for instance, concerning to such a problem, there is a still unsettled conjecture by Meeks III ([8], Conjecture 3.15) stating that: *any properly immersed minimal surface in $\mathbb{R}^3$ of quadratic area growth has a unique tangent cone at infinity.*

Proposition 2.15. *Let $Z \subset \mathbb{R}^n$ be an unbounded semialgebraic set. A vector $w \in \mathbb{R}^n$ is a tangent vector of Z at infinity if and only if there exists a continuous semialgebraic curve $\gamma\colon (\varepsilon,+\infty) \to Z$ such that $\lim\limits_{t\to+\infty} |\gamma(t)| = +\infty$ and $\gamma(t) = tw + o_\infty(t)$, where $g(t) = o_\infty(t)$ means $\lim\limits_{t\to+\infty} \frac{g(t)}{t} = 0$.*

Proof. Suppose that $w \in \mathbb{R}^n$ is a tangent vector of Z at infinity. Let us consider the semialgebraic mapping $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $\phi(x) = \frac{x}{\|x\|^2}$ and denote $X = \phi(Z \setminus \{0\})$. Since Z is an unbounded set, the origin is a non-isolated point of $\overline{X}$. Let $\rho : \mathbb{S}^{n-1} \times [0,+\infty) \to \mathbb{R}^n \setminus \{0\}$ be the mapping given by $\rho(x,t) = tx$. We see that $\rho|_{\mathbb{S}^{n-1}\times(0,+\infty)} : \mathbb{S}^{n-1} \times (0,+\infty) \to \mathbb{R}^n \setminus \{0\}$ is a semialgebraic homeomorphism with inverse mapping $\rho^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1} \times (0,+\infty)$ given by $\rho^{-1}(x) = (\frac{x}{\|x\|}, \|x\|)$. Therefore, the set $Y = \rho^{-1}(X) \subset \mathbb{S}^{n-1} \times [0,+\infty)$ and $\overline{Y}$ are semialgebraic sets. We are going to consider two cases:

1) Case $w \neq 0$. Since w is a tangent vector of Z at infinity, there are a sequence $\{s_k\}_{k\in\mathbb{N}}$ of positive real numbers and a sequence $\{z_k\}_{k\in\mathbb{N}} \subset Z$ such that $\lim\limits_{k\to+\infty} \|z_k\| = +\infty$ and $\lim\limits_{k\to+\infty} \frac{1}{s_k} z_k = w$. Thus, for each $k \in \mathbb{N}$, let us define $x_k = \phi(z_k)$. In this case, $v := \lim\limits_{k\to\infty} s_k \cdot x_k = \frac{w}{\|w\|^2}$. In particular, $\lim\limits_{k\to\infty} \frac{x_k}{\|x_k\|} = \frac{w}{\|w\|} = \frac{v}{\|v\|}$ and $u = (\frac{v}{\|v\|}, 0) \in \overline{Y}$. Then by Curve Selection Lemma (Lemma 2.11), there exists a continuous semialgebraic curve $\beta : [0,\delta) \to \overline{Y}$ such that $\beta(0) = u$ and $\beta((0,\delta)) \subset Y$.

By writing $\beta(t)=(x(t),s(t))$, we get $s:[0,\delta)\to\mathbb{R}$ is a semialgebraic and non-constant function such that $s(0)=0$ and $s(t)>0$ if $t\in(0,\delta)$. By Lemma 2.10, one can suppose that s is analytic in the domain $(0,\delta)$ and strictly increasing. Hence, $s:[0,\delta/2]\to[0,\delta']$ is a semialgebraic homeomorphism, where $\delta'=s(\frac{\delta}{2})$. Let us define $\alpha:[0,r)\to\overline{X}$ by

$$\alpha(t)=\rho\circ\beta\circ s^{-1}(t\|v\|)=\rho(x(s^{-1}(t\|v\|)),s(s^{-1}(t\|v\|)))=t\|v\|x(s^{-1}(t\|v\|)),$$

where $r=\min\{\frac{\delta'}{\|v\|},\delta'\}$. Therefore,

$$\lim_{t\to 0^+}\frac{\alpha(t)}{t}=\lim_{t\to 0^+}\frac{t\|v\|x(s^{-1}(t))}{t}=\lim_{t\to 0^+}\|v\|x(s^{-1}(t))=\|v\|x(0)=v,$$

and, thus, $\alpha(t)=tv+o(t)$. Finally, by defining $\gamma:(\frac{1}{r},+\infty)\to Z$ in this way $\gamma(t)=\phi^{-1}(\alpha(\frac{1}{t}))$, we get

$$\begin{aligned}\gamma(t)&=\frac{\frac{1}{t}v+o(\frac{1}{t})}{\|\frac{1}{t}v+o(\frac{1}{t})\|^2}=t\frac{v}{\|v\|^2}+o_\infty(t)\\&=tw+o_\infty(t).\end{aligned}$$

Since γ is a composition of continuous semialgebraic mappings, γ is a continuous semialgebraic mapping as well.

2) Case $w=0$. In this case, let $\{x_k\}_{k\in\mathbb{N}}\subset Z$ be a sequence such that $\lim\limits_{k\to+\infty}\|x_k\|=+\infty$ (this sequence exists, because Z is unbounded). Thus, $\{\frac{x_k}{\|x_k\|}\}_{k\in\mathbb{N}}$ is, up to take subsequence, a convergent sequence. Let $v\in\mathbb{R}^n$ be the limit of this sequence, i.e., $\lim\limits_{k\to\infty}\frac{x_k}{\|x_k\|}=v$. Likewise as it was done in the Case 1, one can show that there exists a continuous semialgebraic curve $\gamma\colon(\varepsilon,+\infty)\to Z$ such that $\gamma(t)=tv+o_\infty(t)$. Let us define $\widetilde{\gamma}\colon(\varepsilon^2,+\infty)\to Z$ by $\widetilde{\gamma}(t)=\gamma(t^{\frac{1}{2}})$. Thus, we have $\widetilde{\gamma}(t)=o_\infty(t)=tw+o_\infty(t)$.

Reciprocally, if there exists a continuous semialgebraic curve $\gamma:(\varepsilon,+\infty)\to Z$ such that $\lim\limits_{t\to+\infty}|\gamma(t)|=+\infty$ and $\gamma(t)=tw+o_\infty(t)\}$, then for each $k\in\mathbb{N}$ we define $s_k=\varepsilon+k+1$ and $z_k=\gamma(s_k)$. Thus, it is clear that w is a tangent vector of Z at infinity, since $\lim\limits_{k\to+\infty}\|z_k\|=+\infty$ and $\lim\limits_{k\to+\infty}\frac{1}{s_k}z_k=w$. □

In particular, Proposition 2.15 give us the following consequence

Corollary 2.16. *Let $Z\subset\mathbb{R}^n$ be an unbounded semialgebraic set. Then Z has a unique tangent cone at infinity and, moreover, $C_\infty(Z)=\{v\in\mathbb{R}^n;\ \exists\gamma:(\varepsilon,+\infty)\to Z$ C^0 semialgebraic such that $\lim\limits_{t\to+\infty}|\gamma(t)|=+\infty$ and $\gamma(t)=tv+o_\infty(t)\}$.*

Let us remind the definition of tangent cone of a semialgebraic set.

Definition 2.17. *Let $X\subset\mathbb{R}^m$ be a semialgebraic set such that 0 is a non-isolated point of $\overline{X}$. We define the tangent cone of X at 0 to be the set $C_0(X)=\{v\in$*

$\mathbb{R}^n$; $\exists\gamma : [0,\varepsilon) \to \overline{X}$ *continuous and semialgebraic such that* $\gamma((0,\varepsilon)) \subset X$ *and* $\lim\limits_{t\to 0^+} \frac{\gamma(t)}{t} = v\}$.

Thus, we obtain also the following

Corollary 2.18. *Let* $Z \subset \mathbb{R}^n$ *be an unbounded semialgebraic set. Let* $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ *be the semialgebraic mapping given by* $\phi(x) = \frac{x}{\|x\|^2}$ *and denote* $X = \phi(Z\setminus\{0\})$. *Then* $C_\infty(Z)$ *is a semialgebraic set satisfying* $C_\infty(Z) = C_0(X)$ *and* $\dim_{\mathbb{R}} C_\infty(Z) \leq \dim_{\mathbb{R}} Z$.

Proof. Since $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a semialgebraic homeomorphism, we have that X is a semialgebraic set and $\dim_{\mathbb{R}} X = \dim_{\mathbb{R}} Z$. Therefore, by Lemme 1.2 in [6], we get that $C_0(X)$ is a semialgebraic set and $\dim_{\mathbb{R}} C_0(X) \leq \dim_{\mathbb{R}} X$. Moreover, it follows of the proof of Proposition 2.15 that $C_\infty(Z) = C_0(X)$. Thus, $\dim_{\mathbb{R}} C_\infty(Z) \leq \dim_{\mathbb{R}} Z$, which finish the proof. □

The next result is a version at infinity of Theorem 3.2 in [11], where the second named author of this paper proved that bi-Lipschitz homeomorphic subanalytic subsets have bi-Lipschitz homeomorphic tangent cones.

Theorem 2.19. *Let* $A \subset \mathbb{R}^m$ *and* $B \subset \mathbb{R}^n$ *be closed unbounded semialgebraic subsets. If* A *and* B *are bi-Lipschitz homeomorphic at infinity, then their tangent cones at infinity* $C_\infty(A)$ *and* $C_\infty(B)$ *are bi-Lipschitz homeomorphic.*

Proof. Let $\widetilde{K}_1 \subset \mathbb{R}^m$ and $\widetilde{K}_2 \subset \mathbb{R}^n$ be compact subsets such that there exists a bi-Lipschitz homeomorphism $\phi\colon A \setminus \widetilde{K}_1 \to B \setminus \widetilde{K}_2$. Let us denote $X = A \setminus \widetilde{K}_1$ and $Y = B \setminus \widetilde{K}_2$. By taking $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ and doing the following identifications:

$$X \leftrightarrow X \times \{0\} \quad \text{and} \quad Y \leftrightarrow \{0\} \times Y$$

one can suppose that $X, Y \subset \mathbb{R}^N$ and there exists a bi-Lipschitz map $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ such that $\varphi(X) = Y$ (see Lemma 3.1 in [11]). Let $K > 0$ be a constant such that

$$\frac{1}{K}\|x - y\| \leq \|\varphi(x) - \varphi(y)\| \leq K\|x - y\|, \quad \forall x, y \in \mathbb{R}^N. \tag{1}$$

For each $k \in \mathbb{N}$, let us define the mappings $\varphi_k, \psi_k : \mathbb{R}^N \to \mathbb{R}^N$ given by $\varphi_k(v) = \frac{1}{k}\varphi(kv)$ and $\psi_k(v) = \frac{1}{k}\varphi^{-1}(kv)$. For each integer $m \geq 1$, let us define $\varphi_{k,m} := \varphi_k|_{\overline{B}_m} : \overline{B}_m \to \mathbb{R}^N$ and $\psi_{k,m} := \psi_k|_{\overline{B}_{mK}} : \overline{B}_{mK} \to \mathbb{R}^N$, where $\overline{B}_r$ denotes the Euclidean closed ball of radius r and with center at the origin in $\mathbb{R}^N$. Since

$$\frac{1}{K}\|x - y\| \leq \|\varphi_{k,1}(x) - \varphi_{k,1}(y)\| \leq K\|x - y\|, \quad \forall x, y \in \overline{B}_1,\ \forall k \in \mathbb{N}$$

and

$$\frac{1}{K}\|u - v\| \leq \|\psi_{k,1}(u) - \psi_{k,1}(v)\| \leq K\|u - v\|, \quad u, v \in \overline{B}_K,\ \forall k \in \mathbb{N},$$

there exist a subsequence $\{k_{j,1}\}_{j\in\mathbb{N}} \subset \mathbb{N}$ and Lipschitz mappings $d\varphi_1 : \overline{B}_1 \to \mathbb{R}^N$ and $d\psi_1 : \overline{B}_K \to \mathbb{R}^N$ such that $\varphi_{k_{j,1},1} \to d\varphi_1$ uniformly on $\overline{B}_1$ and $\psi_{k_{j,1},1} \to d\psi_1$ uniformly on $\overline{B}_K$ (notice that $\{\varphi_{k,1}\}_{k\in\mathbb{N}}$ and $\{\psi_{k,1}\}_{k\in\mathbb{N}}$ have uniform Lipschitz constants). Furthermore, it is clear that

$$\frac{1}{K}\|u-v\| \le \|d\varphi_1(u) - d\varphi_1(v)\| \le K\|u-v\|, \quad \forall u,v \in \overline{B}_1$$

and

$$\frac{1}{K}\|z-w\| \le \|d\psi_1(z) - d\psi_1(w)\| \le K\|z-w\|, \quad \forall z,w \in \overline{B}_K.$$

Likewise as above, for each $m > 1$, we have

$$\frac{1}{K}\|x-y\| \le \|\varphi_{k,m}(x) - \varphi_{k,m}(y)\| \le K\|x-y\|, \quad x,y \in \overline{B}_m,\ \forall k \in \mathbb{N}$$

and

$$\frac{1}{K}\|u-v\| \le \|\psi_{k,m}(u) - \psi_{k,m}(v)\| \le K\|u-v\|, \quad u,v \in \overline{B}_{mK},\ \forall k \in \mathbb{N}.$$

Therefore, for each $m > 1$, there exist a subsequence $\{k_{j,m}\}_{j\in\mathbb{N}} \subset \{k_{j,m-1}\}_{j\in\mathbb{N}}$ and Lipschitz mappings $d\varphi_m \colon \overline{B}_m \to \mathbb{R}^N$ and $d\psi_m \colon \overline{B}_{mK} \to \mathbb{R}^N$ such that $\varphi_{k_{j,m},m} \to d\varphi_m$ uniformly on $\overline{B}_m$ and $\psi_{k_{j,m},m} \to d\psi_m$ uniformly on $\overline{B}_{mK}$ with $d\varphi_m|_{\overline{B}_{m-1}} = d\varphi_{m-1}$ and $d\psi_m|_{\overline{B}_{(m-1)K}} = d\psi_{m-1}$. Furthermore,

$$\frac{1}{K}\|u-v\| \le \|d\varphi_m(u) - d\varphi_m(v)\| \le K\|u-v\|, \quad \forall u,v \in \overline{B}_m \tag{2}$$

and

$$\frac{1}{K}\|z-w\| \le \|d\psi_m(z) - d\psi_m(w)\| \le K\|z-w\|, \quad \forall z,w \in \overline{B}_{mK}. \tag{3}$$

Let us define $d\varphi, d\psi : \mathbb{R}^N \to \mathbb{R}^N$ by $d\varphi(x) = d\varphi_m(x)$, if $x \in \overline{B}_m$ and $d\psi(x) = d\psi_m(x)$, if $x \in \overline{B}_{mK}$ and, for each $j \in \mathbb{N}$, let $t_j = n_j = k_{j,j}$.

Claim 1. $\varphi_{n_j} \to d\varphi$ and $\psi_{n_j} \to d\psi$ uniformly on compact subsets of $\mathbb{R}^N$.

Let $F \subset \mathbb{R}^N$ be a compact subset. Let us take $m \in \mathbb{N}$ such that $F \subset \overline{B}_m \subset \overline{B}_{mK}$. Thus, $\{n_j\}_{j>m}$ is a subsequence of $\{k_{j,m}\}_{j\in\mathbb{N}}$ and, since $\varphi_{k_{j,m},m} \to d\varphi_m$ uniformly on $\overline{B}_m$ and $\psi_{k_{j,m},m} \to d\psi_m$ uniformly on $\overline{B}_{mK}$, it follows that $\varphi_{n_j} \to d\varphi$ and $\psi_{n_j} \to d\psi$ uniformly on F.

Claim 2. $d\varphi : \mathbb{R}^N \to \mathbb{R}^N$ bi-Lipschitz homeomorphism and $d\psi = (d\varphi)^{-1}$.

It follows from inequalities (2) and (3) that $d\varphi, d\psi : \mathbb{R}^N \to \mathbb{R}^N$ are Lipschitz mappings. Therefore, it is enough to show that $d\psi = (d\varphi)^{-1}$. In order to do that, let $v \in \mathbb{R}^N$ and $w = d\varphi(v) = \lim\limits_{j\to\infty} \frac{\varphi(t_j v)}{t_j}$. Thus,

$$\begin{aligned} \|d\psi(w) - v\| &= \left\| \lim_{j\to\infty} \frac{\psi(t_j w)}{t_j} - v \right\| &&= \lim_{j\to\infty} \left\| \frac{\psi(t_j w)}{t_j} - \frac{t_j v}{t_j} \right\| \\ &= \lim_{j\to\infty} \frac{1}{t_j} \left\| \psi(t_j w) - t_j v \right\| &&= \lim_{j\to\infty} \frac{1}{t_j} \left\| \psi(t_j w) - \psi(\varphi(t_j v)) \right\| \\ &\le \lim_{j\to\infty} \frac{K}{t_j} \left\| t_j w - \varphi(t_j v) \right\| &&= \lim_{j\to\infty} K \left\| w - \frac{\varphi(t_j v)}{t_j} \right\| \\ &= 0. \end{aligned}$$

Then, $d\psi(w) = d\psi(d\varphi(v)) = v$, for all $v \in \mathbb{R}^N$, i.e., $d\psi \circ d\varphi = \mathrm{id}_{\mathbb{R}^N}$. Analogously, one can show that $d\varphi \circ d\psi = \mathrm{id}_{\mathbb{R}^N}$.

Claim 3. $d\varphi(C_\infty(X)) = C_\infty(Y)$.

By Claim 2, it is enough to verify that $d\varphi(C_\infty(X)) \subset C_\infty(Y)$. In order to do that, let $v \in C_\infty(X)$. Then, there is $\alpha\colon (\varepsilon, \infty) \to X$ such that $\alpha(t) = tv + o_\infty(t)$. Thus, $\varphi(\alpha(t)) = \varphi(tv) + o_\infty(t)$, since φ is a Lipschitz mapping. However, $\varphi(t_j v) = t_j d\varphi(v) + o_\infty(t_j)$ and then

$$d\varphi(v) = \lim_{j\to\infty} \varphi_{n_j}(v) = \lim_{j\to\infty} \frac{\varphi(t_j v)}{t_j} = \lim_{j\to\infty} \frac{\varphi(\alpha(t_j))}{t_j} \in C_\infty(Y).$$

Therefore, $d\varphi : C_\infty(X) \to C_\infty(Y)$ is a bi-Lipschitz homeomorphism. We finish the proof by remarking that $C_\infty(A) = C_\infty(X)$ and $C_\infty(B) = C_\infty(Y)$.

□

Corollary 2.20. *Let $X \subset \mathbb{C}^n$ be a complex algebraic subset. If X is Lipschitz regular at infinity, then the tangent cone of X at infinity is Lipschitz regular at infinity.*

Let $X \subset \mathbb{C}^n$ be a complex algebraic subset. Let $\mathcal{I}(X)$ be the ideal of $\mathbb{C}[x_1, ..., x_n]$ given by the polynomials which vanishes on X. For each $f \in \mathbb{C}[x_1, ..., x_n]$, let us denote by f^* the homogeneous polynomial composed of the monomials in f of maximum degree.

Proposition 2.21 (Theorem 1.1 in [7]). *Let $X \subset \mathbb{C}^n$ be a complex algebraic subset. Then, $C_\infty(X)$ is the affine variety $V(\langle f^*; f \in \mathcal{I}(X)\rangle)$.*

3. Proof of Theorem 1.1

Let us begin this section by recalling some basic facts about degree of complex algebraic sets. More precisely, we are going to see that affine linear subspaces of $\mathbb{C}^n$ are characterized among algebraic subsets of $\mathbb{C}^n$ by being those of degree 1. In such a direction, let $\iota\colon \mathbb{C}^n \hookrightarrow \mathbb{P}^n$ be the embedding given by $\iota(x_1, ..., x_n) = [1 : x_1 : ... : x_n]$ and let $p\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection mapping given by $p(x_0, x_1, \ldots, x_n) = [x_0 : x_1 : ... : x_n]$.

Remark 3.1. Let A be an algebraic set in $\mathbb{P}^n$ and X be an algebraic set in $\mathbb{C}^n$. Then $\widetilde{A} = p^{-1}(A) \cup \{0\}$ is a homogeneous complex algebraic set in $\mathbb{C}^{n+1}$ and the closure $\overline{\iota(X)}$ of $\iota(X)$ in $\mathbb{P}^n$ is an algebraic set in $\mathbb{P}^n$.

Definition 3.2. *Let $Y \subset \mathbb{C}^n$ be a pure p-dimensional analytic set such that $0 \in Y$ and let $L \in G(n-p,n)$ such that $L \cap C(Y,0) = \{0\}$. Let $\pi\colon \mathbb{C}^n \to \mathbb{C}^p$ be the orthogonal projection such that $L = \pi^{-1}(0)$. Therefore, there exist an open neighborhood U of 0 and a proper analytic subset $\sigma \subset \mathbb{C}^p$ such that $m = \#(\pi^{-1}(t) \cap (Y \cap U))$ does not depend of $t \in \pi(U) \setminus \sigma$ (see subsection 10.1 in* [4]*). Moreover, by Proposition 2 in (*[4]*, page 122), m does not depend also of L. Thus, we define* **the multiplicity of Y at** 0 *to be $m(Y,0) = m$. If A is a pure dimensional algebraic set in $\mathbb{P}^n$ and X is a pure dimensional algebraic set in $\mathbb{C}^n$, we define* **the degree of** A *by $\deg(A) = m(\widetilde{A},0)$ and* **the degree of** X *by $\deg(X) = \deg(\overline{\iota(X)})$.*

Proposition 3.3. *Let $X \subset \mathbb{C}^n$ be a pure dimensional algebraic subset. Then, $\deg(X) = 1$ if and only if X is an affine linear subspace of $\mathbb{C}^n$.*

Proof. Let $A = \overline{\iota(X)}$ be the closure of $\iota(X)$ in $\mathbb{P}^n$. By definition, $\deg(X) = \deg(A)$ and $\deg(A) = m(\widetilde{A},0)$, where $\widetilde{A} = p^{-1}(A) \cup \{0\}$. Thus, $\deg(X) = 1$ if and only if $m(\widetilde{A},0) = 1$. However, as $\widetilde{A}$ is a homogeneous complex algebraic set in $\mathbb{C}^{n+1}$, $m(\widetilde{A},0) = 1$ if and only if $\widetilde{A}$ is a complex linear subspace. In order to finish the proof, we remark that $\widetilde{A}$ is a complex linear subspace in $\mathbb{C}^{n+1}$ if and only if A is a complex projective plane in $\mathbb{P}^n$. □

At this moment, we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Since $C_\infty(X)$ is a linear subspace of $\mathbb{C}^n$, one can consider the orthogonal projection $\pi\colon \mathbb{C}^n \to C_\infty(X)$. Let us choose linear coordinates (x,y) in $\mathbb{C}^n$ such that

$$C_\infty(X) = \{(x,y) \in \mathbb{C}^n;\ y = 0\}.$$

Claim 1. There exist positive constants C and ρ such that $X \subset \{(x,y); \|y\| < C\|x\|\} \cup B_\rho$.

Indeed, if Claim 1 is not true, there exists a sequence $\{(x_k,y_k)\}_{k\in\mathbb{N}} \subset X$ such that $\lim\limits_{k\to+\infty} \|(x_k,y_k)\| = +\infty$ and $\|y_k\| \geq k\|x_k\|$. Thus, up to a subsequence, one can suppose that $\lim\limits_{k\to+\infty} \frac{y_k}{\|y_k\|} = y_0$. Since $\frac{\|x_k\|}{\|y_k\|} \leq \frac{1}{k}$, $(0,y_0) \in C_\infty(X)$, which is a contradiction, because $y_0 \neq 0$. Therefore, Claim 1 is true.

Now, by Theorem 2 in ([4], page 77), X is an algebraic set.

Claim 2. If $\gamma : (\varepsilon,\infty) \to X$ is an arc such that $\lim\limits_{t\to+\infty} \|\gamma(t)\| = +\infty$ and $\pi \circ \gamma(t) = tv + o_\infty(t)$, then $\gamma(t) = tv + o_\infty(t)$.

In order to prove Claim 2, let us write $\gamma(t) = (x(t),y(t))$. By Claim 1, there exists $t_0 > 0$ such that $\|y(t)\| \leq C\|x(t)\|$ for all $t \geq t_0$, since $\lim\limits_{t\to+\infty} \|\gamma(t)\| = +\infty$. Thus, since $\frac{x(t)}{t}$ is bounded, $\frac{y(t)}{t}$ is bounded. Let us suppose that $y(t) \neq o_\infty(t)$. Then, there exist a sequence $\{t_k\}_{k\in\mathbb{N}}$ and $r > 0$ such that $t_k \to +\infty$ and $\frac{\|y(t_k)\|}{t_k} \geq r$ for all k. Since $\left\{\frac{y(t_k)}{t_k}\right\}_{k\in\mathbb{N}}$ is bounded, up to a subsequence, one can suppose that $\lim\limits_{k\to+\infty} \frac{y(t_k)}{t_k} = y_0$. Therefore, $\lim\limits_{k\to+\infty} \frac{\gamma(t_k)}{t_k} = (v', y_0) \in C_\infty(X)$, where $v = (v', 0)$. However, this is a contradiction, since $\|y_0\| \geq r > 0$ and this implies that $y_0 \neq 0$. Then, $y(t) = o_\infty(t)$ and, therefore, $\gamma(t) = tv + o_\infty(t)$.

Let $L = \pi^{-1}(0)$. By Claim 1, one can see that $\overline{\iota(X)} \cap \overline{\iota(L)} \cap (\mathbb{P}^n \setminus \iota(\mathbb{C}^n)) = \emptyset$. Therefore, $\pi|_X \colon X \to C_\infty(X)$ is a ramified cover with degree equal to $\deg(X)$ (see [4], Corollary 1 in the page 126). Moreover, the ramification locus of $\pi|_X$ is a codimension ≥ 1 complex algebraic subset Σ of the linear space $C_\infty(X)$.

Let us suppose that the degree $\deg(X)$ is strictly greater than 1. Since Σ is a codimension ≥ 1 complex algebraic subset of the space $C_\infty(X)$, then by Corollary 2.18, $\dim_\mathbb{R} C_\infty(\Sigma) \leq \dim_\mathbb{R} \Sigma < \dim_\mathbb{R} C_\infty(X)$ and, thus, there exists a unit tangent vector $v_0 \in C_\infty(X) \setminus C_\infty(\Sigma)$.

Since v_0 is not tangent to Σ at infinity, there exist positive real numbers λ and R such that

$$C_{\lambda,R} = \{v \in C_\infty(X);\ \|v - tv_0\| < \lambda t,\ \text{for some } t > R\}$$

does not intersect the set Σ. Since we have assumed that the degree $\deg(X) \geq 2$, we have at least two different liftings $\gamma_1(t)$ and $\gamma_2(t)$ of the half-line $r(t) = tv_0$, i.e. $\pi(\gamma_1(t)) = \pi(\gamma_2(t)) = tv_0$. Since π is the orthogonal projection on $C_\infty(X)$ and the vector v_0 is the unit tangent vector at infinity to the images $\pi \circ \gamma_1$ and $\pi \circ \gamma_2$, then v_0 is the tangent vector at infinity to the arcs γ_1 and γ_2. By construction, we have $\operatorname{dist}(\gamma_i(t), \pi|_X{}^{-1}(\Sigma)) \geq \lambda t$ for $i = 1, 2$, where by dist we mean the Euclidean distance.

On the other hand, any path in X connecting $\gamma_1(t)$ to $\gamma_2(t)$ is the lifting of a loop, based at the point tv_0 which is not contractible in $C_\infty(X) \setminus \Sigma$. Thus, the length of such a path must be at least $2\lambda t$. It implies that the inner distance, $\mathrm{d_X}(\gamma_1(\mathrm{t}), \gamma_2(\mathrm{t}))$, in X, between $\gamma_1(t)$ and $\gamma_2(t)$, is at least $2\lambda t$. But, by Claim 2, $\gamma_1(t)$ and $\gamma_2(t)$ are tangent at infinity, that is,

$$\frac{\|\gamma_1(t) - \gamma_2(t)\|}{t} \to 0 \quad \text{as} \quad t \to +\infty,$$

and $\lambda > 0$, we obtain that X is not Lipschitz normally embedded at infinity. Otherwise there will be $\widetilde{C} > 0$ and a compact subset $K \subset \mathbb{C}^n$ such that:

$$d_X(x_1, x_2) \leq \widetilde{C}\|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in X \setminus K,$$

hence:

$$\begin{aligned} 2\lambda &\leq \frac{\mathrm{d_X}(\gamma_1(\mathrm{t}),\gamma_2(\mathrm{t}))}{t} \\ &\leq \widetilde{C}\frac{\|\gamma_1(t)-\gamma_2(t)\|}{t} \to 0 \ \text{ as } t\to+\infty, \end{aligned}$$

which is a contradiction. We have concluded that $\deg(X)=1$ and, by Proposition 3.3, it follows that X is an affine linear subspace. □

Let us remark that the real version of Theorem 1.1 does not hold true in general, as it is shown bellow.

Example 3.4. *The set $X=\{(x,y,z)\in\mathbb{R}^3;\ z=\sin(x+y)\}$ is Lipschitz normally embedded, since it is bi-Lipschitz homeomorphic to $\mathbb{R}^2$ and, moreover, X has a unique tangent cone at infinity with $C_\infty X=\{(x,y,z)\in\mathbb{R}^3;\ z=0\}$. However, X is not an algebraic set and, in particular, it is not a linear subspace of $\mathbb{R}^3$.*

As a direct consequence of Theorem 1.1 we obtain the following

Corollary 3.5. *Let $X\subset\mathbb{C}^n$ be a pure d-dimensional algebraic subset such that $C_\infty(X)$ is a complex linear subspace of $\mathbb{C}^n$. If X is Lipschitz normally embedded at infinity, then X is an affine linear subspace of $\mathbb{C}^n$.*

Notice that the assumptions in Corollary 3.5 are sharp in the sense that, in order to get the same conclusion, none of those assumptions can be removed, as we can see bellow.

Example 3.6. *The complex plane curve $X=\{(x,y)\in\mathbb{C}^2;\ y=x^2\}$ has linear tangent cone at infinity and is not Lipschitz normally embedded at infinity; X is not an affine linear subset of $\mathbb{C}^2$. As another example, $Z=\{(x,y,z)\in\mathbb{C}^3;\ x^2+y^2+z^2=0\}$. We see that Z is Lipschitz normally embedded and $C_\infty(Z)$ is Z itself, which is not a linear subspace; Z is not an affine linear subset of $\mathbb{C}^3$.*

It follows from Proposition 2.21 that the tangent cone at infinity of a complex algebraic subset of $\mathbb{C}^n$ is a homogeneous complex algebraic subset and, therefore, it is a complex cone in the following sense: an algebraic subset of $\mathbb{C}^n$ is called a *complex cone* if it is a union of one-dimensional complex linear subspaces of $\mathbb{C}^n$. The next result was proved by David Prill in [10]:

Lemma 3.7 (Theorem in [10]). *Let $V\subset\mathbb{C}^n$ be a complex cone. If $0\in V$ has a neighborhood homeomorphic to a Euclidean ball, then V is a linear subspace of $\mathbb{C}^n$*

Theorem 3.8. *Let $X\subset\mathbb{C}^n$ be a pure dimensional complex algebraic subset. If X is Lipschitz regular at infinity, then X is an affine linear subspace of $\mathbb{C}^n$.*

Proof. Let us suppose that the complex algebraic subset $X \subset \mathbb{C}^n$ is Lipschitz regular at infinity. Thus, let $h\colon U \to \mathbb{R}^N \setminus B$ be a bi-Lipschitz homeomorphism, where U is an open neighborhood of the infinity in X (i.e. a complement of a compact subset in X) and $B \subset \mathbb{R}^N$ is a closed Euclidean ball centered at the origin $0 \in \mathbb{R}^N$. Let $C_\infty(X)$ be the tangent cone at infinity of X. It comes from Theorem 2.19 that there exists a bi-Lipschitz homeomorphism $dh\colon C_\infty(X) \to C_\infty(\mathbb{R}^N \setminus B) = \mathbb{R}^N$. In particular, $C_\infty(X)$ is a topological manifold. By Prill's Theorem (Lemma 3.7), it follows that $C_\infty(X)$ is a complex linear subspace of $\mathbb{C}^n$. By Corollary 2.9, X is Lipschitz normally embedded at infinity and, by Corollary 3.5, it follows that X is an affine linear subspace of $\mathbb{C}^n$. □

We would like to remark that if we remove the assumption that $X \subset \mathbb{C}^n$ is pure dimensional in Theorem 3.8, we obtain that X is a union of an affine linear subspace of $\mathbb{C}^n$ with a 0-dimensional complex algebraic subset, since by Geometric form of Noether's normalization lemma in ([9], page 42), any non-zero dimensional complex algebraic set is an unbounded set. Finally, we would like to mention that Theorem 3.8 does not hold true (with same assumptions) for real algebraic sets (cf. Example 2.4).

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