

# Decay of Complex-time Determinantal and Pfaffian Correlation Functionals in Lattices

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## Abstract

We supplement the determinantal and Pfaffian bounds of [4] for many-body localization of quasi-free fermions, by considering the high dimensional case and complex-time correlations. Our proof uses the analyticity of correlation functions via the Hadamard three-line theorem. We show that the dynamical localization for the one-particle system yields the dynamical localization for the many-point fermionic correlation functions, with respect to the Hausdorff distance in the determinantal case. In [4], a stronger notion of decay for many-particle configurations was used but only at dimension one and for real times. Considering determinantal and pfaffian correlation functionals for complex times is important in the study of weakly interacting fermions.

**Keywords:** disordered fermion system, many-body localization, determinant bound

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## 1 Introduction

Since a few years, the problem of (Anderson) localization in many-body systems is garnering attention. The mathematical understanding of this phenomenon for *interacting* quantum particles, as addressed in 2006 by [1] for weakly interacting fermions at small densities, is a long-term goal. In 2009, [2, 3] contributed first rigorous results. In 2016, an exponential decay of many-particle correlations was proven for quasi-free fermions in *one-dimensional* lattices with disorder [4]. Via the Jordan-Wigner transformation, this includes the disordered *XY* spin chains. This last paper has attracted much attention and it has already been cited many times in one and a half year. See, e.g., [5–13].

As pointed out in [4], it is an interesting open question (a) whether [4, Theorems 1.1 and 1.2] can be generalized to higher dimensions. Another open question (b) is their generalization for *complex-time* correlation functions. This last point is relevant because such correlation functions (of quasi-free fermions) can be useful to study localization of *weakly interacting* fermion systems on lattices. In fact, (quasi-free) complex-time correlation functions appear in the perturbative expansion of (full) correlations for weakly interacting systems. See, for instance, [14, Section 5.4.1].

By considering the many-body localization in the sense of the Hausdorff distance, like in [3], we propose an answer to both questions (a) and (b), using the Hadamard three-line theorem (Section 4). See Corollary 2.3, which, together with Theorem 2.2, is our main result on determinantal correlation functionals. In a similar way, we also prove the decay of complex-time Pfaffian correlation functionals with respect to a splitting width (like [3, Equation (5.9)]) of particle configurations. This is a version of [4, Theorem 1.4] which holds true at any dimension  $d \in \mathbb{N}$ . See Theorem 3.1.

## 2 Decay of Determinantal Correlation Functionals

### 2.1 Setup of the Problem and Main Results

(i): Let  $d \in \mathbb{N}$ . For a fixed parameter  $\epsilon \in (0, 1]$ , we define

$$\mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{X}_2) \doteq \max \left\{ \max_{x_1 \in \mathcal{X}_1} \min_{x_2 \in \mathcal{X}_2} |x_1 - x_2|^\epsilon, \max_{x_2 \in \mathcal{X}_2} \min_{x_1 \in \mathcal{X}_1} |x_1 - x_2|^\epsilon \right\}, \quad \mathcal{X}_1, \mathcal{X}_2 \subset \mathbb{Z}^d, \quad (1)$$

which is the well-known Hausdorff distance between the two sets, associated with the metric  $(x_1, x_2) \mapsto |x_1 - x_2|^\epsilon$  on  $\mathbb{Z}^d$ .

(ii): We consider (non-relativistic) fermions in the lattice  $\mathbb{Z}^d$  with arbitrary finite spin set  $S$ . Thus, we define the one-particle Hilbert space to be  $\mathfrak{h} \doteq \ell^2(\mathbb{Z}^d; \mathbb{C}^S)$ , the canonical orthonormal basis  $\{\mathfrak{e}_{x,\sigma}\}_{(x,\sigma) \in \mathbb{Z}^d \times S}$  of which is

$$\mathfrak{e}_{x_0, \sigma_0}(x, \sigma) \doteq \delta_{x, x_0} \delta_{\sigma, \sigma_0}, \quad x, x_0 \in \mathbb{Z}^d, \quad \sigma, \sigma_0 \in S. \quad (2)$$

(iii): Let  $(\Omega, \mathfrak{F}, \mathfrak{a})$  be a standard<sup>1</sup> probability space. As is usual,  $\mathbb{E}[\cdot]$  denotes the expectation value associated with the probability measure  $\mathfrak{a}$ . We consider  $\mathfrak{F}$ -measurable families  $\{H_\omega\}_{\omega \in \Omega} \subset \mathcal{B}(\mathfrak{h})$  of bounded one-particle Hamiltonians satisfying the following (one-body localization) assumption, at fixed  $\beta \in \mathbb{R}^+$ :

#### Condition 2.1

There is a Borel set  $I \subset \mathbb{R}$  as well as constants  $\epsilon \in (0, 1]$ ,  $D$  and  $\mu \in \mathbb{R}^+$  such that, for all  $x_1 \in \mathbb{Z}^d$  and  $R > 0$ ,

$$\sum_{x_2 \in \mathbb{Z}^d: |x_1 - x_2|^\epsilon \geq R} \mathbb{E} \left[ \sup_{z \in \mathbb{S}_\beta} \max_{\sigma_1, \sigma_2 \in S} \left| \left\langle \mathfrak{e}_{x_1, \sigma_1}, \frac{e^{izH_\omega} \chi_I(H_\omega)}{1 + e^{\beta H_\omega}} \mathfrak{e}_{x_2, \sigma_2} \right\rangle_{\mathfrak{h}} \right| \right] \leq D e^{-\mu R}, \quad (3)$$

where

$$\mathbb{S}_\beta \doteq \mathbb{R} - i[0, \beta], \quad \beta \in \mathbb{R}^+, \quad (4)$$

$\chi_I$  is the characteristic function of the set  $I$ , and  $|x_1 - x_2|$  the euclidean distance between the lattice points  $x_1, x_2 \in \mathbb{Z}^d$ .

This assumption is similar to the so-called strong exponential dynamical localization in  $I$ , see, e.g., [15, Definition 7.1]. Note that, for  $\epsilon \in (0, 1]$ ,  $(x_1, x_2) \mapsto |x_1 - x_2|^\epsilon$  defines a translation invariant metric on the lattice  $\mathbb{Z}^d$ . Observe also that, for all  $\beta \in \mathbb{R}^+$  and  $z \in \mathbb{S}_\beta$ , the function  $\lambda \mapsto |e^{z\lambda} (1 + e^{\beta\lambda})^{-1}|$  on  $\mathbb{R}$  is bounded by 1. In particular, the left-hand side of (3) is bounded by the eigenfunction correlator [15, Eq. (7.1)]. Condition 2.1 replaces [4, Eq. (1.19)], noting that

$$\rho(s, t) = \frac{e^{i(t-s)H_\omega}}{1 + e^{\beta H_\omega}}, \quad s, t \in \mathbb{R}, \quad (5)$$

is the main example they have in mind [4, Eq. (2.37)].

(iv): Let  $\text{CAR}(\mathfrak{h})$  be the CAR  $C^*$ -algebra generated by the unit  $\mathbf{1}$  and  $\{a(\varphi)\}_{\varphi \in \mathfrak{h}}$ . For any  $A_1, A_2 \in \text{CAR}(\mathfrak{h})$  and any  $z_1, z_2 \in \mathbb{C}$ , we define

$$\mathbb{O}_{z_1, z_2}(A_1, A_2) \doteq \begin{cases} A_1 A_2 & \text{if } \text{Im}(z_1) \leq \text{Im}(z_2), \\ -A_2 A_1 & \text{if } \text{Im}(z_1) > \text{Im}(z_2). \end{cases} \quad (6)$$

<sup>1</sup>I.e.,  $\mathfrak{F}$  is the Borel  $\sigma$ -algebra of a Polish space  $\Omega$ .

(v): For any  $\beta \in \mathbb{R}^+$  and  $\omega \in \Omega$ , we define the (gauge invariant) quasi-free state  $\rho_\omega \equiv \rho_{\beta, \omega}$  by the condition

$$\rho_\omega (a(\varphi_1)^* a(\varphi_2)) = \left\langle \varphi_2, \frac{1}{1 + e^{\beta H_\omega}} \varphi_1 \right\rangle_{\mathfrak{h}}, \quad \varphi_1, \varphi_2 \in \mathfrak{h}. \quad (7)$$

This state is the unique KMS state at inverse temperature  $\beta \in \mathbb{R}^+$  associated with the unique strongly continuous group  $\{\tau_t^{(\omega)}\}_{t \in \mathbb{R}}$  of (Bogoliubov) automorphisms of  $\text{CAR}(\mathfrak{h})$  satisfying

$$\tau_t^{(\omega)} (a(\varphi)) = a(e^{itH_\omega} \varphi), \quad t \in \mathbb{R}, \varphi \in \mathfrak{h}. \quad (8)$$

Note that, for all  $\varphi \in \mathfrak{h}$ , the maps

$$t \mapsto \tau_t^{(\omega)} (a(\varphi)) \quad \text{and} \quad t \mapsto \tau_t^{(\omega)} (a(\varphi)^*)$$

on  $\mathbb{R}$  uniquely extend to entire functions: For any  $z \in \mathbb{C}$  and  $\varphi \in \mathfrak{h}$ ,

$$\tau_z^{(\omega)} (a(\varphi)^*) \doteq a(e^{izH_\omega} \varphi)^* \quad \text{and} \quad \tau_z^{(\omega)} (a(\varphi)) \doteq a(e^{i\bar{z}H_\omega} \varphi). \quad (9)$$

Observe additionally that, for any  $z_1, z_2 \in \mathbb{C}$  and  $\varphi_1, \varphi_2 \in \mathfrak{h}$ ,

$$\begin{aligned} & \rho_\omega \left( \mathbb{O}_{z_1, z_2} \left( \tau_{z_1}^{(\omega)} (a(\varphi_1)^*), \tau_{z_2}^{(\omega)} (a(\varphi_2)) \right) \right) \\ &= \begin{cases} \left\langle \varphi_2, \frac{e^{i(z_1 - z_2)H_\omega}}{1 + e^{\beta H_\omega}} \varphi_1 \right\rangle_{\mathfrak{h}} & \text{if } \text{Im}(z_1) \leq \text{Im}(z_2), \\ - \left\langle \varphi_2, \frac{e^{(\beta + i(z_1 - z_2))H_\omega}}{1 + e^{\beta H_\omega}} \varphi_1 \right\rangle_{\mathfrak{h}} & \text{if } \text{Im}(z_1) > \text{Im}(z_2). \end{cases} \end{aligned} \quad (10)$$

Below, we show that strong one-body localization, in the sense of Condition 2.1, yields the corresponding many-body localization for the quasi-free state  $\rho_\omega$ , in the sense of the Hausdorff distance, as stated in Corollary 2.3. This is achieved by estimating, in Theorem 4.1, determinants of the form

$$\det \left[ G_\omega \left( (\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}) \right) \right]_{k,l=1}^N \quad (11)$$

in terms of the entries of one single row or column. In (11),  $\beta \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$ ,  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$  are normalized vectors,  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$  and

$$G_\omega \left( (\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}) \right) \doteq \rho_\omega \left( \mathbb{O}_{z_k, z_{N+l}} \left( \tau_{z_k}^{(\omega)} (a(\varphi_k)^*), \tau_{z_{N+l}}^{(\omega)} (a(\varphi_{N+l})) \right) \right)$$

is the two-point, complex-time-ordered correlation function associated with the quasi-free state  $\rho_\omega$ .

### Theorem 2.2

Let  $\{H_\omega\}_{\omega \in \Omega} \subset \mathcal{B}(\mathfrak{h})$  be a family of bounded Hamiltonians. For all  $\omega \in \Omega$ ,  $\beta \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$ , norm-one vectors  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$ , and  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$  (see (4))

$$\begin{aligned} & \left| \det \left[ G_\omega \left( (\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}) \right) \right]_{k,l=1}^N \right| \\ & \leq \min \left\{ \min_{k \in \{1, \dots, N\}} \sum_{l=1}^N |G_\omega \left( (\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}) \right)|, \min_{l \in \{1, \dots, N\}} \sum_{k=1}^N |G_\omega \left( (\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}) \right)| \right\}. \end{aligned}$$

**Proof.** Fix all parameters of the theorem. By expanding the determinant along a fixed row or column, for any  $m \in \{1, \dots, N\}$ ,

$$\begin{aligned}
& \det [G_\omega ((\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}))]_{k,l=1}^N \tag{12} \\
&= \sum_{n=1}^N (-1)^{m+n} G_\omega ((\varphi_m, z_m), (\varphi_{N+n}, z_{N+n})) \\
&\quad \times \det [G_\omega ((\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}))]_{\substack{k \in \{1, \dots, N\} \setminus \{m\} \\ l \in \{1, \dots, N\} \setminus \{n\}}} \\
&= \sum_{n=1}^N (-1)^{m+n} G_\omega ((\varphi_n, z_n), (\varphi_{N+m}, z_{N+m})) \\
&\quad \times \det [G_\omega ((\varphi_k, z_k), (\varphi_{N+l}, z_{N+l}))]_{\substack{k \in \{1, \dots, N\} \setminus \{n\} \\ l \in \{1, \dots, N\} \setminus \{m\}}}
\end{aligned}$$

Then, the assertion directly follows from Lemma 2.5. ■

### Corollary 2.3

If Condition 2.1 holds true then, for all  $\beta \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$ ,  $\mathcal{X}_1 = \{x_1, \dots, x_N\}$ ,  $\mathcal{X}_2 = \{x_{N+1}, \dots, x_{2N}\} \subset \mathbb{Z}^d$  such that  $|\mathcal{X}_1| = |\mathcal{X}_2| = N$ , and  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$ ,

$$\mathbb{E} \left[ \max_{\sigma_1, \dots, \sigma_{2N}} \left| \det [G_\omega ((\chi_I(H_\omega) \mathbf{e}_{x_k, \sigma_k}, z_k), (\chi_I(H_\omega) \mathbf{e}_{x_{N+l}, \sigma_{N+l}}, z_{N+l}))]_{k,l=1}^N \right| \right] \leq D e^{-\mu \mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{X}_2)},$$

where  $\mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{X}_2)$  is the Hausdorff distance (1) between the  $N$ -particle configurations  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . Recall that  $\chi_I$  is the characteristic function of the Borel set  $I$  and note that the constants  $\epsilon$ ,  $D$  and  $\mu$  are exactly the same as in Condition 2.1.

**Proof.** Combine Condition 2.1 and Theorem 2.2 with Equations (9) and (10). ■

The analogue of [4, Theorem 1.1], i.e., an estimate like Corollary 2.3 for the many-point correlation functions at fixed  $\omega \in \Omega$ , instead of an estimate for their expectation values, easily follows by replacing Condition 2.1 with a similar bound for a fixed  $\omega \in \Omega$ . We omit the details.

The bound of Corollary 2.3 is a version of [4, Theorem 1.2] which holds at any dimension  $d \in \mathbb{N}$  and for any complex times within the strip  $\mathbb{S}_\beta$ . However, two observations in relation with [4] are important to mention:

- Since, for any  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2 \subset \mathbb{Z}^d$ ,

$$\mathfrak{d}_\epsilon(\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) \leq \max \{ \mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{Y}_1), \mathfrak{d}_\epsilon(\mathcal{X}_2, \mathcal{Y}_2) \},$$

we have

$$\mathfrak{d}_\epsilon(\mathcal{X}, \mathcal{Y}) \leq \mathfrak{d}_\epsilon^{(S)}(\mathcal{X}, \mathcal{Y}) \doteq \min_{\pi \in \mathcal{S}_N} \max_{j \in \{1, \dots, N\}} |x_j - y_{\pi(j)}|^\epsilon \tag{13}$$

for any set  $\mathcal{X} = \{x_1, \dots, x_N\} \subset \mathbb{Z}^d$  and  $\mathcal{Y} = \{y_1, \dots, y_N\} \subset \mathbb{Z}^d$  of  $N \in \mathbb{N}$  (different) lattice points. Here,  $\mathcal{S}_N$  is the set of all permutations  $\pi$  of  $N$  elements. The distance we use, i.e., the Hausdorff distance (1), is therefore weaker than the symmetrized configuration distance  $\mathfrak{d}_\epsilon^{(S)}$  [4, Equation (1.13) and remarks below]. Nevertheless, Corollary 2.3 yields the main features of localization. Whether Corollary 2.3 holds true, at any dimension, when  $\mathfrak{d}_\epsilon$  is replaced with  $\mathfrak{d}_\epsilon^{(S)}$  is an open question. See also discussions of [3, Section 1.3].

- The proofs of [4, Theorems 1.1 and 1.2] use that, for all  $N \in \mathbb{N}$ ,  $x_1, \dots, x_{2N} \in \mathbb{Z}^d$ ,  $\sigma_1, \dots, \sigma_{2N} \in \mathbb{S}$ , and  $t_1, \dots, t_{2N} \in \mathbb{R}$ , the  $N \times N$  matrix

$$\mathbf{M} \doteq \left[ \left\langle \mathbf{e}_{x_{N+l}, \sigma_{N+l}}, \rho(t_{N+l}, t_k) \mathbf{e}_{x_k, \sigma_k} \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^N$$

defines an operator on  $\mathbb{C}^N$  of norm at most 1. This is true even for complex times, provided that

$$z_1 = \dots = z_N \in \mathbb{S}_\beta, \quad z_{N+1} = \dots = z_{2N} \in \mathbb{S}_\beta, \quad \text{Im}(z_N) \leq \text{Im}(z_{N+1}). \quad (14)$$

(cf. [4, Erratum]). However, this is generally not true when  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$  are different from each other. For this reason, instead of a bound on the norm of  $\mathbf{M}$ , our proof uses (in an essential way) the analyticity of correlation functions with respect to complex times.

The results of this section are also reminiscent of [3, Theorem 1.1] where a bound like Corollary 2.3, with the Hausdorff distance but for complex times satisfying (14), can be found for  $n$ -particle correlation functions. Note, additionally, that in [3] a particle interaction is included, but *no* particle statistics is taken into account: The  $n$ -particle Hilbert space is the full space  $\ell^2(\mathbb{Z}^{nd})$ . By contrast, we consider many-fermion systems, which would correspond in [3, Theorem 1.1] to restrict  $\ell^2(\mathbb{Z}^{nd})$  to its subspace of antisymmetric functions. In this situation, the one-particle localization theory cannot be directly used, even in the quasi-free fermion case. Moreover, we do not fix the particle number, by using the grand-canonical setting.

Finally, observe that *quasi-free*, complex-time-ordered, many-point correlations appear in the perturbative expansion of *interacting* correlation functions. See, e.g., [14, Section 5.4.1]. Therefore, as a first step towards the proof of localization in fully interacting fermion systems, it is important to establish localization for these correlations, as stated in Corollary 2.3. For instance, by combining Corollary 2.3 with [14, Theorem 5.4.4], one can show that a local, weak interaction cannot destroy the (static) localization of the thermal, many-point correlation functions of quasi-free fermions in lattices.

## 2.2 Universal Bounds on Determinants from the Hadamard Three-line Theorem

For any permutation  $\pi \in \mathcal{S}_n$  of  $n \in \mathbb{N}$  elements with sign  $(-1)^\pi$ , we define the monomial  $\mathbb{O}_\pi(A_1, \dots, A_n) \in \text{CAR}(\mathfrak{h})$  in  $A_1, \dots, A_n \in \text{CAR}(\mathfrak{h})$  by the product

$$\mathbb{O}_\pi(A_1, \dots, A_n) \doteq (-1)^\pi A_{\pi^{-1}(1)} \cdots A_{\pi^{-1}(n)}. \quad (15)$$

In other words,  $\mathbb{O}_\pi$  places the operator  $A_k$  at the  $\pi(k)$ th position in the monomial  $(-1)^\pi A_{\pi^{-1}(1)} \cdots A_{\pi^{-1}(n)}$ . Further, for all  $k, l \in \{1, \dots, n\}$ ,  $k \neq l$ ,

$$\pi_{k,l} : \{1, 2\} \rightarrow \{1, 2\} \quad (16)$$

is the identity function if  $\pi(k) < \pi(l)$ , otherwise  $\pi_{k,l}$  interchanges 1 and 2. (Remark that  $\mathbb{O}_{z_k, z_l}$  (6) is equal to  $\mathbb{O}_{\pi_{k,l}}$  for a conveniently chosen permutation  $\pi$ .) Then, the following identities holds true for quasi-free states:

### Lemma 2.4

Let  $\rho$  be a quasi-free state on  $\text{CAR}(\mathfrak{h})$ . For any  $N \in \mathbb{N}$ , all permutations  $\pi \in \mathcal{S}_{2N}$  and  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$ ,

$$\begin{aligned} & \det \left[ \rho \left( \mathbb{O}_{\pi_{k, N+l}} \left( a(\varphi_k)^*, a(\varphi_{N+l}) \right) \right) \right]_{k,l=1}^N \\ &= \rho \left( \mathbb{O}_\pi \left( a(\varphi_1)^*, \dots, a(\varphi_N)^*, a(\varphi_{2N}), \dots, a(\varphi_{N+1}) \right) \right). \end{aligned} \quad (17)$$

**Proof.** See [18, Lemma 3.1]. Compare with (28). ■

Using Lemma 2.4 and the Hadamard three-line theorem (via Corollary 4.2), we obtain a universal bound on determinants of the form (11):

**Lemma 2.5**

Fix  $H = H^* \in \mathcal{B}(\mathfrak{h})$ . Let the quasi-free state  $\rho$  on  $\text{CAR}(\mathfrak{h})$  be the unique KMS state at inverse temperature  $\beta \in \mathbb{R}^+$  associated with the unique strongly continuous group  $\{\tau_t\}_{t \in \mathbb{R}}$  of automorphisms of  $\text{CAR}(\mathfrak{h})$  satisfying (8)-(9) for  $H_\omega = H$ . Then, for any  $N \in \mathbb{N}$ ,  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$  and  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$  (see (4)),

$$\left| \det \left[ \rho \left( \mathbb{O}_{z_k, z_{N+l}} \left( \tau_{z_k} (a(\varphi_k)^*), \tau_{z_{N+l}} (a(\varphi_{N+l})) \right) \right) \right]_{k,l=1}^N \right| \leq \prod_{k=1}^{2N} \|\varphi_k\|_{\mathfrak{h}}.$$

**Proof.** Fix all parameters of the lemma and choose any permutation  $\pi \in \mathcal{S}_{2N}$  such that, for all  $k, l \in \{1, \dots, N\}$ ,

$$\text{Im}(z_k) \leq \text{Im}(z_{N+l}) \Leftrightarrow \pi(k) < \pi(N+l). \quad (18)$$

Then, by Lemma 2.4,

$$\begin{aligned} & \det \left[ \rho \left( \mathbb{O}_{z_k, z_{N+l}} \left( \tau_{z_k} (a(\varphi_k)^*), \tau_{z_{N+l}} (a(\varphi_{N+l})) \right) \right) \right]_{k,l=1}^N \\ &= \rho \left( \mathbb{O}_\pi \left( \tau_{z_1} (a(\varphi_1)^*), \dots, \tau_{z_N} (a(\varphi_N)^*), \tau_{z_{2N}} (a(\varphi_{2N})), \dots, \tau_{z_{N+1}} (a(\varphi_{N+1})) \right) \right). \end{aligned} \quad (19)$$

Define the entire analytic map  $\Upsilon$  from  $\mathbb{C}^{2N}$  to  $\mathbb{C}$  by

$$\begin{aligned} \Upsilon(\xi_1, \dots, \xi_{2N}) &\doteq \rho \left( \mathbb{O}_\pi \left( \tau_{\xi_1 + \dots + \xi_{2N - \pi(1) + 1}} (a(\varphi_1)^*), \dots, \tau_{\xi_1 + \dots + \xi_{2N - \pi(N) + 1}} (a(\varphi_N)^*), \right. \right. \\ &\quad \left. \left. \tau_{\xi_1 + \dots + \xi_{2N - \pi(2N) + 1}} (a(\varphi_{2N})), \dots, \tau_{\xi_1 + \dots + \xi_{2N - \pi(N+1) + 1}} (a(\varphi_{N+1})) \right) \right). \end{aligned} \quad (20)$$

Now, impose additionally that the permutation  $\pi$  of  $2N$  elements used in (19)-(20) satisfies, for any  $k, l \in \{1, \dots, N\}$ ,  $k \neq l$ , the conditions

$$\text{Im}(z_k) < \text{Im}(z_l) \Leftrightarrow \pi(k) < \pi(l); \quad \text{Im}(z_{2N-k}) < \text{Im}(z_{2N-l}) \Leftrightarrow \pi(2N-k) < \pi(2N-l).$$

Ergo, by (18),

$$\text{Im}(z_{\pi^{-1}(1)}) \leq \dots \leq \text{Im}(z_{\pi^{-1}(N)}) \leq \text{Im}(z_{\pi^{-1}(2N)}) \leq \dots \leq \text{Im}(z_{\pi^{-1}(N+1)}) \quad (21)$$

and, by (19)-(20), the assertion follows if we can bound the function  $\Upsilon$  on the tube  $\mathfrak{T}_{2N}$  defined below by (33) for  $n = 2N$ . Since  $\Upsilon$  is uniformly bounded on  $\mathfrak{T}_{2N}$ , it suffices to bound the function  $\Upsilon$  on the boundary

$$\partial \mathfrak{T}_{2N} \doteq \left\{ (\xi_1, \dots, \xi_{2N}) \in \mathbb{C}^{2N} : \forall j \in \{1, \dots, 2N\}, \text{Im}(\xi_j) \in \{-\beta, 0\}, \sum_{j=1}^{2N} \text{Im}(\xi_j) \in \{-\beta, 0\} \right\},$$

by Corollary 4.2. By the KMS property [14, Section 5.3.1], note that, for all  $t_1, \dots, t_{2N} \in \mathbb{R}$  and  $k \in \{1, \dots, 2N\}$ ,

$$\Upsilon(t_1, \dots, t_{k-1}, t_k - i\beta, t_{k+1}, \dots, t_{2N}) = \Upsilon(t_{k+1}, \dots, t_{2N}, t_1, \dots, t_k)$$

while

$$\sup_{(\xi_1, \dots, \xi_{2N}) \in \mathbb{R}^{2N}} |\Upsilon(\xi_1, \dots, \xi_{2N})| \leq \prod_{k=1}^{2N} \|\varphi_k\|_{\mathfrak{h}}.$$

As a consequence,

$$\sup_{(\xi_1, \dots, \xi_{2N}) \in \mathfrak{X}_{2N}} |\Upsilon(\xi_1, \dots, \xi_{2N})| = \sup_{(\xi_1, \dots, \xi_{2N}) \in \partial \mathfrak{X}_{2N}} |\Upsilon(\xi_1, \dots, \xi_{2N})| \leq \prod_{k=1}^{2N} \|\varphi_k\|_{\mathfrak{h}} \quad (22)$$

and the assertion follows from (19), (20) and (33). ■

Observe that estimates like (22) are related to the generalization of the Hölder inequality to non-commutative  $L^p$ -spaces. See, e.g., [19].

### 3 Decay of Pfaffian Correlation Functionals

An estimate similar to Corollary 2.3 can be obtained for Pfaffians of the two-point correlation functions on the  $d$ -dimensional square lattice  $\mathbb{Z}^d$ , by the same methods, because they also can be seen, like in the proof of Lemma 2.5, as many-point correlation functions of quasi-free fermions.

(i): For a fixed parameter  $\epsilon \in (0, 1]$  and any subset  $\mathcal{X} \subset \mathbb{Z}^d$  we define the quantity

$$\ell_\epsilon(\mathcal{X}) \doteq \max_{x \in \mathcal{X}} \min_{y \in \mathcal{X} \setminus \{x\}} |x - y|^\epsilon. \quad (23)$$

It is a kind of splitting width of the configuration  $\mathcal{X}$  with respect to the metric  $(x, y) \mapsto |x - y|^\epsilon$ : This quantity is large whenever isolated points of  $\mathcal{X}$  are spread in space, but it stays small if the points are packed in clusters containing at least two points. It is used here to quantify the localization of Pfaffian correlation functionals. Observe that  $\ell_\epsilon$  is similar to the splitting width of a configuration defined by [3, Equation (5.9)].

(ii): For any  $N \in \mathbb{N}$ , the Pfaffian of a  $2N \times 2N$  skew-symmetric complex matrix  $M$  is defined by

$$\text{Pf} [M_{k,l}]_{k,l=1}^{2N} \doteq \frac{1}{2^N N!} \sum_{\pi \in \mathcal{S}_{2N}} (-1)^\pi \prod_{j=1}^N M_{\pi(2j-1), \pi(2j)}, \quad (24)$$

where we recall that  $\mathcal{S}_{2N}$  is the set of all permutations of  $2N$  elements.

(iii): Let the field operators be defined by

$$B(\varphi) \doteq a(\varphi)^* + a(\varphi), \quad \varphi \in \mathfrak{h}.$$

For  $(x, \sigma) \in \mathbb{Z}^d \times \mathbb{S}$  and  $\varphi = \mathbf{e}_{x,\sigma}$  or  $\varphi = i\mathbf{e}_{x,\sigma}$ , we obtain the on-site Majorana fermions of [4, Equation (1.22)].

Below, we show that strong one-body localization, in the sense of Condition 2.1, yields the localization of many-point correlations of field operators with respect to the quantity (23). This is achieved by estimating, in Theorem 3.1, Pfaffians of the form

$$\text{Pf} [\mathcal{G}_\omega((\varphi_k, z_k), (\varphi_l, z_l))]_{k,l=1}^{2N} \quad (25)$$

in terms of the entries of one single row. In (25),  $\beta \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$ ,  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$  are normalized vectors,  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$  and

$$\mathcal{G}_\omega((\varphi_k, z_k), (\varphi_l, z_l)) \doteq \rho_\omega(\mathbb{O}_{z_k, z_l}(\tau_{z_k}^{(\omega)}(B(\varphi_k)), \tau_{z_l}^{(\omega)}(B(\varphi_l))))$$

is the two-point, complex-time-ordered correlation function of field operators associated with the quasi-free state  $\rho_\omega$ . See Section 2.1. Observe that the matrix in the Pfaffian of (25) is skew-symmetric, by construction.

### Theorem 3.1

Let  $\{H_\omega\}_{\omega \in \Omega} \subset \mathcal{B}(\mathfrak{h})$  be a  $\mathfrak{F}$ -measurable family  $\{H_\omega\}_{\omega \in \Omega} \subset \mathcal{B}(\mathfrak{h})$  of bounded (one-particle) Hamiltonians satisfying Condition 2.1. Then, for all  $\omega \in \Omega$ ,  $\beta \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$ ,  $\mathcal{X} = \{x_1, \dots, x_{2N}\} \subset \mathbb{Z}^d$  such that  $|\mathcal{X}| = 2N$ , and  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$  (see (4)),

$$\mathbb{E} \left[ \max_{\substack{p_1, \dots, p_{2N} \in \{0,1\} \\ \sigma_1, \dots, \sigma_{2N} \in \mathbb{S}}} \text{Pf} [\mathcal{G}_\omega ((i^{p_k} \chi_I(H_\omega) \mathbf{e}_{x_k, \sigma_k}, z_k), (i^{p_l} \chi_I(H_\omega) \mathbf{e}_{x_l, \sigma_l}, z_l))]_{k,l=1}^{2N} \right] \leq 2D e^{-\mu \ell_\epsilon(\mathcal{X})}.$$

The constants  $\epsilon$ ,  $D$  and  $\mu$  are exactly the same as in Condition 2.1.

**Proof.** The proof uses similar arguments as for determinantal correlation functionals. We present them in four steps:

Step 1: Similar to determinants, Pfaffians have a Laplace expansion with respect to any row of its matrix:

$$\begin{aligned} \text{Pf} [\mathcal{G}_\omega ((\varphi_k, z_k), (\varphi_l, z_l))]_{k,l=1}^{2N} &= \sum_{n=1, n \neq m}^{2N} (-1)^{m+n+1+\theta(m-n)} \mathcal{G}_\omega ((\varphi_m, z_m), (\varphi_n, z_n)) \\ &\quad \times \text{Pf} [\mathcal{G}_\omega ((\varphi_k, z_k), (\varphi_l, z_l))]_{\substack{k \in \{1, \dots, 2N\} \setminus \{m\} \\ l \in \{1, \dots, 2N\} \setminus \{n\}}} \end{aligned} \quad (26)$$

for any  $\beta \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$ ,  $m \in \{1, \dots, 2N\}$ ,  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$  and  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$ , where  $\theta$  is the Heaviside step function. Compare (26) with (12).

Step 2: Since  $\rho_\omega$  is, by definition, a quasi-free state, observe that

$$\rho_\omega (\text{B}(\varphi_1) \cdots \text{B}(\varphi_{2N})) = \text{Pf} [\rho_\omega (\mathbb{O}_{\text{id}_{k,l}} (\text{B}(\varphi_k), \text{B}(\varphi_l)))]_{k,l=1}^{2N}, \quad (27)$$

for all  $N \in \mathbb{N}$  and  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$ , where  $\text{id}_{k,l}$  is defined by (16),  $\pi$  being the neutral element  $\text{id}$  of the permutation group  $\mathcal{S}_{2N}$ . See, e.g., [16, Equations (6.6.9) and (6.6.10)]. For any permutation  $\pi \in \mathcal{S}_{2N}$  ( $N \in \mathbb{N}$ ), Equation (27) can be written as

$$\rho_\omega (\mathbb{O}_\pi (\text{B}(\varphi_1), \dots, \text{B}(\varphi_{2N}))) = \text{Pf} [\rho_\omega (\mathbb{O}_{\pi_{k,l}} (\text{B}(\varphi_k), \text{B}(\varphi_l)))]_{k,l=1}^{2N}, \quad (28)$$

where  $\mathbb{O}_\pi$  and the permutation  $\pi_{k,l}$  are defined by (15) and (16), respectively. See, e.g., [17, Proposition B.2]. Compare (28) with Lemma 2.4.

Step 3: Then, given  $2N \in \mathbb{N}$  complex numbers  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$  ( $\beta \in \mathbb{R}^+$ ), similar to (21), we choose a permutation  $\pi \in \mathcal{S}_{2N}$  such that, for any  $k, l \in \{1, \dots, 2N\}$ ,  $k \neq l$ ,

$$\pi(k) < \pi(l) \Leftrightarrow [\text{Im}(z_k) < \text{Im}(z_l)] \vee [(\text{Im}(z_k) = \text{Im}(z_l)) \wedge (k < l)].$$

Using the Hadamard three-line theorem (via Corollary 4.2), we thus obtain a universal bound on Pfaffians of the form

$$\left| \text{Pf} \left[ \rho_\omega (\mathbb{O}_{z_k, z_l} (\tau_{z_k} (\text{B}(\varphi_k)), \tau_{z_l} (\text{B}(\varphi_l)))) \right]_{k,l=1}^{2N} \right| \leq \prod_{k=1}^{2N} \|\varphi_k\|_{\mathfrak{h}} \quad (29)$$

for any  $N \in \mathbb{N}$ ,  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$  and  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$ . To get this inequality, we have used that

$$\|\text{B}(\varphi)\|_{\text{CAR}(\mathfrak{h})} = \|\varphi\|_{\mathfrak{h}}, \quad \varphi \in \mathfrak{h}.$$

Compare (29) with Lemma 2.5.



Step 4: We infer from (26) and (29) that

$$\left| \text{Pf} [\mathcal{G}_\omega ((\varphi_k, z_k), (\varphi_l, z_l))]_{k,l=1}^{2N} \right| \leq \sum_{n=1, n \neq m}^{2N} |\mathcal{G}_\omega ((\varphi_m, z_m), (\varphi_n, z_n))| \quad (30)$$

for any  $\beta \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$ ,  $m \in \{1, \dots, 2N\}$ ,  $\varphi_1, \dots, \varphi_{2N} \in \mathfrak{h}$  and  $z_1, \dots, z_{2N} \in \mathbb{S}_\beta$ . By gauge invariance, Condition 2.1 yield the inequality

$$\sum_{x_2 \in \mathbb{Z}^d: |x_1 - x_2|^\epsilon \geq R} \mathbb{E} \left[ \sup_{z_1, z_2 \in \mathbb{S}_\beta} \max_{\substack{p_1, p_2 \in \{0,1\} \\ \sigma_1, \sigma_2 \in \mathbb{S}}} |\mathcal{G}_\omega ((i^{p_1} \mathbf{e}_{x_1, \sigma_1}, z_1), (i^{p_2} \mathbf{e}_{x_2, \sigma_2}, z_2))| \right] \leq 2D e^{-\mu R}. \quad (31)$$

Therefore, the assertion is a direct consequence of Inequalities (30) and (31). ■

Theorem 3.1 is a version of [4, Theorem 1.4] which holds true at any dimension  $d \in \mathbb{N}$  and for any complex times within the strip  $\mathbb{S}_\beta$ . A result similar to [4, Theorem 1.3] for the many-point correlation functions of field operators at fixed  $\omega \in \Omega$ , instead of an estimate for their expectation values, easily follows by replacing Condition 2.1 with a similar bound for a fixed  $\omega \in \Omega$ . We again omit the details.

One observation in relation with [4, Theorems 1.3 and 1.4] is important to mention: For any disjoint partition  $\mathcal{X}_1, \mathcal{X}_2$  of  $\mathcal{X} \subset \mathbb{Z}^d$ , we deduce from (1) and (23) that

$$\ell_\epsilon(\mathcal{X}) \leq \mathfrak{d}_\epsilon(\mathcal{X}_1, \mathcal{X}_2). \quad (32)$$

By (13) it follows that, for any disjoint partition  $\mathcal{X}_1, \mathcal{X}_2$  of  $\mathcal{X} \subset \mathbb{Z}^d$  such that  $|\mathcal{X}_1| = |\mathcal{X}_2|$ ,

$$\ell_\epsilon(\mathcal{X}) \leq \mathfrak{d}_\epsilon^{(S)}(\mathcal{X}_1, \mathcal{X}_2).$$

Note that the right hand side of the above inequality corresponds to [4, Equation (1.27)] when  $\mathcal{X}_1 = \{x_1, x_3, \dots, x_{2N-1}\} \subset \mathbb{Z}^{d=1}$  and  $\mathcal{X}_2 = \{x_2, x_4, \dots, x_{2N}\} \subset \mathbb{Z}^{d=1}$  for  $2N \in \mathbb{N}$  (different) lattice points. Therefore, our notion of localization for Pfaffian correlation functionals is weaker than the one used in [4, Theorems 1.3 and 1.4]. Note, however, that, like  $\ell_\epsilon(\mathcal{X})$ , the quantity [4, Equation (1.27)] stays small if the points of  $\mathcal{X}$  are packed in clusters containing exactly two points  $\{x_k, x_{k+1}\}$ ,  $k = 1, 3, \dots, 2N$ , independently of how far-apart from each other the clusters are. Therefore, our notion of localization captures qualitatively the behavior of the one used in [4, Theorems 1.3 and 1.4].

## 4 Appendix: Log convexity of Multivariable Analytic Functions on Tubes

Fix  $\beta \in \mathbb{R}^+$ . Let

$$\mathfrak{T}_1 \doteq \{\xi \in \mathbb{C} : \text{Im} \{\xi\} \in [-\beta, 0]\} = \mathbb{S}_\beta,$$

(see (4)) and  $f : \mathfrak{T}_1 \rightarrow \mathbb{C}$  be a bounded continuous function. Define the map  $B_f^{(1)} : [-\beta, 0] \rightarrow [-\infty, \infty)$  by

$$B_f^{(1)}(s) \doteq \ln \left( \sup_{t \in \mathbb{R}} |f(t + is)| \right).$$

We use the convention  $\ln 0 \doteq -\infty$  and  $0 \cdot (-\infty) \doteq -\infty$ . Then, the Hadamard three-line theorem [20, Theorem 12.3] states:

### Theorem 4.1

Let  $\beta \in \mathbb{R}^+$  and  $f : \mathfrak{T}_1 \rightarrow \mathbb{C}$  be a bounded continuous function. If  $f$  is holomorphic in the interior of  $\mathfrak{T}_1$  then  $B_f^{(1)}$  is a convex function.

This theorem has the following generalization to holomorphic functions in several variables: For all  $n \in \mathbb{N}$ , let  $K_n \subset \mathbb{R}^n$  be the simplex

$$K_n \doteq \{(s_1, \dots, s_n) : s_1, \dots, s_n \in [-\beta, 0], s_1 + \dots + s_n \geq -\beta\}$$

and define, for all  $n \in \mathbb{N}$ , the ‘‘tube’’

$$\mathfrak{T}_n \doteq \{(\xi_1, \dots, \xi_n) \in \mathbb{C}^n : (\operatorname{Im} \{\xi_1\}, \dots, \operatorname{Im} \{\xi_n\}) \in K_n\}. \quad (33)$$

Define further the map  $B_f^{(n)} : K_n \rightarrow [-\infty, \infty)$  by

$$B_f^{(n)}(s_1, \dots, s_n) \doteq \ln \left( \sup_{(t_1, \dots, t_n) \in \mathbb{R}^n} |f(t_1 + is_1, \dots, t_n + is_n)| \right)$$

with  $f : \mathfrak{T}_n \rightarrow \mathbb{C}$  being a bounded continuous function. Then, we obtain the following corollary:

**Corollary 4.2**

Let  $\beta \in \mathbb{R}^+$ ,  $n \in \mathbb{N}$  and  $f : \mathfrak{T}_n \rightarrow \mathbb{C}$  be a bounded continuous function. If  $f$  is holomorphic in the interior of  $\mathfrak{T}_n$  then  $B_f^{(n)}$  is a convex function.

**Proof.** Fix all parameters of the corollary and assume that  $f$  is holomorphic in the interior of  $\mathfrak{T}_n$ . Take  $(s_1, \dots, s_n) \in K_n$  and  $(s'_1, \dots, s'_n) \in K_n$ . For all  $(t_1, \dots, t_n) \in \mathbb{R}^n$ , define the function  $F_{(t_1, \dots, t_n)} : \mathfrak{T}_1 \rightarrow \mathbb{C}$  by

$$F_{(t_1, \dots, t_n)}(\xi) \doteq f(t_1 + i(s_1(1 + \xi\beta^{-1}) - s'_1\xi\beta^{-1}), \dots, t_n + i(s_n(1 + \xi\beta^{-1}) - s'_n\xi\beta^{-1})).$$

For all  $\xi \in \mathfrak{T}_1$ , note that

$$(t_1 + i(s_1(1 + \xi\beta^{-1}) - s'_1\xi\beta^{-1}), \dots, t_n + i(s_n(1 + \xi\beta^{-1}) - s'_n\xi\beta^{-1})) \in \mathfrak{T}_n,$$

by convexity of  $K_n$ . This function is bounded and continuous on  $\mathfrak{T}_1$ , and holomorphic in the interior of  $\mathfrak{T}_1$ . Hence, by Theorem 4.1, for all  $\alpha \in [0, 1]$ ,

$$\begin{aligned} \ln \left( \sup_{t \in \mathbb{R}} |F_{(t_1, \dots, t_n)}(t - i\alpha\beta)| \right) &\leq \alpha \ln \left( \sup_{t \in \mathbb{R}} |F_{(t_1, \dots, t_n)}(t - i\beta)| \right) \\ &+ (1 - \alpha) \ln \left( \sup_{t \in \mathbb{R}} |F_{(t_1, \dots, t_n)}(t)| \right). \end{aligned} \quad (34)$$

Since  $\ln$  is a monotonically increasing, continuous function, for all  $\alpha \in [0, 1]$ ,

$$\begin{aligned} B_f^{(n)}(\alpha s'_1 + (1 - \alpha)s_1, \dots, \alpha s'_n + (1 - \alpha)s_n) &= \ln \left( \sup_{(t_1, \dots, t_n) \in \mathbb{R}^n} \sup_{t \in \mathbb{R}} |F_{(t_1, \dots, t_n)}(t - i\alpha\beta)| \right) \\ &= \sup_{(t_1, \dots, t_n) \in \mathbb{R}^n} \ln \left( \sup_{t \in \mathbb{R}} |F_{(t_1, \dots, t_n)}(t - i\alpha\beta)| \right), \end{aligned}$$

which, by (34), in turn implies that

$$B_f^{(n)}(\alpha s'_1 + (1 - \alpha)s_1, \dots, \alpha s'_n + (1 - \alpha)s_n) \leq (1 - \alpha) B_f^{(n)}(s_1, \dots, s_n) + \alpha B_f^{(n)}(s'_1, \dots, s'_n)$$

for all  $\alpha \in [0, 1]$ . ■

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## References

- [1] D.M. Basko, I.L. Aleiner, B.L. Altshuler, Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states. *Ann. Phys.* **321** (2006) 1126–1205
- [2] V. Chulaevsky and Y.M. Suhov, Eigenfunctions in a Two-Particle Anderson Tight Binding Model, *Commun. Math. Phys.* **289** (2009) 701–723
- [3] M. Aizenman and S. Warzel, Localization Bounds for Multiparticle Systems, *Commun. Math. Phys.* **290**(3) (2009) 903–934
- [4] R. Sims and S. Warzel, Decay of Determinantal and Pfaffian Correlation Functionals in One-Dimensional Lattices, *Commun. Math. Phys.* **347** (2016) 903–931; Erratum, *Commun. Math. Phys.* In preparation (2017).
- [5] H. Abdul-Rahmana and G. Stolz, A uniform area law for the entanglement of eigenstates in the disordered XY chain, *J. Math. Phys.* **56** (2015) 121901
- [6] R. Seiringer and S. Warzel, Decay of correlations and absence of superfluidity in the disordered Tonks–Girardeau gas, *New J. Phys.* **18** (2016) 035002 (14pp)
- [7] M. Gebert and M. Lemm, On Polynomial Lieb–Robinson Bounds for the XY Chain in a Decaying Random Field *J Stat Phys* **164** (2016) 667–679
- [8] H. Abdul-Rahman, B. Nachtergaele, R. Sims, G. Stolz, Entanglement Dynamics of Disordered Quantum XY Chains. *Lett. Math. Phys.* **106**(5) (2016) 649–674.
- [9] G. Stolz, Many-body localization for disordered Bosons, *New J. Phys.* **18** (2016) 031002 (3 pp)
- [10] R. Mavi, Localization for the Ising model in a transverse field with generic aperiodic disorder, arXiv:1605.06514 (2016)
- [11] H. Abdul-Rahman, B. Nachtergaele, R. Sims, G. Stolz, Localization properties of the disordered XY spin chain, *Ann. Phys. (Berlin)* 1600280 (2017) (17pp)
- [12] V. Beaud and S. Warzel, Low-energy Fock-space localization for attractive hard-core particles in disorder, arXiv:1703.02465 (2017)
- [13] P. Le Doussal, S.N. Majumdar, G. Schehr, Periodic Airy process and equilibrium dynamics of edge fermions in a trap, arXiv:1702.06931 (2017)
- [14] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol. II, 2nd ed.* Springer-Verlag, New York, 1996
- [15] M. Aizenman and S. Warzel, *Random Operators: Disorder Effects on Quantum Spectra and Dynamics*, GSM 168, 2016
- [16] D.E. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Mathematical Monographs, Oxford University Press, 1998
- [17] J. Feldman, H. Knorrer and E. Trubowitz, Fermionic Functional Integrals and the Renormalization Group, CRM Monograph Series, Volume 16 (American Mathematical Society, 2002).
- [18] J.-B. Bru and W. de Siqueira Pedra, Universal Bounds for Large Determinants from Non-Commutative Hölder Inequalities in Fermionic Constructive Quantum Field Theory, *M3AS: Mathematical Models and Methods in Applied Sciences* **12**(27) (2017)

- [19] H. Araki and T. Masuda: Positive Cones and  $L_p$ -Spaces for von Neumann Algebras, *Publ. RIMS, Kyoto Univ.***18** (1982) 339–411
- [20] B. Simon, *Convexity: An Analytic Viewpoint*, Cambridge Tracts in Mathematics 187, Cambridge University Press, 2011