Vector-valued operators, optimal weighted estimates and the C_p condition

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Abstract

In this paper some new results concerning the C_p classes introduced by Muckenhoupt [28] and later extended by Sawyer [39], are provided. In particular we extend the result to the full range expected $p > 0$, to the weak norm, to other operators and to their vector-valued extensions. Some of those results rely upon sparse domination results that in some cases we provide as well. We will also provide sharp weighted estimates for vector valued extensions relying on those sparse domination results.

1. Introduction

1.1. The C_p condition

We recall that a weight w, that is, a non-negative locally integrable function, belongs to the Muckenhoupt A_p class for $1 < p < \infty$ if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the axes. And in the case $p = 1$ we say that $w \in A_1$ if

$$Mw \leq \kappa w \quad \text{a.e.}$$

and we define $[w]_{A_1} = \inf\{\kappa > 0 : Mw \leq \kappa w \quad \text{a.e.}\}$. The quantity $[w]_{A_p}$ is called the A_p constant or characteristic of the weight w. We say that $w \in A_\infty$ if

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)}\int_Q M(w\chi_Q) < \infty.$$

Calderón-Zygmund principle states that for each singular operator there exists a maximal operator that "controls" it. A paradigmatic example of that principle is the Coifman-Fefferman estimate, namely, for each $0 < p < \infty$ and every $w \in A_\infty$ there exists $c = c_{n,w,p} > 0$ such that

$$\|T^*f\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)}. \tag{1.1}$$

where T^* stands for the maximal Calderón-Zygmund operator (see Subsection 2.1 for the precise definition). This kind of estimate plays a central role in modern euclidean Harmonic Analysis. In particular we emphasize its key role in the main result in [23].

The estimate in (1.1) leads to a natural question. Is the A_∞ condition necessary for (1.1) to hold? B. Muckenhoupt [28] provided a negative answer to the question. He proved that in the case when T is

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[1]Supported by the Basque Government through the BERC 2018-2021 program and by Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa excellence accreditation SEV-2013-0323

[2]Supported by the project MTM2017-82160-C2-1-P funded by (AEI/FEDER, UE) and acronym "HAQMEC".

[3]grants UNLP 11/X752 and PICT 2014-1771 ANPCYT, Argentina.

[4]Supported by Juan de la Cierva - Formación 2015 FJCI-2015-24547

[5]Supported by CONICET PIP 11220130100329CO, Argentina.

the Hilbert transform, (1.1) does not imply that w satisfies the A_∞ condition. He showed that if (1.1) holds with $p > 1$ and T is the Hilbert transform, then $w \in C_p$, that is if there exist $c, \delta > 0$ such that for every cube Q and every subset $E \subseteq Q$ we have that

$$w(E) \leq c\left(\frac{|E|}{|Q|}\right)^{\delta} \int_{\mathbb{R}^n} M(\chi_Q)^p w.$$

Observe that that $A_\infty \subset C_p$ for every $p > 0$. B. Muckenhoupt showed, in dimension one, that if $w \in A_p$, $1 < p < \infty$, then $w\chi_{[0,\infty)} \in C_p$. In the same paper it was conjectured that the C_p condition is also sufficient for (1.1) to hold, which is still open. Not much later, the necessity of the C_p condition was extended to arbitrary dimension and a converse result was provided by E.T. Sawyer [39]. More precisely he proved the following result.

Theorem I (E. Sawyer [39]). *Let $1 < p < \infty$ and let $w \in C_{p+\epsilon}$ for some $\epsilon > 0$. Then*

$$\|T^*(f)\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)}. \tag{1.2}$$

Also, relying upon Sawyer's techniques, K. Yabuta [40, Theorem 2] established the following result extending the classical result of C. Fefferman and E. Stein relating M and the sharp maximal $M^\#$ function [10, 17].

Theorem II (K. Yabuta [40]). *Let $1 < p < \infty$ and let $w \in C_{p+\epsilon}$ for some $\epsilon > 0$. Then*

$$\|M(f)\|_{L^p(w)} \leq c\|M^\# f\|_{L^p(w)}. \tag{1.3}$$

The proof of this result, although based on a key lemma from [39], is simpler than the proof of (1.2) by Sawyer. In this paper we will present a different approach for proving (1.2) based on Yabuta's lemma which is conceptually much simpler and much flexible. Furthermore, we extend estimate (1.2) to the full expected range, namely $0 < p < \infty$ and to some vector-valued operators. We remark that in the last case, the classical good-λ seems not be applicable. None of the known methods yield this result.

Remark 1. *We remark that we do not know how to extend Theorem II to the full range $0 < p < \infty$ as in Theorems 1 and 2 below. However, this lemma is the key to prove those theorems in the full range.*

We remark that more recently, A. Lerner [19] provided another proof of Yabuta's result (1.3) improving it slightly. He established, using a different argument, that if a weight w satisfies the following estimate

$$w(E) \leq \left(\frac{|E|}{|Q|}\right)^{\delta} \int_{\mathbb{R}^n} \varphi_p\left(M(\chi_Q)\right) w,$$

where

$$\int_0^1 \varphi_p(t)\frac{dt}{t^{p+1}} < \infty$$

then (1.3) holds.

Let us now turn attention to our contribution. We say that an operator T satisfies the condition (D) if there are some constants, $\delta \in (0, 1)$ and $c > 0$ such that for all f,

$$M_\delta^\#(Tf)(x) \leq cMf(x). \tag{D}$$

Some examples of operators satisfying condition (D) are:

- **Calderón-Zygmund operators** These operators are generalization of the regular singular integral operators as defined above. This was observed in [2].
- **Weakly strongly singular integral operators** These operators were considered by C. Fefferman in [9].
- **Pseudo-differential operators**. To be more precise, the pseudo-differential operators satisfying condition (D) are those that belong to the Hörmander class ([13]).
- **Oscillatory integral operators** These operators were by introduced by Phong and Stein [35].

The proof that last three cases satisfy condition (D) can be found in [1].

It is also possible to consider a suitable variant of condition (D) which will allow us to treat some vector-valued operators. We recall that given an operator G, $1 < q < \infty$ and $\boldsymbol{f} = \{f_j\}_{j=1}^{\infty}$ we define the vector-valued extension $\overline{G}_q$ by

$$\overline{G}_q \boldsymbol{f}(x) = \left(\sum_{j=1}^{\infty} |G(f_j)(x)|^q \right)^{\frac{1}{q}}$$

We say that an operator T satisfies the (D_q) condition with $1 < q < \infty$ if for every $0 < \delta < 1$ there exists a finite constant $c = C_{\delta,q,T} >$ such that

$$M_\delta^{\#}\left(\overline{T}_q \boldsymbol{f}\right)(x) \leq c\, M(|\boldsymbol{f}|_q)(x) \tag{D_q}$$

where $|\boldsymbol{f}|_q(x) = \left(\sum_{j=1}^{\infty} |f_j(x)|^q\right)^{\frac{1}{q}}$. Two examples of operators satisfying the (D_q) condition are the Hardy-Littlewood maximal operator ([6] Proposition 4.4) and any Calderón-Zygmund operator ([34] Lemma 3.1).

Next theorems extend and improve the main result from [39] since we are able to provide some answers for the range $0 < p \leq 1$ and to consider vector-valued extensions. It is not clear that the method of [39] can be extended to cover both situations. Furthermore, we can extend this result to the multilinear context and other operators like fractional integrals.

Theorem 1. *Let T be an operator satisfying the (D) condition. Let $0 < p < \infty$ and let $w \in C_{\max\{1,p\}+\epsilon}$ for some $\epsilon > 0$. Then*

$$\|Tf\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)}. \tag{1.4}$$

Additionally, if $1 < q < \infty$ and T satisfies the (D_q) condition then

$$\|\overline{T}_q \boldsymbol{f}\|_{L^p(w)} \leq c\|M(|\boldsymbol{f}|_q)\|_{L^p(w)}.$$

Remark 2. *We don't know how to extend* (1.4) *to rough singular integral operators or to the Bochner–Riesz multiplier at the critical index. Indeed, it is not known whether any of these operators satisfies condition (D) above.*

Remark 3. *Following a similar strategy as in the proof of* (1.4) *the following result holds. Let I_α, $0 < \alpha < n$, be a fractional operator and let $1 < p < \infty$. Let $w \in C_{p+\epsilon}$ for some $\epsilon > 0$. Then*

$$\|I_\alpha f\|_{L^p(w)} \leq c\|M_\alpha f\|_{L^p(w)}.$$

It is possible to extend these kind of results to the multilinear setting as follows. Following [12], we say that T is an m-linear Calderón-Zygmund operator if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m$$

for all $x \notin \cap_{j=1}^m \operatorname{supp} f_j$; and also satisfies similar size and regularity conditions as that in Section 2.1.

It was shown in [24], following the Calderón-Zygmund principle mentioned above, that the right maximal operator that "controls" these m-linear Calderón-Zygmund operators is defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

where $\vec{f} = (f_1, \ldots, f_m)$ and where the supremum is taken over all cubes Q containing x. In fact, these m-linear Calderón-Zygmund operators satisfy a version of the (D) condition mentioned above as can be found in [24, Theorem 3.2].

Lemma 1. *Let T be an m-linear Calderón-Zygmund operator and $\delta \in (0, \frac{1}{m})$. Then, there is a constant c such that*

$$M_\delta^{\#}(T(\vec{f}))(x) \leq c\, \mathcal{M}(\vec{f})(x). \tag{1.5}$$

This estimate is very sharp since it is false in the case $\delta = \frac{1}{m}$. Also this estimate is quite useful since one can deduce the following multilinear version of Coifman-Fefferman estimate (1.1),

$$\|T(\vec{f})\|_{L^p(w)} \leq c\,\|\mathcal{M}(\vec{f})\|_{L^p(w)} \qquad 0 < p < \infty,\, w \in A_\infty,$$

which can be found in [24] leading to the characterization of the class of (multilinear) weights for which any mulitilinear Calderón-Zygmund operators are bounded.

Relying upon the pointwise estimate (1.5) it is possible to establish the following extension of (1.4).

Theorem 2. *Let T be an m-linear Calderón-Zygmund operator, and let $0 < p < \infty$. Also let $w \in C_{\max\{1,mp\}+\epsilon}$ for some $\epsilon > 0$. Then*

$$\|T(\vec{f})\|_{L^p(w)} \leq c\,\|\mathcal{M}(\vec{f})\|_{L^p(w)}.$$

We emphasize that the method of Sawyer in [39] does not produce the preceding result even for the case $p > 1$.

For commutators, the following estimates are known (see [30, 34]). For every $0 < \varepsilon < \delta < 1$,

$$M_\varepsilon^\#([b,T]f)(x) \quad \leq \quad c_{\delta,T}\|b\|_{BMO}\left(M_\delta(Tf) + M^2(f)(x)\right), \tag{1.6}$$

$$M_\varepsilon^\#(\overline{[b,T]}_q\boldsymbol{f})(x) \quad \leq \quad c_{\delta,T}\|b\|_{BMO}\left(M_\delta(\overline{T}_q\boldsymbol{f}) + M^2(|\boldsymbol{f}|_q)(x)\right), \qquad 1 < q < \infty, \tag{1.7}$$

where T is a Calderón-Zygmund operator satisfying a log-Dini condition. Relying upon them we obtain the following result.

Theorem 3. *Let T be an ω-Calderón-Zygmund operator with ω satisfying a log-Dini condition and let $b \in \mathrm{BMO}$. Let $0 < p < \infty$ and let $w \in C_{\max\{1,p\}+\epsilon}$ for some $\epsilon > 0$. Then there is a constant c depending on the $C_{\max\{1,p\}+\epsilon}$ condition such that*

$$\|[b,T]f\|_{L^p(w)} \leq c\,\|b\|_{\mathrm{BMO}}\,\|M^2 f\|_{L^p(w)}.$$

Additionally, if $1 < q < \infty$ then Then there is a constant c depending on the $C_{\max\{1,p\}+\epsilon}$ condition such that

$$\|\overline{[b,T]}_q\boldsymbol{f}\|_{L^p(w)} \leq c\,\|b\|_{\mathrm{BMO}}\,\|M^2(|\boldsymbol{f}|_q)\|_{L^p(w)}.$$

Remark 4. *We remark that a similar estimate can be derived for the general k-th iterated commutator: let $0 < p < \infty$ and let $w \in C_{\max\{1,p\}+\epsilon}$ for some $\epsilon > 0$. Then there is a constant c depending on the $C_{\max\{1,p\}+\epsilon}$ condition such that*

$$\|T_b^k f\|_{L^p(w)} \leq c\,\|b\|_{\mathrm{BMO}}^k\,\|M^{k+1} f\|_{L^p(w)}.$$

In the following results we observe that rephrasing Sawyer's method [39] in combination with sparse domination results, that in the vector-valued we settle in section 1.2, we obtain estimates like (1.2) where the strong norm $\|\cdot\|_{L^p(w)}$ is replaced by the weak norm $\|\cdot\|_{L^{p,\infty}(w)}$. The disadvantage of this approach is that we have to restrict ourselves to the range $1 < p < \infty$.

Theorem 4. *Let T be an ω-Calderón-Zygmund operator with ω satisfying the Dini condition. Let $1 < p < \infty$ and let $w \in C_{p+\epsilon}$ for some $\epsilon > 0$. Then there exists $c = c_{T,p,\epsilon,w}$ such that*

$$\|Tf\|_{L^{p,\infty}(w)} \leq c\|Mf\|_{L^{p,\infty}(w)}.$$

If aditionally $1 < q < \infty$ then

$$\|\overline{T}_q\boldsymbol{f}\|_{L^{p,\infty}(w)} \leq c\|M(|\boldsymbol{f}|_q)\|_{L^{p,\infty}(w)}.$$

We also obtain some results for commutators which are completely new in both the scalar and the vector-valued case.

Theorem 5. *Let T be an ω-Calderón-Zygmund operator with ω satisfying a Dini condition and $b \in \mathrm{BMO}$. Let $1 < p < \infty$ and $w \in C_{p+\epsilon}$ for some $\epsilon > 0$. Then there exists $c = c_{T,p,\epsilon,w}$ such that*

$$\|[b,T]f\|_{L^p(w)} \leq c\|b\|_{\mathrm{BMO}}\|M^2 f\|_{L^p(w)},$$
$$\|[b,T]f\|_{L^{p,\infty}(w)} \leq c\|b\|_{\mathrm{BMO}}\|M^2 f\|_{L^{p,\infty}(w)}.$$

If aditionally $1 < q < \infty$ then

$$\|\overline{[b,T]}_q\boldsymbol{f}\|_{L^p(w)} \leq c\|b\|_{\mathrm{BMO}}\|M^2(|\boldsymbol{f}|_q)\|_{L^p(w)},$$
$$\|\overline{[b,T]}_q\boldsymbol{f}\|_{L^{p,\infty}(w)} \leq c\|b\|_{\mathrm{BMO}}\|M^2(|\boldsymbol{f}|_q)\|_{L^{p,\infty}(w)}.$$

We would like to note that the preceding result extends results based in the $M^\sharp$ approach that hold for Calderón-Zygmund operators satisfying a log-Dini condition to operators satisfying just a Dini condition.

1.2. Sparse domination for vector-valued extensions

In the recent years a number of authors have exploited the sparse domination approach to provide quantitative weighted estimates. Our contribution in that direction in this paper is to settle some domination results for vector-valued extensions that we state in the following results. First we summarize some pointwise domination results.

Theorem 6. *Let $1<q<\infty$ and $\boldsymbol{f}=\{f_j\}_{j=1}^\infty$ such that $|\boldsymbol{f}|_q\in L^\infty_c$. There exist 3^n dyadic lattices $\mathcal{D}_j$ and sparse families $\mathcal{S}_j\subseteq\mathcal{D}_j$, such that*

- **Maximal function.**
$$|\overline{M}_q\boldsymbol{f}(x)|\leq c_{n,q}\sum_{k=1}^{3^n}\mathcal{A}^q_{\mathcal{S}_k}|\boldsymbol{f}|_q(x),$$
where
$$\mathcal{A}^q_{\mathcal{S}}f(x)=\left(\sum_{Q\in\mathcal{S}_k}\langle|f|\rangle_Q^q\chi_Q(x)\right)^{\frac{1}{q}}.$$

- **Calderón-Zygmund operators.**
$$\left|\overline{T}_q\boldsymbol{f}(x)\right|\leq c_nC_T\sum_{k=1}^{3^n}\mathcal{A}_{\mathcal{S}_k}|\boldsymbol{f}|_q(x),$$
where
$$\mathcal{A}_{\mathcal{S}}f(x)=\sum_{Q\in\mathcal{S}}\langle|f|\rangle_Q\chi_Q(x)$$
and $C_T=C_K+\|\omega\|_{Dini}+\|T\|_{L^2\to L^2}$.

- **Commutators.** *If additionally $b\in L^1_{loc}$*
$$|\overline{[b,T]}_q\boldsymbol{f}(x)|\leq c_nC_T\sum_{j=1}^{3^n}\left(\mathcal{T}_{\mathcal{S},b}|\boldsymbol{f}|_q(x)+\mathcal{T}^*_{\mathcal{S},b}|\boldsymbol{f}|_q(x)\right),$$
where
$$\mathcal{T}_{\mathcal{S},b}f(x)=\sum_{Q\in\mathcal{S}}|b(x)-b_Q|\langle|f|\rangle_Q\chi_Q(x),$$
$$\mathcal{T}^*_{\mathcal{S},b}f(x)=\sum_{Q\in\mathcal{S}}\langle|b-b_Q||f|\rangle_Q\chi_Q(x).$$

We recall that if $\Omega\in L^1(\mathbb{S}^{n-1})$ satisfies $\int_{\mathbb{S}^{n-1}}\Omega=0.$ we define the rough singular integral operator T_Ω as
$$T_\Omega f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(x')}{|x|^n}f(x-y)dy,$$
where $x'=x/|x|$ and the associated maximal operator by
$$T^*_\Omega f(x)=\sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}\frac{\Omega(x')}{|x|^n}f(x-y)dy\right|.$$
We also recall that the operator $B_{(n-1)/2}$, the Bochner–Riesz multiplier at the critical index, which is defined by
$$\widehat{B_{(n-1)/2}(f)}(\xi)=(1-|\xi|^2)_+^{(n-1)/2}\hat{f}(\xi).$$
In our next Theorem we present our sparse domination results for vector-valued extensions of those kind of operators and commutators.

Theorem 7. *Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$. If T is T_Ω or $B_{(n-1)/2}$ and $1 \leq s < \frac{q'+1}{2}$, then there exists a sparse collection $\mathcal{S}$ such that*

$$\Big|\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} T(f_j)g_j dx\Big| \leq c_{n,q}C_T s' \sum_{Q\in\mathcal{S}}\langle|\mathbf{f}|_q\rangle_Q\langle|\mathbf{g}|_{q'}\rangle_{s,Q}|Q|.$$

If $1 < s < \frac{\min\{q',q\}+1}{2}$, then there exists a sparse collection $\mathcal{S}$ such that

$$\Big|\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} T^*_\Omega(f_j)g_j dx\Big| \leq c_{n,q}\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} s' \sum_{Q\in\mathcal{S}}\langle|\mathbf{f}|_q\rangle_{s,Q}\langle|\mathbf{g}|_{q'}\rangle_{s,Q}|Q|.$$

If $1 < s < \frac{q'+1}{2}$, $1 < r < \frac{q+1}{2}$ and $b \in \mathrm{BMO}$ then

$$\Big|\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} [b,T_\Omega](f_j)g_j dx\Big| \leq c_{n,q}\|b\|_{\mathrm{BMO}}\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} s' \max\{r',s'\} \sum_{Q\in\mathcal{S}}\langle|\mathbf{f}|_q\rangle_{r,Q}\langle|\mathbf{g}|_{q'}\rangle_{s,Q}|Q|.$$

The rest of the paper is organised as follows. In Section 2 we gather some preliminary results and definitions needed in the rest of the paper. Sections 3 and 4 are devoted to settle sparse domination results. Additionally we provide two appendices. In Appendix A we gather some quantitative estimates that follow from the sparse domination results. Finally, in Appendix B we collect some quantitative versions of unweighted estimates that are needed to obtain some of the sparse domination results.

2. Preliminaries

2.1. Notations and basic definitions

In this Section we fix the notation that we will use in the rest of the paper. First we recall the definition of ω-Calderón-Zygmund operator.

Definition 1. A ω-Calderón-Zygmund operator T is a linear operator bounded on $L^2(\mathbb{R}^n)$ that admits the following representation

$$Tf(x) = \int K(x,y)f(y)dy$$

with $f \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$ and where $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \longrightarrow \mathbb{R}$ has the following properties

Size condition $|K(x,y)| \leq C_K \frac{1}{|x-y|^n}, \qquad x \neq 0.$

Smoothness condition Provided that $|y-z| < \frac{1}{2}|x-y|$, then

$$|K(x,y)-K(x,z)| + |K(x,y)-K(z,y)| \leq \frac{1}{|x-y|^n}\omega\left(\frac{|y-z|}{|x-y|}\right),$$

where the modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ is a subaditive, increasing function such that $\omega(0) = 0$.

It is possible to impose different conditions on the modulus of continuity ω. The most general one is the Dini condition. We say that a modulus of continuity ω satisfies a Dini condition if

$$\|\omega\|_{\mathrm{Dini}} = \int_0^1 \omega(t)\frac{dt}{t} < \infty.$$

We will say that the modulus of continuity ω satisfies a log-Dini condition if

$$\|\omega\|_{\text{log-Dini}} = \int_0^1 \omega(t)\log\left(\frac{1}{t}\right)\frac{dt}{t} < \infty.$$

Clearly $\|\omega\|_{\mathrm{Dini}} \leq \|\omega\|_{\text{log-Dini}}$ We recall also that if $\omega(t) = ct^\delta$ we are in the case of the classical Hölder-Lipschitz condition.

Definition 2. Let ω be a modulus of continuity and K be a kernel satisfying the properties in the preceding definition. We define the maximal Calderón-Zygmund operator T^* as

$$T^*f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K(x,y)f(y)dy \right|.$$

To end the Section we would like to recall also the definitions of some variants and generalizations of the Hardy-Littlewood maximal function. We will denote $M_s f(x) = M(|f|^s)(x)^{\frac{1}{s}}$, $M^\sharp f(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f - \langle f\rangle_Q|$, and $M_s^\sharp f(x) = M^\sharp(|f|^s)(x)^{\frac{1}{s}}$, where $s > 0$.

Now we recall that we say that Φ is a Young function if it is a continuous, convex increasing function that satisfies $\Phi(0) = 0$ and such that $\Phi(t) \to \infty$ as $t \to \infty$.

Let f be a measurable function defined on a set $E \subset \mathbb{R}^n$ with finite Lebesgue measure. The Φ-norm of f over E is defined by

$$\|f\|_{\Phi(L),E} := \inf\left\{\lambda > 0 \,:\, \frac{1}{|E|}\int_E \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Using this Φ-norm we define, in the natural way, the Orlicz maximal operator $M_{\Phi(L)}$ by

$$M_{\Phi(L)}f(x) = \sup_{x\in Q} \|f\|_{\Phi(L),Q}.$$

Some particular cases of interest are

- M_r for $r > 1$ given by the Young function $\Phi(t) = t^r$.
- $M_{L(\log L)^\delta}$ with $\delta > 0$ given by the Young function $\Phi(t) = t\log(e+t)^\delta$. It is a well known fact that

  $$M^{(k+1)}f \simeq M_{L(\log L)^k} f,$$

  where $M^k = M\circ \overset{(k)}{\cdots} \circ M$.
- $M_{L(\log\log L)^\delta}$ with $\delta > 0$ given by the Young function $\Phi(t) = t(\log\log(e^e+t))^\delta$.
- $M_{L(\log L)(\log\log L)^\delta}$ with $\delta > 0$ given by the function $\Phi(t) = t\log(e+t)(\log\log(e^e+t))^\delta$.

One basic fact about this kind of maximal operators that follows from the definition of the norm is the following. Given Ψ and Φ Young functions such that for some $\kappa, c > 0$ $\Psi(t) \leq \kappa\Phi(t)$, then

$$\|f\|_{\Psi(L),Q} \leq (\Psi(c)+\kappa)\|f\|_{\Phi(L),Q},$$

and consequently

$$M_{\Psi(L)}f(x) \leq (\Psi(c)+\kappa)M_{\Phi(L)}f(x).$$

Associated to each Young function A there exists a complementary function $\bar{A}$ that can be defined as follows

$$\bar{A}(t) = \sup_{s>0}\{st - A(s)\}.$$

That complementary function is a Young function as well and it satisfies the following pointwise estimate

$$t \leq A^{-1}(t)\bar{A}^{-1}(t) \leq 2t.$$

An interesting property of this associated function is that the following estimate holds

$$\frac{1}{|Q|}\int_Q |fg|dx \leq 2\|f\|_{A,Q}\|g\|_{\bar{A},Q}.$$

A case of interest for us is the case $A(t) = t\log(e+t)$. In that case we have that

$$\frac{1}{|Q|}\int_Q |fg|dx \leq c\|f\|_{L\log L,Q}\|g\|_{\exp(L),Q}.$$

From that estimate taking into account John-Nirenberg's theorem, if $b \in \text{BMO}$, then

$$\frac{1}{|Q|}\int_Q |f(b-b_Q)|dx \leq c\|f\|_{L\log L,Q}\|b-b_Q\|_{\exp(L),Q} \leq c\|f\|_{L\log L,Q}\|b\|_{\text{BMO}}. \tag{2.1}$$

For a detailed account about the ideas presented in the end of this Section we refer the reader to [36, 37].

2.2. Lerner-Nazarov formula

In this Section we recall the definitions of the local oscillation and the Lerner-Nazarov oscillation and we show that the latter is controlled by the former. Built upon Lerner-Nazarov oscillation we will also introduce formula, which will be a quite useful tool for us. Most of the ideas covered in this Section are borrowed from [22]. Among them, we start with the definition of dyadic lattice.

Let us call $\mathcal{D}(Q)$ the dyadic grid obtained repeatedly subdividing Q and its descendents in 2^n cubes with the same side length.

Definition 3. A dyadic lattice $\mathcal{D}$ in $\mathbb{R}^n$ is a family of cubes that satisfies the following properties

1. If $Q \in \mathcal{D}$ then each descendant of Q is in $\mathcal{D}$ as well.
2. For every 2 cubes Q_1, Q_2 we can find a common ancestor, that is, a cube $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$.
3. For every compact set K there exists a cube $Q \in \mathcal{D}$ such that $K \subseteq Q$.

A way to build such a structure is to consider an increasing sequence of cubes $\{Q_j\}$ expanding each time from a different vertex. That choice of cubes gives that $\mathbb{R}^n = \cup_j Q_j$ and it is not hard to check that

$$\mathcal{D} = \bigcup_j \{Q \in \mathcal{D}(Q_j)\}$$

is a dyadic lattice.

Lemma 2. *Given a dyadic lattice $\mathcal{D}$ there exist 3^n dyadic lattices $\mathcal{D}_j$ such that*

$$\{3Q \, : \, Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n} \mathcal{D}_j$$

and for every cube $Q \in \mathcal{D}$ we can find a cube R_Q in each $\mathcal{D}_j$ such that $Q \subseteq R_Q$ and $3l_Q = l_{R_Q}$

Remark 5. *Fix $\mathcal{D}$. For an arbitrary cube $Q \subseteq \mathbb{R}^n$ there is a cube $Q' \in \mathcal{D}$ such that $\frac{l_Q}{2} < l_{Q'} \leq l_Q$ and $Q \subseteq 3Q'$. It suffices to take the cube Q' that contains the center of Q . From the lemma above it follows that $3Q' = P \in \mathcal{D}_j$ for some $j \in \{1, \dots, 3^n\}$. Therefore, for every cube $Q \subseteq \mathbb{R}^n$ there exists $P \in \mathcal{D}_j$ such that $Q \subseteq P$ and $l_P \leq 3l_Q$. From this follows that $|Q| \leq |P| \leq 3^n|Q|$*

Definition 4. $\mathcal{S} \subseteq \mathcal{D}$ is a η-sparse family with $\eta \in (0,1)$ if for each $Q \in \mathcal{S}$ we can find a measurable subset $E_Q \subseteq Q$ such that

$$\eta|Q| \leq |E_Q|$$

and all the E_Q are pairwise disjoint.

We also recall here the definition of Carleson family.

Definition 5. We say that a family $\mathcal{S} \subseteq \mathcal{D}$ is Λ-Carleson with $\Lambda > 1$ if for each $Q \in \mathcal{S}$ we have that

$$\sum_{P \in \mathcal{S}, \, P \subseteq Q} |P| \leq \Lambda|Q|.$$

The following result that establishes the relationship between Carleson and sparse families was obtained in [22] and reads as follows.

Lemma 3. *If $\mathcal{S} \subseteq \mathcal{D}$ is a η-sparse family then it is a $\frac{1}{\eta}$-Carleson family. Conversely if $\mathcal{S}$is Λ-Carleson then it is $\frac{1}{\Lambda}$-sparse.*

Now we turn to recall the definition of the local oscillation [18] which is given in terms of decreasing rearrangements.

Definition 6 (Local oscillation)**.** Given $\lambda \in (0,1)$, a measurable function f and a cube Q. We define

$$\tilde{w}_\lambda(f; Q) := \inf_{c \in \mathbb{R}} \left((f - c)\chi_Q\right)^* (\lambda|Q|).$$

For any function g, its decreasing rearrangement g^* is given by

$$g^*(t) = \inf\left\{\alpha > 0 \,:\, |\{x \in \mathbb{R}^n \,:\, |g| > \alpha\}| \leq t\right\}.$$

In particular,

$$\left((f-c)\chi_Q\right)^*(\lambda|Q|) = \inf\left\{\alpha > 0 \,:\, |\{x \in Q \,:\, |f-c| > \alpha\}| \leq \lambda|Q|\right\}.$$

Now we define Lerner-Nazarov oscillation [22]. We would like to observe that decreasing rearrangements are not involved in the definition.

Definition 7 (Lerner-Nazarov oscillation)**.** Given $\lambda \in (0,1)$, a measurable function f and a cube Q. We define the λ-oscillation of f on Q as

$$w_\lambda(f;Q) := \inf\left\{w(f;E) \,:\, E \subseteq Q,\, |E| \geq (1-\lambda)|Q|\right\},$$

where

$$w(f;E) = \sup_E f - \inf_E f.$$

It is not hard to check that Lerner-Nazarov oscillation is controlled by the local oscillation.

Lemma 4. *Given a measurable function f we have that for every $\lambda \in (0,1)$,*

$$w(f;Q) \leq 2\tilde{w}_\lambda(f;Q).$$

Theorem III (Lerner-Nazarov formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that for each $\varepsilon > 0$*

$$|\{x \in [-R,R]^n \,:\, |f(x)| > \varepsilon\}| = o(R^n) \text{ as } R \to \infty.$$

Then for each dyadic lattice $\mathcal{D}$ and every $\lambda \in (0, 2^{-n-2}]$ we can find a regular $\frac{1}{6}$-sparse family of cubes $\mathcal{S} \subseteq \mathcal{D}$ (depending on f) such that

$$|f(x)| \leq \sum_{Q \in \mathcal{S}} w_\lambda(f;Q)\chi_Q(x) \qquad a.e.$$

3. Proof of Theorem 6

3.1. Hardy-Littlewood Maximal operator

We are going to prove

$$\overline{M}_q \boldsymbol{f}(x) \leq c_{n,q} \sum_{k=1}^{3^n} \left(\sum_{Q \in \mathcal{S}_k} \left(\frac{1}{|Q|} \int_Q |\boldsymbol{f}|_q \right)^q \chi_Q(x) \right)^{\frac{1}{q}}.$$

First we observe that from Remark 5 it readily follows that

$$Mf(x) \leq c_n \sum_{k=1}^{3^n} M^{\mathcal{D}_k} f(x).$$

Taking that into account it is clear that

$$\overline{M}_q \boldsymbol{f}(x) \leq c_n \sum_{k=1}^{3^n} \overline{M}_q^{\mathcal{D}_k} \boldsymbol{f}(x). \tag{3.1}$$

The following estimate for local oscillations

$$\tilde{w}_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} \boldsymbol{f} \right)^q ; Q \right) \leq \frac{c_{n,q}}{\lambda^q} \left(\frac{1}{|Q|} \int_Q |\boldsymbol{f}|_q \right)^q,$$

was established in [4, Lemma 8.1]. Now we recall that by Lemma 4

$$w_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} \boldsymbol{f} \right)^q ; Q \right) \leq 2\tilde{w}_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} \boldsymbol{f} \right)^q ; Q \right).$$

Then

$$w_\lambda\left(\left(\overline{M}_q^{\mathcal{D}}\boldsymbol{f}\right)^q;Q\right)\le\frac{c_{n,q}}{\lambda^q}\left(\frac{1}{|Q|}\int_Q|\boldsymbol{f}|_q\right)^q.$$

Using now Lerner-Nazarov formula (Theorem III) there exists a $\frac{1}{6}$-sparse family $\mathcal{S}\subset\mathcal{D}$ such that

$$\begin{aligned}\overline{M}_q^{\mathcal{D}}\boldsymbol{f}(x)^q&\le\sum_{Q\in\mathcal{S}}w_\lambda\left(\left(\overline{M}_q^{\mathcal{D}}\boldsymbol{f}\right)^q;Q\right)\chi_Q(x)\\&\le\frac{2c_{n,q}}{\lambda^q}\sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q|\boldsymbol{f}|_q\right)^q\chi_Q(x).\end{aligned}$$

Consequently

$$\overline{M}_q^{\mathcal{D}}\boldsymbol{f}(x)\le c_{n,q}\left(\sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q|\boldsymbol{f}|_q\right)^q\chi_Q(x)\right)^{\frac{1}{q}}.$$

Applying this to each $\overline{M}_q^{\mathcal{D}_k}\boldsymbol{f}(x)$ in (3.1) we obtain the desired estimate.

3.2. Calderón-Zygmund operators and commutators

To settle this case we borrow ideas from [20] and [25]. Let T be an ω-CZO with ω satisfying Dini condition and $1<q<\infty$. We define the grand maximal truncated operator $\mathcal{M}_{T_q}$ by

$$\mathcal{M}_{\overline{T}_q}\boldsymbol{f}(x)=\sup_{Q\ni x}\operatorname*{ess\,sup}_{\xi\in Q}\left|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\right|.$$

We also consider a local version of this operator

$$\mathcal{M}_{\overline{T}_q,Q_0}\boldsymbol{f}(x)=\sup_{x\in Q\subseteq Q_0}\operatorname*{ess\,sup}_{\xi\in Q}\left|\overline{T}_q(\boldsymbol{f}\chi_{3Q_0\setminus 3Q})(\xi)\right|.$$

Lemma 5. *Let T be an ω-CZO with ω satisfying Dini condition and $1<q<\infty$. The following pointwise estimates hold:*

1. *For a.e. $x\in Q_0$*

$$|\overline{T}_q(\boldsymbol{f}\chi_{3Q_0})(x)|\le c_n\|\overline{T}_q\|_{L^1\to L^{1,\infty}}|\boldsymbol{f}|_q(x)+\mathcal{M}_{\overline{T}_q,Q_0}f(x).$$

2. *For all $x\in\mathbb{R}^n$*

$$\mathcal{M}_{\overline{T}_q}\boldsymbol{f}(x)\le c_{n,q}(\|\omega\|_{Dini}+C_K)M_qf(x)+\overline{T^*}_q\boldsymbol{f}(x).\tag{3.2}$$

   Furthermore

$$\left\|\mathcal{M}_{\overline{T}_q}\right\|_{L^1\to L^{1,\infty}}\le c_{n,q}C_T,$$

   where $C_T=C_K+\|\omega\|_{Dini}+\|T\|_{L^2\to L^2}$.

Proof. Both estimates essentially follow from adapting arguments in [20] so we will establish just (3.2). Let $x,\xi\in Q$. Denote by B_x the closed ball centered at x of radius $2\text{diam}Q$. Then $3Q\subset B_x$, and we obtain

$$\begin{aligned}|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|&\le|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(\xi)+\overline{T}_q(\boldsymbol{f}\chi_{B_x\setminus 3Q})(\xi)|\\&\le|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(x)|\\&\quad+|\overline{T}_q(\boldsymbol{f}\chi_{B_x\setminus 3Q})(\xi)|+|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(x)|.\end{aligned}$$

By the smoothness condition, since $||a|^r-|b|^r|\le 2^{\max\{1,r\}-1}|a-b|^r$ for every $r>0$ we have that

$$\begin{aligned}&|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(x)|\\&=\left|\left(\sum_{j=1}^\infty|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(\xi)|^q\right)^{\frac{1}{q}}-\left(\sum_{j=1}^\infty|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(x)|^q\right)^{\frac{1}{q}}\right|\\&\le\left(\sum_{j=1}^\infty\left|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(\xi)-T(f_j\chi_{\mathbb{R}^n\setminus B_x})(x)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Now using the smoothness condition (see [20, Proof of Lemma 3.2 (ii)]

$$\left|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(\xi)-T(f_j\chi_{\mathbb{R}^n\setminus B_x})(x)\right|\leq c_n\|\omega\|_{\text{Dini}}Mf_j(x),$$

and then we have that

$$|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(x)|\leq c_{n,q}\|\omega\|_{\text{Dini}}\left(\sum_{j=1}^{\infty}|Mf_j(x)|^q\right)^{\frac{1}{q}}=c_{n,q}\|\omega\|_{\text{Dini}}\overline{M}_q\boldsymbol{f}(x).$$

On the other hand the size condition of the kernel yields

$$\begin{aligned}|\overline{T}_q(\boldsymbol{f}\chi_{B_x\setminus 3Q})(\xi)|&\leq\left|\left(\sum_{j=1}^{\infty}|T(f_j\chi_{B_x\setminus 3Q})(\xi)|^q\right)^{\frac{1}{q}}\right|\\&\leq c_nC_K\left|\left(\sum_{j=1}^{\infty}\left(\frac{1}{|B_x|}\int_{B_x}|f_j|\right)^q\right)^{\frac{1}{q}}\right|\\&\leq c_nC_K\left|\left(\sum_{j=1}^{\infty}(Mf_j(x))^q\right)^{\frac{1}{q}}\right|\leq c_nC_K\overline{M}_q\boldsymbol{f}(x).\end{aligned}$$

To end the proof of the pointwise estimate we observe that

$$|\overline{T}_q(\boldsymbol{f}\chi_{\mathbb{R}^n\setminus B_x})(x)|\leq\overline{T}_q^*\boldsymbol{f}(x).$$

Now, taking into account the pointwise estimate we have just obtained and Theorem 18 below it is clear that

$$\left\|\mathcal{M}_{\overline{T}_q}\right\|_{L^1\to L^{1,\infty}}\leq c_{n,q}C_T.$$

This ends the proof. □

Having the results above at our disposal now we sketch the proofs of the case of Calderón-Zygmund operators and commutators in Theorem 6. Since the case of Calderón-Zygmund operators is simpler, we just show the the case of commutators, to make clear how ideas in [20, 25], need to be adapted to the case of vector-valued extensions.

From Remark 5 it follows that there exist 3^n dyadic lattices such that for every cube Q of $\mathbb{R}^n$ there is a cube $R_Q\in\mathcal{D}_j$ for some j for which $3Q\subset R_Q$ and $|R_Q|\leq 9^n|Q|$

Let us fix a cube $Q_0\subset\mathbb{R}^n$. We claim that there exists a $\frac{1}{2}$-sparse family $\mathcal{F}\subseteq\mathcal{D}(Q_0)$ such that for a.e. $x\in Q_0$

$$\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0})(x)\right|\leq c_nC_T\sum_{Q\in\mathcal{F}}\left(|b(x)-b_{R_Q}|\langle|\boldsymbol{f}|_q\rangle_{3Q}+\langle|(b-b_{R_Q})||\boldsymbol{f}|_q\rangle_{3Q}\right)\chi_Q(x).\tag{3.3}$$

Arguing as in [25] from (3.3) it follows that there exists a $\frac{1}{2}$-sparse family $\mathcal{F}$ such that for every $x\in\mathbb{R}^n$,

$$\left|\overline{[b,T]}_q\boldsymbol{f}(x)\right|\leq c_nC_T\sum_{Q\in\mathcal{F}}\left(|b(x)-b_{R_Q}|\langle|\boldsymbol{f}|_q\rangle_{3Q}+\langle|(b-b_{R_Q})||\boldsymbol{f}|_q\rangle_{3Q}\right)\chi_Q(x).$$

Now we observe that since $3Q\subset R_Q$ and $|R_Q|\leq 3^n|3Q|$ we have that $|h|_{3Q}\leq c_n|h|_{R_Q}$. Setting

$$\mathcal{S}_j=\{R_Q\in\mathcal{D}_j\,:\,Q\in\mathcal{F}\}$$

and using that $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that each family $\mathcal{S}_j$ is $\frac{1}{2\cdot 9^n}$-sparse. Then we have that

$$\left|\overline{[b,T]}_q\boldsymbol{f}(x)\right|\leq c_nC_T\sum_{j=1}^{3^n}\sum_{R\in\mathcal{S}_j}\left(|b(x)-b_R|\langle|\boldsymbol{f}|_q\rangle_R+\langle|(b-b_R)||\boldsymbol{f}|_q\rangle_R\right)\chi_R(x).$$

To prove the claim it suffices to prove the following recursive estimate: There exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \le \frac{1}{2}|Q_0|$ and for a.e. $x \in Q_0$,

$$\begin{aligned}&\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0})(x)\right|\chi_{Q_0}(x)\\ &\le c_n C_T\left(|b(x)-b_{R_{Q_0}}|\langle|\boldsymbol{f}|_q\rangle_{3Q_0}+\langle|(b-b_{R_{Q_0}})||\boldsymbol{f}|_q\rangle_{3Q_0}\right)+\sum_j\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3P_j})(x)\right|\chi_{P_j}(x).\end{aligned}$$

Iterating this estimate we obtain the claim with $\mathcal{F}=\{P_j^k\}$ where $\{P_j^0\}=\{Q_0\}$, $\{P_j^1\}=\{P_j\}$ and $\{P_j^k\}$ are the cubes obtained at the k-th stage of the iterative process. Now we observe that for any arbitrary family of disjoint cubes $P_j \in \mathcal{D}(Q_0)$ we have that by the sublinearity of $\overline{[b,T]}_q$,

$$\begin{aligned}\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0})(x)\right|\chi_{Q_0}(x) &\le \left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0})(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)\\ +\sum_j\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)&+\sum_j\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3P_j})(x)\right|\chi_{P_j}(x).\end{aligned}$$

So it suffices to show that we can choose a family of pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ with $\sum_j |P_j| \le \frac{1}{2}|Q_0|$ and such that for a.e. $x \in Q_0$,

$$\begin{aligned}&\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0})(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\\ &\le c_n C_T\left(|b(x)-b_{R_{Q_0}}|\langle|\boldsymbol{f}|_q\rangle_{3Q_0}+\langle|(b-b_{R_{Q_0}})||\boldsymbol{f}|_q\rangle_{3Q_0}\right).\end{aligned}$$

Now we recall that $[b,T]f=[b-c,T]f=(b-c)Tf-T((b-c)f)$ for every $c \in \mathbb{R}$. Then

$$\begin{aligned}\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0})(x)\chi_{Q_0\setminus\bigcup_j P_j}(x) &= \left(\sum_{k=1}^{\infty}\left|[b-b_{R_{Q_0}},T](f_k\chi_{3Q_0})(x)\right|^q\right)^{\frac{1}{q}}\chi_{Q_0\setminus\bigcup_j P_j}(x)\\ &\le\left(\sum_{k=1}^{\infty}\left|\left(b(x)-b_{R_{Q_0}}\right)T(f_k\chi_{3Q_0})(x)-T\left(\left(b-b_{R_{Q_0}}\right)f_k\chi_{3Q_0}\right)(x)\right|^q\right)^{\frac{1}{q}}\chi_{Q_0\setminus\bigcup_j P_j}(x)\\ &\le\left(\sum_{k=1}^{\infty}\left(\left|\left(b(x)-b_{R_{Q_0}}\right)T(f_k\chi_{3Q_0})(x)\right|+\left|T\left(\left(b-b_{R_{Q_0}}\right)f_k\chi_{3Q_0}\right)(x)\right|\right)^q\right)^{\frac{1}{q}}\chi_{Q_0\setminus\bigcup_j P_j}(x)\\ &=\left|b(x)-b_{R_{Q_0}}\right|\overline{T}_q\left(\boldsymbol{f}\chi_{3Q_0}\right)(x)\chi_{Q_0\setminus\bigcup_j P_j}(x)+\overline{T}_q\left(\left(b-b_{R_{Q_0}}\right)\boldsymbol{f}\chi_{3Q_0}\right)(x)\chi_{Q_0\setminus\bigcup_j P_j}(x).\end{aligned}$$

Analogously we also have that

$$\begin{aligned}&\sum_j\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\\ &\le\sum_j\left(\left|b(x)-b_{R_{Q_0}}\right|\overline{T}_q\left(\boldsymbol{f}\chi_{3Q_0\setminus 3P_j}\right)(x)+\overline{T}_q\left(\left(b-b_{R_{Q_0}}\right)\boldsymbol{f}\chi_{3Q_0\setminus 3P_j}\right)\right)\chi_{P_j}(x).\end{aligned}$$

And combining both estimates

$$\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0})(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|\overline{[b,T]}_q(\boldsymbol{f}\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\le I_1+I_2,$$

where

$$I_1=\left|b(x)-b_{R_{Q_0}}\right|\left(\left|\overline{T}_q(\boldsymbol{f}\chi_{3Q_0})(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|\overline{T}_q(\boldsymbol{f}\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\right)$$

and

$$I_2=\left|\overline{T}_q\left((b-b_{R_{Q_0}})\boldsymbol{f}\chi_{3Q_0}\right)(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|\overline{T}_q\left((b-b_{R_{Q_0}})\boldsymbol{f}\chi_{3Q_0\setminus 3P_j}\right)(x)\right|\chi_{P_j}(x).$$

Now we define the set $E = E_1 \cup E_2$ where

$$E_1 = \{x \in Q_0 \,:\, |\boldsymbol{f}|_q > \alpha_n \langle |\boldsymbol{f}|_q \rangle_{3Q_0}\} \cup \left\{x \in Q_0 \,:\, \mathcal{M}_{T_q,Q_0}\boldsymbol{f} > \alpha_n C_T \langle |\boldsymbol{f}|_q \rangle_{3Q_0}\right\}$$

and

$$\begin{aligned} E_2 = &\left\{x \in Q_0 \,:\, |b - b_{R_{Q_0}}||\boldsymbol{f}|_q > \alpha_n \langle |b - b_{R_{Q_0}}||\boldsymbol{f}|_q \rangle_{3Q_0}\right\} \\ &\cup \left\{x \in Q_0 \,:\, \mathcal{M}_{T,Q_0}\left((b - b_{R_{Q_0}})\boldsymbol{f}\right) > \alpha_n C_T \langle |b - b_{R_{Q_0}}||\boldsymbol{f}|_q \rangle_{3Q_0}\right\}. \end{aligned}$$

Since $\mathcal{M}_{\overline{T}_q}$ is of weak type $(1,1)$ with

$$\|\mathcal{M}_{\overline{T}_q}\|_{L^1 \to L^{1,\infty}} \leq c_n C_T.$$

from this point it suffices to follow the arguments given in [25, Theorem 1.1] taking into account Lemma 5 to end the proof.

4. Proof of Theorem 7

To settle Theorem 7, unlike our previous approach, we do not need to go through the original proof. This is due to a very nice observation by Culiuc, Di Plinio and Ou [5], combined with the corresponding results for the scalar setting. Let us recall first those results.

Theorem IV ([3, Theorems A and B])**.** *Let T be T_Ω or $B_{(n-1)/2}$. Then for all $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, we have that*

$$\left| \int_{\mathbb{R}^n} T(f) g dx \right| \leq c_n C_T s' \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q \langle |g| \rangle_{s,Q} |Q|,$$

where $\mathcal{S}$ is a sparse family of some dyadic lattice $\mathcal{D}$,

$$\begin{cases} 1 < s < \infty & \text{if } T = B_{(n-1)/2} \text{ or } T = T_\Omega \text{ with } \Omega \in L^\infty(\mathbb{S}^{n-1}) \\ q' \leq s < \infty & \text{if } T = T_\Omega \text{ with } \Omega \in L^{q,1}\log L(\mathbb{S}^{n-1}) \end{cases}$$

and

$$C_T = \begin{cases} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}, & \text{if } T = T_\Omega \text{ with } \Omega \in L^\infty(\mathbb{S}^{n-1}) \\ \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} & \text{if } \Omega \in L^{q,1}\log L(\mathbb{S}^{n-1}) \\ 1 & \text{if } T = B_{(n-1)/2}. \end{cases}$$

For T_Ω^* with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ the following sparse domination was provided in [7].

$$\left| \int_{\mathbb{R}^n} T_\Omega^*(f) g \right| \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} s' \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{s,Q} \langle |g| \rangle_{s,Q} |Q| \qquad 1 < s < \infty. \tag{4.1}$$

In the case of commutators, the following result was recently obtained in [38], hinging upon techniques in [21].

Theorem V. *Let T_Ω be a rough homogeneous singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then, for every compactly supported $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ every $b \in \mathrm{BMO}$ and $1 < p < \infty$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that*

$$|\langle [b, T_\Omega] f, g \rangle| \leq C_n s' \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} \left(\mathcal{T}_{\mathcal{S}_j,1,s}(b,f,g) + \mathcal{T}^*_{\mathcal{S}_j,1,s}(b,f,g) \right), \tag{4.2}$$

where

$$\begin{aligned} \mathcal{T}_{\mathcal{S}_j,r,s}(b,f,g) &= \sum_{Q \in \mathcal{S}_j} \langle |f| \rangle_{r,Q} \langle |(b - b_Q) g| \rangle_{s,Q} |Q| \\ \mathcal{T}^*_{\mathcal{S}_j,r,s}(b,f,g) &= \sum_{Q \in \mathcal{S}_j} \langle |(b - b_Q) f| \rangle_{r,Q} \langle |g| \rangle_{s,Q} |Q|. \end{aligned}$$

Analogously as we did in the preceding sections, if $1 < q < \infty$ and T is T_Ω or $B_{(n-1)/2}$ and $b \in L^1_{\text{loc}}$, we consider the corresponding vector-valued versions of T and $[b,T]$ that are defined as follows

$$\overline{T}_q \mathbf{f}(x) = \Big(\sum_{j=1}^{\infty} |T(f_j)|^q\Big)^{\frac{1}{q}}$$

$$\overline{[b,T]}_q \mathbf{f}(x) = \Big(\sum_{j=1}^{\infty} |[b,T](f_j)|^q\Big)^{\frac{1}{q}}.$$

Having those results at our disposal, we have

$$\begin{aligned}\Big|\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} T(f_j)g_j dx\Big| &\le c_n C_T s' \sum_j \sum_{Q\in\mathcal{S}_j} \langle |f_j|\rangle_Q \langle |g_j|\rangle_{s,Q}|Q| \\ &\le 2c_n C_T s' \int_{\mathbb{R}^n} \sum_j \mathcal{M}_{1,s}(f_j,g_j)(x)dx,\end{aligned}$$

where

$$\mathcal{M}_{r,s}(f,g)(x) = \sup_{Q\ni x} \langle |f|\rangle_{r,Q}\langle |g|\rangle_{s,Q},$$

and $\langle |h|\rangle_{u,Q} = \langle |h|^u\rangle_Q^{\frac{1}{u}}$ with $u > 1$.

In the case of T^*_Ω, taking into account (4.1) and arguing as above

$$\Big|\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} T^*_\Omega(f_j)g_j dx\Big| \le 2c_n C_T s' \int_{\mathbb{R}^n} \sum_j \mathcal{M}_{s,s}(f_j,g_j)(x)dx.$$

For the commutator $[b,T_\Omega]$ with $b \in \text{BMO}$ and $\Omega \in L^\infty(\mathbb{S}^{n-1})$, taking into account Theorem V, we observe that choosing $u = \frac{s+1}{2}$ then $u' \le 2s'$ and we have that

$$\begin{aligned}\sum_{Q\in\mathcal{S}_j} \langle f\rangle_Q \langle (b-b_Q)g\rangle_{u,Q}|Q| &\le \sum_{Q\in\mathcal{S}_j} \langle f\rangle_Q \langle b-b_Q\rangle_{u(\frac{s}{u})',Q}\langle g\rangle_{s,Q}|Q| \\ &\le c_n u\left(\frac{s}{u}\right)' \|b\|_{BMO} \sum_{Q\in\mathcal{S}_j} \langle f\rangle_Q \langle g\rangle_{s,Q}|Q| \\ &\le c_n u\left(\frac{s}{u}\right)' \|b\|_{BMO} \sum_{Q\in\mathcal{S}_j} \langle f\rangle_{r,Q} \langle g\rangle_{s,Q}|Q| \\ &\le c_n s' \|b\|_{BMO} \sum_{Q\in\mathcal{S}_j} \langle f\rangle_{r,Q} \langle g\rangle_{s,Q}|Q|.\end{aligned}$$

On the other hand

$$\sum_{Q\in\mathcal{S}_j} \langle (b-b_Q)f\rangle_Q \langle g\rangle_{u,Q}|Q| \le c_n r' \|b\|_{BMO} \sum_{Q\in\mathcal{S}_j} \langle f\rangle_{r,Q}\langle g\rangle_{s,Q}|Q|,$$

from which it readily follows that

$$|\langle [b,T_\Omega]f,g\rangle| \le c_n s'(s'+r')\|b\|_{\text{BMO}} \sum_{Q\in\mathcal{S}_j} \langle f\rangle_{r,Q}\langle g\rangle_{s,Q}|Q|.$$

Consequently

$$\Big|\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} [b,T](f_j)g_j dx\Big| \le c_n s' \max\{s',r'\}\|b\|_{\text{BMO}} \int_{\mathbb{R}^n}\sum_j \mathcal{M}_{r,s}(f_j,g_j)(x)dx.$$

where

$$\mathcal{M}_{r,s}(f,g)(x) = \sup_{Q\ni x} \langle |f|\rangle_{r,Q}\langle |g|\rangle_{s,Q}.$$

The consideratons above reduce the proof of Theorem 7 to provide a sparse domination for $\overline{(\mathcal{M}_{r,s})}_1(\mathbf{f},\mathbf{g})$. That was already done in [5]. Here we would like to track the constants, so present an alternative proof.

Lemma 6. *Let* $1 < q < \infty$, $1 \le s < \frac{q'+1}{2}$ *and* $1 \le r < \frac{q+1}{2}$. *Then there exists a sparse family of dyadic cubes* $\mathcal{S}$ *such that*

$$\overline{(\mathcal{M}_{1,s})}_1(\mathbf{f},\mathbf{g}) \le c_n q q' \sum_{Q\in\mathcal{S}} \langle |\mathbf{f}|_q\rangle_{r,Q}\langle |\mathbf{g}|_{q'}\rangle_{s,Q}\chi_Q.$$

Proof. Again, we use the three lattice theorem to reduce the problem to study the related dyadic maximal operator. Namely, we shall prove

$$\overline{(\mathcal{M}^{\mathcal{D}}_{1,s})}_1(\mathbf{f},\mathbf{g}) \le c_n q q' \sum_{Q\in\mathcal{S}} \langle |\mathbf{f}|_q\rangle_{r,Q}\langle |\mathbf{g}|_{q'}\rangle_{s,Q}\chi_Q,$$

where $\mathcal{D}$ is a dyadic grid and

$$\mathcal{M}^{\mathcal{D}}_{1,s}(f,g)(x) = \sup_{\substack{Q\ni x\\ Q\in\mathcal{D}}} \langle |f|\rangle_{r,Q}\langle |g|\rangle_{s,Q}.$$

We shall use the Lerner-Nazarov formula. So we only need to calculate the local mean oscillation. For every $x \in Q_0$, notice that

$$\mathcal{M}^{\mathcal{D}}_{r,s}(f,g)(x) = \max\{\mathcal{M}^{\mathcal{D}}_{r,s}(f\chi_{Q_0}, g\chi_{Q_0})(x), \sup_{\substack{Q\in\mathcal{D}\\ Q\supset Q_0}} \langle |f|\rangle_{r,Q}\langle |g|\rangle_{s,Q}\}.$$

the second term on the right is constant, so based on this we define

$$K_0 = \sum_{j\in\mathbb{Z}} \sup_{\substack{Q\in\mathcal{D}\\ Q\supset Q_0}} \langle |f_j|\rangle_{r,Q}\langle |g_j|\rangle_{s,Q}.$$

Then

$$\left|\{x\in Q_0 : |\overline{(\mathcal{M}^{\mathcal{D}}_{1,s})}_1(\mathbf{f},\mathbf{g})(x) - K_0| > t\}\right| \le \left|\{x\in Q_0 : |\overline{(\mathcal{M}^{\mathcal{D}}_{1,s})}_1(\mathbf{f}\chi_{Q_0},\mathbf{g}\chi_{Q_0})(x)| > t\}\right|.$$

Now we are in the position to apply the Fefferman-Stein inequality for vector-valued maximal operators. Since we need to track the constants, here we use the version in Grafakos' book [11, Theorem 5.6.6]:

$$\|\overline{M}_q(\mathbf{f})\|_{L^{1,\infty}} \le c_n q' \||\mathbf{f}|_q\|_{L^1}. \tag{4.3}$$

We also need the Hölder's inequality for the weak type spaces, which can also be found in [11, p. 16]:

$$\|f_1\cdots f_k\|_{L^{p,\infty}} \le p^{-\frac{1}{p}} \prod_{i=1}^k p_i^{\frac{1}{p_i}} \|f_i\|_{L^{p_i,\infty}}, \tag{4.4}$$

where $\frac{1}{p} = \sum_{i=1}^k \frac{1}{p_i}$ and $0 < p_i < \infty$. With (4.3) and (4.4) at hand, we have that since $1 < r, s < \infty$,

$$\begin{aligned}
\|\overline{(\mathcal{M}_{1,s})}^{\mathcal{D}}_1(\mathbf{f},\mathbf{g})\|_{L^{\frac{rs}{r+s},\infty}} &\le r^{\frac1r} s^{\frac1s}\left(\frac{r+s}{rs}\right)^{\frac{r+s}{rs}} \|\overline{M}_q\mathbf{f}\|_{L^{r,\infty}} \|\overline{(M_s)}_{q'}\mathbf{g}\|_{L^{s,\infty}}\\
&\le r^{\frac1r} s^{\frac1s}\left(\frac{r+s}{rs}\right)^{\frac{r+s}{rs}} \|\overline{(M_r)}_q\mathbf{f}\|_{L^{r,\infty}} \|\overline{(M_s)}_{q'}\mathbf{g}\|_{L^{s,\infty}}\\
&\le r^{\frac1r} s^{\frac1s}\left(\frac{r+s}{rs}\right)^{\frac{r+s}{rs}} \|\overline{M}_{q/r}|\mathbf{f}|^r\|^{\frac1r}_{L^{1,\infty}} \|\overline{M}_{q'/s}|\mathbf{g}|^{\mathbf{s}}\|^{\frac1s}_{L^{1,\infty}}\\
&\le c_n r^{\frac1r} s^{\frac1s}\left(\frac{r+s}{rs}\right)^{\frac{r+s}{rs}} \left(\frac{q}{r}\right)' \left(\frac{q'}{s}\right)' \||\mathbf{f}|^r_q\|^{\frac1r}_{L^1} \||\mathbf{g}|^s_{q'}\|^{\frac1s}_{L^1}.
\end{aligned}$$

Now we observe that $c_n r^{\frac1r} s^{\frac1s}\left(\frac{r+s}{rs}\right)^{\frac{r+s}{rs}} \left(\frac{q}{r}\right)' \left(\frac{q'}{s}\right)' \le c_n q q' = \kappa$. Then,

$$\left|\{x\in Q_0 : |\overline{(\mathcal{M}^{\mathcal{D}}_{1,s})}_1(\mathbf{f}\chi_{Q_0},\mathbf{g}\chi_{Q_0})(x)| > t\}\right| \le \frac{\kappa^{\frac{rs}{r+s}}}{t^{\frac{rs}{r+s}}} \left(\int_{Q_0} |\mathbf{f}|^r_q\right)^{\frac1r\frac{rs}{r+s}} \left(\int_{Q_0} |\mathbf{g}|^s_{q'}\right)^{\frac1s\frac{rs}{r+s}}.$$

Taking into account the preceding estimates, we have that

$$\begin{aligned}\omega_\lambda((\overline{\mathcal{M}^{\mathcal{P}}_{1,s}})_1(\mathbf{f},\mathbf{g}),Q_0) &\le \left((\overline{\mathcal{M}^{\mathcal{P}}_{1,s}})_1(\mathbf{f},\mathbf{g})-K_0\right)^*(\lambda|Q_0|)\\ &\le c_n q q' \lambda^{-\frac{r+s}{rs}}\langle|\mathbf{f}|_q\rangle_{r,Q_0}\langle|\mathbf{g}|_{q'}\rangle_{s,Q_0}\chi_{Q_0}\\ &\le c_n q q' \lambda^{-2}\langle|\mathbf{f}|_q\rangle_{r,Q_0}\langle|\mathbf{g}|_{q'}\rangle_{s,Q_0}\chi_{Q_0},\end{aligned}$$

where the last inequality holds since $0<\lambda<1$. From this point, a direct application of Lerner-Nazarov formula (Theorem III) together with the 3^n-dyadic lattices trick ends the proof. □

5. Proofs of C_p condition estimates

5.1. Proofs of $M^\sharp$ approach results. Theorems 1, 2 and 3

Proof of Theorem 1. Let $\delta\in(0,1)$ be a parameter to be chosen. Then, by the Lebesgue differentiation theorem

$$\|T(f)\|_{L^p(w)}\le\|M(T(f)^\delta)^{\frac{1}{\delta}}\|_{L^p(w)}=\|M(T(f)^\delta)\|^{\frac{1}{\delta}}_{L^{p/\delta}(w)}.$$

Now we choose $\delta\in(0,1)$ such that

$$\max\{1,p\}<\frac{p}{\delta}<\max\{1,p\}+\varepsilon.$$

If we denote $\varepsilon_1=\max\{1,p\}+\varepsilon-\frac{p}{\delta}$ then, since $w\in C_{\max\{1,p\}+\varepsilon}$, we have that $w\in C_{p/\delta+\varepsilon_1}$ and a direct application of Lemma II combined with the (D) condition yields

$$\|T(f)\|_{L^p(w)}\le c\,\|M^{\#}(T(f)^\delta)\|^{\frac{1}{\delta}}_{L^{p/\delta}(w)}=c\,\|M^{\#}_\delta(T(f))\|_{L^p(w)}\le c\,\|Mf\|_{L^p(w)},$$

which is the desired result. The vector-valued case is analogous, assuming the (D_q) condition instead so we omit the proof.

□

Proof of Theorem 2. The proof is similar to the case $m=1$. Let $\delta\in(0,\frac{1}{m})$ be a parameter to be chosen. Then, as above

$$\|T(\vec{f})\|_{L^p(w)}\le\|M(|T(\vec{f})|^\delta)^{\frac{1}{\delta}}\|_{L^p(w)}.$$

Now we choose $\delta\in(0,\frac{1}{m})$ such that

$$\max\{1,mp\}<\frac{p}{\delta}<\max\{1,mp\}+\varepsilon.$$

If we denote $\varepsilon_m=\max\{1,mp\}+\varepsilon-\frac{p}{\delta}$ then, since $w\in C_{\max\{1,mp\}+\varepsilon}$, we have that $w\in C_{p/\delta+\varepsilon_m}$ and a direct application of Lemma II combined with (1.5) yields

$$\|T(\vec{f})\|_{L^p(w)}\le c\,\|M^{\#}(|T(\vec{f})|^\delta)\|^{\frac{1}{\delta}}_{L^{p/\delta}(w)}=c\,\|M^{\#}_\delta(T(\vec{f}))\|_{L^p(w)}\le c\,\|\mathcal{M}(\vec{f})\|_{L^p(w)},$$

as we wanted to prove. □

Proof of Theorem 3. We will use the key pointwise estimate (1.6): if $0<\delta<\delta_1$: there exists a positive constant $c=c_{\delta,\delta_1,T}$ such that,

$$M^{\#}_\delta([b,T]f)(x)\le c_{\delta,\delta_1,T}\|b\|_{BMO}\left(M_{\delta_1}(Tf)+M^2(f)(x)\right).$$

By the Lebesgue differentiation theorem,

$$\|[b,T]f\|_{L^p(w)}\le\|M(|[b,T]f|^\delta)^{\frac{1}{\delta}}\|_{L^p(w)}.$$

We choose $0<\delta<\delta_1<1$ such that

$$\max\{1,p\}<\frac{p}{\delta_1}<\frac{p}{\delta}<\max\{1,p\}+\varepsilon.$$

Now, if we denote $\varepsilon_1=\max\{1,p\}+\varepsilon-\frac{p}{\delta}$ then, since $w\in C_{\max\{1,p\}+\varepsilon}$, we have that $w\in C_{p/\delta+\varepsilon_1}$ and a direct application of Lemma II yields

$$\|[b,T]\|_{L^p(w)}\le c\|M^{\#}(|[b,T]|^\delta)^{\frac{1}{\delta}}\|_{L^p(w)}.$$

Combining the preceding estimate with (1.6),

$$\|[b,T]\|_{L^p(w)} \leq c\,\|b\|_{BMO}\left(\|M_{\delta_1}(Tf)\|_{L^p(w)} + \|M^2 f\|_{L^p(w)}\right).$$

For the second term we are done, whilst for first one, taking into account our choice for δ_1 and arguing as in the proof of Theorem 1,

$$\|M_{\delta_1}(Tf)\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)}$$

and we are done. Taking into account (1.7) the vector-valued case is analogous so we omit the proof. □

5.2. Proofs of Theorems 4 and 5

The proof of Theorem 4 is actually a consequence of the sparse domination combined with the following Theorem.

Theorem 8. *Let $1 < p < q < \infty$. Let $\mathcal{S}$ be a sparse family and $w \in C_q$. Then*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)},$$
$$\|\mathcal{A}_{\mathcal{S}} f\|_{L^{p,\infty}(w)} \leq c\|Mf\|_{L^{p,\infty}(w)}.$$

Something analogous happens with Therorem 5. It is a consequence of the sparse domination combined with the following result.

Theorem 9. *Let $1 < p < q < \infty$. Let $\mathcal{S}$ be a sparse family and $w \in C_q$, and $b \in \mathrm{BMO}$. Then*

$$\|\mathcal{T}_{b,\mathcal{S}} f\|_{L^p(w)} \leq c\|b\|_{\mathrm{BMO}}\|Mf\|_{L^p(w)},$$
$$\|\mathcal{T}_{b,\mathcal{S}} f\|_{L^{p,\infty}(w)} \leq c\|b\|_{\mathrm{BMO}}\|Mf\|_{L^{p,\infty}(w)}.$$

and

$$\|\mathcal{T}^*_{b,\mathcal{S}} f\|_{L^p(w)} \leq c\|b\|_{\mathrm{BMO}}\|M^2 f\|_{L^p(w)},$$
$$\|\mathcal{T}^*_{b,\mathcal{S}} f\|_{L^{p,\infty}(w)} \leq c\|b\|_{\mathrm{BMO}}\|M^2 f\|_{L^{p,\infty}(w)}.$$

To establish the preceding results we will rely upon some Lemmas that are based on ideas of [39].

5.2.1. Lemmata

In this section we present the technical lemmas needed to establish Theorems 4 and 5. Results here are essentially an elaboration of Sawyer's arguments [39].

Let $\Omega_k := \{f > 2^k\}$ and define

$$(M_{k,p,q}(f)(x))^p = 2^{kp}\int_{\Omega_k} \frac{d(y,\Omega_k^c)^{n(q-1)}}{d(y,\Omega_k^c)^{nq} + |x-y|^{nq}}\,dy.$$

When Ω_k is open let $\Omega_k = \cup_j Q_j^k$ be the Whitney decomposition, i.e., Q_j^k are pairwise disjoint and

$$8 < \frac{\mathrm{dist}(Q_j^k,\Omega_k^c)}{\mathrm{diam} Q_j^k} \leq 10, \quad \sum_j \chi_{6Q_j^k} \leq C_n \chi_{\Omega_k},$$

then it is easy to check that

$$M_{k,p,q}(f)^p \approx 2^{kp}\sum_j M(\chi_{Q_j^k})^q.$$

Our key lemma is the following

Lemma 7. *Suppose that $1 < p < q < \infty$ and that w satisfies the C_q condition. Then for all compactly supported f,*

$$\sup_k \int (M_{k,p,q}(Mf))^p w \leq C\|Mf\|^p_{L^{p,\infty}(w)}.$$

Proof. Let $\Omega_k := \{Mf > 2^k\} = \cup_j Q_j^k$ be the Whitney decomposition. Let N be a positive integer (to be chosen later) and fix a Whitney cube Q_i^{k-N}. We now claim

$$|\Omega_k \cap 3Q_i^{k-N}| \leq C_n 2^{-N} |Q_i^{k-N}|. \tag{5.1}$$

Indeed, let $g = f\chi_{22\sqrt{n}Q_i^{k-N}}$ and $h = f - g$. Let $x_0 \in 22\sqrt{n}Q_i^{k-N} \setminus \Omega_{k-N}$. It is easy to check that for any $x \in 3Q_i^{k-N}$, we have

$$M(h)(x) \leq c_n M(f)(x_0) \leq c_n 2^{k-N}.$$

Let N be sufficiently large such that $c_n 2^{-N} \leq 1/2$. Then

$$\begin{aligned}
|\Omega_k \cap 3Q_i^{k-N}| &= |\{x \in 3Q_i^{k-N} : M(f) > 2^k\}| \\
&\leq |\{x \in 3Q_i^{k-N} : M(g) > 2^{k-1}\}| \\
&\leq 2^{1-k} c_n \int g \leq 2^{1-k} c_n |22\sqrt{n}Q_i^{k-N}| M(f)(x_0) \\
&\leq C_n 2^{-N} |Q_i^{k-N}|.
\end{aligned}$$

As that in [39], define $S(k) = 2^{kp} \sum_j \int M(\chi_{Q_j^k})^q w$ and $S(k; N, i) = 2^{kp} \sum_j \int M(\chi_{Q_j^k})^q w$, where the latter sum is taken over those j for which $Q_j^k \cap Q_i^{k-N} \neq \emptyset$. Since $Q_j^k \cap Q_i^{k-N} \neq \emptyset$ together with (5.1) implies $\ell(Q_j^k) \leq \ell(Q_i^{k-N})$ for large N, and this further implies $Q_j^k \subset Q_i^{k-N}$, we have

$$\begin{aligned}
S(k; N, i) &\leq \int 2^{kp} \sum_{j : Q_j^k \subset 3Q_i^{k-N}} |M\chi_{Q_j^k}|^q w \\
&= \int_{5Q_i^{k-N}} + \int_{\mathbb{R}^n \setminus 5Q_i^{k-N}} := I + II \quad \text{for } N \text{ large.}
\end{aligned}$$

By the argument in [39], we know

$$I \leq C_\delta 2^{kp} w(5Q_i^{k-N}) + \delta 2^{kp} \int |M\chi_{Q_i^{k-N}}|^q w,$$

where we have used $M(\chi_{6Q_i^{k-N}}) \eqsim M(\chi_{Q_i^{k-N}})$. Next we estimate II, we have

$$\begin{aligned}
II &\leq c_n 2^{kp} \int_{\mathbb{R}^n \setminus 5Q_i^{k-N}} \frac{\sum |Q_j^k|^q}{|x - c_{Q_i^{k-N}}|^{nq}} w(x) dx \\
&\leq c_{n,q} 2^{kp} \int_{\mathbb{R}^n \setminus 5Q_i^{k-N}} \Big(\frac{2^{-N} |Q_i^{k-N}|}{|x - c_{Q_i^{k-N}}|^n} \Big)^q w(x) dx \\
&\leq c_{n,q} 2^{N(p-q)} 2^{(k-N)p} \int |M\chi_{Q_i^{k-N}}|^q w.
\end{aligned}$$

Thus for N large (depending on p, q),

$$\begin{aligned}
S(k) &\leq \sum_i S(k; N, i) \\
&\leq C_\delta c_n 2^{kp} w(\Omega_{k-N}) + (\delta 2^{Np} + c_{n,q} 2^{N(p-q)}) S(k - N) \\
&\leq C_{n,\delta} 2^{kp} w(\Omega_{k-N}) + \frac{1}{2} S(k - N).
\end{aligned}$$

Taking the supremum over $k \leq M$, we get

$$\sup_{k \leq M} \int (M_{k,p,q}(Mf))^p w \leq c_{n,p,q} \|M(f)\|_{L^{p,\infty}(w)}^p,$$

provided that

$$\sup_{k \leq M} \int (M_{k,p,q}(Mf))^p w < \infty.$$

By monotone convergence, we can assume that f has compact support, say supp $f \subset Q$. Without loss of generality, assume $f \geq 0$ and $2^s < \langle f \rangle_Q \leq 2^{s+1}$. Then it is easy to check that

$$M(f) \gtrsim 2^s M(\chi_Q).$$

Moreover, for $k \geq s+1$, $\Omega_k \subset 3Q$ and we have

$$\begin{aligned}
&\sup_{s+1<k\leq M} 2^{kp} \sum_j \int M(\chi_{Q_j^k})^q w \\
&\leq \sup_{s+1<k\leq M} 2^{kp} \int M(\chi_Q)^q w \\
&= \sup_{s+1<k\leq M} 2^{kp} \sum_{\ell\geq 1} \int_{2^{\ell+1}Q\setminus 2^\ell Q} M(\chi_Q)^q w + \sup_{s+1<k\leq M} 2^{kp} \int_{2Q} M(\chi_Q)^q w \\
&= I + II.
\end{aligned}$$

First, we estimate II. We have

$$\begin{aligned}
II &\leq 2^{Mp} w(2Q) \leq 2^{Mp} w(\{x : M\chi_Q(x) \geq \frac{1}{2^n}\}) \\
&\leq 2^{Mp} c_{n,p} \|M\chi_Q\|^p_{L^{p,\infty}(w)} \leq 2^{Mp-sp} c_{n,p} \|Mf\|^p_{L^{p,\infty}(w)} < \infty.
\end{aligned}$$

Next we estimate I. Direct calculations give us

$$\begin{aligned}
I &\leq \sup_{s+1<k\leq M} 2^{kp} \sum_{\ell\geq 1} c_{n,q} 2^{-nq\ell} w(2^{\ell+1}Q \setminus 2^\ell Q) \\
&\leq \sup_{s+1<k\leq M} 2^{kp} \sum_{\ell\geq 1} c_{n,q} 2^{-nq\ell} w(\{x : M(\chi_Q) \geq \frac{1}{2^{(\ell+1)n}}\}) \\
&\leq 2^{Mp} c_{n,q} \sum_{\ell\geq 1} 2^{-n(q-p)\ell} \|M(\chi_Q)\|^p_{L^{p,\infty}(w)} \\
&\leq c_{n,p,q} 2^{Mp} 2^{-sp} \|Mf\|^p_{L^{p,\infty}(w)} < \infty.
\end{aligned}$$

It remains to consider the case $k \leq s$. We still follow the idea of Sawyer, but with slight changes. In this case, $\Omega_k \subset (2^{\frac{s-k+2}{n}} + 1)Q$. Then again,

$$\begin{aligned}
\sup_{k\leq s} 2^{kp} \sum_j \int |M(\chi_{Q_j^k})|^q w &\leq \sup_{k\leq s} 2^{kp} c_{n,q} \int |M\chi_{2^{\frac{s-k}{n}}Q}|^q w \\
&= 2^{sp} \sup_{m\geq 0} 2^{-mp} c_{n,q} \int |M\chi_{2^{\frac{m}{n}}Q}|^q w \\
&\leq 2^{sp} \sup_{m\geq 0} 2^{-mp} c_{n,q} \int_{2^{\frac{m}{n}+1}Q} |M\chi_{2^{\frac{m}{n}}Q}|^q w \\
&+ 2^{sp} \sup_{m\geq 0} 2^{-mp} c_{n,q} \sum_{\ell\geq 1} \int_{2^{\frac{m}{n}+\ell+1}Q\setminus 2^{\frac{m}{n}+\ell}Q} |M\chi_{2^{\frac{m}{n}}Q}|^q w \\
&\leq c_{n,p,q} \|Mf\|^p_{L^{p,\infty}(w)} < \infty,
\end{aligned}$$

where the last step follows from similar calculations as the ones above. Now

$$\sup_{k\leq M} \int (M_{k,p,q}(Mf))^p w \leq c_{n,p,q} \|M(f)\|^p_{L^{p,\infty}(w)}$$

and taking the supremum over M we conclude the proof. □

Our last result in this subsection is the following technical lemma.

Lemma 8. *Let $\{Q_j^k\}_j$ be a collection of disjoint cubes in $\{Mf > 2^k\}$, then*

$$2^{kp} \sum_j M(\chi_{Q_j^k})^q \lesssim M_{k,p,q}(Mf)^p.$$

Proof. The proof is straightforward. Indeed, let $c_{Q_j^k}$ be the center of Q_j^k, and P be the cube from the Whitney decomposition of $\{Mf > 2^k\}$ which contains $c_{Q_j^k}$. Of course by the Whitney property, $Q_j^k \subset c_n P$ for some dimensional constant c_n. Then

$$M(\chi_{Q_j^k}) \leq M(\chi_{c_n P}) \leq c_n' M(\chi_P)$$

and the result follows. □

5.2.2. Proof of Theorem 8

We only provide the proof for the strong type (p,p) estimate, since the weak type (p,p) is analogous. Let $\gamma > 0$ a small parameter that will be chosen, then we have that

$$\begin{aligned}
\|\mathcal{A}_{\mathcal{S}} f\|_{L^p(w)}^p &\leq \sum_{k\in\mathbb{Z}} 2^{(k+1)p} w(\{x : 2^k < \mathcal{A}_{\mathcal{S}} f(x) \leq 2^{k+1}\}) \\
&\leq c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{\mathcal{S}} f(x) > 2^k\}) \\
&\leq c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{\mathcal{S}} f(x) > 2^k, M(f)(x) \leq \gamma 2^k\}) \\
&+ c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : M(f)(x) > \gamma 2^k\}) \\
&\leq c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{\mathcal{S}} f(x) > 2^k, M(f)(x) \leq \gamma 2^k\}) + c_{p,\gamma} \|Mf\|_{L^p(w)}^p.
\end{aligned}$$

So we only need to estimate

$$\sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{\mathcal{S}} f(x) > 2^k, M(f)(x) \leq \gamma 2^k\}).$$

Split $\mathcal{S} = \cup_m \mathcal{S}_m$, where

$$\mathcal{S}_m := \{Q \in \mathcal{S} : 2^m < \langle f \rangle_Q \leq 2^{m+1}\}.$$

It is easy to see that, if $2^m \geq \gamma 2^k$, then for $x \in Q \in \mathcal{S}_m$, $Mf(x) > \gamma 2^k$. Set $m_0 = \lfloor \log_2(\frac{1}{\gamma}) \rfloor + 1$, then we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{\mathcal{S}} f(x) > 2^k,\ M(f)(x) \leq \gamma 2^k\}) \\
&= \sum_{k\in\mathbb{Z}} 2^{kp} w\left(\left\{x : \sum_{m\leq k-m_0} \mathcal{A}_{\mathcal{S}_m} f(x) > 2^k (1 - \frac{1}{\sqrt{2}}) \sum_{m \leq k-m_0} 2^{\frac{m-k+m_0}{2}},\ M(f)(x) \leq \gamma 2^k\right\}\right) \\
&\leq \sum_{k\in\mathbb{Z}} 2^{kp} \sum_{m\leq k-m_0} w\left(\left\{x : \mathcal{A}_{\mathcal{S}_m} f(x) > (1 - \frac{1}{\sqrt{2}}) 2^{\frac{m+k+m_0}{2}},\ M(f)(x) \leq \gamma 2^k\right\}\right).
\end{aligned}$$

Denote $b_m = \sum_{Q\in\mathcal{S}_m} \chi_Q$, then $\mathcal{A}_{\mathcal{S}_m} f \leq 2^{m+1} b_m$. and therefore, if we denote $\mathcal{S}_m^*$ is the collection of maximal dyadic cubes in $\mathcal{S}_m$, taking into account the local exponential decay for sparse operators (see for instance [29]),

$$\begin{aligned}
&\left|\left\{\mathcal{A}_{\mathcal{S}_m} f(x) > (1 - \frac{1}{\sqrt{2}}) 2^{\frac{m+k+m_0}{2}}\right\}\right| \leq \left|\left\{b_m > \frac{\sqrt{2}-1}{2\sqrt{2}} 2^{\frac{-m+k+m_0}{2}}\right\}\right| \\
&\leq \sum_{Q\in\mathcal{S}_m^*} \left|\left\{x \in Q : b_m > \frac{\sqrt{2}-1}{2\sqrt{2}} 2^{\frac{-m+k+m_0}{2}}\right\}\right| \leq \exp(-c 2^{\frac{-m+k+m_0}{2}}) \sum_{Q\in\mathcal{S}_m^*} |Q|.
\end{aligned}$$

Now, by the C_q condition, we have

$$\begin{aligned}
&w\left(\left\{\mathcal{A}_{\mathcal{S}_m} f(x) > (1 - \frac{1}{\sqrt{2}}) 2^{\frac{m+k+m_0}{2}}\right\}\right) \\
&= \sum_{Q\in\mathcal{S}_m^*} w\left(\left\{x \in Q : \mathcal{A}_{\mathcal{S}_m} f(x) > (1 - \frac{1}{\sqrt{2}}) 2^{\frac{m+k+m_0}{2}}\right\}\right) \\
&\leq \exp(-c\epsilon 2^{\frac{-m+k+m_0}{2}}) \sum_{Q\in\mathcal{S}_m^*} \int M(\chi_Q)^q w.
\end{aligned}$$

Since $\cup_{Q\in\mathcal{S}_m^*} Q \subset \{x : Mf(x) > 2^m\}$, a combined application of Lemmas 7 and 8 yields the desired result.

5.2.3. Proof of Theorem 9

We may assume that $\|b\|_{\mathrm{BMO}} = 1$ Again we just settle the strong type estimate, since the weak-weak type (p,p) estimate is analogous.

First we note that using (2.1)

$$\mathcal{T}_{b,\mathcal{S}}^* f(x) = \sum_{Q\in\mathcal{S}} \frac{1}{|Q|}\int_Q |b-b_Q||f|\chi_Q \lesssim \|b\|_{\mathrm{BMO}} \sum_{Q\in\mathcal{S}} \|f\|_{L\log L,Q}\chi_Q = \|b\|_{\mathrm{BMO}}\, \mathcal{A}_{L\log L,\mathcal{S}} f.$$

Now we observe that we have that

$$\begin{aligned}
\|\mathcal{A}_{L\log L,\mathcal{S}} f\|_{L^p(w)} &\leq \sum_{k\in\mathbb{Z}} 2^{(k+1)p} w(\{x : 2^k < \mathcal{A}_{L\log L,\mathcal{S}} f \leq 2^{k+1}\})\\
&\leq c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{L\log L,\mathcal{S}} f(x) > 2^k\})\\
&\leq c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{L\log L,\mathcal{S}} f(x) > 2^k, M_{L\log L} f(x) \leq \gamma 2^k\})\\
&+ c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : M_{L\log L} f(x) > \gamma 2^k\})\\
&\leq c_p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{L\log L,\mathcal{S}} f(x) > 2^k, M_{L\log L} f(x) \leq \gamma 2^k\})\\
&+ c_{p,\gamma} \|M_{L\log L} f\|_{L^p(w)}^p.
\end{aligned}$$

So we only need to estimate

$$\sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{L\log L,\mathcal{S}} f(x) > 2^k, M_{L\log L} f(x) \leq \gamma 2^k\}).$$

Split $\mathcal{S} = \cup_m \mathcal{S}_m$, where

$$\mathcal{S}_m := \{Q \in \mathcal{S} : 2^m < \|f\|_{L\log L,Q} \leq 2^{m+1}\}.$$

It is easy to see that, if $2^m \geq \gamma 2^k$, then for $x \in Q \in \mathcal{S}_m$, $M_{L\log L} f(x) > \gamma 2^k$. Set $m_0 = \lfloor \log_2(\frac{1}{\gamma})\rfloor + 1$, we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \mathcal{A}_{L\log L,\mathcal{S}} f(x) > 2^k, M_{L\log L} f(x) \leq \gamma 2^k\})\\
&= \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x : \sum_{m\leq k-m_0} \mathcal{A}_{L\log L,\mathcal{S}_m} f(x) > 2^k(1-\frac{1}{\sqrt{2}}) \sum_{m\leq k-m_0} 2^{\frac{m-k+m_0}{2}}, M_{L\log L} f(x) \leq \gamma 2^k\})\\
&\leq \sum_{k\in\mathbb{Z}} 2^{kp} \sum_{m\leq k-m_0} w(\{x : \mathcal{A}_{L\log L,\mathcal{S}_m} f(x) > (1-\frac{1}{\sqrt{2}}) 2^{\frac{m+k+m_0}{2}}, M_{L\log L} f(x) \leq \gamma 2^k\}).
\end{aligned}$$

Denote $b_m = \sum_{Q\in\mathcal{S}_m} \chi_Q$, then $\mathcal{A}_{L\log L,\mathcal{S}_m} f \leq 2^{m+1} b_m$. and therefore, by sparseness,

$$\begin{aligned}
&\left|\left\{\mathcal{A}_{L\log L,\mathcal{S}_m} f(x) > (1-\frac{1}{\sqrt{2}}) 2^{\frac{m+k+m_0}{2}}\right\}\right|\\
&\leq \left|\left\{b_m > \frac{\sqrt{2}-1}{2\sqrt{2}} 2^{\frac{-m+k+m_0}{2}}\right\}\right| \leq \exp(-c 2^{\frac{-m+k+m_0}{2}}) \sum_{Q\in\mathcal{S}_m^*} |Q|,
\end{aligned}$$

where $\mathcal{S}_m^*$ is the collection of maximal dyadic cubes in $\mathcal{S}_m$. By the C_q condition, we have

$$\begin{aligned}
&w\left(\left\{\mathcal{A}_{L\log L,\mathcal{S}_m} f(x) > (1-\frac{1}{\sqrt{2}}) 2^{\frac{m+k+m_0}{2}}\right\}\right)\\
&\leq \exp(-c\epsilon 2^{\frac{-m+k+m_0}{2}}) \sum_{Q\in\mathcal{S}_m^*} \int M(\chi_Q)^q w.
\end{aligned}$$

Since $\cup_{Q\in\mathcal{S}_m^*}Q\subset\{x:M_{L\log L}f(x)>2^m\}\subseteq\{x:M(Mf)(x)>2^{m-n}\}$ then we have that using Lemma 8

$$\sum_{Q\in\mathcal{S}_m^*}M(\chi_Q)^q w\lesssim M_{m-n,p,q}(M(Mf))^p$$

and consequently

$$\sum_{Q\in\mathcal{S}_m^*}\int M(\chi_Q)^q w\lesssim\int M_{m-n,p,q}(M(Mf))^p.$$

Hence taking into account Lemma 7

$$\begin{aligned}w\left(\left\{\mathcal{A}_{L\log L,\mathcal{S}_m}f(x)>(1-\frac{1}{\sqrt{2}})2^{\frac{m+k+m_0}{2}}\right\}\right)&\lesssim\exp(-c\epsilon 2^{\frac{-m+k+m_0}{2}})\int M_{m-n,p,q}(M(Mf))^p\\&\lesssim\exp(-c\epsilon 2^{\frac{-m+k+m_0}{2}})\|M(Mf)\|_{L^p(w)}^p.\\&\simeq\exp(-c\epsilon 2^{\frac{-m+k+m_0}{2}})\|M_{L\log L}f\|_{L^p(w)}^p.\end{aligned}$$

This yields

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}2^{kp}w(\{x:\mathcal{A}_{L\log L,\mathcal{S}}f(x)>2^k,M_{L\log L}f(x)\le\gamma 2^k\})\\&\lesssim\sum_{k\in\mathbb{Z}}2^{kp}\sum_{m\le k-m_0}\exp(-c\epsilon 2^{\frac{-m+k+m_0}{2}})\|M_{L\log L}f\|_{L^p(w)}^p\end{aligned}$$

and we are done.

Now we turn our attention to $\mathcal{T}_{b,\mathcal{S}}f(x)$. We observe that we have that arguing as before

$$\begin{aligned}\|\mathcal{T}_{b,\mathcal{S}}f(x)f\|_{L^p(w)}^p&\le\sum_{k\in\mathbb{Z}}2^{(k+1)p}w(\{x:2^k<\mathcal{T}_{b,\mathcal{S}}f(x)\le 2^{k+1}\})\\&\le c_p\sum_{k\in\mathbb{Z}}2^{kp}w(\{x:\mathcal{T}_{b,\mathcal{S}}f(x)>2^k,M(f)(x)\le\gamma 2^k\})+c_{p,\gamma}\|Mf\|_{L^p(w)}^p.\end{aligned}$$

So we only need to estimate

$$\sum_{k\in\mathbb{Z}}2^{kp}w(\{x:\mathcal{T}_{b,\mathcal{S}}f(x)>2^k,M(f)(x)\le\gamma 2^k\}).$$

Split $\mathcal{S}=\cup_m\mathcal{S}_m$, where

$$\mathcal{S}_m:=\{Q\in\mathcal{S}:2^m<\langle f\rangle_Q\le 2^{m+1}\}.$$

It is easy to see that, if $2^m\ge\gamma 2^k$, then for $x\in Q\in\mathcal{S}_m$, $Mf(x)>\gamma 2^k$. Set $m_0=\lfloor\log_2(\frac{1}{\gamma})\rfloor+1$, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}2^{kp}w(\{x:\mathcal{T}_{b,\mathcal{S}}f(x)>2^k,M(f)(x)\le\gamma 2^k\})\\&=\sum_{k\in\mathbb{Z}}2^{kp}w\left(\left\{x:\sum_{m\le k-m_0}\mathcal{T}_{b,\mathcal{S}_m}f(x)>2^k(1-\frac{1}{\sqrt{2}})\sum_{m\le k-m_0}2^{\frac{m-k+m_0}{2}},M(f)(x)\le\gamma 2^k\right\}\right)\\&\le\sum_{k\in\mathbb{Z}}2^{kp}\sum_{m\le k-m_0}w\left(\left\{x:\mathcal{T}_{b,\mathcal{S}_m}f(x)>(1-\frac{1}{\sqrt{2}})2^{\frac{m+k+m_0}{2}},M(f)(x)\le\gamma 2^k\right\}\right).\end{aligned}$$

Now we observe that $\mathcal{T}_{b,\mathcal{S}_m}f(x)\le 2^{m+1}\sum_{Q\in\mathcal{S}_m}|b(x)-b_Q|\chi_Q$, therefore

$$\begin{aligned}&\left|\left\{\mathcal{T}_{b,\mathcal{S}_m}f(x)>(1-\frac{1}{\sqrt{2}})2^{\frac{m+k+m_0}{2}}\right\}\right|\\&\le\left|\left\{\sum_{Q\in\mathcal{S}_m}|b(x)-b_Q|\chi_Q>\frac{\sqrt{2}-1}{2\sqrt{2}}2^{\frac{-m+k+m_0}{2}}\right\}\right|\\&=\sum_{Q\in\mathcal{S}_m^*}\left|\left\{x\in Q:\sum_{P\in\mathcal{S}_m,P\subseteq Q}|b(x)-b_P|\chi_P>\frac{\sqrt{2}-1}{2\sqrt{2}}2^{\frac{-m+k+m_0}{2}}\right\}\right|,\end{aligned}$$

where $\mathcal{S}_m^*$ is the collection of maximal dyadic cubes in $\mathcal{S}_m$. Now taking into account [25, Lemma 5.1], we have that there exists a sparse family $\widetilde{\mathcal{S}_m}$ containing $\mathcal{S}_m$ such that

$$|b(x)-b_P|\chi_P(x)\le \|b\|_{\mathrm{BMO}}c_n\sum_{R\subseteq P,P\in\mathcal{S}_m}\chi_R(x)=c_n\sum_{R\subseteq P,P\in\mathcal{S}_m}\chi_R(x).$$

Taking that into account we can continue the preceding computation as follows

$$\begin{aligned}&\sum_{Q\in\mathcal{S}_m^*}\left|\left\{x\in Q:\sum_{P\in\mathcal{S}_m,P\subseteq Q}\left(c_n\sum_{R\subseteq P,P\in\widetilde{\mathcal{S}_m}}\chi_R(x)\right)\chi_P>\frac{\sqrt{2}-1}{2\sqrt{2}}2^{\frac{-m+k+m_0}{2}}\right\}\right|\\ &\sum_{Q\in\mathcal{S}_m^*}\left|\left\{x\in Q:\left(\sum_{P\in\widetilde{\mathcal{S}_m},P\subseteq Q}\chi_P\right)^2>c\frac{\sqrt{2}-1}{2\sqrt{2}}2^{\frac{-m+k+m_0}{2}}\right\}\right|\\ &\le\exp(-c2^{\frac{-m+k+m_0}{4}})\sum_{Q\in\mathcal{S}_m^*}|Q|.\end{aligned}$$

Hence, combining the preceding estimates and using the C_q condition, we have

$$\begin{aligned}&w\left(\left\{\mathcal{T}_{b,\mathcal{S}_m}f(x)>(1-\frac{1}{\sqrt{2}})2^{\frac{m+k+m_0}{2}}\right\}\right)\\ &\le\exp(-c\epsilon2^{\frac{-m+k+m_0}{4}})\sum_{Q\in\mathcal{S}_m^*}\int M(\chi_Q)^qw.\end{aligned}$$

Since $\cup_{Q\in\mathcal{S}_m^*}Q\subset\{x:Mf(x)>2^m\}$ then we have that using Lemma 8

$$\sum_{Q\in\mathcal{S}_m^*}M(\chi_Q)^qw\lesssim M_{m,p,q}(Mf)^p$$

and consequently

$$\sum_{Q\in\mathcal{S}_m^*}\int M(\chi_Q)^qw\lesssim\int M_{m,p,q}(Mf)^p.$$

Hence taking into account Lemma 7

$$\begin{aligned}w\left\{\mathcal{T}_{b,\mathcal{S}_m}f(x)>(1-\frac{1}{\sqrt{2}})2^{\frac{m+k+m_0}{2}}\right\}&\lesssim\exp(-c\epsilon2^{\frac{-m+k+m_0}{4}})\int M_{m,p,q}(Mf)^p\\ &\lesssim\exp(-c\epsilon2^{\frac{-m+k+m_0}{4}})\|Mf\|_{L^p(w)}^p.\end{aligned}$$

This yields

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}2^{kp}w(\{x:\mathcal{T}_{b,\mathcal{S}_m}f(x)>2^k,M(f)(x)\le\gamma2^k\})\\ &\lesssim\sum_{k\in\mathbb{Z}}2^{kp}\sum_{m\le k-m_0}\exp(-c\epsilon2^{\frac{-m+k+m_0}{4}})\|Mf\|_{L^p(w)}^p\end{aligned}$$

and we are done.

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Appendix A. Consequences of sparse domination results

The following results follow combining the sparse domination results provided above and ideas in [14, 25] and a suitable adaption of the conjugation method in the case of the one weighted setting for the commutator.

Theorem 10 (A_p weak and strong type estimates)**.** *Let $1 < p, q < \infty$ and $w \in A_p$. Then*

- **Maximal function**

$$\|\overline{M}_q(\sigma \boldsymbol{f})\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p}} \left([w]_{A_\infty}^{\left(\frac{1}{q}-\frac{1}{p}\right)_+} + [\sigma]_{A_\infty}^{\frac{1}{p}}\right) \||\boldsymbol{f}|_q\|_{L^p(\sigma)}$$

$$\|\overline{M}_q(\sigma \boldsymbol{f})\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\frac{1}{p}} [w]_{A_\infty}^{\left(\frac{1}{q}-\frac{1}{p}\right)_+} \||\boldsymbol{f}|_q\|_{L^p(\sigma)} \qquad p \neq q.$$

- **Calderón-Zygmund operators**

$$\|\overline{T}_q(\sigma \boldsymbol{f})\|_{L^p(w)} \le c_{n,p,q} C_T [w]_{A_p}^{\frac{1}{p}} \left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}} \right) \||\boldsymbol{f}|_q\|_{L^p(\sigma)}$$
$$\|\overline{T}_q(\sigma \boldsymbol{f})\|_{L^{p,\infty}(w)} \le c_{n,p,q} C_T [w]_{A_p}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(\sigma)}.$$

- **Commutators of Calderón-Zygmund operators**

$$\|\overline{[b,T]}_q \boldsymbol{f}\|_{L^p(w)} \le c_{n,p,q} C_T [w]_{A_p}^{\frac{1}{p}} \left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}} \right) \left([w]_{A_\infty} + [\sigma]_{A_\infty} \right) \|\boldsymbol{f}\|_{L^p(w)}.$$

If $\mu, \lambda \in A_p$, *and* $\nu = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p}}$. *If* $b \in \mathrm{BMO}_\nu$, *namely if* $\|b\|_{\mathrm{BMO}_\nu} = \sup_Q \frac{1}{\nu(Q)} \int_Q |b - b_Q| dx < \infty$, *then*
$$\|\overline{[b,T]}_q \boldsymbol{f}\|_{L^p(\lambda)} \le c_{n,p,q} C_T \max\left\{ [\mu]_{A_p} [\lambda]_{A_p} \right\}^{\max\left\{1, \frac{1}{p-1}\right\}} \|b\|_{\mathrm{BMO}_\nu} \||\boldsymbol{f}|_q\|_{L^p(\mu)}.$$

- **Rough singular integrals, commutators and** $B_{(n-1)/2}$

$$\|\overline{T}_q\|_{L^p(w)} \le c_{n,p,q}\ c_T [w]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}) \min\{[\sigma]_{A_\infty}, [w]_{A_\infty}\},$$
$$\|\overline{(T_\Omega^*)}_q\|_{L^p(w)} \le c_{n,p,q} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}) \max\{[\sigma]_{A_\infty}, [w]_{A_\infty}\},$$
$$\|\overline{[b,T_\Omega]}_q\|_{L^p(w)} \le c_{n,p,q} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}} [w]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}) \max\{[\sigma]_{A_\infty}, [w]_{A_\infty}\}^2.$$

The following estimates can be obtained using the proofs for sparse operators contained in [8, 25].

Theorem 11 (Endpoint estimates). *Let* $1 < p, q < \infty$, w *a weight and* $v \in A_1$. *Then*

- **Calderón-Zygmund operators**

$$\|\overline{T}_q(\boldsymbol{f})\|_{L^{1,\infty}(w)} \lesssim c_\Phi \int_{\mathbb{R}^n} |\boldsymbol{f}(x)|_q M_\Phi w(x) dx,$$

where $c_\Phi = \int_1^\infty \frac{\Phi^{-1}(t)}{t^2 \log(e+t)} dt$. *From this estimate we derive the following*

$$\|\overline{T}_q(\boldsymbol{f})\|_{L^{1,\infty}(v)} \lesssim [v]_{A_1} \log(e + [v]_{A_\infty}) \int_{\mathbb{R}^n} |\boldsymbol{f}(x)|_q v(x) dx.$$

- **Commutators**

$$w\left(\left\{ x \in \mathbb{R}^n \,:\, \overline{[b,T]}_q \boldsymbol{f}(x) > t \right\} \right) \lesssim c_T \frac{c_\varphi}{t} \int_{\mathbb{R}^n} \Phi\left(\|b\|_{\mathrm{BMO}} \frac{|\boldsymbol{f}(x)|_q}{t} \right) M_{(\Phi\circ\varphi)(L)} w(x) dx,$$

where $\Phi(t) = t\log(e+t)$ *and* $c_\varphi = \int_1^\infty \frac{\varphi^{-1}(t)}{t^2 \log(e+t)} dt$. *From this estimate it follows that*

$$v\left(\left\{ x \in \mathbb{R}^n \,:\, \overline{[b,T]}_q \boldsymbol{f}(x) > t \right\} \right) \lesssim [v]_{A_1} [v]_{A_\infty} \log(e + [v]_{A_\infty}) \int_{\mathbb{R}^n} \Phi\left(\frac{|\boldsymbol{f}|_q \|b\|_{\mathrm{BMO}}}{t} \right) v dx.$$

Using results for sparse operators contained in [7, 26, 27, 33, 37] we obtain the following result.

Theorem 12 (Fefferman-Stein type inequalities). *Let* w *a weight,* $1 < p < \infty$ *and* $r > 1$ *small enough. Then*

- **Calderón-Zygmund operators and commutators**

$$||\overline{[b,T]}_q \boldsymbol{f}||_{L^p(w)} \le c_{n,q} C_T \|b\|_{\mathrm{BMO}} (pp')^2 (r')^{1+\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(M_r w)}$$
$$||\overline{T}_q \boldsymbol{f}||_{L^p(w)} \le c_{n,q} C_T pp' (r')^{\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(M_r w)}.$$

- **Rough singular integrals, commutators and $B_{(n-1)/2}$**

$$\|\overline{T}_q(\boldsymbol{f})\|_{L^p(w)} \leq c_{n,p,q} c_T (r')^{\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(M_r w)},$$
$$\|\overline{(T_\Omega^*)}_q(\boldsymbol{f})\|_{L^p(w)} \leq c_{n,p,q} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} (r')^{1+\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(M_r w)},$$
$$\|\overline{[b,T_\Omega]}_q\|_{L^p(w)} \leq c_{n,p,q} \|b\|_{\mathrm{BMO}} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} (r')^{2+\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(M_r w)}.$$

Theorem 13. *Let $1 \leq s < p < \infty$, $r > 1$ small enough and $w \in A_s$. Then*

- **Calderón-Zygmund operators and commutators**

$$||\overline{T}_q \boldsymbol{f}||_{L^p(w)} \leq c_{n,q} C_T p p' [w]_{A_s}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(w)},$$
$$||\overline{[b,T]}_q \boldsymbol{f}||_{L^p(w)} \leq c_{n,q} C_T \|b\|_{\mathrm{BMO}} (pp')^2 [w]_{A_s}^{\frac{1}{p}} [w]_{A_\infty}^{1+\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(w)},$$

- **Rough singular integrals, commutators and $B_{(n-1)/2}$**

$$\|\overline{T}_q(\boldsymbol{f})\|_{L^p(w)} \leq c_{n,p,q} [w]_{A_s}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(w)},$$
$$\|\overline{(T_\Omega^*)}_q(\boldsymbol{f})\|_{L^p(w)} \leq c_{n,p,q} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_s}^{\frac{1}{p}} [w]_{A_\infty}^{1+\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(w)},$$
$$\|\overline{[b,T_\Omega]}_q(\boldsymbol{f})\|_{L^p(w)} \leq c_{n,p,q} \|b\|_{\mathrm{BMO}} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_s}^{\frac{1}{p}} [w]_{A_\infty}^{2+\frac{1}{p'}} \||\boldsymbol{f}|_q\|_{L^p(w)}.$$

In the following Theorems we gather some estimates in the spirit of [29], some of them already contained there, that can be settled combining sparse domination results with ideas in [16, 32].

Theorem 14. *Let $1 < q < \infty$, T be an ω-Calderón-Zygmund operator with ω satisfying the Dini condition and $b \in \mathrm{BMO}$. Assume also that* $\operatorname{supp} |f|_q \subseteq Q$. *Then*

$$\left|\left\{x \in Q : \overline{M_q}\boldsymbol{f}(x) > tM\left(|\boldsymbol{f}|_q\right)(x)\right\}\right| \leq c_1 e^{-c_2 t^q} |Q|,$$
$$\left|\left\{x \in Q : \overline{T_q}\boldsymbol{f}(x) > tM\left(|\boldsymbol{f}|_q\right)(x)\right\}\right| \leq c_1 e^{-c_2 t} |Q|,$$
$$\left|\left\{x \in Q : \left|\overline{[b,T]_q}\boldsymbol{f}(x)\right| > tM^2\left(|\boldsymbol{f}|_q\right)(x)\right\}\right| \leq c_1 e^{-\sqrt{c_2 \frac{t}{\|b\|_{\mathrm{BMO}}}}} |Q|.$$

We will finish this section with a similar type result for rough singular integrals.

Theorem 15. *Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $T = T_\Omega$ or $B_{(n-1)/2}$. Let also Q be a cube and f such that* $\operatorname{supp} f \subseteq Q$, *then there exist some constants $c, \alpha > 0$ such that*

$$|\{x \in Q : |Tf(x)| > tMf(x)\}| \leq ce^{-\sqrt{\alpha t}} |Q|, \qquad t > 0.$$

Remark 6. *We believe that the preceding estimate is not sharp, we conjecture that the decay should be exponential instead of subexponential.*

Appendix B. Unweighted quantitative estimates

In this appendix we collect some quantitative unweighted estimates for Calderón-Zygmund satisfying Dini condition and their vector-valued counterparts. These estimates are somehow implicit in the literature and are a basic ingredient for our fully-quantitative sparse domination results. Our first result provides a quantitative pointwise estimate involving $M_\delta^\sharp$ and T. It can be obtained following the strategy devised in [2] it is not hard to check that the following estimate holds.

Proposition 1. *Let T be an ω-Calderón-Zygmund operator satisfying a Dini condition. For each $0 < \delta < 1$ we have that*

$$M_\delta^\sharp(Tf)(x_0) \leq 2^{n+1} \left(\frac{1}{1-\delta}\right)^{\frac{1}{\delta}} \left(\|T\|_{L^2 \to L^2} + \|\omega\|_{Dini}\right) Mf(x_0).$$

Our next result provides quantitative control of $\|\overline{T}_q\|_{L^1 \to L^{1,\infty}}$

Proposition 2. *Let $1 < q < \infty$ and T be an ω-Calderón-Zygmund operator satisfying a Dini condition. Then*

$$\|\overline{T}_q\|_{L^1 \to L^{1,\infty}} \leq c_n(\|\omega\|_{Dini} + \|T\|_{L^q \to L^q}).$$

Furthermore, since $\|T\|_{L^q \to L^q} \leq c_n \left(\|\omega\|_{Dini} + \|T\|_{L^2 \to L^2}\right)$

$$\|\overline{T}_q\|_{L^1 \to L^{1,\infty}} \leq c_n(\|\omega\|_{Dini} + \|T\|_{L^2 \to L^2}).$$

Proof. It suffices to follow the proof in [34], but considering the Calderón-Zygmund decomposition with respect to the level $\alpha\lambda$ and then optimize in α. □

Appendix B.1. Boundedness of $\overline{M}_q$ on $L^{p,\infty}$

In this Section we prove that $\overline{M}_q : L^{p,\infty} \to L^{p,\infty}$. For that purpose we will use the following Fefferman-Stein type estimate obtained in [31, Theorem 1.1]

Theorem 16. *Let $1 < p < q < \infty$ then, if g is a locally integrable function, we have that*

$$\int_{\mathbb{R}^n} \overline{M}_q \boldsymbol{f} g \leq \int_{\mathbb{R}^n} |\boldsymbol{f}|_q M g.$$

As we anounced, using the estimate in Theorem 16, we can obtain the following result.

Theorem 17. *Let $1 < p, q < \infty$. Then*

$$\left\|\overline{M}_q \boldsymbol{f}\right\|_{L^{p,\infty}} \leq c_{n,q} \left\||\boldsymbol{f}|_q\right\|_{L^{p,\infty}}.$$

Proof. Let us fix $1 < r < \min\{p, q\}$. Then

$$\left\|\overline{M}_q \boldsymbol{f}\right\|_{L^{p,\infty}} = \left\|\left(\overline{M}_q \boldsymbol{f}\right)^{\frac{r}{r}}\right\|_{L^{p,\infty}} = \left\|\left(\overline{M}_q \boldsymbol{f}\right)^r\right\|_{L^{\frac{p}{r},\infty}}^{\frac{1}{r}}.$$

Now by duality

$$\left\|\left(\overline{M}_q \boldsymbol{f}\right)^r\right\|_{L^{\frac{p}{r},\infty}}^{\frac{1}{r}} = \left(\sup_{\|g\|_{L^{\left(\frac{p}{r}\right)',1}}=1} \left|\int_{\mathbb{R}^n} \left(M_q \boldsymbol{f}\right)^r g\right|\right)^{\frac{1}{r}},$$

and using Theorem 16 together with Hölder's inequality in the context of Lorentz spaces we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \left(\overline{M}_q \boldsymbol{f}\right)^r g\right| &\leq \int_{\mathbb{R}^n} \left|\left(\overline{M}_q \boldsymbol{f}\right)^r g\right| \leq \int_{\mathbb{R}^n} |\boldsymbol{f}|_q^r \, |Mg| \\
&\leq \| \, |\boldsymbol{f}|_q^r \, \|_{L^{\frac{p}{r},\infty}} \|Mg\|_{L^{\left(\frac{p}{r}\right)',1}} \\
&\leq c_{n,p,q} \| \, |\boldsymbol{f}|_q \, \|_{L^{p,\infty}}^r \|g\|_{L^{\left(\frac{p}{r}\right)',1}} \leq c_{n,p,q} \| \, |\boldsymbol{f}|_q \, \|_{L^{p,\infty}}^r.
\end{aligned}$$

Summarizing

$$\left\|\overline{M}_q \boldsymbol{f}\right\|_{L^{p,\infty}} = \left\|\left(\overline{M}_q \boldsymbol{f}\right)^r\right\|_{L^{\frac{p}{r},\infty}}^{\frac{1}{r}} \leq \left(c_{n,p,q} \| \, |\boldsymbol{f}|_q \, \|_{L^{p,\infty}}^r\right)^{\frac{1}{r}} \leq c_{n,p,q} \| \, |\boldsymbol{f}|_q \, \|_{L^{p,\infty}}.$$

□

Appendix B.2. Weak type $(1,1)$ of $\overline{T^*}_q$

In this Section we present a fully quantitative estimate of the weak-type $(1,1)$ of $\overline{T^*}_q$ via a suitable pointwise Cotlar inequality.

Now we recall Cotlar's inequality for T^*. In [15, Theorem A.2] the following result is obtained

Lemma 9. *Let T be an ω-Calderón-Zygmund operator with ω satisfying a Dini condition and let $\delta \in (0,1)$. Then*

$$T^* f(x) \leq c_{n,\delta} \left(M_\delta(|Tf|)(x) + \left(\|T\|_{L^2 \to L^2} + \|\omega\|_{Dini}\right) Mf(x)\right).$$

Armed with this lemma we are in the position to prove the following pointwise vector-valued Cotlar's inequality.

Lemma 10. *Let T be an ω-Calderón-Zygmund operator with ω satisfying a Dini condition, $\delta \in (0,1)$ and $1 < q < \infty$. Then*

$$\overline{T^*}_q \boldsymbol{f}(x) \leq c_{n,\delta} \left(\overline{M}_{\frac{q}{\delta}} (|\overline{T} \boldsymbol{f}|^\delta)(x)^{\frac{1}{\delta}} + (\|T\|_{L^2 \to L^2} + \|\omega\|_{Dini}) \, \overline{M}_q \boldsymbol{f}(x) \right),$$

where $|\overline{T}\boldsymbol{f}|^\delta$ stands for $\left\{|Tf_j|^\delta\right\}_{j=1}^\infty$.

Proof. It suffices to apply Lemma 9 to each term of the sum. □

Theorem 18. *Let T be an ω-Calderón-Zygmund operator with ω satisfying the Dini condition, and $1 < q < \infty$. Then*

$$\|\overline{T^*}_q \boldsymbol{f}\|_{L^{1,\infty}} \leq c_{n,\delta,q} \left(\|T\|_{L^2 \to L^2} + \|\omega\|_{Dini}\right) \||\boldsymbol{f}|_q\|_{L^1}.$$

Proof. Using the previous lemma

$$\|\overline{T^*}_q \boldsymbol{f}\|_{L^{1,\infty}} \leq c_{n,\delta} \left(\left\| \overline{M}_{\frac{q}{\delta}} (|\overline{T} \boldsymbol{f}|^\delta)^{\frac{1}{\delta}} \right\|_{L^{1,\infty}} + (\|T\|_{L^2 \to L^2} + \|\omega\|_{\text{Dini}}) \left\| \overline{M}_q \boldsymbol{f} \right\|_{L^{1,\infty}} \right).$$

For the second term we have that

$$\left\| \overline{M}_q \boldsymbol{f} \right\|_{L^{1,\infty}} \leq c_{n,q} \||\boldsymbol{f}|_q\|_{L^1}$$

so we only have to deal with the first term.

Using that $\overline{M}_q : L^{p,\infty} \to L^{p,\infty}$ (Theorem 17) we have that

$$\begin{aligned} \left\| \overline{M}_{\frac{q}{\delta}} (|\overline{T} \boldsymbol{f}|^\delta)(x)^{\frac{1}{\delta}} \right\|_{L^{1,\infty}} &= \left\| \overline{M}_{\frac{q}{\delta}} (|\overline{T} \boldsymbol{f}|^\delta) \right\|_{L^{\frac{1}{\delta},\infty}}^{\frac{1}{\delta}} \leq C_{n,\delta,q} \left\| |T\boldsymbol{f}|_{\frac{q}{\delta}}^\delta \right\|_{L^{\frac{1}{\delta},\infty}}^{\frac{1}{\delta}} \\ &= C_{n,\delta,q} \left\| \overline{T}_q \boldsymbol{f} \right\|_{L^{1,\infty}} \leq C_{n,\delta,q} \|\overline{T}_q\|_{L^1 \to L^{1,\infty}} \||\boldsymbol{f}|_q\|_{L^1}. \end{aligned}$$

Now, taking into account Proposition 2 we have that

$$\max \left\{ \|\overline{T}_q\|_{L^1 \to L^{1,\infty}}, \|T\|_{L^2 \to L^2} + \|\omega\|_{\text{Dini}} \right\} \leq c_{n,q} \left(\|T\|_{L^2 \to L^2} + \|\omega\|_{\text{Dini}}\right)$$

and we are done.

□