Measure-valued weak solutions for some kinetic equations with singular kernels for quantum particles

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EXCELENCIA SEVERO OCHOA

Abstract

In this thesis, we present a mathematical study of three problems arising in the kinetic theory of quantum gases.

In the first part, we consider a Boltzmann equation that is used to describe the time evolution of the particle density of a homogeneous and isotropic photon gas, that interacts through Compton scattering with a low-density electron gas at non-relativistic equilibrium. The kernel in the kinetic equation is highly singular, and we introduce a truncation motivated by the very-peaked shape of the kernel along the diagonal. With this modified kernel, the global existence of measure-valued weak solutions is established for a large set of initial data.

We also study a simplified version of this equation, that appears at very low temperatures of the electron gas, where only the quadratic terms are kept. The global existence of measure-valued weak solutions is proved for a large set of initial data, as well as the global existence of L^1 solutions for initial data that satisfy a strong integrability condition near the origin. The long time asymptotic behavior of weak solutions for this simplified equation is also described.

In the second part of the thesis, we consider a system of two coupled kinetic equations related to a simplified model for the time evolution of the particle density of the normal and superfluid components in a homogeneous and isotropic weakly interacting dilute Bose gas. We establish the global existence of measure-valued weak solutions for a large class of initial data. The conservation of mass and energy and the production of moments of all positive order is also proved. Finally, we study some of the properties of the condensate density and establish an integral equation that describes its time evolution.

Contents

Resumen

En esta tesis, se consideran tres problemas relacionados con la teoría cinética de gases para partículas cuánticas.

En el primer problema, se estudian las soluciones débiles de la siguiente ecuación integro-diferencial:

$$\frac{\partial u}{\partial t}(t,x) = \int_0^\infty b(x,y)\Big[u(t,y)\big(x^2+u(t,x)\big)e^{-x} - u(t,x)\big(y^2+u(t,y)\big)e^{-y}\Big]dy, \quad (1)$$

para $t > 0$, $x \geq 0$. El núcleo $b(x,y) \geq 0$ es una función continua y simétrica en $[0,\infty)^2 \setminus \{(0,0)\}$, singular en el origen $(x,y) = (0,0)$, con el soporte contenido en un entorno de la diagonal $\{x = y \geq 0\}$.

En el segundo problema, se considera una versión simplificada de la ecuación (1), donde solo aparecen los términos cuadráticos:

$$\frac{\partial u}{\partial t}(t,x) = u(t,x)\int_0^\infty b(x,y)\big(e^{-x}-e^{-y}\big)u(t,y)dy \qquad t>0,\ x\geq 0. \quad (2)$$

En el tercer y último problema, se considera el siguiente sistema de dos ecuaciones acopladas:

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,x) = n(t)Q(g)(t,x) \qquad t>0,\ x>0, & (3)\\ n'(t) = -n(t)\displaystyle\int_0^\infty Q(g)(t,x)dx \qquad t>0, & (4)\end{cases}$$

$$Q(g)(t,x) = \int_0^x q(g)(t,x,y)dy - 2\int_x^\infty q(g)(t,y,x)dy, \quad (5)$$

$$q(g)(t,x,y) = \frac{g(t,x-y)}{\sqrt{x-y}}\frac{g(t,y)}{\sqrt{y}} - \frac{g(t,x)}{\sqrt{x}}\left(1+\frac{g(t,x-y)}{\sqrt{x-y}}+\frac{g(t,y)}{\sqrt{y}}\right), \quad (6)$$

y se estudian sus soluciones débiles en términos de la medida $G(t,\cdot)$ definida por

$$G(t,\cdot) = n(t)\delta_0(\cdot) + g(t,\cdot), \qquad t\geq 0, \quad (7)$$

donde $\delta_0(\cdot)$ el la delta de Dirac en $x = 0$.

Aunque con un núcleo distinto, que denotaremos en esta sección por $K(x,y)$, la ecuación (1) aparece en la literatura como un modelo simplificado para describir la evolución en tiempo de la la densidad de partículas de un gas de fotones, homogéneo e isotrópico, que interactúa únicamente mediante el efecto Compton con un gas diluido de electrones en equilibrio no relativista (cf. [25], [49], [72], [42]). La función $u(t,x) \geq 0$ representa la densidad de fotones con energía $x \geq 0$ en el gas a tiempo

$t \geq 0$. El núcleo $b(x,y)$ que se considera en (1) es una aproximación de $K(x,y)$, cuya expresión explícita se conoce (ver [57] para una deducción detallada).

La ecuación (1) aparece ya en en el artículo de Kompaneets [49], de 1957. Una deducción de la ecuación, siguiendo un argumento de tipo Boltzmann, puede verse en [25]. En [23] se deriva, mediante un método tipo BBGKY, una ecuación de transporte más general que describe un sistema homogéneo e isotrópico de fotones y de partículas cargadas. También se obtienen los resultados de [25] y se incorporan términos de corrección que representan efectos de correlación. Teorías cinéticas efectivas y generales para describir las dinámicas de quasipartículas en plasmas relativistas pueden consultarse en [12] y en [8].

En cuanto a la literatura matemática, la ecuación (1) ha sido estudiada bajo distintas condiciones sobre el núcleo b. En [29] se prueba la existencia de soluciones para núcleos acotados y para núcleos con un determinado crecimiento exponencial. La ecuación (1) con el núcleo físico $K(x,y)$ ha sido estudiada por M. Chane-Yook y A. Nouri en [22], y E. Ferrari y A. Nouri en [34]. En [34] se prueba que, si bien el problema de Cauchy tiene soluciones débiles globalmente definidas en tiempo para cualquier dato inicial no negativo $u_0 \in L^1(\mathbb{R}_+)$ acotado superiormente por la distribución de Planck en casi todo punto, i.e., tal que $u_0(x) \leq x^2(e^x - 1)^{-1}$ para casi todo $x > 0$, no existe solución, ni siquiera definda localmente en tiempo, si $u_0(x) > x^2(e^x - 1)^{-1}$ para casi todo $x > 0$.

La ecuación simplificada (2) se introduce en [76, 77] para estudiar la evolución de la distribución de fotones a temperatura muy baja, pero finita. A pesar de que (2) es una aproximación de (1) un tanto basta, ciertas propiedades qualitativas de sus soluciones podrían guardar semejanzas con las de (1).

El sistema (3)–(4) es una cierta aproximación de un modelo simplificado que describe la evolución de un gas diluido de bosones, homogéneo e isotrópico, en presencia de un condensado, donde solo se consideran aquellas interacciones en las que interviene el condensado (cf. [26], [47], [73], [66]). En [1] y en [3, 4] se estudian problemas similares para distintas aproximaciones. La función $g(t,x) \geq 0$ representa la densidad de bosones con energía $x > 0$ del gas a tiempo $t \geq 0$, y $n(t) \geq 0$ representa la densidad de condensado a tiempo $t \geq 0$.

El sistema (3)–(4) puede deducirse de ecuaciones más generales y complicadas para partículas cuánticas (cf. [73], [68], [71], [62]), de donde se obtienen ecuaciones de tipo Boltzmann, como la ecuación de Nordheim ([58]). Estas ecuaciones se consideran para el caso en que la matriz de transición de probabilidad se toma proporcional a $\sqrt{n(t)}$ y la energía $E(p)$ de una partícula con momento $p \in \mathbb{R}^3$ y masa m se toma como $E(p) = |p|^2/(2m)$. Para distribuciones espacialmente homogéneas y radialmente simétricas, todas las integrales angulares pueden resolverse explícitamente y se obtienen las ecuaciones (3)–(4).

El sistema (3)–(4) también puede deducirse de la ecuación de Nordheim, teniendo en cuenta únicamente el término de la integral de colisión que involucra al condensado (cf. [67], [66] y Proposition 3.2.1). Es posible que las soluciones del sistema (3)–(4) guarden por tanto ciertas similitudes con las soluciones de la ecuación de Nordheim. La teoría sobre soluciones débiles globales para la ecuación homogénea de Norhdeim ha sido desarrollada por X. Lu en [53, 54, 55, 56], y más recientemente, por X. Lu y W. Li en [51]. Por otro lado, en [32] y [33] se estudia, para el caso isotrópico, el comportamiento singular de algunas soluciones. Varios resultados de existencia en el caso no homogéneo han sido obtenidos recientemente por L. Arkeryd y A. Nouri en

[5, 6, 7]. El problema de Cauchy para una modificación del sistema (3)–(4) ha sido considerado por A. Nouri en [60]. Si en la ecuación (3) se tiran los términos lineales, se obtiene la aproximación de turbulencia débil, que ha sido estudiada en detalle, para $n(t) \equiv 1$, por A. H. M. Kierkels y J. J. L. Velázquez en [45, 46]. En el Capítulo 3 adaptaremos algunos de los resultados que aparecen en [45].

Por consideraciones físicas, se espera que las soluciones de los tres problemas (1), (2) y (3)–(4) satisfagan las siguientes leyes de conservación. En (1) y en (2), la conservación del número total de partículas, que en ambos casos se escribe como

$$\int_0^\infty u(t,x)dx = \text{constante} \qquad \forall t \geq 0,$$

y en el sistema (3)–(4), tanto la conservación del número total de partículas como la energía total, que se escriben, respectivamente,

$$n(t) + \int_0^\infty g(t,x)dx = \text{constante} \qquad \forall t \geq 0,$$

$$\int_0^\infty x\, g(t,x)dx = \text{constante} \qquad \forall t \geq 0.$$

Sin embargo, los argumentos matemáticos para derivar estas leyes de conservación sulen involucrar el teorema de Fubini, cuyas hipótesis no se satisfacen necesariamente en todos los casos, debido a las singularidades de los núcleos que aparecen en (1) y en (5)–(6).

Aunque la interacción de partículas que se consideran (esferas duras, sección de corte Thomson) en la derivación de (1) y de (3)–(4) no son singulares, cuando las ecuaciones se escriben para distribuciones radialmente simétricas, sí que aparecen núcleos singulares debido a los factores geométricos. Con estas singularidades, las integrales de colisión en (1), (2), y en (5)–(6) no están bien definidas en un entorno del origen bajo la hipótesis de que $u(t,\cdot)$ y $g(t,\cdot)$ sean funciones integrables.

Es posible, aun así, considerar las siguientes formulaciones débiles de los problemas (1), (2), y (3)–(4). Para toda $\varphi \in C_b^1([0,\infty))$, la formulación débil de (1) se escribe:

$$\frac{d}{dt}\int_0^\infty \varphi(x)u(t,x)dx = \frac{1}{2}\int_0^\infty\int_0^\infty k_\varphi(x,y)u(t,x)u(t,y)dydx - \frac{1}{2}\int_0^\infty \mathcal{L}_\varphi(x)u(t,x)dx, \tag{8}$$

$$k_\varphi(x,y) = b(x,y)(e^{-x} - e^{-y})(\varphi(x) - \varphi(y)),$$

$$\mathcal{L}_\varphi(x) = \int_0^\infty b(x,y)y^2 e^{-y}(\varphi(x) - \varphi(y))dy,$$

y para el problema (2),

$$\frac{d}{dt}\int_0^\infty \varphi(x)u(t,x)dx = \frac{1}{2}\int_0^\infty\int_0^\infty k_\varphi(x,y)u(t,x)u(t,y)dydx. \tag{9}$$

Por último, en términos de la medida $G(t,\cdot)$ definida en (7), la formulación débil del sistema (3)–(4) se escribe:

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(x)G(t,x)dx = \tag{10}$$

$$= G(t,\{0\})\left[\iint_{[0,\infty)^2} \phi_\varphi(x,y)G(t,x)G(t,y)dxdy - \int_{[0,\infty)} \psi_\varphi(x)G(t,x)dx\right],$$

$$\phi_\varphi(x,y) = \frac{1}{\sqrt{xy}}\Big(\varphi(x+y) + \varphi(|x-y|) - 2\varphi(\text{máx}\{x,y\})\Big),$$
$$\psi_\varphi(x) = \sqrt{x}\big(\varphi(0)+\varphi(x)\big) - \frac{2}{\sqrt{x}}\int_0^x \varphi(y)dy.$$

Para $\varphi \in C_b^1([0,\infty))$, el lado derecho de (8) tiene perfecto sentido si $u(t,\cdot)$ es una función integrable con un momento exponencial finito. Sin embargo, en [29] se prueba que, de entre todas las distribuciones con el mismo número de partíclas $M>0$, los máximos de la entropía H para el problema (1) vienen dados en términos de las distribuciones de Bose-Einstein:

$$u_M(x) = \alpha\delta_0(x) + u_\mu(x), \qquad u_\mu(x) = \frac{x^2}{e^{x-\mu}-1},$$
$$\mu \le 0, \quad \alpha \ge 0, \quad \alpha\mu = 0, \quad M = \alpha + \int_0^\infty u_\mu(x)dx.$$

Vamos entonces a considerar soluciones u del problema (1) tales que $u(t,\cdot)$ es una medida no negativa en $[0,\infty)$. Es necesario entonces que las funciones k_φ y $\mathcal{L}_\varphi$ en (8) sean continuas en $[0,\infty)^2$ y $[0,\infty)$ respectivamente, lo que se consigue para funciones test $\varphi \in C_b^1([0,\infty))$ que satisfagan la condición $\varphi'(0)=0$.

En la ecuación simplificada (2) consideraremos tanto soluciones débiles medida como soluciones con valores en L^1.

Por otra parte, si $\varphi \in C_b^1([0,\infty))$, las funciones ϕ_φ y ψ_φ en (10) son continuas en $[0,\infty)^2$ y $[0,\infty)$ respectivamente, y el lado derecho de (10) está bien definido para medidas no negativas $G(t,\cdot)$ finitas, con un momento de orden $1/2$ finito.

Soluciones con valores en espacios de medida han sido consideradas por otros autores en problemas relacionados (cf., por ejemplo, [29], [45] y [53, 54, 55, 56]).

Pasemos ahora a describir brevemente algunos de los contenidos principales de la tesis. La existencia global de soluciones débiles para los problemas de valor inicial asociados a (1), (2) y (3)–(4) se prueba para datos iniciales generales, y se obtienen estimaciones para algunos de sus momentos. Para ello se utilizan técnicas clásicas, si bien es importante usarlas en la formulación débil expresada en las variables adecuadas. Primero, se considera una sucesión de problemas aproximados para núcleos acotados, y se prueba la existencia de soluciones por medio de un argumento de punto fijo. Después, se obtienen soluciones débiles por paso al límite. La compacidad de la sucesión de soluciones aproximadas se obtiene de las cotas uniformes dadas por las leyes de conservación.

Otro de los temas principales de esta tesis, para el cual se han obtenido resultados parciales, consiste en estudiar la evolución de las funciones

$$u(t,\{0\}) = \int_{\{0\}} u(t,x)dx \qquad \text{y} \qquad G(t,\{0\}) = \int_{\{0\}} G(t,x)dx.$$

Este problema está motivado por la posible relación de (1) y (3)–(4) con las distribuciones de Bose-Einstein. La prueba de los resultados obtenidos involucra el estudio detallado de las integrales

$$\int_{[0,\infty)} \varphi(x)u(t,x)dx \qquad \text{y} \qquad \int_{[0,\infty)} \varphi(x)G(t,x)dx$$

para ciertas clases de funciones test φ (funciones monótonas en (1) y en (2), y funciones convexas en (3)–(4)).

Sobre la ecuación simplificada (2), también se prueba la existencia global de soluciones con valores en L^1 bajo fuertes condiciones de integrabilidad sobre el dato inicial. Por último, se describe el comportamiento asintótico en tiempo de las soluciones débiles con valores medida del problema (2) que se han obtenido.

La estructura de la tesis es la siguiente. En el Capítulo 1 se introducen los tres problemas principales (1), (2), y (3)–(4). Las ecuaciones (1) y (2) se estudian en el Capítulo 2, y el sistema (3)–(4) en el Capítulo 3. Los resultados técnicos del Capítulo 2 se presentan en los Apéndices A y B, y los del Capítulo 3, en el Apéndice C.

Chapter 1

Introduction

In this thesis, we consider three nonlinear problems related to the kinetic theory of quantum gases.

In the first problem, we study the weak solutions of the following integro-differential equation:

$$\frac{\partial u}{\partial t}(t,x) = \int_0^\infty b(x,y)\Big[u(t,y)\big(x^2+u(t,x)\big)e^{-x} - u(t,x)\big(y^2+u(t,y)\big)e^{-y}\Big]dy \quad (1.0.1)$$

for $t > 0$, $x \geq 0$. The kernel $b(x,y) \geq 0$ is a continuous symmetric function on $[0,\infty)^2 \setminus \{(0,0)\}$ with a singularity at the origin $(x,y) = (0,0)$ and supported in a neighbourhood of the diagonal $\{x = y \geq 0\}$.

In the second problem, we study a simplified version of (1.0.1) in which the linear terms are neglected and we consider the quadratic part only:

$$\frac{\partial u}{\partial t}(t,x) = u(t,x)\int_0^\infty b(x,y)\big(e^{-x} - e^{-y}\big)u(t,y)dy \qquad t > 0,\ x \geq 0. \quad (1.0.2)$$

In the third and last problem, we consider the following system of two coupled equations:

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,x) = n(t)Q(g)(t,x) \qquad t > 0,\ x > 0, & (1.0.3) \\ n'(t) = -n(t)\displaystyle\int_0^\infty Q(g)(t,x)dx \qquad t > 0, & (1.0.4) \end{cases}$$

$$Q(g)(t,x) = \int_0^x q(g)(t,x,y)dy - 2\int_x^\infty q(g)(t,y,x)dy, \quad (1.0.5)$$

$$q(g)(t,x,y) = \frac{g(t,x-y)}{\sqrt{x-y}}\frac{g(t,y)}{\sqrt{y}} - \frac{g(t,x)}{\sqrt{x}}\left(1 + \frac{g(t,x-y)}{\sqrt{x-y}} + \frac{g(t,y)}{\sqrt{y}}\right), \quad (1.0.6)$$

and study their weak solutions in terms of the measure-valued function G defined by

$$G(t,\cdot) = n(t)\delta_0(\cdot) + g(t,\cdot), \qquad t \geq 0, \quad (1.0.7)$$

where $\delta_0(\cdot)$ is the Dirac delta in $x = 0$.

The equation (1.0.1) is related to a simplified description for the time evolution of the particle density of a homogeneous and isotropic photon gas that interacts through Compton scattering with a dilute electron gas at nonrelativistic equilibrium (cf. [25], [49], [72], [42]). Up to a constant, the function $u(t,x) \geq 0$ represents the particle density of photons with energy $x \geq 0$ at time $t \geq 0$. The kernel $b(x,y)$ that we consider in (1.0.1) is a certain approximation of the kernel that appears in the literature, denoted in this section by $K(x,y)$. An explicit expression for $K(x,y)$ is known and a derivation is given in [57].

The equation (1.0.1) may already be found in Kompaneets's paper [49], in 1957. A detailed deduction of (1.0.1), using a strictly Boltzmann approach, may be found in [25]. A transport equation to describe a homogeneous isotropic system of charged particles and photons is derived in [23] by means of a BBGKY type method. In that paper, similar results as in [25] are obtained, with the addition of correction terms that represent correlation effects. General effective kinetic theories describing quasiparticle dynamics in relativistic plasmas may be found in [12] and [8].

In the mathematical literature, the equation (1.0.1) has already been studied under different conditions on the kernel b. The existence of solutions for the Cauchy problem was proved in [29] for bounded kernels and for kernels with an exponential growth at infinity. The equation (1.0.1) with the physical kernel $K(x,y)$ has been considered by M. Chane-Yook and A. Nouri in [22], and E. Ferrari and A. Nouri in [34]. It is proved in [34] that the Cauchy problem has a global weak solution for all nonnegative initial data $u_0 \in L^1(\mathbb{R}_+)$ that are bounded from above by the Planck distribution at almost every point, i.e., such that $u_0(x) \leq x^2(e^x - 1)^{-1}$ a.e. $x > 0$, but that there is no weak solution, even local in time, if $u_0(x) > x^2(e^x - 1)^{-1}$ a.e. $x > 0$.

The simplified equation (1.0.2) was introduced in [76, 77] in order to analyze, at low but finite temperature of the electron gas, the evolution of the particle density of photons at low energies. Although (1.0.2) is a coarse approximation of (1.0.1), certain qualitative properties of its solutions could bear a resemblance with those of (1.0.1).

The system (1.0.3)–(1.0.4) is an approximation of a simplified model that describes the evolution of a homogeneous and isotropic dilute Bose gas in presence of a condensate, where only the interactions involving the condensate are taken into account (cf. [26], [47], [73], [66]). Similar problems, where different approximations are considered, may be found in [1] and [3, 4]. The function $g(t,x) \geq 0$ represents, up to a constant, the particle density of bosons with energy $x > 0$ at time $t \geq 0$, and $n(t) \geq 0$ stands for the condensate density at time $t \geq 0$. The problem (1.0.3)–(1.0.4) may be derived from more general and complicated equations for quantum particles (cf. [73], [68], [71], [62]). In the derivation procedure, general Boltzmann equations for quantum particles are obtained, as the so-called Nordheim equation (cf. [58]). These kinetic equations are then considered for probability transition matrices proportional to $\sqrt{n(t)}$, and for a certain approximation of the energy of a particle. Then, for radially symmetric and spatially homogeneous density functions, all the angular integrals can be performed explicitly and the equations (1.0.3)–(1.0.4) are obtained.

The system (1.0.3)–(1.0.4) may also be formally deduced from the Nordheim equation, keeping only the collision integral that corresponds to the interactions involving the condensate (cf. [67], [66], and Proposition 3.2.1). It may then be possible for the solutions to (1.0.3)–(1.0.4) to share some similarities with those of the Nordheim equation. The existence of global weak solutions for the homogeneous Nordheim equation and some of its qualitative properties has been developed by X.

Lu in [53, 54, 55, 56], and more recently by X. Lu and W. Li in [51]. Singular isotropic solutions of the Nordheim equation are described in [32] and [33]. Some existence results in the non homogeneous case have been obtained by L. Arkeryd and A. Nouri in [5, 6, 7]. The Cauchy problem for a modification of the system (1.0.3)–(1.0.4) has been considered by A. Nouri in [60]. If the linear terms in equation (1.0.3) are dropped, one obtains the so-called wave turbulence approximation, that has been studied in detail, for $n(t) \equiv 1$, by A. H. M. Kierkels and J. J. L. Velázquez in [45, 46]. In Chapter 3, we will reproduce and adapt some of the arguments presented in [45].

It is expected from physical considerations that the solutions of (1.0.1), (1.0.2), and (1.0.3)–(1.0.4) satisfy the following conservation laws. The conservation of the total number of particles in (1.0.1) and (1.0.2), that in both cases reads:

$$\int_0^\infty u(t,x)dx = \text{constant} \qquad \forall t \geq 0, \tag{1.0.8}$$

and for the system (1.0.3)–(1.0.4), the conservation of the total number of particles and the total energy, that reads, respectively,

$$n(t) + \int_0^\infty g(t,x)dx = \text{constant} \quad \forall t \geq 0, \tag{1.0.9}$$

$$\int_0^\infty x\, g(t,x)dx = \text{constant} \quad \forall t \geq 0. \tag{1.0.10}$$

However, the arguments to derive these conservation laws involve Fubini's theorem, whose hypothesis are not necessarily fulfilled, due to the singularities of the kernels in (1.0.1) and (1.0.5)–(1.0.6).

Although the particle interactions that are considered (hard spheres, Thomson cross section) in the derivation of (1.0.1) and (1.0.3)–(1.0.4) are not singular, when the equations are written in terms of the densities of particles in energy variables, singular kernels do appear in the collision integrals. Due to these singularities, the collision integrals in (1.0.1) and (1.0.5)–(1.0.6) are not well defined near the origin under the natural assumption that $u(t,\cdot)$ and $g(t,\cdot)$ are integrable functions for all $t \geq 0$.

It is possible, however, to consider the following weak formulation for the equation (1.0.1):

$$\frac{d}{dt}\int_0^\infty \varphi(x)u(t,x)dx = \frac{1}{2}\int_0^\infty\int_0^\infty k_\varphi(x,y)u(t,x)u(t,y)dydx - \frac{1}{2}\int_0^\infty \mathcal{L}_\varphi(x)u(t,x)dx, \tag{1.0.11}$$

$$k_\varphi(x,y) = b(x,y)(e^{-x} - e^{-y})(\varphi(x) - \varphi(y)),$$

$$\mathcal{L}_\varphi(x) = \int_0^\infty b(x,y)y^2e^{-y}(\varphi(x) - \varphi(y))dy,$$

for all $\varphi \in C_b^1([0,\infty))$. The corresponding weak formulation for the simplified equation (1.0.2) reads:

$$\frac{d}{dt}\int_0^\infty \varphi(x)u(t,x)dx = \frac{1}{2}\int_0^\infty\int_0^\infty k_\varphi(x,y)u(t,x)u(t,y)dydx, \tag{1.0.12}$$

and finally, in terms of the nonnegative measure $G(t,\cdot)$ defined by (1.0.7), the weak formulation for the system (1.0.3)–(1.0.4) reads:

$$\frac{d}{dt}\int_{[0,\infty)}\varphi(x)G(t,x)dx = \tag{1.0.13}$$
$$= G(t,\{0\})\Big[\iint_{[0,\infty)^2}\phi_\varphi(x,y)G(t,x)G(t,y)dydx - \int_{[0,\infty)}\psi_\varphi(x)G(t,x)dx\Big],$$

$$\phi_\varphi(x,y) = \frac{1}{\sqrt{xy}}\Big(\varphi(x+y)+\varphi(|x-y|)-2\varphi(\max\{x,y\})\Big),$$
$$\psi_\varphi(x) = \sqrt{x}\big(\varphi(0)+\varphi(x)\big) - \frac{2}{\sqrt{x}}\int_0^x \varphi(y)dy,$$

for all $\varphi \in C_b^1([0,\infty))$.

For $\varphi \in C_b^1([0,\infty))$, the right hand side of (1.0.11) is well defined for $u(t,\cdot)$ an integrable function with a finite exponential moment. However, it was proved in [29] that among all the distributions with the same number of particles $M > 0$, the maxima of the entropy H for the problem (1.0.1) are given in terms of the Bose-Einstein distributions:

$$u_M(x) = \alpha\delta_0(x) + u_\mu(x), \qquad u_\mu(x) = \frac{x^2}{e^{x-\mu}-1},$$
$$\mu \le 0, \quad \alpha \ge 0, \quad \alpha\mu = 0, \quad M = \alpha + \int_0^\infty u_\mu(x)dx.$$

We are then going to consider solutions u of the problem (1.0.1) such that $u(t,\cdot)$ is a nonnegative measure on $[0,\infty)$. The functions k_φ and $\mathcal{L}_\varphi$ in (1.0.11) then need to be continuous on $[0,\infty)^2$ and $[0,\infty)$ respectively. That is achieved for test functions $\varphi \in C_b^1([0,\infty))$ that satisfy the condition $\varphi'(0) = 0$.

In the simplified problem (1.0.2), we will consider measure-valued solutions as well as L^1-valued solutions.

On the other hand, for $\varphi \in C_b^1([0,\infty))$, the functions ϕ_φ and ψ_φ in (1.0.13) are continuous on $[0,\infty)^2$ and $[0,\infty)$ respectively, and the right hand side of (1.0.13) is well defined for finite nonnegative measures $G(t,\cdot)$ with a finite moment of order $1/2$.

Measure-valued solutions have been considered by other authors in related problems (cf., for instance, [29], [45] and [53, 54, 55, 56]).

Let us mention very briefly some of the main contributions of this thesis. The global existence of weak solutions for the initial value problems associated to (1.0.1), (1.0.2), and (1.0.3)–(1.0.4) is established for a large set of initial data. Several estimates on some of their moments are also obtained. This is done using classical arguments, although it is important to use them with the weak formulation in the appropriate variables. We first consider a sequence of approximated problems for bounded kernels, then we prove the existence of solutions by means of a fixed point theorem, and finally obtain weak solutions by passage to the limit. The compactness of the sequence of approximate solutions is deduced from uniform bounds, provided by the conservation laws.

Motivated by the possible relation of problems (1.0.1) and (1.0.3)–(1.0.4) with the Bose-Einstein distributions, we have also been interested in the time evolution of the functions

$$u(t,\{0\}) = \int_{\{0\}} u(t,x)dx \qquad \text{and} \qquad G(t,\{0\}) = \int_{\{0\}} G(t,x)dx,$$

and some partial results have been obtained. Their proofs involve the careful study of the evolution of the integrals

$$\int_{[0,\infty)} \varphi(x)u(t,x)dx \qquad \text{and} \qquad \int_{[0,\infty)} \varphi(x)G(t,x)dx$$

for certain classes of test functions φ (monotone test functions for (1.0.1) and convex for (1.0.3)–(1.0.4)).

Lastly, the existence of L^1-valued solutions for the simplified equation (1.0.2) is proved under a strong integrability condition on the initial data, and the asymptotic behavior of the measure-valued weak solutions to (1.0.2) is also described.

This thesis is organized as follows. In the present Chapter 1 we introduce the main problems (1.0.1), (1.0.2) and (1.0.3)–(1.0.4). The equations (1.0.1) and (1.0.2) are studied in Chapter 2 and the system (1.0.3)–(1.0.4) in Chapter 3. Some technical results of Chapter 2 are presented in the Appendices A and B, and those of Chapter 3 in Appendix C.

We present, in the next two sections, the main results of Chapter 2 and Chapter 3.

1.1 On a Boltzmann equation for Compton scattering with a low-density electron gas at non-relativistic equilibrium

The evolution of the particle density of a photon gas that interacts only through Compton scattering with a low density electron gas at nonrelativistic equilibrium, is usually described by a Boltzmann equation (cf. [61], [42]). When the photon gas is spatially homogeneous and its particle density f isotropic, the equation simplifies to the following expression (cf. [25], [72]):

$$\begin{aligned} k^2\frac{\partial f}{\partial t}(t,k) &= Q_\beta(f,f)(t,k) \qquad t>0,\ k\ge 0, \qquad (1.1.14)\\ Q_\beta(f,f)(t,k) &= \int_0^\infty \Big(f(t,k')\left(1+f(t,k)\right)e^{-\beta k} - \\ &\qquad -f(t,k)(1+f(t,k'))e^{-\beta k'}\Big)\, kk'\mathcal{B}_\beta(k,k')dk', \end{aligned}$$

where $k=|\mathbf{k}|$ denotes the energy of a photon of momentum $\mathbf{k}\in\mathbb{R}^3$ (taking the speed of light c equal to 1), $\beta=(\hbar T)^{-1}$, T is the temperature of the electron gas and $(4\pi/3)k^2f(t,k)\ge 0$ is the particle density. The function $\mathcal{B}_\beta(k,k')$ is the so-called redistribution function and has been deduced in [57]. Previous mathematical results on this equation may be found in [29], [22, 34]. See also [38] and [18] for numerical methods and simulations.

It is common in the physics literature to approximate (1.1.14) by the Kompaneets equation:

$$\frac{\partial f}{\partial t} = \frac{1}{k^2}\frac{\partial}{\partial k}\left(k^4\left(\frac{\partial f}{\partial k} + f^2 + f\right)\right), \tag{1.1.15}$$

(cf. [49]). Equation (1.1.15) has deserved great attention due to its importance in modern cosmology and high energy astrophysics (cf. [11], [42]). However, although this approximation is generally performed under some assumptions on the solution f, no precise mathematical statement for such assumptions are known (for simpler kernels than $\mathcal{B}_\beta$, a rigorous derivation of (1.1.15) may be found in [29]).

It is expected from physical considerations that the total number of particles is conserved by (1.1.14). Since the particle density f is assumed to be isotropic, that with some abuse of notation we express as $f(t,\mathbf{k}) = f(t,k)$, if we introduce the new variable $v(t,k) = k^2 f(t,k)$ that is (up to a constant) the particle density in energy variables, the conservation of the total number of particles then writes

$$\int_0^\infty v(t,k)dk = \text{constant} \qquad \forall t \geq 0.$$

In terms of v, the equation (1.1.14) reads

$$\frac{\partial v}{\partial t}(t,k) = \int_0^\infty \frac{\mathcal{B}_\beta(k,k')}{kk'} q_\beta(v,v)(t,k,k')dk', \tag{1.1.16}$$

$$q_\beta(v,v)(t,k,k') = v(t,k')\big(k^2 + v(t,k)\big)e^{-\beta k} - v(t,k)\big(k'^2 + v(t,k')\big)e^{-\beta k'}. \tag{1.1.17}$$

In the nonrelativistic limit, when $\beta >> (mc^2)^{-1}$, where m is the mass of an electron and c is the speed of light, the differential cross section for Compton scattering is usually approximated by the Thomson differential cross section (cf. [42]), and it is then possible to deduce the following expression for the function $\mathcal{B}_\beta$ (cf. [57]):

$$\mathcal{B}_\beta(k,k') = \sqrt{\beta}\, e^{\beta\frac{(k'+k)}{2}} \int_0^\pi \frac{(1+\cos^2\theta)}{|\mathbf{k}'-\mathbf{k}|} e^{-\beta m \frac{(k'-k)^2 + \frac{|\mathbf{k}'-\mathbf{k}|^4}{4m^2}}{2|\mathbf{k}'-\mathbf{k}|^2}} \sin\theta d\theta.$$

In particular, the function $\mathcal{B}_\beta$ satisfies the following properties (cf. Figure 1.1 and Figure 1.2):

(i) It is singular at the origin:

$$\mathcal{B}_\beta(k,k) = \frac{44}{15}\sqrt{\beta}\left(\frac{1}{k} + \beta\right) + \mathcal{O}(k) \qquad \text{as} \quad k \to 0,$$

(ii) it grows exponentially at infinity:

$$\mathcal{B}_\beta(k,k) = e^{\beta k}\left(\mathcal{O}\left(\frac{1}{k^2}\right) + e^{-\frac{\beta k^2}{2m}}\mathcal{O}\left(\frac{1}{k^3}\right)\right) \qquad \text{as} \quad k \to \infty,$$

(iii) it is strictly positive at the axes:

$$\mathcal{B}_\beta(k,0) = \mathcal{B}_\beta(0,k) = \frac{8}{3}\frac{\sqrt{\beta}}{k} e^{\frac{\beta k}{2}} e^{-\frac{\beta m}{2}\left(1+\frac{k^2}{4m^2}\right)} > 0, \qquad k > 0.$$

The kernel $(kk')^{-1}\mathcal{B}_\beta(k,k')$ in (1.1.16) is therefore quite singular at the axes and, of course, does not satisfy the hypothesis imposed in [29]. Is not even possible to give

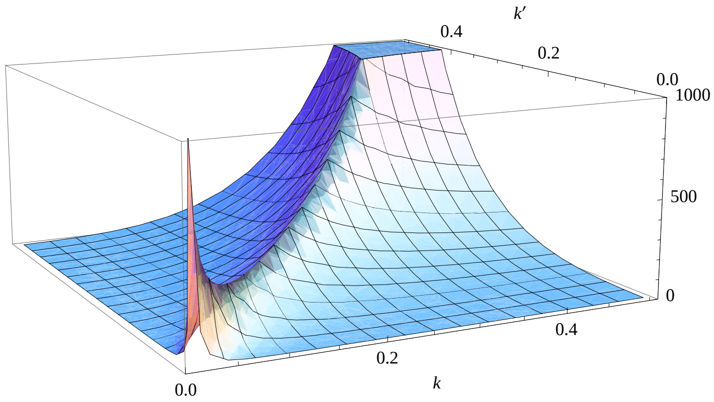

Figure 1.1: The kernel $\mathcal{B}_\beta(k,k')$ for $\beta = 10$, $m = 1$, $(k,k') \in (0,\frac{1}{2})^2$.

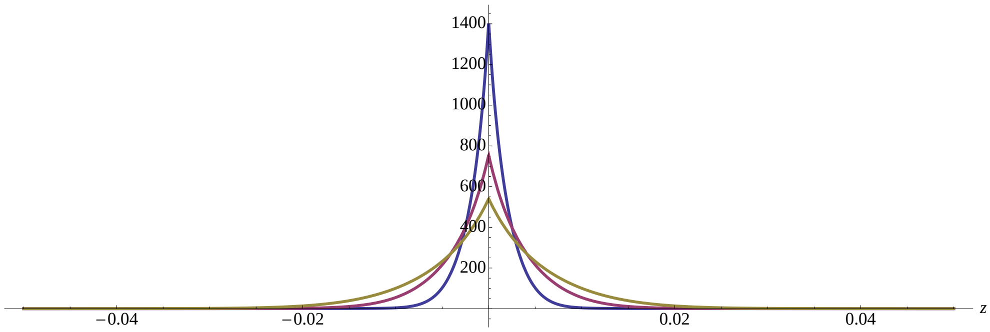

Figure 1.2: Cross sections of $\mathcal{B}_\beta$ perpendicular to the diagonal and very close to the origin. More precisely, let $\widetilde{\mathcal{B}}_\beta(w,z) = \mathcal{B}_\beta(k,k')$ with $w = (k+k')/\sqrt{2}$, $z = (k-k')/\sqrt{2}$. The figure shows, for $\beta = 10$, $m = 1$ and $z \in (-\frac{1}{2},\frac{1}{2})$, the function $z \mapsto \widetilde{\mathcal{B}}_\beta(w,z)$ for the values $w = 0.01$ (blue), $w = 0.02$ (red) and $w = 0.03$ (yellow).

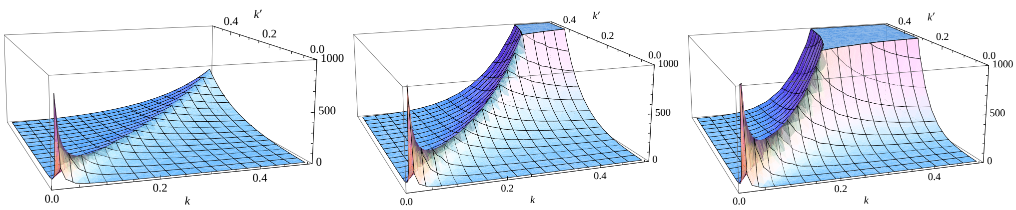

Figure 1.3: From left to right, the kernel $\mathcal{B}_\beta(k,k')$ with $m = 1$, $(k,k') \in (0,\frac{1}{2})^2$ and for $\beta = 7$, $\beta = 10$ and $\beta = 13$.

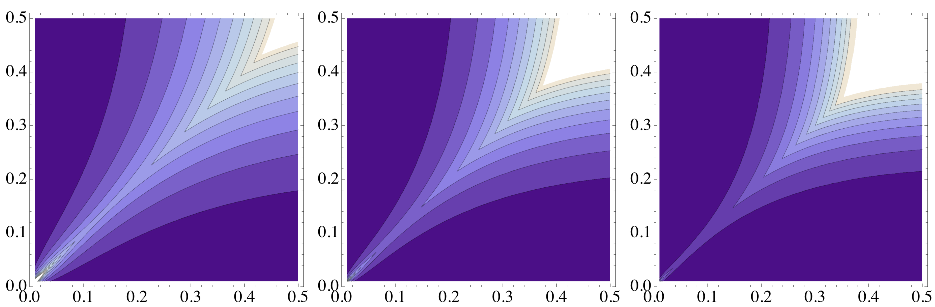

Figure 1.4: From left to right, several level sets $\mathcal{B}_\beta(k,k') =$ constant for $m = 1$, $(k,k') \in (0,\frac{1}{2})^2$ and $\beta = 7$, $\beta = 10$ and $\beta = 13$.

sense to the right hand side of (1.1.16), not even in weak form, for a general finite nonnegative measure $v(t)$.

We then consider the following truncated problem:

$$\frac{\partial v}{\partial t}(t,k) = \int_0^\infty \frac{\mathcal{B}_\beta(k,k')\Phi(k,k')}{kk'} q_\beta(v,v)(t,k,k')dk' \tag{1.1.18}$$

where $q_\beta(v,v)$ is defined in (1.1.17) and Φ is a cut-off function supported in a neighbourhood of the diagonal $\{k = k' > 0\}$ of the form (cf. Figure 1.5):

$$|k - k'| \leq \sqrt{kk'(k+k')} \qquad \text{for} \quad k,\ k' << 1,$$
$$\theta k \leq k' \leq \frac{k}{\theta} \qquad \text{elsewhere, where } \theta \in (0,1) \text{ is fixed.}$$

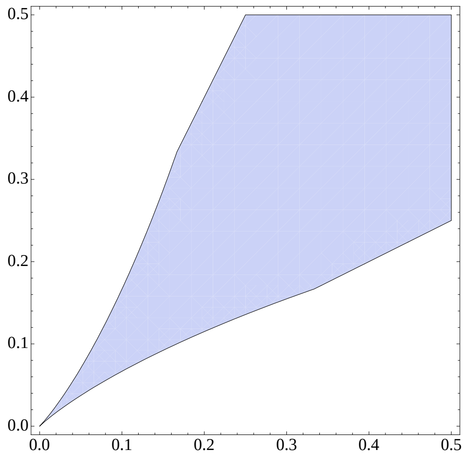

Figure 1.5: The support of $\Phi(k,k')$ for $(k,k') \in [0,\frac{1}{2}]^2$, $\theta = 1/2$.

This truncation, similar to the one proposed in [77], is suggested by the very peaked shape of $\mathcal{B}_\beta$ along the diagonal (cf. Figure 1.1–Figure 1.4). The function $\mathcal{B}_\beta\Phi$ is a first, rough approximation of $\mathcal{B}_\beta$ that keeps, nevertheless, a singular behavior at the origin. A possible extension to this work could be the construction of weak solutions to (1.1.16) through the truncated problem (1.1.18), by passage to the limit in the cut-off function parameters.

Although in this work the value of β remains fixed, and may then be taken equal to 1, certain aspects of the equation appear more clearly in some other variables, rescaled with β. In particular, the fact that $\mathcal{B}_\beta$ is more and more peaked along the diagonal as $\beta \to \infty$ (cf. Figure 1.3 and Figure 1.4). When the time is rescaled to $\beta^3 t$ and the energy to βk, so that the total number of particles is unchanged, the equation (1.1.18) rewrites as (1.0.1), where

$$u(t,x) = \beta^{-1} v(\tilde{t},k) \qquad t = \beta^3 \tilde{t}, \quad x = \beta k. \tag{1.1.19}$$

The function $b(x,y)$ in (1.0.1) that we consider, satisfies general conditions. In particular, these conditions are fulfilled by the truncated kernel $(kk')^{-1}\mathcal{B}_\beta(k,k')\Phi(k,k')$. The function $K(x,y)$ mentioned at the beginning of the introduction corresponds to the kernel $(kk')^{-1}\mathcal{B}_\beta(k,k')$ written in the rescaled variables as in (1.1.19).

In order to present the main results of Chapter 2, let us denote by $\mathscr{M}_+([0,\infty))$ the space of nonnegative finite Radon measures on $[0,\infty)$, endowed with the so-called narrow topology, and denote by $C([0,\infty),\mathscr{M}_+([0,\infty)))$ the space of continuous functions from $[0,\infty)$ onto $\mathscr{M}_+([0,\infty))$. For convenience and unless otherwise is noted, we write $\mu(x)dx$ for every measure μ, even if μ is not absolutely continuous with respect to the Lebesgue measure.

Let us define now the following notion of weak solutions.

Definition 1.1.1. Given $u_0 \in \mathscr{M}_+([0,\infty))$, we say that u is a weak solution of (1.0.1) with initial data u_0 if:

(i) $u \in C([0,\infty),\mathscr{M}_+([0,\infty)))$ and $\int_{[0,\infty)} \varphi(x)u(0,x)dx = \int_{[0,\infty)} \varphi(x)u_0(x)dx$ for all $\varphi \in C_b([0,\infty))$.

(ii) The map $t \mapsto \int_{[0,\infty)} \varphi(x)u(t,x)dx$ belongs to $W^{1,\infty}_{loc}([0,\infty))$ for all $\varphi \in C^1_b([0,\infty))$ with $\varphi'(0)=0$,

(iii) u satisfies the weak formulation (1.0.11) for almost every $t \geq 0$ and for all $\varphi \in C^1_b([0,\infty))$ with $\varphi'(0)=0$.

We may then state our result about the existence of global weak solutions (cf. Theorem 2.1.2).

Theorem. *Given any $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying*

$$\int_{[0,\infty)} e^{\eta x} u_0(x)dx < \infty \tag{1.1.20}$$

for some $\eta \in \left(\frac{1-\theta}{2}, \frac{1}{2}\right)$, there exists $u \in C([0,\infty),\mathscr{M}_+([0,\infty)))$ weak solution of (1.0.1) with initial data u_0. Moreover, u satisfies

$$\int_{[0,\infty)} u(t,x)dx = \int_{[0,\infty)} u_0(x)dx \qquad \forall t \geq 0,$$
$$\int_{[0,\infty)} e^{\eta x} u(t,x)dx \leq e^{C_\eta t} \int_{[0,\infty)} e^{\eta x} u_0(x)dx \qquad \forall t \geq 0,$$

where C_η is a positive constant.

The entropy functional

$$H(u(t)) = \int_{[0,\infty)} h(x,u_r(t,x))dx - \int_{[0,\infty)} x\, u_s(t,x)dx,$$
$$h(x,s) = (x^2+s)\ln(x^2+s) - s\ln s - x^2\ln(x^2) - sx,$$

(cf. [58], [29]) is well defined for the weak solutions thus obtained, where $u = u_r + u_s$ is the Lebesgue decomposition of u into a regular and a singular part (cf. [29] Lemma 3.1 and Lemma 4.1). However, due to the singularity of the kernel b, we do not know how to give sense to the corresponding dissipation of entropy $D(u(t))$ that appears in [29].

We have scarce information about the qualitative properties of the weak solutions of (1.0.1), and in particular, on their long time behavior. It may then be useful to consider the simplified equation (1.0.2), introduced in [76, 77], where the linear terms in the right hand side of (1.0.1) have been dropped. For large values of the parameter

β, this simplification is suggested by the fact that, when (1.1.18) is written in the rescaled variables (1.1.19), the linear terms in the collision integral are formally of lower order (cf. Appendix B.1).

We define now the following notion of weak solutions for the simplified problem (1.0.2).

Definition 1.1.2. Given $u_0 \in \mathscr{M}_+([0,\infty))$, we say that u is a weak solution of (1.0.2) with initial data u_0 if:

(i) $u \in C([0,\infty), \mathscr{M}_+([0,\infty)))$ and $\int_{[0,\infty)} \varphi(x)u(0,x)dx = \int_{[0,\infty)} \varphi(x)u_0(x)dx$ for all $\varphi \in C_b([0,\infty))$.

(ii) The map $t \mapsto \int_{[0,\infty)} \varphi(x)u(t,x)dx$ belongs to $W^{1,\infty}_{loc}([0,\infty))$ for all $\varphi \in C^1_b([0,\infty))$ with $\varphi'(0) = 0$,

(iii) u satisfies the weak formulation (1.0.12) for almost every $t \geq 0$ and for all $\varphi \in C^1_b([0,\infty))$ with $\varphi'(0) = 0$.

The existence of weak solutions for (1.0.2) presented below (cf. Theorem 2.5.1) is proved with the same arguments as for equation (1.0.1).

Theorem. *For any initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (1.1.20) for some $\eta > (1-\theta)/2$, there exists a weak solution $u \in C([0,\infty), \mathscr{M}_+([0,\infty)))$ of (1.0.2) with initial data u_0. Moreover, u also satisfies*

$$\int_{[0,\infty)} u(t,x)dx = \int_{[0,\infty)} u_0(x)dx \qquad \forall t \geq 0,$$

$$\int_{[0,\infty)} e^{\eta x}u(t,x)dx \leq \int_{[0,\infty)} e^{\eta x}u_0(x)dx \qquad \forall t \geq 0,$$

Our next result shows the existence of solutions in L^1 for initial data that satisfy a strong integrability condition (cf. Theorem 2.1.4).

Theorem. *For any nonnegative initial data $u_0 \in L^1(\mathbb{R}_+)$ such that:*

$$\forall r > 0, \quad \int_0^\infty u_0(x)\left(e^{\frac{r}{x^{3/2}}} + e^{\eta x}\right)dx < \infty \tag{1.1.21}$$

for some $\eta > (1-\theta)/2$, there exists a nonnegative weak solution u of (1.0.2) with initial data u_0. Moreover, $u \in C([0,\infty), L^1(\mathbb{R}_+))$, satisfies (1.0.2) in $L^1(\mathbb{R}_+)$ for almost every $t > 0$ and

$$\|u(t)\|_1 = \|u_0\|_1 \qquad \forall t \geq 0,$$

$$u(t,x) \leq u_0(x)e^{\frac{tC_0}{x^{3/2}}} \qquad \forall t \geq 0,\ a.e.\ x > 0,$$

where C_0 is a positive constant.

The condition (1.1.21) on the initial data is then a sufficient condition to have a global solution in L^1, and therefore, to prevent the formation of any Dirac delta in finite time.

The weak solutions of (1.0.2) satisfy two properties that may be loosely described as follows:

(i) the support of the solution is invariant in time (cf. Proposition 2.5.4) and

(ii) the distribution of mass tends to "move" to lower values on the energy spectrum (cf. Proposition 2.5.5).

These two properties have the following consequence on the long time behavior of the solutions. Suppose, for example, that the support of the initial data u_0 is an interval (a_0, a_1), with $a_0 > 0$. If $a_1 - a_0$ is sufficiently small, then a weak solution of (1.0.2) converges to $M\delta_{a_0}$, where M is the integral of the initial data u_0. Even if $a_1 - a_0$ is large, or the support of u_0 was the disjoint union of two intervals (a_0, a_1), (a_2, a_3) with $a_0 > 0$, due to the non local nature of (1.0.2), one could still expect all the mass to move or *jump* to the lower endpoint a_0 of the left interval. However, due to the lack of strict positivity of the kernel b that we consider, particles that are sufficiently far away from each other do not interact. As a consequence, if the distance between the two intervals (a_0, a_1) and (a_2, a_3) is sufficiently large, the mass on each of the intervals is constant in time, and the solution converges to a sum of, at least, two Dirac measures, one at the point a_0 and other at the point a_2. This is stated in the following result (cf. Theorem 2.1.8).

Theorem. *Let u be a weak solution of (1.0.2) for an initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (1.1.20) for some $\eta \geq (1-\theta)/2$. Then there exist a sequence of nonnegatives numbers $\{(m_i, k_i)\}_{i\in\mathbb{N}}$ such that u converges in $C([0,\infty), \mathscr{M}_+([0,\infty)))$ to the measure*

$$\mu = \sum_{i=0}^{\infty} m_i \delta_{k_i}. \tag{1.1.22}$$

This Theorem shows that the asymptotic state of the simplified equation (1.0.2) is formed by an at most countable set of Dirac measures. This suggests the possibility for the solutions of the complete equation (1.0.1) to develop more than one peak that could remain for very long times.

1.2 On a system of two coupled equations for the normal fluid - condensate interaction in a Bose gas

A kinetic description of a weakly interacting dilute Bose gas below the critical temperature, when a condensate is present, was considered in [26] and [47]. The expressions of the collision integral for the interactions between particles in the normal component of the gas, and for the interaction of the condensate with particles in the normal component of the gas where deduced. Their results were then generalized in [73], where the authors derived coupled equations for the distribution functions of the normal and superfluid components.

The system that only takes into account the interaction of the condensate with particles in the normal fluid for a homogeneous isotropic gas has been treated in [3], where the existence of global solutions is proved for several approximations of the scattering amplitude and the energy of the particles. This problem has also been considered in [1], where the transition probability matrix is taken proportional to $|p||p_1||p_2|$ and the energy of a particle with momentum $p \in \mathbb{R}^3$ is proportional to $|p|\sqrt{n(t)}$.

The purpose of this chapter is to consider one of the regimes introduced in [26] and [47], where the square of the transition probability matrix in the kinetic equation is taken proportional to the condensate density $n(t)$, and where the energy of a particle with momentum $p \in \mathbb{R}^3$ and mass m is taken to be $E(p) = |p|^2/(2m)$. With this approximations and for a spatially homogeneous and isotropic gas, the system may be written as (1.0.3)–(1.0.4) (cf. [66]). The same approximation, but for a non isotropic gas with periodic spatial dependence has been considered in [4] for initial data close to equilibrium The existence of weak solutions to (1.0.3) with a simplified equation for $n(t)$ has been proved in [60].

Let us present now our main results on the weak solutions of (1.0.3)–(1.0.4) (cf. Definition 3.1.2) that are studied in Chapter 3 in terms of the measure $G(t)$ defined in (1.0.7). We start with the global existence of weak solutions (cf. Theorem 3.1.3 and Theorem 3.1.4) and some properties of their moments $M_\alpha(G)$, where

$$M_\alpha(G) = \int_{[0,\infty)} x^\alpha G(x)dx \qquad \alpha \in \mathbb{R},$$
$$\mathscr{M}_+^\alpha([0,\infty)) = \{G \in \mathscr{M}_+([0,\infty)) : \ M_\alpha(G) < \infty\}.$$

Theorem. *Suppose that $G_0 \in \mathscr{M}_+^1([0,\infty))$ satisfies $G_0(\{0\}) > 0$. Then, there exists a weak solution $G \in C\big([0,\infty), \mathscr{M}_+^1([0,\infty))\big)$ to (1.0.3)–(1.0.4) with initial data G_0 that satisfies the following properties:*

(i) G conserves the total number of particles N and energy E:

$$M_0(G(t)) = M_0(G_0) = N \qquad \forall t \geq 0,$$
$$M_1(G(t)) = M_1(G_0) = E \qquad \forall t \geq 0.$$

(ii) For all $\alpha \geq 3$, if $M_\alpha(G_0) < \infty$, then $G \in C\big((0,\infty), \mathscr{M}_+^\alpha([0,\infty))\big)$ and

$$M_\alpha(G(t)) \leq \Big(M_\alpha(G_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau(t)\Big)^{\frac{\alpha-1}{2}} \qquad \forall t > 0,$$

where $\tau(t) = \int_0^t G(s,\{0\})ds$.

(iii) For all $\alpha \geq 3$,

$$M_\alpha(G(t)) \leq C(\alpha, E)\left(\frac{1}{1 - e^{-\gamma(\alpha,E)\tau(t)}}\right)^{2(\alpha-1)} \qquad \forall t > 0,$$

where the constants $C(\alpha,E)$ and $\gamma(\alpha,E)$ are defined in Theorem 3.3.1.

(iv) If $\alpha \in (1,3]$ and

$$E > C(\alpha)N^{5/3},$$

where $C(\alpha) > 0$ is an explicit constant, then $M_\alpha(G(t))$ is a decreasing function on $(0,\infty)$.

(v) For all $T > 0$, $R > 0$ and $\alpha \in \big(-\frac{1}{2},\infty\big)$,

$$\int_0^T G(t,\{0\}) \int_{(0,R]} x^\alpha G(t,x)dxdt \leq$$
$$\leq \frac{2R^{\frac{1}{2}+\alpha}}{1 - \big(\frac{2}{3}\big)^{\frac{1}{2}+\alpha}} \left(\int_0^T G(t,\{0\})dt\right)^{\frac{1}{2}} \left(\frac{\sqrt{E}}{2}\int_0^T G(t,\{0\})dt + \sqrt{N}\right). \quad (1.2.23)$$

Some remarks can be made about the result above.

(a) The condition $G_0(\{0\}) > 0$ just reflects the initial presence of a condensate in the system, and it is necessary in view of the particular form of (1.0.3)-(1.0.4) and (1.0.13).

(b) An estimate for the moment of order $\alpha \geq 3$ of the solution is provided in point (*ii*), under the assumption that the same moment is finite for the initial data. In point (*ii*), on the contrary, no assumption is made on the initial data. It is shown the instantaneous gain of moments of any order $\alpha \geq 3$. The estimate in (*iii*) blows-up as $t \to 0$, as it should.

(c) If an algebraic behavior around the origin like $G(t,x) \sim x^\beta$ was known for some $\beta \in \mathbb{R}$, then we would need $\beta \geq -1/2$, so that the left hand side of (1.2.23) is integrable for all $\alpha > -1/2$. Of course, no such algebraic behavior of G is known to hold.

Some properties of the function $n(t)$ are given in the next two results (cf. Theorem 3.1.7 and Theorem 3.1.8).

Theorem. *Let G be a weak solution to (1.0.3)–(1.0.4) constructed above, and decompose it as*

$$G(t,x) = n(t)\delta_0(x) + g(t,x), \qquad n(t) = G(t,\{0\}).$$

Then, the function n is right continuous and a.e. differentiable on $[0,\infty)$. Moreover, there exists a positive measure μ on $[0,\infty)$ such that

$$0 < \mu((0,t]) < \infty \qquad \forall t > 0 \tag{1.2.24}$$

and

$$n'(t) = \frac{d}{dt}\mu((0,t]) - n(t)M_{1/2}(g(t)) \qquad a.e.\ t > 0. \tag{1.2.25}$$

The value $\mu((0,t])$ is a limit that involves the behavior of g near the origin. This term appears because the collision operator $Q(g)(t,x)$ in (1.0.5) is not integrable at the origin. If we had $g \in L^1(\mathbb{R}_+)$ and $M_{-1/2}(g(s)) < \infty$ for a.e. $s \in [0,t]$, that would imply $\mu((0,t]) = 0$ (cf. Proposition 3.6.4), and the function n would follow, according to (1.2.26), the equation

$$n'(t) = -n(t)M_{1/2}(g(t)) \qquad a.e.\, t > 0. \tag{1.2.26}$$

The function n would then be monotonically decreasing on $[0,\infty)$. However, the property (1.2.24) shows that this can not be the case and that, in particular, for every $t > 0$ there exists a subset $E \subset [0,t)$ with $|E| > 0$ such that $M_{-1/2}(g(s)) = \infty$ for all $s \in E$, even if the moment of order $-1/2$ of the initial data g_0 is finite. Then g loses instantaneously the moment $M_{-1/2}(g(s))$ and the behavior of $n(t)$ is more involved.

This is quite different from the results obtained in [1], where the authors have shown the existence of a global solution (n,g) that satisfies

$$\frac{d}{dt}(n^2(t)) = -2\int_0^\infty P(g)(t,x)dx \qquad \forall t > 0, \tag{1.2.27}$$

where $P(g)(t,x)$ is the corresponding collision integral in the equation for g, and the integral in the right hand side of (1.2.27) is absolutely convergent.

The property (1.2.24) gives some further insight about the possible behavior of G near the origin. If $G(t,x) \sim x^\beta$ around the origin for some $\beta \in \mathbb{R}$, we saw above that necessarily $\beta \geq -1/2$. Suppose, moreover, that $G(t)$ has no atoms on $(0,\infty)$ for all $t \in [0,T]$ for some $T > 0$. Then, if $\beta > -1/2$, we would have

$$\sup_{t\in[0,T]} \int_{(0,\infty)} \frac{G(t,x)}{\sqrt{x}} dx < \infty,$$

and by Proposition 3.6.4 we would obtain $\mu((0,t]) = 0$ for all $t \in (0,T]$, in contradiction with (1.2.24). This observation implies that in case G had the algebraic behavior $G(t,x) \sim x^\beta$ near the origin, either $\beta = -1/2$ or, if $\beta > -1/2$, then $G(t)$ would have atoms on $(0,\infty)$ for $t \in A \subset [0,T]$ with $|A| > 0$.

In the last result of this section, we prove that, under some conditions on the two first moments of the initial data G_0, the function $n(t)$ vanishes as $t \to \infty$, sufficiently fast to be integrable.

Theorem. *Let G be a weak solution to (1.0.3)–(1.0.4) with mass N and energy E constructed above. If, for some $\alpha \in (1,3]$,*

$$E > C(\alpha) N^{5/3}, \tag{1.2.28}$$

where $C(\alpha) > 0$ is an explicit constant, then

$$\lim_{t\to\infty} n(t) = 0$$

and there is an explicit constant $C(N,E,\alpha)$ such that, for all $t_0 > 0$,

$$\int_{t_0}^\infty n(t)dt \leq C(N,E,\alpha) M_\alpha(G(t_0)).$$

According to [53] and the references therein, given an initial datum G_0 with total mass N and energy E, the kinetic temperature T and the critical kinetic temperature T_c (cf. [53] for a definition) satisfy the following relation

$$\frac{T}{T_c} = \kappa \frac{E}{N^{5/3}}, \qquad \kappa = \frac{(2\pi)^{\frac{1}{3}}}{3} \frac{\zeta(3/2)^{5/3}}{\zeta(5/2)}.$$

Using this identity, the condition (1.2.28) reads in terms of the variable T:

$$T > \kappa C(\alpha) T_c.$$

The value $\kappa C(\alpha)$ increases with α and takes its minimum at $\alpha = 1$, $\kappa C(1) \approx 4.48403$ (cf. Remark 3.1.12). The result above then could be interpreted as follows: the density $n(t)$ converges to zero if the kinetic temperature of the initial datum G_0 is $\kappa C(1)$ times above of the critical kinetic temperature.

1.3 Conclusions

In this thesis, we have studied three problems, (1.0.1), (1.0.2) and (1.0.3)–(1.0.4), related to the mathematical description of dilute gases for quantum particles. These problems are given by kinetic equations that, when written in terms of the natural density in radial variables, have singular kernels. The singularities do not come from the particle interactions that are considered, but rather from the angular integrations and the geometric factors that appear when considering radially symmetric solutions.

The global existence of measure-valued weak solutions for a large class of initial data has been proved for these three problems. In the case of (1.0.2), we have also obtained a sufficient condition on the initial data in order to have global L^1-valued solutions.

Some properties of the weak solutions have been obtained. For example, we have described the long time behavior of the weak solutions of (1.0.2), proving their convergence to a countable sum of Dirac masses. In problem (1.0.3)–(1.0.4), we have obtained estimates on the time evolution of some moments of G, as well as some properties of the function $n(t)$. In particular, the evolution of $n(t)$ has been proved to follow a non trivial integral equation, involving the local behavior of the measure g around the origin. The long time behavior of the function $n(t)$ has also been studied under certain conditions.

Several questions related to these results remain open and could be the object of future research. One interesting question is, for example, the existence (or non existence) of weak solutions to (1.0.1), where the kernel $b(x, y)$ is replaced by $K(x, y)$, for general initial data. It would also be interesting to understand the long time behavior of the weak solutions u and G of the problems (1.0.1) and (1.0.3)–(1.0.4) respectively, as well as to know under what conditions, if any, the weak solutions take their values $u(t)$ and $G(t)$ in $L^1(0, \infty)$ or in $\mathscr{M}_+([0, \infty))$. Lastly, it would be worthwhile to extend the analysis of Chapter 3 to the whole Nordheim equation by considering also the collision integral that accounts for the interactions between particles with strictly positive energy.

Chapter 2

On a Boltzmann equation for Compton scattering with a low-density electron gas at non-relativistic equilibrium

2.1 Introduction.

When only Compton scattering events are considered, the evolution of the particle density of a gas of photons that interact with electrons at non relativistic equilibrium is usually described by means of a Boltzmann equation that may be found in [25], [49], [72] and many others. For a spatially homogeneous isotropic gas of photons and non relativistic electrons at equilibrium, the equation simplifies to the following expression:

$$k^2 \frac{\partial f}{\partial t}(t,k) = Q_\beta(f,f)(t,k), \qquad t>0,\ k\geq 0, \tag{2.1.1}$$

$$Q_\beta(f,f)(t,k) = \int_0^\infty \Big(f(t,k')\,(1+f(t,k))e^{-\beta k} - \\ -f(t,k)(1+f(t,k'))e^{-\beta k'} \Big)\, kk'\mathcal{B}_\beta(k,k')dk'. \tag{2.1.2}$$

The variable $k=|\mathbf{k}|$ denotes the energy of a photon of momentum $\mathbf{k}\in\mathbb{R}^3$ (taking the speed of light c equal to 1), $\beta=(\hbar T)^{-1}$, T is the temperature of the gas of electrons, $(4\pi/3)k^2 f(t,k)\geq 0$ is the particle density, and $\mathcal{B}_\beta(k,k')$ is a function called sometimes the redistribution function.

We emphasize that only elastic collisions of one photon and one electron giving rise to one photon and one electron are considered in this equation, and no radiation effects are taken into account. As shown in [17], the cross section for emission of an additional photon of energy k diverges as k approaches zero, and so the probability of a Compton process unaccompanied by such emission is zero. It follows that the equation (2.1.1), (2.1.2) can not take accurately into account photons with too small energy.

When the speed of light c is taken into account, the corresponding equation (2.1.1), (2.1.2) is very often approximated by a nonlinear Fokker Planck equation (cf. [49]). For $\beta >> (mc^2)^{-1}$ (that corresponds to non relativistic electrons with mass m), the scattering cross section of photons with energies $k << mc^2$ may be approximated by the Thompson scattering cross section. It is then possible to deduce the following expression of $\mathcal{B}_\beta(k,k')$:

$$\mathcal{B}_\beta(k,k') = \sqrt{\beta}\, e^{\beta\frac{(k'+k)}{2}} \int_0^\pi \frac{(1+\cos^2\theta)}{|\mathbf{k}'-\mathbf{k}|} e^{-\beta\frac{\Delta^2+\frac{m^2v^4}{4}}{2mv^2}} d\cos\theta, \tag{2.1.3}$$

$$v = \frac{1}{m}|\mathbf{k}'-\mathbf{k}|, \quad \Delta = k'-k, \tag{2.1.4}$$

(cf. [57] and [30]). It is then argued (cf. [49] for example) that $\mathcal{B}_\beta(k,k')$ is strongly peaked in the region

$$\left\{k>0,\ k'>0:\ |k-k'| << \min\{k,k'\}\right\} \tag{2.1.5}$$

for large values of β, (cf. Figure 1.3) and then, if the variations of f are not too large, it is possible to expand the integrand of (2.1.1) around k and, after a suitable rescaling of the time variable, the equation (2.1.1), (2.1.2) is approximated by:

$$k^2\frac{\partial f}{\partial t} = \frac{\partial}{\partial k}\left(k^4\left(\frac{\partial f}{\partial k}+f^2+f\right)\right), \tag{2.1.6}$$

the Kompaneets equation ([49]). However, it is difficult to determine under what conditions on the initial data and in what range of photon energies k, is this approximation correct.

Due in particular to its importance in modern cosmology and high energy astrophysics, the Kompaneets equation (2.1.6) has received great attention in the literature of physics (cf. the review [11]). It has also been studied from a more strictly mathematical point of view [20, 28, 43], and several of its possible approximations have also been considered [9, 50]. It was first observed in [75] that for a large class of initial data, as t increases, the solutions of (2.1.6) may develop steep profiles, very close to a shock wave, near $k=0$. This was proved to happen in [28] for some of the solutions, for k in a neighborhood of the origin and at large times.

On the basis of the equilibrium distributions F_M of (2.1.1), (2.1.2) given by

$$k^2F_M = k^2f_\mu + \alpha\delta_0, \quad \mu\le 0, \quad \alpha\ge 0, \quad \alpha\mu = 0, \tag{2.1.7}$$

$$f_\mu(k) = \frac{1}{e^{k-\mu}-1}, \quad \int_0^\infty k^2 f_\mu(k)dk = M_\mu, \quad M = \alpha + M_\mu, \tag{2.1.8}$$

some of its unsteady solutions are also expected to develop, asymptotically in time, very large values and strong variation in very small regions near the origin. This was proved to be true in [29] where, under the assumptions that $e^{-\eta(k+k')}(kk')^{-1}\mathcal{B}_\beta(k,k')$ is a bounded function on $[0,\infty)^2$ for some $\eta\in[0,1)$, it is shown that, as $t\to\infty$, certain solutions form a Dirac delta at the origin. A detailed description of this formation was later given in [31], assuming $\mathcal{B}_\beta(k,k')(kk')^{-1}\equiv 1$ and for some classes of initial data. Of course, in the region where this delta formation takes place, the equation (2.1.1), (2.1.2) can not be approximated by the Kompaneets equation (2.1.6).

It is obvious however that the function $\mathcal{B}_\beta(k,k')$ in (2.1.3), (2.1.4) does not satisfies the conditions imposed in [29] or [31]. On the other hand, the Boltzmann equation (2.1.1), (2.1.2) with the kernel (2.1.3), (2.1.4) was considered in [60] and

[34]. Local existence for small initial data with a moment of order -1 was proved in [60]. It was proved in [34] that, although the Cauchy problem is globally solvable in time for initial data bounded from above by the Planck distribution, there is no solution, even local in time, for initial data greater that the Planck distribution. This seems to be an effect of the very small values of k and k' with respect to $|k-k'|$ in the collision integral, and indicates that some truncation is needed in order to have a reasonable theory for the Cauchy problem. (cf. Section 2.1.1 below).

In this article, we consider first the Cauchy problem for an equation where the kernel (2.1.3), (2.1.4) is truncated in a region where k or k' are much smaller than $|k-k'|$, although keeping the strong singularity at the origin $k=k'=0$. This is achieved by multiplying the kernel $\mathcal{B}_\beta$ by a suitable cut off function $\Phi(k,k')$,

$$k^2\frac{\partial f}{\partial t}(t,k)=\widetilde{Q}_\beta(f,f)(t,k) \tag{2.1.9}$$

$$\widetilde{Q}_\beta(f,f)(t,k)=\int_0^\infty\Big(f(t,k')\,(1+f(t,k))e^{-\beta k}- -f(t,k)(1+f(t,k'))e^{-\beta k'}\Big)\,kk'\Phi(k,k')\mathcal{B}_\beta(k,k')dk' \tag{2.1.10}$$

The Cauchy problem for (2.1.9), (2.1.10) proved to have weak solutions for a large class of initial data in the space of non negative measures. Because of some difficulties coming from the kernel $\mathcal{B}_\beta$ and its truncation, it is not possible to perform the same analysis as in [29] or [31], where the asymptotic behavior of the solutions was described.

In order to obtain some further insight, a simplified equation was proposed in [76] and [77], where the authors suggest to keep only the quadratic terms in (2.1.2) when $f>>1$ (or when the function f has a large derivative) and consider,

$$k^2\frac{\partial f}{\partial t}(t,k)=f(t,k)\int_0^\infty f(t,k')\big(e^{-\beta k}-e^{-\beta k'}\big)kk'\mathcal{B}_\beta(k,k')dk'. \tag{2.1.11}$$

This equation may be formally obtained in the limit $\beta\to\infty$ and βk of order one (cf. Appendix B). If the reasoning leading from equation (2.1.1) to the Kompaneets equation is applied to the equation (2.1.11) we obtain the non linear first order equation,

$$k^2\frac{\partial f}{\partial t}=\frac{\partial}{\partial k}\left(k^4f^2\right). \tag{2.1.12}$$

For the same reasons as for the equation (2.1.1), we shall consider the equation with the truncated redistribution function,

$$k^2\frac{\partial f}{\partial t}(t,k)=f(t,k)\int_0^\infty f(t,k')\big(e^{-\beta k}-e^{-\beta k'}\big)kk'\Phi(k,k')\mathcal{B}_\beta(k,k')dk'. \tag{2.1.13}$$

As for the equation (2.1.9), (2.1.10), equation (2.1.13) has weak solutions for a large set of initial data. Moreover, if the initial data is an integrable function sufficiently flat around the origin, it has a global solution that remains for all time an integrable function flat around the origin. The weak solutions of (2.1.13) converge to a limit as t tends to infinity that may be almost completely characterized. It is formed by an at most countable number of Dirac masses, whose locations are determined by the way in which the mass of the initial data is distributed. This suggests a possible transient behavior for the solutions of the complete equation (2.1.9), where large and concentrated peaks could form and remain for some time.

We refer to [39] for recent numerical simulations on the behavior of the solutions of the equation (2.1.1) and the Kompaneets approximation. The anisotropic case has also been recently considered in [19].

We describe now our results in more detail.

2.1.1 The function $\mathcal{B}_\beta(k,k')$. Weak formulation.

Due to the k^2 factor in the left hand side of (2.1.1), it is natural to introduce the new variable

$$v(t,k) = k^2 f(t,k). \tag{2.1.14}$$

This variable v is now, up to a constant, the photon density in the radial variables, and equation (2.1.1), (2.1.2) reads,

$$\frac{\partial v}{\partial t}(t,k) = \mathcal{Q}_\beta(v,v)(t,k), \qquad t>0,\ k\geq 0, \tag{2.1.15}$$

$$\mathcal{Q}_\beta(v,v)(t,k) = \int_0^\infty q_\beta(v,v')\frac{\mathcal{B}_\beta(k,k')}{kk'}dk', \tag{2.1.16}$$

$$q_\beta(v,v) = v'(k^2+v)e^{-\beta k} - v(k'^2+v')e^{-\beta k'}, \tag{2.1.17}$$

where we use the common notation $v = v(t,k)$ and $v' = v(t,k')$. As a consequence of the change of variables (2.1.14), the factor kk' in the collision integral has been changed to $(kk')^{-1}$.

An expression of $\mathcal{B}_\beta(k,k')$ may be obtained at low density of electrons and using the non relativistic approximation of the Compton scattering cross section (cf. [57, 30]). It may be seen in particular that $\mathcal{B}_\beta(k,0)>0$ for all $k>0$, and

$$\mathcal{B}_\beta(k,k') = \frac{44}{15}\left(\frac{2}{k+k'}+1\right) + \mathcal{O}(k+k'), \quad k+k'\to 0, \tag{2.1.18}$$

$$\mathcal{B}_\beta(k,0) = \frac{8}{3}\frac{\sqrt{\beta}}{k}e^{\frac{\beta k}{2}}e^{-\frac{\beta m}{2}\left(1+\frac{k^2}{4m^2}\right)}. \tag{2.1.19}$$

The kernel $\mathcal{B}_\beta(k,k')(kk')^{-1}$ is then rather singular near the axes, and the collision integral $\mathcal{Q}_\beta(v,v)$ is not defined for $v(t)$ a general non negative bounded measure. In order to overcome this problem, it is usual to introduce weak solutions. A natural definition of weak solution is:

$$\frac{d}{dt}\int_{[0,\infty)} v(t,k)\varphi(k)dk = \frac{1}{2}\iint_{[0,\infty)^2}\big(\varphi(k)-\varphi(k')\big)q_\beta(v,v')\frac{\mathcal{B}_\beta(k,k')}{kk'}dkdk' \tag{2.1.20}$$

for a suitable space of test functions φ. Again, we use the notation $\varphi = \varphi(k)$ and $\varphi' = \varphi(k')$. Since $\mathcal{B}_\beta(k,0)>0$ for all $k>0$, the integral in the right hand side of (2.1.20) may still diverge. It was actually proved in [34] that for initial data v_0 such that

$$v_0(x) > \frac{x^2}{e^x-1}, \quad \forall x\geq 0,$$

the (2.1.20) has no solution in $C([0,T),\mathscr{M}_+^1([0,\infty)))$, for any $T>0$.

Kernels with that kind of singularities have been considered in coagulation equations. One possible way to overcome this difficulty is to impose test functions φ

compactly supported on $(0, \infty)$, like in [59], or such that $\varphi(x) \sim x^\alpha$ as $x \to 0$ for some α large enough, like for example in [36], (but in that case we could not expect to obtain any information on what happens near the origin), or also to look for solutions v in suitable weighted spaces like in [10] and [21] (but that would exclude the Dirac delta at the origin). In all these cases, the propagation of negative moments for all $t > 0$ is necessary. That property does not seem to hold true for (2.1.9), cf. Remark 2.2.13 for the local propagation of some negative moments. See Remark 2.1.5 and Remark 2.5.8 for the equation (2.1.13).

Truncated kernel: why and how.

As we have already mentioned, the equation (2.1.1), (2.1.2) does not describe the Compton scattering if "too" low energy photons are considered, since in that case the spontaneous emission of photons must be taken into account (cf. [17]). At this level of description then, some cut off seems necessary for a coherent description, where only collisions of one photon and one electron giving one photon and one electron are considered.

In view of the properties of the function $\mathcal{B}_\beta$ for β large presented in Appendix B, and since no precise indication is available in the literature of physics, we use a mathematical criteria as follows:

(i) - We truncate the kernel $\mathcal{B}_\beta$, down to zero, out of the following subset of $[0, \infty) \times [0, \infty)$:

$$\forall (k, k') \in [0, \delta_*]^2, \quad |k - k'| \le \rho_* (kk')^{\alpha_1} (k + k')^{\alpha_2}, \tag{2.1.21}$$

$$\forall (k, k') \in [0, \infty)^2 \setminus [0, \delta_*]^2, \quad \theta k \le k' \le \theta^{-1} k, \tag{2.1.22}$$

for some constants $\delta_* > 0$, $\rho_* > 0$, $\alpha_1 \ge 1/2$, $2\alpha_2 \ge 3 - 4\alpha$, and $\theta \in (0, 1)$.

(ii) - In order to minimize the region of this truncation, we choose $\alpha_1 = \alpha_2 = 1/2$.

(iii) - We leave $\mathcal{B}_\beta$ unchanged as much as possible inside that region, but at the same time we want the resulting truncated kernel to belong to $C((0, \infty) \times (0, \infty))$.

Remark 2.1.1. It is suggested in [74] that for very large values of β, the support of $\mathcal{B}_\beta$ is a subdomain of $|k - k'| < 2k^2/mc^2$ for small values of k and k'. That would be a stronger truncation than in (ii).

Then we multiply $\mathcal{B}_\beta(k, k')$ by $\Phi(k, k')$, where:

1. $\Phi(k, k') = \Phi(k', k)$ for all $k > 0$, $k' > 0$,
2. $\Phi \in C([0, \infty)^2 \setminus \{(0, 0\})$,
3. $\mathrm{supp}(\Phi) = D$, where $(k, k') \in D$ if and only if (2.1.21) and (2.1.22) hold for $\alpha_1 = \alpha_2 = 1/2$, and some constants $\theta \in (0, 1)$, $\delta_* > 0$ and $\rho_* = \rho_*(\theta, \delta_*)$.
4. $\Phi(k, k') = 1 \; \forall (k, k') \in D_1 \subset D$, where $(k, k') \in D_1$ if and only if,

$$|k - k'| \le \rho_1 \sqrt{kk'(k + k')} \quad \text{if} \quad (k, k') \in [0, \delta_*]^2,$$

$$\theta_1 k \le k' \le \theta_1^{-1} k \qquad \text{if} \quad (k, k') \in [0, \infty)^2 \setminus [0, \delta_*]^2,$$

for some $\theta_1 \in (\theta, 1)$ and $\rho_1 = \rho_1(\theta_1, \delta_*) > 0$. (Cf. also (2.2.5)–(2.2.11)). Then, for all $\varphi \in C^1([0, \infty))$,

$$\left(e^{-\beta k} - e^{-\beta k'}\right)\left(\varphi(k) - \varphi(k')\right) \frac{\mathcal{B}_\beta(k, k')}{kk'} \Phi(k, k') \in L^\infty_{loc}([0, \infty) \times [0, \infty)),$$

and if $\varphi'(0) = 0$,

$$\left(e^{-\beta k} - e^{-\beta k'}\right)\left(\varphi(k) - \varphi(k')\right)\frac{\mathcal{B}_\beta(k,k')}{kk'}\Phi(k,k') \in C([0,\infty) \times [0,\infty))$$

(cf. Lemma A.0.1, Lemma A.0.3 and (2.2.29)).

In the first part of this work, we then consider the problem

$$\frac{\partial v}{\partial t}(t,k) = \int_{[0,\infty)} q_\beta(v,v')\frac{\mathcal{B}_\beta(k,k')\Phi(k,k')}{kk'}dk'. \tag{2.1.23}$$

We need the following notations:

$C_b^1([0,\infty))$ is the space of bounded continuous functions, with continuous bounded derivative, on $[0,\infty)$.

The space of nonnegative bounded Radon measures is denoted $\mathscr{M}_+([0,\infty))$, and

$$\mathscr{M}_+^\rho([0,\infty)) = \{v \in \mathscr{M}_+([0,\infty)) : M_\rho(v) < \infty\}, \ \ \forall \rho \in \mathbb{R},$$

$$M_\rho(v) = \int_{[0,\infty)} k^\rho v(k)dk \quad \text{(moment of order } \rho), \tag{2.1.24}$$

$$X_\rho(v) = \int_{[0,\infty)} e^{\rho k} v(k)dk. \tag{2.1.25}$$

We use the notation $\int v(k)dk$ instead of $\int dv(k)$, even if the measure v is not absolutely continuous with respect to the Lebesgue measure.

Unless stated otherwise, in $\mathscr{M}_+([0,\infty))$ we consider the narrow topology. We recall that the narrow topology is generated by the metric $d_0(\mu,\nu) = \|\mu - \nu\|_0$, where (cf. [15], Theorem 8.3.2),

$$\|\mu\|_0 = \sup\left\{\int_{[0,\infty)} \varphi d\mu : \varphi \in \mathrm{Lip}_1([0,\infty)),\ \|\varphi\|_\infty \leq 1\right\}, \tag{2.1.26}$$

$$\mathrm{Lip}_1([0,\infty)) = \{\varphi : [0,\infty) \to \mathbb{R} : |\varphi(x) - \varphi(y)| \leq |x-y|\}. \tag{2.1.27}$$

The following is an existence result for the problem (2.1.23).

Theorem 2.1.2. *Given any $v_0 \in \mathscr{M}_+([0,\infty))$ satisfying*

$$X_\eta(v) < \infty, \tag{2.1.28}$$

for some $\eta \in \left(\frac{1-\theta}{2}, \frac{1}{2}\right)$, then there exists $v \in C([0,\infty), \mathscr{M}_+([0,\infty)))$ weak solution of (2.1.23), i.e., such that satisfies the following (i)-(ii):

(i) For all $\varphi \in C_b([0,\infty))$,

$$\int_{[0,\infty)} v(\cdot,k)\varphi(k)dk \in C([0,\infty);\mathbb{R}), \tag{2.1.29}$$

$$\int_{[0,\infty)} v(0,k)\varphi(k)dk = \int_{[0,\infty)} v_0(k)\varphi(k)dk, \tag{2.1.30}$$

(ii) For all $\varphi \in C_b^1([0,\infty))$ with $\varphi'(0) = 0$,

$$\int_{[0,\infty)} v(\cdot,k)\varphi(k)dk \in W^{1,\infty}_{loc}([0,\infty);\mathbb{R}), \tag{2.1.31}$$

and for almost every $t > 0$,

$$\frac{d}{dt}\int\limits_{[0,\infty)} v(t,k)\varphi(k)dk = \frac{1}{2}\iint\limits_{[0,\infty)^2} \frac{\Phi\mathcal{B}_\beta}{kk'} q_\beta(v,v')(\varphi-\varphi')dkdk'. \tag{2.1.32}$$

The measure $v(t)$ also satisfies, for all $t \geq 0$,

$$M_0(v(t)) = M_0(v_0) \tag{2.1.33}$$

$$X_\eta(v(t)) \leq e^{C_\eta t} X_\eta(v_0), \tag{2.1.34}$$

where

$$C_\eta = \frac{C_*}{2\theta^2}\frac{(1-\theta)}{(1+\theta)}\frac{\eta}{\left(\frac{1}{2}-\eta\right)}, \quad C_* > 0. \tag{2.1.35}$$

Remark 2.1.3. Theorem 2.1.2 does not precludes the formation, in finite time, of a Dirac measure at the origin in the weak solutions of (2.1.23) with integrable initial data. Such a possibility was actually considered for the solutions of the Kompaneets equation (cf. [74, 75, 77] and others). It was proved in [28] and [29] that, for large sets of initial data, this does not happen, neither in the Kompaneets equation, nor in equation (2.1.15) with a very simplified kernel. But it is not known yet if it may happen for the equation (2.1.15) with the kernel $\Phi(k,k')\mathcal{B}_\beta(k,k')$.

Given a weak solution $\{u(t)\}_{t>0}$ of (2.1.23) whose Lebesgue decomposition is $u(t) = g(t) + G(t)$, with $g(t) \in L^1([0,\infty))$, the natural physical entropy is

$$H(u(t)) = \int_{(0,\infty)} h(x,g(t,x))dx - \int_{(0,\infty)} xG(t,x)dx, \tag{2.1.36}$$

$$h(x,s) = (x^2+s)\log(x^2+s) - s\log s - x^2\log x^2 - sx. \tag{2.1.37}$$

But the corresponding dissipation of entropy used, for example, in [29], is not defined due to the singularity of the kernel $\frac{\Phi\mathcal{B}_\beta}{kk'}$ at the origin. The study of the long time behavior of the weak solutions obtained in Theorem 2.1.2 seems then to be more involved than in [29], (cf. Section 2.4 also).

2.1.2 The simplified equation

In view of the exponential terms in (2.1.2), it is very natural to consider the scaled variable $\beta k = x$, then scale the time variable too as $\beta^3 t = \tau$, and the dependent variable as $\beta^{-1}k^2 f(t,k) = u(\tau,x)$ in order to let the total number of particles to be unchanged (cf. Section B.1). When this is done, it appears that the linear term is formally of lower order in $\beta >> 1$:

$$\begin{aligned}\frac{\partial u}{\partial \tau}(\tau,x) &= \int_0^\infty \frac{B_\beta(x,y)}{xy}\left(e^{-x}-e^{-y}\right)u(\tau,x)u(\tau,y)dy+\\ &+\beta^{-3}\int_0^\infty \frac{B_\beta(x,y)}{xy}\left(u(\tau,y)x^2e^{-x}-u(\tau,x)y^2e^{-y}\right)dy,\end{aligned} \tag{2.1.38}$$

where $B_\beta(x,y) = \beta^{-1}\mathcal{B}_\beta(k,k')$.

If only the quadratic term is kept in (2.1.23), the following equation is obtained

$$\frac{\partial v}{\partial t}(t,k)=v(t,k)\int_{[0,\infty)}v(t,k')\big(e^{-\beta k}-e^{-\beta k'}\big)\frac{\mathcal{B}_\beta(k,k')\Phi(k,k')}{kk'}dk'. \tag{2.1.39}$$

Weak solutions $u\in C([0,\infty),\mathscr{M}_+([0,\infty)))$ to (2.1.39) for all initial data $u_0\in\mathscr{M}_+([0,\infty))$ satisfying (2.1.28) are proved to exist (cf. Theorem 2.5.1) with similar arguments as for the complete equation.

But equation (2.1.39) also has solutions $v\in C([0,\infty),L^1([0,\infty)))$ for initial data $v_0\in L^1([0,\infty))$ that are sufficiently flat around the origin. This "flatness" condition happens then to be sufficient to prevent the finite time formation of a Dirac measure at the origin in the solutions of (2.1.39).

Theorem 2.1.4. *For any nonnegative initial data $v_0\in L^1([0,\infty))$ such that:*

$$\forall r>0,\qquad \int_0^\infty v_0(k)\left(e^{\frac{r}{k^{3/2}}}+e^{\eta k}\right)dk<\infty, \tag{2.1.40}$$

for some $\eta>(1-\theta)/2$, there exists a nonnegative global weak solution $v\in C([0,\infty),L^1([0,\infty)))$ of (2.1.39) that also satisfies

$$v(t,k)=v_0(k)e^{\int_0^t\int_0^\infty\left(e^{-\beta k}-e^{-\beta k'}\right)\frac{\mathcal{B}_\beta(k,k')\Phi(k,k')}{kk'}v(s,k')\,dk'\,ds}, \tag{2.1.41}$$

for all $t>0$, and a.e. $k>0$. Moreover, for all $t>0$,

$$\|v(t)\|_1=\|v_0\|_1, \tag{2.1.42}$$

$$v(t,k)\le v_0(k)e^{\frac{tC_0}{k^{3/2}}},\ \forall t>0,\ \textit{and}\ \ a.e.\,k>0, \tag{2.1.43}$$

where $C_0=\frac{\rho_*C_*}{\sqrt{\theta(1+\theta)}}X_\eta(v_0)$.

Remark 2.1.5. It follows from (2.1.43) that for the solution v obtained in Theorem 2.1.4, $v(t)$ satisfies (2.1.40) for almost every $t>0$. That property is then propagated globally in time.

For a solution v to equation (2.1.39), the moment $M_\rho(v(t))$ defined in (2.1.24) is proved to be a Lyapunov function on $[0,\infty)$ for all $\rho\ge 1$ (cf. Lemma 2.5.10). With some abuse of language, we sometimes refer to $M_\rho(v)$ as an entropy functional for equation (2.1.39). It is possible to characterize the nonnegative measures that minimize $M_\rho(v)$ for $\rho>1$, or satisfy $D_\rho(v)=0$, where

$$D_\rho(v)=\iint_{(0,\infty)^2}\frac{\Phi\mathcal{B}_\beta}{kk'}\big(e^{-\beta k}-e^{-\beta k'}\big)(k^\rho-k'^\rho)v(k)v(k')dkdk'. \tag{2.1.44}$$

This question is solved, usually, at mass M and energy E fixed. Because of the truncated kernel $\Phi\mathcal{B}_\beta$, it is also necessary to introduce the following property about the support of the measure v in connection with the support of the kernel $\Phi\mathcal{B}_\beta$.

Given a measure $v\in\mathscr{M}_+([0,\infty))$, we denote $\{A_n(v)\}_{n\in\mathbb{N}}$ the, at most, countable collection of disjoint closed subsets of the support of v such that,

$$\begin{aligned}&(k,k')\in A_n\times A_n \text{ for some } n\in\mathbb{N},\ \text{ if and only if, } \Phi(k,k')\neq 0 \text{ or}\\ &\exists\{k_n\}_{n\in\mathbb{N}}\subset[k,k');\ k_1=k,\ \lim_{n\to\infty}k_n=k',\Phi(k_n,k_{n+1})\neq 0\ \forall n\in\mathbb{N}.\end{aligned} \tag{2.1.45}$$

(cf. Section 2.5.3 for a precise definition of $A_n(v)$).

Let us define now, for any countable collection $\mathcal{C} = \{C_n, M_n\}_{n\in\mathbb{N}}$ of disjoint, closed subsets $C_n \subset [0,\infty)$ enjoying the property (2.1.45), and positive real numbers M_n, the following family of non negative measures,

$$\mathcal{F}_{\mathcal{C},\alpha} = \left\{ v \in \mathscr{M}_+^\alpha([0,\infty)) : C_n = A_n(v),\ M_n = \int_{A_n(v)} v(k)dk \right\}.$$

Theorem 2.1.6. *For any $\mathcal{C}$ and $\alpha > 1$ as above, the following statements are equivalent:*

(i) $v \in \mathcal{F}_{\mathcal{C},\alpha}$ *and* $D_\alpha(v) = 0$.

(ii) $M_\alpha(v) = \min\{M_\alpha(v) : v \in \mathcal{F}_{\mathcal{C},\alpha}\}$.

(iii) $v = \sum_{n=0}^\infty M_n \delta_{k_n}$, *where* $k_n = \min\{k \in A_n\}$.

Remark 2.1.7. For any sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n > 0$, $x_n \to 0$ as $n \to \infty$, and $\Phi(x_n, x_m) = 0$ for all $n \neq m$, the measure

$$u = \sum_{n=0}^\infty \alpha_n \delta_{x_n}$$

satisfies the conditions (i)–(ii) in Theorem 2.1.6. Although $0 \in \operatorname{supp} u$, there is no Dirac measure at the origin.

The long time behavior of the weak solutions of (2.1.39) is yet only partially understood as shows the following Theorem,

Theorem 2.1.8. *Let v be a weak solution of (2.1.39) constructed in Theorem 2.5.1 for an initial data $v_0 \in \mathscr{M}_+([0,\infty))$ satisfying $X_\eta(v_0) < \infty$ for some $\eta \geq (1-\theta)/2$.*

Then, as $t \to \infty$, $v(t)$ converges in $C([0,\infty), \mathscr{M}_+([0,\infty)))$ to the measure

$$\mu = \sum_{i=0}^\infty M_i' \delta_{k_i'}, \tag{2.1.46}$$

where $M_i' \geq 0$, $k_i' \geq 0$ satisfy the following properties:

1. $k_i' \in \operatorname{supp}(v_0)$ *for all* $i \in \mathbb{N}$,

2. $\Phi(k_i', k_j') = 0$ *for all* $i \neq j$,

3. *If we define* $\mathcal{J}_n = \{i \in \mathbb{N} : k_i' \in A_n(v_0),\ M_i' > 0\}$ *for all* $n \in \mathbb{N}$,

$$\sum_{i\in\mathcal{J}_n} M_i' = M_n, \tag{2.1.47}$$

4. *For all $n \in \mathbb{N}$, if $k_n = \min\{k \in A_n(v_0)\} > 0$, then there exists k_i' such that $k_i' = k_n$.*

Remark 2.1.9. If in Point 4 of Theorem 2.1.8, $k_n = \min\{k \in A_n(v_0)\} = 0$ for some $n \in \mathbb{N}$, but v_0 has no Dirac measure at $k = 0$, we do not know if $k_i' = 0$ for some $i \in \mathbb{N}$, even if the origin belongs to the support of the limit measure μ (cf. Remark 2.1.7 for example).

The measure μ is of course determined by the initial data v_0, but its complete description (i.e. the values of k_i' and m_i') is not known, only the locations k_i' of some of the Dirac masses. For example, it is possible to have $k_i' = k_n = \min\{k \in A_n(v_0)\}$ and $k_i' < k_j' \in A_n(v_0)$ for some n, i, j in $\mathbb{N}$, and k_i', k_j' not seeing each other, i.e., $\Phi(k_i', k_j') = 0$ (cf. Example 1, Section 2.5.4). The location of a Dirac measure at $k_i' = x_n$ is just given by the support of the initial data, but the appearance of a Dirac measure at k_j' is more difficult to be determined.

The long time behaviour that is proved in Theorem 2.1.8 for the solutions of the simplified equation (2.1.39) can not be expected, of course, to hold for the solutions of the complete equation (2.1.23). But in combination with the equation (2.1.38) for β large, it could indicate that the solutions of the complete problem (2.1.23) also undergo the formation of large an concentrated peaks, that could remain for some long, although finite, time.

2.1.3 General comment

The main results of the article are stated in this Introduction in terms of the original variables, t, k, and $v(t,k) = k^2 f(t,k)$. However, in order to make clearly appear some important aspects of the equation, it is useful to introduce τ, x, and $u(\tau, x)$, variables scaled with the parameter β. This is a natural parameter since it is related with the inverse of the temperature of the gas of electrons.. This scaling makes clearly appear two features of the equation for $\beta >> 1$, namely, the fact that $\mathcal{B}_\beta$ is very much peaked along the diagonal, and the different scaling properties of the quadratic and linear part of the collision integral in (2.1.15) (cf. Section B.1 for details).

However, since in all this work the value of the parameter β remains fixed, it is taken equal to one, without any loss of generality. Therefore, except in Section B, we have $\tau \equiv t$, $x \equiv k$ and $u \equiv v$. In particular, for the sake of brevity, we do not re-write again the main results in terms of the variables x and u, although the proofs will be written in those terms.

The main results are actually proved for general kernels B satisfying some of the properties that the truncated kernel $\Phi(k,k')\mathcal{B}_\beta(k,k')$ is proved to enjoy, and that are sufficient for our purpose.

2.2 Existence of weak solutions.

In this Section we prove existence of weak solutions to the following problem:

$$\frac{\partial u}{\partial t}(t,x) = Q(u,u) = \int_{[0,\infty)} b(x,y) q(u,u) dy, \tag{2.2.1}$$

$$u(0) = u_0 \in \mathscr{M}_+([0,\infty)), \tag{2.2.2}$$

where $t > 0$, $x \geq 0$,

$$q(u,u) = u(t,y)(x^2 + u(t,x))e^{-x} - u(t,x)(y^2 + u(t,y))e^{-y}, \tag{2.2.3}$$

$$b(x,y) = \frac{B(x,y)}{xy}, \tag{2.2.4}$$

under the following assumptions on the kernel B:

(i) $B(x,y)\geq 0$ for all $(x,y)\in[0,\infty)^2$, (2.2.5)

(ii) $B(x,y)=B(y,x)$ for all $(x,y)\in[0,\infty)^2$, (2.2.6)

(iii) $B\in C([0,\infty)^2\setminus\{(0,0)\})$, (2.2.7)

(iv) There exist $\theta\in(0,1)$, $\delta_*>0$ and $\rho_*=\rho_*(\theta,\delta_*)>0$ such that

$$\operatorname{supp}(B)=\Gamma=\Gamma_1\cup\Gamma_2, \tag{2.2.8}$$

$$\Gamma_1=\left\{(x,y)\in[0,\infty)^2\setminus[0,\delta_*]^2:\theta x\leq y\leq\theta^{-1}x\right\}, \tag{2.2.9}$$

$$\Gamma_2=\left\{(x,y)\in[0,\delta_*]^2:|x-y|\leq\rho_*\sqrt{xy(x+y)}\right\} \tag{2.2.10}$$

(v) There exists a constant $C_*>0$ such that, for all $(x,y)\in\Gamma$,

$$B(x,y)\leq B\left(\frac{x+y}{2},\frac{x+y}{2}\right)\leq\frac{C_*e^{\frac{x+y}{2}}}{x+y}. \tag{2.2.11}$$

Remark 2.2.1. The region Γ in (2.2.8)–(2.2.10) is such that:

$$\Gamma=\left\{(x,y)\in[0,\infty)^2:y\in\big(\gamma_1(x),\gamma_2(x)\big)\right\}, \tag{2.2.12}$$

where

$$\gamma_1(x)=\begin{cases}\frac{2x+\rho_*^2x^2-\rho_*x^{3/2}\sqrt{\rho_*^2x+8}}{2(1-\rho_*^2x)} & \text{if } x\in[0,\delta_*]\\ \theta x & \text{if } x\in(\delta_*,\infty),\end{cases} \tag{2.2.13}$$

$$\gamma_2(x)=\begin{cases}\frac{2x+\rho_*^2x^2+\rho_*x^{3/2}\sqrt{\rho_*^2x+8}}{2(1-\rho_*^2x)} & \text{if } x\in[0,\theta\delta_*]\\ \theta^{-1}x & \text{if } x\in(\theta\delta_*,\infty).\end{cases} \tag{2.2.14}$$

In particular $\theta x\leq\gamma_1(x)\leq x\leq\gamma_2(x)\leq\theta^{-1}x$ for all $x\geq 0$. The value of $\rho_*=\rho_*(\theta,\delta_*)$ is chosen so that γ_1 and γ_2 are continuous.

Definition 2.2.2. We say that a map $u:[0,\infty)\to\mathscr{M}_+([0,\infty))$ is a weak solution of (2.2.1)–(2.2.2) if

$$(i)\ \forall\varphi\in C_b([0,\infty)),\ \int_{[0,\infty)}u(\cdot,x)\varphi(x)dx\in C([0,\infty);\mathbb{R}) \tag{2.2.15}$$

$$\text{and}\qquad \int_{[0,\infty)}u(0,x)\varphi(x)dx=\int_{[0,\infty)}u_0(x)\varphi(x)dx, \tag{2.2.16}$$

$$(ii)\ \forall\varphi\in C_b^1([0,\infty)),\ \varphi'(0)=0, \tag{2.2.17}$$

$$\int_{[0,\infty)}u(\cdot,x)\varphi(x)dx\in W^{1,\infty}_{loc}([0,\infty);\mathbb{R}), \tag{2.2.18}$$

$$\frac{d}{dt}\int_{[0,\infty)}u(t,x)\varphi(x)dx=\frac{1}{2}\iint_{[0,\infty)^2}b(x,y)q(u,u)(\varphi(x)-\varphi(y))dydx. \tag{2.2.19}$$

The existence of weak solutions for the problem (2.2.1), (2.2.2) was proved in [29] under conditions on the kernel b not fulfilled in our case. In order to use that result in [29], we first consider a regularised version of (2.2.1), with a truncated function $b_n\in L^\infty([0,\infty)\times[0,\infty))$.

It is not possible to define the dissipation of entropy for the weak solutions of (2.2.1) as in [29], for the same reason as for the equation (2.1.23). However, it may be defined for the solutions u_n of the regularised version of (2.2.1), with the truncated kernel b_n,

$$D^{(n)}(u_n) = \frac{1}{2}D_1^{(n)}(g_n) + D_2^{(n)}(g_n, G_n) + \frac{1}{2}D_3^{(n)}(G_n), \tag{2.2.20}$$

$$D_1^{(n)}(g_n) = \iint_{(0,\infty)^2} b_n(x,y) j\left((x^2+g_n)e^{-x}g_n', (y^2+g_n')e^{-y}g_n\right) dydx, \tag{2.2.21}$$

$$D_2^{(n)}(g_n, G_n) = \iint_{(0,\infty)^2} b_n(x,y) j\left((x^2+g_n)e^{-x}, g_n e^{-y}\right) G_n(y) dydx, \tag{2.2.22}$$

$$D_3^{(n)}(G_n) = \iint_{(0,\infty)^2} b_n(x,y) j\left(e^{-x}, e^{-y}\right) G_n(y) G_n(x) dydx, \tag{2.2.23}$$

$$j(a,b) = (a-b)(\ln a - \ln b), \quad \forall a > 0, b > 0, \tag{2.2.24}$$

where $u_n = g_n + G_n$ is the Lebesgue's decomposition of u_n.

2.2.1 Regularised problem

For $n \in \mathbb{N}$, let $\phi_n \in C_c((0,\infty))$ be such that $0 \le \phi_n(x) \le x^{-1}$ for all $x \ge 0$, $\text{supp}(\phi_n) = [1/(n+1), n+1]$ and $\phi_n(x) = x^{-1}$ for $x \in [1/n, n]$, so that $\lim_{n\to\infty} \phi_n(x) = x^{-1}$. Then we define

$$b_n(x,y) = B(x,y)\phi_n(x)\phi_n(y), \tag{2.2.25}$$

and consider the problem

$$\frac{\partial u_n}{\partial t}(t,x) = Q_n(u_n, u_n) = \int_{[0,\infty)} b_n(x,y) q(u_n, u_n) dy, \tag{2.2.26}$$

$$u_n(0) = u_0 \in \mathscr{M}_+([0,\infty)). \tag{2.2.27}$$

If we denote

$$K_\varphi(u,u) = \frac{1}{2}\iint_{[0,\infty)^2} k_\varphi(x,y) u(t,x) u(t,y) dydx, \tag{2.2.28}$$

$$k_\varphi(x,y) = b(x,y)(e^{-x} - e^{-y})(\varphi(x) - \varphi(y)), \tag{2.2.29}$$

$$L_\varphi(u) = \frac{1}{2}\int_{[0,\infty)} \mathcal{L}_\varphi(x) u(t,x) dx, \tag{2.2.30}$$

$$\mathcal{L}_\varphi(x) = \int_0^\infty \ell_\varphi(x,y) dy, \tag{2.2.31}$$

$$\ell_\varphi(x,y) = b(x,y) y^2 e^{-y}(\varphi(x) - \varphi(y)), \tag{2.2.32}$$

then (2.2.19) reads

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(x) u(t,x) = K_\varphi(u,u) - L_\varphi(u), \tag{2.2.33}$$

and the weak formulation of (2.2.26) reads

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(x) u_n(t,x) = K_{\varphi,n}(u_n, u_n) - L_{\varphi,n}(u_n), \tag{2.2.34}$$

where b is replaced by b_n in the formulas (2.2.28)–(2.2.32). Since $b_n \in L^\infty([0,\infty)^2)$ for all $n \in \mathbb{N}$, Theorem 3 in [29] may be applied (cf. Proposition 2.2.4). For any $u \in \mathscr{M}_+([0,\infty))$, we denote $u = u_r + u_s$ the Lebesgue decomposition of u into an absolutely continuous measure with respect to the Lebesgue measure, u_r, and a singular measure, u_s.

Remark 2.2.3. By symmetry and Lemma A.0.3, for all $\varphi \in C_b^1([0,\infty))$,

$$K_\varphi(u,u) = \int_{[0,\infty)} \int_{[0,x)} k_\varphi(x,y)u(t,x)u(t,y)dydx.$$

Proposition 2.2.4. *For any $n \in \mathbb{N}$ and any initial data $u_0 = u_{0,r} + u_{0,s} \in \mathscr{M}_+^1([0,\infty))$, there exists a unique weak solution $u_n = u_{n,r} + u_{n,s} \in C([0,\infty), \mathscr{M}_+^1([0,\infty)))$ to (2.2.26), (2.2.27) that satisfies*

$$M_0(u_n(t)) = M_0(u_0) \qquad \forall t \geq 0, \tag{2.2.35}$$

$$\operatorname{supp}(u_{n,s}(t)) \subset \operatorname{supp}(u_{0,s}) \qquad \forall t \geq 0, \tag{2.2.36}$$

and for all $\varphi \in C_c([0,\infty) \times [0,\infty))$,

$$\int_{[0,\infty)} \varphi(t,x)u_n(t,x)dx = \int_{[0,\infty)} \varphi(0,x)u_0(x)dx \tag{2.2.37}$$
$$+ \int_0^t \int_{[0,\infty)} \varphi_t(t,x)u(t,x)dxds + \int_0^t \int_{[0,\infty)} Q_n(u_n,u_n)\varphi(s,x)dxds,$$

and for all t_1 and t_2 with $t_2 \geq t_1 \geq 0$,

$$\int_{t_1}^{t_2} D^{(n)}(u_n(t))dt = H(u_n(t_1)) - H(u_n(t_2)). \tag{2.2.38}$$

Moreover, if $u_0 \in L^1([0,\infty))$ then $u_n \in C([0,\infty), L^1([0,\infty)))$.

Proof. Theorem 3 in [29]. □

Remark 2.2.5. In Proposition 2.2.4, the space $\mathscr{M}_+^1([0,\infty))$ is endowed with the total variation norm.

Corollary 2.2.6. *Let u_n be as in Proposition 2.2.4 for $n \in \mathbb{N}$. Then (2.2.34) holds for all $t > 0$ and for all nonnegative $\varphi \in C([0,\infty))$ such that $\int_{[0,\infty)} \varphi(x)u_0(x)dx < \infty$.*

Proof. Given a nonnegative function $\varphi \in C([0,\infty))$ such that $\int_{[0,\infty)} \varphi(x)u_0(x)dx < \infty$, let $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_c([0,\infty))$ be such that $\varphi_k(x) \to \varphi(x)$ as $k \to \infty$ for all $x \in [0,\infty)$, and $\varphi_k \leq \varphi_{k+1} \leq \varphi$ for all $k \in \mathbb{N}$. By (2.2.37) with test function φ_k, and recalling that ϕ_n is compactly supported, it is easy to deduce using Fubini's theorem, the symmetry of B, and the antisymmetry of $q(u_n,u_n)$, that for all $k \in \mathbb{N}$,

$$\int_{[0,\infty)} \varphi_k(x)u_n(t,x)dx = \int_{[0,\infty)} \varphi_k(x)u_0(x)dx$$
$$+ \int_0^t \big(I_{\varphi_k,n}(u_n,u_n) - L_{\varphi_k,n}(u_n)\big)ds. \tag{2.2.39}$$

Using again that ϕ_n is compactly supported, we can pass to the limit as $k \to \infty$ in (2.2.39) by monotone and dominated convergence theorems to obtain (2.2.39) with φ instead of φ_k.

Now, since $u_n \in C([0,\infty), \mathscr{M}_+^1([0,\infty)))$, where the topology on $\mathscr{M}_+^1([0,\infty))$ is the total variation norm, it follows that the maps

$$t \mapsto K_{\varphi,n}(u_n(t), u_n(t)), \quad t \mapsto L_{\varphi,n}(u_n(t))$$

are continuous for all $n \in \mathbb{N}$ and all $t > 0$. Then (2.2.34) follows from (2.2.39) with φ istead of φ_k, by the fundamental theorem of calculus. □

2.2.2 The limit $n \to \infty$

The goal now is to pass to the limit as $n \to \infty$ in (2.2.34) and obtain a weak solution of (2.2.1)–(2.2.11). We start with the following uniform estimate.

Proposition 2.2.7. *Let u_n and u_0 be as in Proposition 2.2.4. If $X_\eta(u_0) < \infty$ for some $\eta \in (0, 1/2)$, then for all $t > 0$ and all $n \in \mathbb{N}$,*

$$X_\eta(u_n(t)) \le e^{C_\eta t} X_\eta(u_0) \tag{2.2.40}$$

where C_η is defined in (2.1.35).

Proof. Let $\eta \in (0, 1/2)$ and take $\varphi(x) = e^{\eta x}$ in (2.2.34), which is allowed by Proposition 2.2.6. If we drop all the negative terms in (2.2.34), we use (A.0.2) in Appendix A (for C^1 functions instead of Lipschitz functions), and $\phi_n(x) \le x^{-1}$, then

$$\begin{aligned}\frac{d}{dt}\int_{[0,\infty)} e^{\eta x} u_n(t,x)dx &\le \frac{1}{2}\int_{[0,\infty)} u_n(t,x)\int_x^\infty |\ell_\varphi(x,y)|dydx \\ &\le \frac{C_*(1-\theta)}{2\theta^2(1+\theta)}\int_{[0,\infty)} u_n(t,x)e^{\frac{x}{2}}\int_x^\infty \varphi'(y)e^{-\frac{y}{2}}dydx \\ &\le C_\eta \int_{[0,\infty)} e^{\eta x} u_n(t,x)dx,\end{aligned}$$

from where (2.2.40) follows using Gronwall's inequality. □

We prove now the following pre-compactness result of $\{u_n(t)\}_{n\in\mathbb{N}}$ for any fixed $t > 0$.

Proposition 2.2.8. *Let u_n and u_0 be as in Proposition 2.2.4. Then, for every fixed $t > 0$, there exist a subsequence of $\{u_n(t)\}_{n\in\mathbb{N}}$ (not relabelled) and $U \in \mathscr{M}_+([0,\infty))$ such that, for all $\varphi \in C_0([0,\infty))$,*

$$\lim_{n\to\infty}\int_{[0,\infty)} \varphi(x)u_n(t,x)dx = \int_{[0,\infty)} \varphi(x)U(x)dx. \tag{2.2.41}$$

Moreover, if u_0 satisfies $X_\eta(u_0) < \infty$ for some $\eta \in (0, 1/2)$, then

$$X_\eta(U) \le e^{C_\eta t} X_\eta(u_0), \tag{2.2.42}$$

where C_η is defined in (2.1.35), and (2.2.41) holds for all $\varphi \in C([0,\infty))$ satisfying the growth condition

$$|\varphi(x)| \le ce^{\alpha x} \quad \forall x \in [0,\infty),\ c > 0,\ 0 \le \alpha < \eta. \tag{2.2.43}$$

Proof. By (2.2.35), the sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathscr{M}_+([0,\infty))$, and thus has a subsequence, still denoted $u_n(t)$, that converges to some $U \in \mathscr{M}_+([0,\infty))$ in $\sigma(\mathscr{M}([0,\infty)), C_0([0,\infty)))$ (the weak* topology), i.e., (2.2.41) holds for all $\varphi \in C_0([0,\infty))$. Moreover, if $\zeta_j \in C_c([0,\infty))$ is such that $0 \le \zeta_j \le 1$, $\zeta_j(x) = 1$ for all $x \in [0,j]$ and $\zeta_j(x) = 0$ for all $x \ge j+1$, so that $\zeta_j \to 1$, then by weak* convergence and (2.2.35),

$$\int_{[0,\infty)} \zeta_j(x)U(x)dx = \lim_{n\to\infty} \int_{[0,\infty)} \zeta_j(x)u_n(t,x)dx \le \lim_{n\to\infty} \int_{[0,\infty)} u_n(t,x)dx = \int_{[0,\infty)} u_0(x)dx,$$

and then, as $j \to \infty$,

$$\int_{[0,\infty)} U(x)dx \le \int_{[0,\infty)} u_0(x)dx. \tag{2.2.44}$$

Suppose now that u_0 satisfies (2.1.28) for some $\eta \in (0,1/2)$, and let $\psi(x) = e^{\eta x}$ and $\psi_j = \psi\zeta_j$, where ζ_j is as before. Then, by weak* convergence and Proposition 2.2.7,

$$\int_{[0,\infty)} \psi_j(x)U(x)dx = \lim_{n\to\infty} \int_{[0,\infty)} \psi_j(x)u_n(t,x)dx \le \liminf_{n\to\infty} \int_{[0,\infty)} e^{\eta x}u_n(t,x)dx \le e^{C_\eta t} \int_{[0,\infty)} e^{\eta x}u_0(x)dx,$$

and letting $j \to \infty$, (2.2.42) holds.

Let now $\varphi \in C([0,\infty))$ satisfying (2.2.43), and define $\varphi_j = \varphi\zeta_j$, with ζ_j as before, so that $\varphi_j \to \varphi$ pointwise as $j \to \infty$. Then, for all $j \in \mathbb{N}$,

$$\begin{aligned} &\left| \int_{[0,\infty)} \varphi(x)u_n(t,x)dx - \int_{[0,\infty)} \varphi(x)U(x)dx \right| \\ &\le \left| \int_{[0,\infty)} \varphi_j(x)u_n(t,x)dx - \int_{[0,\infty)} \varphi_j(x)U(x)dx \right| \\ &+ \int_{[0,\infty)} |\varphi(x) - \varphi_j(x)|u_n(t,x)dx + \int_{[0,\infty)} |\varphi(x) - \varphi_j(x)|U(x)dx. \end{aligned} \tag{2.2.45}$$

By (2.2.41), the first term in the right hand side above converges to zero as $n \to \infty$ for all $j \in \mathbb{N}$. We just need to prove that the second and the third terms are arbitrarily small (for j large enough). Both terms are treated in the same way. We use that $\varphi_j = \varphi$ on $[0,j]$, (2.2.43), and Proposition 2.2.7 to obtain

$$\begin{aligned} &\int_{[0,\infty)} |\varphi(x) - \varphi_j(x)|u_n(t,x)dx = \int_{(j,\infty)} |\varphi(x) - \varphi_j(x)|u_n(t,x)dx \\ &\le 2\int_{(j,\infty)} |\varphi(x)|u_n(t,x)dx \le 2c\int_{(j,\infty)} e^{\alpha x}u_n(t,x)dx \\ &\le 2ce^{(\alpha-\eta)j} \int_{(j,\infty)} e^{\eta x}u_n(t,x)dx \le 2ce^{(\alpha-\eta)j}e^{C_\eta t} \int_{[0,\infty)} e^{\eta x}u_0(x)dx, \end{aligned}$$

and by similar estimates, and (2.2.42),

$$\int_{[0,\infty)} |\varphi(x) - \varphi_j(x)| U(x) dx \le 2ce^{(\alpha-\eta)j} e^{C_\eta t} \int_{[0,\infty)} e^{\eta x} u_0(x) dx.$$

Since $\alpha < \eta$, both terms converges to zero as $j \to \infty$. □

The equicontinuity of $\{u_n\}_{n\in\mathbb{N}}$ in the narrow topology is proved in the following Proposition.

Proposition 2.2.9. *Let u_n and u_0 be as in Proposition 2.2.4, and suppose that $X_\eta(u_0) < \infty$ for some $\eta \in \left[\frac{1-\theta}{2}, \frac{1}{2}\right)$. Then, for all $n \in \mathbb{N}$, φ L-Lipschitz on $[0,\infty)$, $0 < T < \infty$ and t, $t_0 \in [0,T]$,*

$$\left| \int_{[0,\infty)} \varphi(x) u_n(t,x) dx - \int_{[0,\infty)} \varphi(x) u_n(t_0,x) dx \right| \le C(u_0,T)|t - t_0|, \tag{2.2.46}$$

where

$$C(u_0,T) = LC_* \left[AM_0(u_0) + \frac{(1-\theta)}{2\theta^2(1+\theta)} \right] e^{TC_{\frac{(1-\theta)}{2}}} X_{\frac{(1-\theta)}{2}}(u_0),$$

and A is given in (A.0.1). In particular, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is equicontinuous from $[0,\infty)$ into $\mathscr{M}_+([0,\infty)$ with the narrow topology.

Proof. Let φ be L-Lipschitz, $0 < T < \infty$ and let t, $t_0 \in [0,T]$ with $t_0 \le t$. By (2.2.34)

$$\begin{aligned} &\left| \int_{[0,\infty)} \varphi(x) u_n(t,x) dx - \int_{[0,\infty)} \varphi(x) u_n(t_0,x) dx \right| \\ &\le \int_{t_0}^{t} \Big(|K_{\varphi,n}(u_n(s), u_n(s))| + |L_{\varphi,n}(u_n(s))| \Big) ds. \end{aligned} \tag{2.2.47}$$

By (A.0.5), Remark A.0.5, (2.2.35), and Proposition 2.2.7,

$$\begin{aligned} \int_{t_0}^{t} |K_{\varphi,n}(u_n(s), u_n(s))| ds &\le LC_* AM_0(u_0) \int_{t_0}^{t} X_{\frac{(1-\theta)}{2}}(u_n(s)) ds \\ &\le LC_* AM_0(u_0) e^{tC_{\frac{(1-\theta)}{2}}} X_{\frac{(1-\theta)}{2}}(u_0)(t - t_0), \end{aligned} \tag{2.2.48}$$

and by (A.0.6) (positive part only), Remark A.0.5 and Proposition 2.2.7,

$$\begin{aligned} \int_{t_0}^{t} |L_{\varphi,n}(u_n(s))| ds &\le \frac{LC_*(1-\theta)}{2\theta^2(1+\theta)} \int_{t_0}^{t} X_{\frac{(1-\theta)}{2}}(u_n(s)) ds \\ &\le \frac{LC_*(1-\theta)}{2\theta^2(1+\theta)} e^{tC_{\frac{(1-\theta)}{2}}} X_{\frac{(1-\theta)}{2}}(u_0)(t - t_0). \end{aligned} \tag{2.2.49}$$

Using (2.2.48) and (2.2.49) in (2.2.47), the estimate (2.2.46) follows. For the equicontinuity, let $\varepsilon > 0$ and consider $\delta < \varepsilon / C(u_0,T)$. By (2.1.26), (2.1.27), if we take the supremum on (2.2.46) among all $\varphi \in \mathrm{Lip}_1([0,\infty))$ with $\|\varphi\|_\infty \le 1$, we deduce that for all $t \in [0,T]$, $t_0 \in [0,T]$ such that $|t - t_0| < \delta$, then $d_0(u_n(t), u_n(t_0)) < \varepsilon$ for all $n \in \mathbb{N}$, that is, $\{u_n\}_{n\in\mathbb{N}}$ is equicontinuous on $[0,T]$. □

As a Corollary of Proposition 2.2.8 and Proposition 2.2.9, we obtain that a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ converges to a limit u in the space $C([0,\infty), \mathscr{M}_+([0,\infty)))$.

Corollary 2.2.10. *Let u_n and u_0 be as in Proposition 2.2.4, and suppose that $X_\eta(u_0) < \infty$ for some $\eta \in \left[\frac{1-\theta}{2}, \frac{1}{2}\right)$. Then there exist a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (not relabelled) and $u \in C([0,\infty), \mathscr{M}_+([0,\infty)))$ such that*

$$\lim_{n\to\infty} d_0(u_n(t), u(t)) = 0 \quad \forall t \geq 0, \tag{2.2.50}$$

and the convergence is uniform on the compact sets of $[0,\infty)$. Moreover,

$$X_\eta(u(t)) \leq e^{C_\eta t} X_\eta(u_0) \qquad \forall t \geq 0, \tag{2.2.51}$$

where C_η is given in (2.1.35), and for all $\varphi \in C([0,\infty))$ satisfying (2.2.43),

$$\lim_{n\to\infty} \int_{[0,\infty)} \varphi(x) u_n(t,x)dx = \int_{[0,\infty)} \varphi(x) u(t,x)dx \quad \forall t \geq 0. \tag{2.2.52}$$

Remark 2.2.11. (2.2.50) implies that, for every $\varphi \in C_b([0,\infty))$,

$$\lim_{n\to\infty} \sup_{t_1\leq t\leq t_2} \left| \int_{[0,\infty)} u_n(t,x)\varphi(x)dx - \int_{[0,\infty)} u(t,x)\varphi(x)dx \right| = 0. \tag{2.2.53}$$

Proof. By Proposition 2.2.8, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is relatively compact on $(\mathscr{M}_+([0,\infty)), d_0)$, and by Proposition 2.2.9, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is equicontinuous from $[0,\infty)$ into $(\mathscr{M}_+([0,\infty)), d_0)$. Then, from Arzelà-Ascoli theorem, u_n converges pointwise (for all $t \geq 0$) to a continuous function u, and the convergence is uniform on compact sets. Since the metric d_0 generates the narrow topology, and the convergence in (2.2.50) is uniform on compact sets, then (2.2.53) follows. The estimate (2.2.51) and the limit (2.2.52) are obtained as in Proposition 2.2.8, since the time t is fixed. □

We prove now that the limit u of the sequence $\{u_n\}_{n\in\mathbb{N}}$ is indeed a weak solution of (2.2.1)–(2.2.2).

Corollary 2.2.12. *Given any $v_0 \in \mathscr{M}_+([0,\infty))$ satisfying (2.1.28) for some $\eta \in \left(\frac{1-\theta}{2}, \frac{1}{2}\right)$, there exists $v \in C([0,\infty), \mathscr{M}_+([0,\infty)))$ weak solution of (2.2.1)–(2.2.2), that also satisfies (2.1.33) and (2.1.34).*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of solutions for the regularised problem (2.2.26), (2.2.27). By Corollary 2.2.10, a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ converges to a limit $u \in C([0,\infty), \mathscr{M}_+([0,\infty)))$. Since u is continuous from $[0,\infty)$ to $(\mathscr{M}_+([0,\infty)), d_0)$ and d_0 generates the narrow topology, then (2.2.15) holds. Next, we prove that u satisfies (2.2.16)–(2.2.19). To this end, let $\varphi \in C_b^1([0,\infty))$ with $\varphi'(0) = 0$. By (2.2.34), for all $n \in \mathbb{N}$ and all $t \geq 0$,

$$\int_{[0,\infty)} \varphi(x) u_n(t,x)dx = \int_{[0,\infty)} \varphi(x) u_0(x)dx + \int_0^t \Big(K_{\varphi,n}(u_n(s), u_n(s)) + L_{\varphi,n}(u_n(s)) \Big) ds, \tag{2.2.54}$$

and our goal is now to pass to the limit as $n \to \infty$ term by term. By (2.2.53), for all $t \geq 0$,

$$\lim_{n\to\infty} \int_{[0,\infty)} \varphi(x) u_n(t,x)dx = \int_{[0,\infty)} \varphi(x) u(t,x)dx. \tag{2.2.55}$$

Let us prove that for all $t \geq 0$,

$$\lim_{n\to\infty} L_{\varphi,n}(u_n(t)) = L_\varphi(u(t)), \tag{2.2.56}$$

$$\lim_{n\to\infty} K_{\varphi,n}(u_n(t), u_n(t)) = K_\varphi(u(t), n(t)). \tag{2.2.57}$$

Starting with (2.2.56), we have

$$\big|L_\varphi(u) - L_{\varphi,n}(u_n)\big| \leq \big|L_\varphi(u) - L_\varphi(u_n)\big| + |L_\varphi(u_n) - L_{\varphi,n}(u_n)|. \tag{2.2.58}$$

Since $\mathcal{L}_\varphi \in C([0,\infty))$ and $\mathcal{L}_\varphi$ satisfies the growth condition (2.2.43) with $\alpha = (1-\theta)/2$, (cf. Lemma A.0.1), then by (2.2.52) the first term in the right hand side of (2.2.58) converges to zero as $n \to \infty$. For the second term we have, for any $R > 0$,

$$|L_\varphi(u_n) - L_{\varphi,n}(u_n)| \leq \int_{[0,R]} |\mathcal{L}_\varphi(x) - \mathcal{L}_{\varphi,n}(x)| u_n(t,x)dx + \int_{(R,\infty)} |\mathcal{L}_\varphi(x) - \mathcal{L}_{\varphi,n}(x)| u_n(t,x)dx.$$

On the one hand, using (2.2.35),

$$\int_{[0,R]} |\mathcal{L}_\varphi(x) - \mathcal{L}_{\varphi,n}(x)| u_n(t,x)dx \leq M_0(u_0)\|\mathcal{L}_\varphi - \mathcal{L}_{\varphi,n}\|_{C([0,R])},$$

which converges to zero as $n \to \infty$ by Lemma A.0.6. On the other hand, by (A.0.3),

$$\int_{(R,\infty)} |\mathcal{L}_\varphi(x) - \mathcal{L}_{\varphi,n}(x)| u_n(t,x)dx \leq 2\int_{(R,\infty)} |\mathcal{L}_\varphi(x)| u_n(t,x)dx$$
$$\leq C\int_{(R,\infty)} e^{\frac{(1-\theta)x}{2}} u_n(t,x)dx \leq Ce^{R\left(\frac{1-\theta}{2}-\eta\right)} \int_{(R,\infty)} e^{\eta x} u_n(t,x)dx, \tag{2.2.59}$$

where $C = \frac{LC_*(1-\theta)}{\theta^2(1+\theta)}$, and by Proposition 2.2.7 we deduce that (2.2.59) converges to zero as $R \to \infty$. That concludes the proof of (2.2.56).

In order to prove (2.2.57), we use

$$|K_\varphi(u,u) - K_{\varphi,n}(u_n,u_n)| \leq |K_\varphi(u,u) - K_\varphi(u_n,u_n)| + |K_\varphi(u_n,u_n) - K_{\varphi,n}(u_n,u_n)|. \tag{2.2.60}$$

Then, for the first term in the right hand side of (2.2.60), given $R > 0$, we use

$$\iint_{[0,\infty)^2} k_\varphi(x,y)u(x)u(y)dydx$$
$$\leq \left(\iint_{[0,R]^2} + \iint_{[\gamma_1(R),\infty)^2} - \iint_{[\gamma_1(R),R]}\right) k_\varphi(x,y)u(x)u(y)dydx,$$

to deduce

$$|K_\varphi(u,u) - K_\varphi(u_n,u_n)| \leq I_1 + I_2 + I_3, \tag{2.2.61}$$

$$I_1 = \left|\iint_{[0,R]^2} k_\varphi u(x)u(y)dydx - \iint_{[0,R]^2} k_\varphi u_n(x)u_n(y)dydx\right|,$$

$$I_2 = \left|\iint_{[\gamma_1(R),R]^2} k_\varphi u(x)u(y)dydx - \iint_{[\gamma_1(R),R]^2} k_\varphi u_n(x)u_n(y)dydx\right|,$$

$$I_3 = \left|\iint_{[\gamma_1(R),\infty)^2} k_\varphi u(x)u(y)dydx - \iint_{[\gamma_1(R),\infty)^2} k_\varphi u_n(x)u_n(y)dydx\right|.$$

Since $k_\varphi \in C([0,\infty)^2)$ (cf. Lemma A.0.3), then by Stone-Weierstrass theorem, $k_\varphi(x,y)$ can be approximated on any compact subset $X \subset [0,\infty)^2$ by functions of the form $\psi_1(x)\psi_2(y)$, with $\psi_i \in C(X)$ for $i = 1,2$. By Tietze extension theorem we may assume that $\psi_i \in C([0,\infty))$ for $i = 1,2$. Then, using that u_n converges narrowly to u, we deduce that for any $\varepsilon > 0$, $R > 0$, there exists $n_* \in \mathbb{N}$ such that for all $n \geq n_*$

$$I_1 < \varepsilon, \qquad I_2 < \varepsilon. \tag{2.2.62}$$

Then, for I_3 we have the following.

$$I_3 \leq \iint_{[\gamma_1(R),\infty)^2} k_\varphi(x,y)\big(u(x)u(y) + u_n(x)u_n(y)\big)dydx, \tag{2.2.63}$$

and by (A.0.1), calling $C = \|\varphi'\|_\infty C_* A$,

$$\begin{aligned}
&\iint_{[\gamma_1(R),\infty)^2} k_\varphi u(t,x)u(t,y)dydx \leq C \iint_{[\gamma_1(R),\infty)^2} e^{\frac{|x-y|}{2}} u(t,x)u(t,y)dydx \\
&\leq 2C \int_{[\gamma_1(R),\infty)} e^{\frac{(1-\theta)x}{2}} u(t,x) \int_{[\gamma_1(R),x]} u(t,y)dydx \\
&\leq 2CX_\eta(u(t)) \int_{[\gamma_1(R),\infty)} u(t,y)dy.
\end{aligned} \tag{2.2.64}$$

We now use that for all $x > 0$, $t > 0$, there exists $R > 0$ such that

$$\begin{aligned}
\int_{[\gamma_1(R),x]} u(t,y)dy &\leq \frac{1}{\gamma_1(R)} \int_{[\gamma_1(R),\infty)} yu(t,y)dy \\
&\leq \frac{1}{\gamma_1(R)} \int_{[\gamma_1(R),\infty)} e^{\eta y}u(t,y)dy \leq \frac{e^{C_\eta t}X_\eta(u_0)}{\gamma_1(R)},
\end{aligned} \tag{2.2.65}$$

where we have used (2.2.51). Using (2.2.65) in (2.2.64), and (2.2.51) again,

$$\iint_{[\gamma_1(R),\infty)^2} k_\varphi u(t,x)u(t,y)dydx \leq \frac{2Ce^{2C_\eta t}(X_\eta(u_0))^2}{\gamma_1(R)},$$

and the same estimate holds when u is replaced by u_n. We then obtain from (2.2.63) that, for any $\varepsilon > 0$, there exists $R > 0$ such that $I_3 < \varepsilon$ for all $n \in \mathbb{N}$. Combining this with (2.2.62), we then deduce from (2.2.61) that for all $t > 0$

$$\lim_{n\to\infty} |K_\varphi(u(t),u(t)) - K_\varphi(u_n(t),u_n(t))| = 0. \tag{2.2.66}$$

Now, for the second term in the right hand side of (2.2.60), we have

$$\begin{aligned}
&|K_\varphi(u_n,u_n) - K_{\varphi,n}(u_n,u_n)| \\
&\leq \int_{[0,\infty)} \int_{[0,x]} |k_\varphi(x,y)||1 - xy\phi_n(x)\phi_n(y)|u_n(t,x)u_n(t,y)dydx,
\end{aligned}$$

and we decompose the integral above as follows:

$$\int_{[0,\infty)} \int_{[0,x]} = \int_{\frac{1}{n}}^{n} \int_{\frac{1}{n}}^{x} + \int_{n}^{\infty} \int_{0}^{x} + \int_{0}^{n} \int_{0}^{\min\{x,\frac{1}{n}\}}. \tag{2.2.67}$$

It plays no role in the argument whether the limits of integration are open or closed, so we use the standard notation for integrals. By definition $\phi_n(x) = x^{-1}$ for all $x \in [1/n, n]$, and then

$$\int_{\frac{1}{n}}^{n} \int_{\frac{1}{n}}^{x} |k_\varphi(x,y)||1 - xy\phi_n(x)\phi_n(y)|u_n(t,x)u_n(t,y)dydx = 0.$$

Now, by (A.0.5) and (2.2.35),

$$\begin{aligned}&\int_{n}^{\infty} \int_{0}^{x} |k_\varphi(x,y)|u_n(t,x)u_n(t,y)dydx\\ &\leq LC_*AM_0(u_0)\int_n^\infty e^{\frac{(1-\theta)x}{2}}u_n(t,x)dx\\ &\leq LC_*AM_0(u_0)e^{n(\frac{1-\theta}{2}-\eta)}\int_n^\infty e^{\eta x}u_n(t,x)dx,\end{aligned}$$

and from Proposition 2.2.7 we deduce that it converges to zero as $n \to \infty$. For the las term in the right hand side of (2.2.67), we argue as follows. Let us define $x_n = \gamma_2(1/n)$ and $D_n = [0, x_n] \times [0, 1/n]$. Notice that $x_n \to 0$ as $n \to \infty$. Then by (2.2.35)

$$\begin{aligned}\int_0^n \int_0^{\min\{x,\frac{1}{n}\}} |k_\varphi(x,y)||1 - xy\phi_n(x)\phi_n(y)|u_n(t,x)u_n(t,y)dydx&\\ \leq \max_{(x,y)\in D_n} |k_\varphi(x,y)|M_0(u_0)^2.&\end{aligned}$$

Since $k_\varphi(0,0) = 0$ and k_φ is continuous (cf. Lemma A.0.3), then $k_\varphi(x,y) \to 0$ for all $(x,y) \in D_n$ as $n \to \infty$. That concludes the proof of (2.2.57).

From the limits (2.2.56), (2.2.57), the uniform bounds (2.2.48), (2.2.49), dominated convergence theorem and (2.2.55), we obtain

$$\begin{aligned}\int_{[0,\infty)} \varphi(x)u(t,x)dx = &\int_{[0,\infty)} \varphi(x)u_0(x)dx\\ &+ \int_0^t \Big(K_\varphi(u(s),u(s)) + L_\varphi(u(s))\Big)ds. \qquad (2.2.68)\end{aligned}$$

The identity (2.2.16) then follows from (2.2.68) for $t = 0$. It follows from Proposition 2.2.9, by passage to the limit as $n \to \infty$, that for any $\varphi \in C_b^1([0,\infty))$ with $\varphi'(0) = 0$, the map $t \mapsto \int_{[0,\infty)} u(t,x)\varphi(x)dx$ is locally Lipschitz on $[0,\infty)$, i.e., (2.2.17) holds, and then from (2.2.68), the weak formulation (2.2.19) follows. Taking $\varphi = 1$ in (2.2.19), we obtain (2.1.33). The estimate (2.1.34) is just (2.2.51). □

Remark 2.2.13. Because of the exponential growth of the kernel B, an exponential moment is required on the initial data u_0. This exponential moment is propagated to the solution for all $t > 0$. Using that exponential it easily follows that for any $\rho \geq 1$, if $M_{-\rho}(u_0) < \infty$, there exists a constant $C_1 > 0$, and a non negative locally bounded $C_2(t)$ such that,

$$\frac{d}{dt}\int_{[0,\infty)} u(t,x)x^{-\rho}dx \leq C_1\left(\int_{[0,\infty)} u(t,x)x^{-\rho}dx\right)^2 + C_2(t),$$

from where it follows that $M_{-\rho}(u(t)) < \infty$ for t in a bounded interval.

Proof of Theorem 2.1.2 . Theorem 2.1.2 follows from Corollary 2.2.12 since the function $b(k,k') = \frac{\Phi\mathcal{B}_\beta}{kk'}$ satisfies (2.2.5)–(2.2.11). □

2.3 The singular part of the solution.

If u is a weak solution of (2.2.1)–(2.2.11) obtained in Theorem 2.1.2, for all $t > 0$, the measure $u(t)$ may now be decomposed by the Lebesgue's decomposition Theorem as

$$u(t) = g(t) + \alpha(t)\delta_0 + G(t) \tag{2.3.1}$$

$$g(t) \in L^1([0,\infty)),\ \alpha \geq 0,\ G(t) \perp dx,\ G(t, \{0\}) = 0 \tag{2.3.2}$$

In this Section we give some properties of u, α, and G.

We first notice that the weak solution u of (2.2.1)–(2.2.11) obtained in Theorem 2.1.2, satisfies the equation (2.2.1) in the sense of distributions. This follows from the properties of the support of the function B and Fubini's Theorem. A similar argument may be used for slightly more general test functions φ. To be more precise, let us define the set

$$\mathscr{C} = \Big\{\varphi \in C_b([0,\infty)) : \sup_{x\geq 0} \frac{|\varphi(x)|}{x^{3/2}} < \infty\Big\}. \tag{2.3.3}$$

Proposition 2.3.1. *Let u be a solution of (2.2.1)–(2.2.11) obtained in Theorem 2.1.2. Then, for almost every $t > 0$, $\partial u/\partial t \in \mathscr{D}'((0,\infty))$, $Q(u(t), u(t)) \in \mathscr{D}'((0,\infty))$, and*

$$\forall \varphi \in C_c((0,\infty)), \quad \frac{d}{dt}\langle u(t), \varphi\rangle = \langle Q(u(t), u(t)), \varphi\rangle. \tag{2.3.4}$$

Moreover,

$$\forall \varphi \in \mathscr{C}, \quad \frac{d}{dt}\langle u(t), \varphi\rangle = \langle \mathcal{Q}(u(t), u(t)), \varphi\rangle, \tag{2.3.5}$$

where

$$\mathcal{Q}(u(t), u(t)) = \int_{[0,\infty)} b(x,y)\Big[(e^{-x} - e^{-y})u(t,x)u(t,y) - u(t,x)y^2e^{-y} + u(t,y)x^2e^{-x}\Big]dy. \tag{2.3.6}$$

Remark 2.3.2. Notice that in (2.3.6), the integral containing the factor $(e^{-x} - e^{-y})$ is convergent near the origin even for test functions $\varphi \in \mathscr{C} \setminus C_c((0,\infty))$. That is not true anymore if we consider each of the terms e^{-x} and e^{-y} separately.

Proof. By (2.2.8)–(2.2.11) and (2.1.33),

$$\int_{[0,\infty)} |\varphi(x)| \int_{[0,\infty)} \frac{B(x,y)}{xy} E(x,y)dydx < \infty,$$

where $E(x,y)$ is one the functions in

$$\Big\{u(y)x^2e^{-x}, u(x)y^2e^{-y}, u(x)u(y)e^{-x}, u(x)u(y)e^{-y}\Big\}$$

when $\varphi \in C_c^1((0,\infty))$, or, one of the functions in

$$\Big\{u(x)u(y)|e^{-x} - e^{-y}|, u(x)y^2e^{-y}, u(y)x^2e^{-x}\Big\}$$

when $\varphi \in \mathscr{C} \cap C_b^1([0,\infty))$. Since u is a weak solution and satisfies (2.2.19), we deduce from Fubini's Theorem the identity (2.3.4) for $\varphi \in C_c^1((0,\infty))$, and the identities (2.3.5)-(2.3.6) for $\varphi \in \mathscr{C} \cap C_b^1([0,\infty))$. By a density argument the Proposition follows. □

We may prove now the following property of the singular measure $G(t)$.

Theorem 2.3.3. *Let u be a weak solution of (2.2.1)–(2.2.11) obtained in Theorem 2.1.2, and consider the decomposition (2.3.1), (2.3.2). If $G(0) = 0$ in $\mathscr{D}'((0,\infty))$, then $G(t) = 0$ in $\mathscr{D}'((0,\infty))$ for all $t > 0$.*

Proof. By (2.3.4), for a.e. $t > 0$ and for all $\varphi \in C_c((0,\infty))$,

$$\frac{d}{dt}\int_{[0,\infty)} u(t,x)\varphi(x)dx = \int_{[0,\infty)} \varphi(x)Q(u(t),u(t))(x)dx,$$

and then, after integration in time:

$$\int_{[0,\infty)} \left\{u(t,x) - u(0,x) - \int_0^t Q(u(s),u(s))(x)ds\right\}\varphi(x)dx = 0$$

for a.e. $t > 0$. If we plug now $u = g + \alpha\delta_0 + G$ in this formula and use that $\varphi \in C_c((0,\infty))$, we obtain for a.e. $t > 0$,

$$\int_{[0,\infty)} \Big\{g(t) + G(t) - g(0) - G(0) - R(t) - S(t)\Big\}\varphi(x)dx = 0,$$

where

$$R(t,x) = \int_0^t \left(g(s,x)W(s,x) + x^2e^{-x}\int_{[0,\infty)} b(x,y)u(s,y)dy\right)ds, \tag{2.3.7}$$

$$S(t,x) = \int_0^t G(s,x)W(s,x)ds, \tag{2.3.8}$$

$$W(s,x) = \int_{[0,\infty)} b(x,y)(e^{-x} - e^{-y})u(s,y)dy - \int_{[0,\infty)} b(x,y)y^2e^{-y}dy. \tag{2.3.9}$$

It follows that, for a.e. $t > 0$,

$$g(t) + G(t) - g(0) - G(0) - R(t) - S(t) = 0 \text{ in } \mathscr{D}'((0,\infty)). \tag{2.3.10}$$

Let us prove now that $R(t,\cdot) \in L^1_{loc}((0,\infty))$ for all $t \geq 0$. To this end, we first show that $W(t,\cdot) \in L^\infty_{loc}((0,\infty))$ for all $t \geq 0$. Let then x be in a compact set $[a,c]$, with $0 < a < c < \infty$, and let $t \geq 0$. Using that $\text{supp}(b) = \Gamma \subset \{(x,y) \in [0,\infty)^2 : \theta x \leq y \leq \theta^{-1}x\}$, the bound (2.2.11), and that $x \in [a,c]$, it is easily proved that there exists a constant $0 < C < \infty$ that depends only on a, c, θ and C_*, such that for all $(x,y) \in \Gamma$ with $x \in [a,c]$,

$$b(x,y)|e^{-x} - e^{-y}| \leq C \qquad \text{and} \qquad b(x,y)\max\{x^2,y^2\} \leq C. \tag{2.3.11}$$

We then obtain from (2.3.9) that for all $t \geq 0$, $x \in [a,c]$,

$$|W(t,x)| \leq C(M_0(u(t)) + 1),$$

and by the conservation of mass (2.1.33),

$$\sup_{t\geq 0}\|W(t,\cdot)\|_{L^\infty([a,c])}\leq C(M_0(u_0)+1). \tag{2.3.12}$$

Using now (2.3.11), (2.3.12) and (2.1.33), we deduce from (2.3.7) that for all $t\geq 0$, $x\in[a,c]$,

$$|R(t,x)|\leq C(M_0(u_0)+1)\int_0^t g(s,x)ds+CM_0(u_0)t. \tag{2.3.13}$$

Then, since $\sup_{t\geq 0}\|g(t,\cdot)\|_{L^1([a,c])}\leq\sup_{t\geq 0}M_0(u(t))=M_0(u_0)$, it follows from (2.3.13) that

$$\|R(t,\cdot)\|_{L^1([a,c])}\leq C(M_0(u_0)+1)M_0(u_0)t+(c-a)CM_0(u_0)t. \tag{2.3.14}$$

On the other hand, using the Lebesgue decomposition Theorem, we have for all $t\geq 0$:

$$S(t)=S_{ac}(t)+S_s(t),\qquad S_{ac}(t)\in L^1([0,\infty)),\quad S_s(t)\perp dx.$$

Using this decomposition in (2.3.10), we deduce that for a.e. $t>0$,

$$g(t)-g(0)-R(t)-S_{ac}(t)=-G(t)+G(0)+S_s(t)\quad\text{in }\mathscr{D}'((0,\infty)).$$

Since the left hand side is absolutely continuous with respect to the Lebesgue measure and the right hand side is singular, we then obtain for a.e. $t>0$,

$$\begin{aligned}&g(t)=g(0)+R(t)+S_{ac}(t)\qquad\text{in}\quad\mathscr{D}'((0,\infty)),\\&G(t)=G(0)+S_s(t)\qquad\text{in}\quad\mathscr{D}'((0,\infty)).\end{aligned}$$

Then, for all $\varphi\in C_c((0,\infty))$ and a.e. $t>0$,

$$\int_{[0,\infty)}\varphi(x)G(t,x)dx=\int_{[0,\infty)}\varphi(x)G(0,x)dx+\int_{[0,\infty)}\varphi(x)S_s(t,x)dx. \tag{2.3.15}$$

We use now that for all nonnegative $\varphi\in C_c((0,\infty))$, $t\geq 0$,

$$\int_{[0,\infty)}\varphi(x)S_s(t,x)dx\leq\int_{[0,\infty)}\varphi(x)|S_s(t,x)|dx\leq\int_{[0,\infty)}\varphi(x)|S(t,x)|dx, \tag{2.3.16}$$

where $|S_s(t)|$ and $|S(t)|$ are the total variation measures of $S_s(t)$ and $S(t)$ respectively. Then, if $\varphi\geq 0$ and $\operatorname{supp}(\varphi)\subset[a,c]$ for finite $c>a>0$, we deduce from (2.3.8), (2.3.12) and (2.3.16) that

$$\begin{aligned}\int_{[0,\infty)}\varphi(x)S_s(t,x)dx&\leq\int_0^t\|W(s,\cdot)\|_{L^\infty([a,c])}\int_{[0,\infty)}\varphi(x)G(s,x)dxds\\&\leq C(M_0(u_0)+1)\int_0^t\int_{[0,\infty)}\varphi(x)G(s,x)dxds,\end{aligned}$$

and then, we obtain from (2.3.15) that, for a.e. $t>0$,

$$\begin{aligned}\int_{[0,\infty)}\varphi(x)G(t,x)dx\leq&\int_{[0,\infty)}\varphi(x)G(0,x)dx\\&+C(M_0(u_0)+1)\int_0^t\int_{[0,\infty)}\varphi(x)G(s,x)dxds.\end{aligned}$$

Then by Gronwall's Lemma,

$$\int_{[0,\infty)} \varphi(x)G(t,x)dx \leq$$
$$\leq \left(\int_{[0,\infty)} \varphi(x)G(0,x)dx\right)\left(1 + C(M_0(u_0)+1)te^{C(M_0(u_0)+1)t}\right).$$

We deduce that, if $G(0) = 0$, then $\int_{[0,\infty)} \varphi(x)G(t,x)dx = 0$ for every $\varphi \in C_c((0,\infty))$ and then, $G(t) = 0$ in $\mathscr{D}'((0,\infty))$ for a.e. $t > 0$. □

Remark 2.3.4. If $\alpha(0) = 0$, we do not know whether or not $\alpha(t) = 0$ for a.e. $t > 0$.

2.3.1 An equation for the mass at the origin

We can obtain information of the measure at the origin $u(t,\{0\})$ from the weak formulation (2.2.19), by choosing test functions like in the following Remark.

Remark 2.3.5. Let $\varphi \in C_b^1([0,\infty))$ be nonincreasing with $\operatorname{supp}\varphi = [0,1]$, $\varphi(0) = 1$ and $\varphi'(0) = 0$. Then, let $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$ for $\varepsilon > 0$. It follows from (2.1.33) and dominated convergence that for all $t \geq 0$,

$$\lim_{\varepsilon\to 0} \int_{[0,\infty)} \varphi_\varepsilon(x)u(t,x)dx = u(t,\{0\}). \tag{2.3.17}$$

Proposition 2.3.6. *Let u be a weak solution of (2.2.1)–(2.2.11) obtained in Theorem 2.1.2, and denote $\alpha(t) = u(t,\{0\})$. Then α is right continuous, nondecreasing and a.e. differentiable on $[0,\infty)$. Moreover, for all t and t_0 with $t \geq t_0 \geq 0$, and all φ_ε as in Remark 2.3.5, the following limit exists:*

$$\lim_{\varepsilon\to 0} \int_{t_0}^{t} K_{\varphi_\varepsilon}(u(s),u(s))ds, \tag{2.3.18}$$

and

$$\alpha(t) = \alpha(t_0) + \lim_{\varepsilon\to 0} \int_{t_0}^{t} K_{\varphi_\varepsilon}(u(s),u(s))ds. \tag{2.3.19}$$

Proof. Let us prove first (2.3.19). Using φ_ε in (2.2.33), we deduce by (2.3.17) that for all t and t_0 with $t \geq t_0 \geq 0$, the following limit exists:

$$\lim_{\varepsilon\to 0} \int_{t_0}^{t} \big(K_{\varphi_\varepsilon}(u(s),u(s)) - L_{\varphi_\varepsilon}(u(s))\big)ds,$$

and moreover

$$\alpha(t) = \alpha(t_0) + \lim_{\varepsilon\to 0} \int_{t_0}^{t} \big(K_{\varphi_\varepsilon}(u(s),u(s)) - L_{\varphi_\varepsilon}(u(s))\big)ds. \tag{2.3.20}$$

We claim

$$\lim_{\varepsilon\to 0} \int_{t_0}^{t} L_{\varphi_\varepsilon}(u(s))ds = 0. \tag{2.3.21}$$

In order to prove (2.3.21), we first obtain an integrable majorant of $L_{\varphi_\varepsilon}(u(s))$, and then we show

$$\lim_{\varepsilon\to 0} L_{\varphi_\varepsilon}(u(s)) = 0 \quad \forall s \geq 0. \tag{2.3.22}$$

Taking into account Γ, the support of $\Psi_\varepsilon(x,y) = \varphi_\varepsilon(x) - \varphi_\varepsilon(y)$, and using $\mathcal{L}_{\varphi_\varepsilon}(0) = 0$ (cf. Lemma A.0.1), we have

$$\begin{aligned} |L_{\varphi_\varepsilon}(u(s))| \leq & \int_{(0,\varepsilon)} u(s,x) \int_{\theta x}^{\theta^{-1}x} |\ell_{\varphi_\varepsilon}(x,y)| dy dx \\ & + \int_{\left[\varepsilon, \frac{\varepsilon}{\theta}\right]} u(s,x) \int_{\theta x}^{\varepsilon} |\ell_{\varphi_\varepsilon}(x,y)| dy dx. \end{aligned} \tag{2.3.23}$$

Since

$$\varphi_\varepsilon(x) - \varphi_\varepsilon(y) = \int_{\frac{y}{\varepsilon}}^{\frac{x}{\varepsilon}} \varphi'(z) dz \leq \frac{\|\varphi'\|_\infty}{\varepsilon} |x-y|,$$

then by (A.0.2)

$$|\ell_{\varphi_\varepsilon}(x,y)| \leq \frac{c}{\varepsilon} e^{\frac{x-y}{2}}, \quad c = \frac{C_*(1-\theta)}{\theta^2(1+\theta)} \|\varphi'\|_\infty,$$

and from (2.3.23) we deduce

$$\begin{aligned} |L_{\varphi_\varepsilon}(u(s))| \leq \frac{2c}{\varepsilon} \Bigg[& \int_{(0,\varepsilon)} u(s,x) \big(e^{\frac{(1-\theta)x}{2}} - e^{\frac{(1-\theta^{-1}x)}{2}} \big) dx \\ & + \int_{\left[\varepsilon, \frac{\varepsilon}{\theta}\right]} u(s,x) \big(e^{\frac{(1-\theta)x}{2}} - e^{\frac{x-\varepsilon}{2}} \big) dx \Bigg]. \end{aligned}$$

We now use $e^{\frac{(1-\theta)x}{2}} - e^{\frac{(1-\theta^{-1}x)}{2}} \leq \frac{(\theta^{-1}-\theta)x}{2} e^{\frac{(1-\theta)x}{2}}$, $e^{\frac{(1-\theta)x}{2}} - e^{\frac{x-\varepsilon}{2}} \leq \frac{(\varepsilon-\theta x)}{2} e^{\frac{(1-\theta)x}{2}}$, and (2.2.51) to obtain, for all $\varepsilon > 0$,

$$\begin{aligned} |L_{\varphi_\varepsilon}(u(s))| & \leq c(\theta^{-1}-\theta) \int_{(0,\varepsilon)} u(s,x) e^{\frac{(1-\theta)x}{2}} dx \\ & \quad + c(1-\theta) \int_{\left[\varepsilon, \frac{\varepsilon}{\theta}\right]} u(s,x) e^{\frac{(1-\theta)x}{2}} dx \\ & \leq c(\theta^{-1}-\theta) \int_{\left(0, \frac{\varepsilon}{\theta}\right]} e^{\frac{(1-\theta)x}{2}} u(s,x) dx \\ & \leq c(\theta^{-1}-\theta) e^{sC_{(1-\theta)/2}} \int_{[0,\infty)} e^{\frac{(1-\theta)x}{2}} u_0(x) dx. \end{aligned} \tag{2.3.24}$$

The right hand side above is independent of ε, and it is clearly integrable on $[0,t]$, for all $t > 0$.

Let us prove now (2.3.22). If we prove

$$\lim_{\varepsilon\to 0} \mathcal{L}_{\varphi_\varepsilon}(x) = 0 \quad \forall x \geq 0, \tag{2.3.25}$$

then by (2.3.24) and dominated convergence, (2.3.22) follows. Therefore we are left to prove (2.3.25). On the one hand, since $\mathcal{L}_{\varphi_\varepsilon}(0) = 0$ for all $\varepsilon > 0$ (cf. Lemma

A.0.1), then $\lim_{\varepsilon\to 0}\mathcal{L}_{\varphi_\varepsilon}(0)=0$. On the other hand, for all $x>0$ and $y\in[0,\infty)$, the function $\ell_{\varphi_\varepsilon}(x,y)$ is well defined and

$$\lim_{\varepsilon\to 0}\ell_{\varphi_\varepsilon}(x,y)=0. \tag{2.3.26}$$

Moreoever, by (2.2.11)

$$|\ell_{\varphi_\varepsilon}(x,y)|\leq B(x,y)\frac{y}{x}e^{-y}(\varphi_\varepsilon(x)+\varphi_\varepsilon(y))\leq 2C_*\frac{ye^{\frac{x-y}{2}}}{x(x+y)}\mathbb{1}_\Gamma(x,y),$$

and then

$$\begin{aligned}\int_0^\infty|\ell_{\varphi_\varepsilon}(x,y)|dy&\leq 2C_*e^{\frac{(1-\theta)x}{2}}\frac{1}{x}\int_{\theta x}^{\theta^{-1}x}\frac{y}{x+y}dy\\&=2C_*e^{\frac{(1-\theta)x}{2}}\int_\theta^{\theta^{-1}}\frac{z}{1+z}dz<+\infty.\end{aligned}\tag{2.3.27}$$

It follows from (2.3.26), (2.3.27) and dominated convergence that $\mathcal{L}_{\varphi_\varepsilon}(x)\to 0$ as $\varepsilon\to 0$ for all $x>0$, and then (2.3.25) holds. That proves (2.3.22), which combined with (2.3.24) and dominated convergence, finally proves (2.3.21). Using (2.3.21) in (2.3.20), then the limit in (2.3.18) exists and (2.3.19) holds.

Since $K_{\varphi_\varepsilon}(u,u)\geq 0$ for all $\varepsilon>0$, it follows from (2.3.19) that α is monotone nondecreasing, and then a.e. differentiable by Lebesgue Theorem.

We are left to prove the right continuity of α. Since α is nondecreasing, we already know

$$\alpha(t)\leq\liminf_{h\to 0^+}\alpha(t+h), \tag{2.3.28}$$

so it is sufficient to prove

$$\limsup_{h\to 0^+}\alpha(t+h)\leq\alpha(t). \tag{2.3.29}$$

To this end, let φ_ε as in Remark 2.3.5. Using $\alpha(t+h)\leq\int_{[0,\infty)}\varphi_\varepsilon(x)u(t+h,x)dx$ and (2.2.33) with φ_ε, we have

$$\alpha(t+h)\leq\int_{[0,\infty)}\varphi_\varepsilon(x)u(t,x)dx+\int_t^{t+h}\big(K_{\varphi_\varepsilon}(u(s),u(s))+L_{\varphi_\varepsilon}(u(s))\big)ds.$$

From Proposition A.0.4 and (2.2.51), we deduce that $K_{\varphi_\varepsilon}(u(s),u(s))$ and $L_{\varphi_\varepsilon}(u(s))$ are locally integrable in time for every fixed $\varepsilon>0$, so letting $h\to 0$ above, and then $\varepsilon\to 0$, we finally obtain (2.3.29). The right continuity then follows from (2.3.28) and (2.3.29). □

Remark 2.3.7. By a standard approximation argument, it is possible to use $\mathbb{1}_{[0,\varepsilon)}$ as a test function in (2.2.33). Then, by similar arguments as in the proof of Proposition 2.3.6, it can be seen that equation (2.3.19) also holds when φ_ε is replaced by $\mathbb{1}_{[0,\varepsilon)}$, and then, for all $t\geq t_0\geq 0$,

$$\alpha(t)=\alpha(t_0)+\lim_{\varepsilon\to 0}\int_{t_0}^t\iint_{D_\varepsilon}\frac{B(x,y)}{xy}(e^{-x}-e^{-y})u(s,x)u(s,y)dydxds, \tag{2.3.30}$$

where $D_\varepsilon=[\varepsilon,\gamma_2(\varepsilon))\times(\gamma_1(x),\varepsilon)$.

2.4 On entropy and entropy dissipation.

Suppose that u is the weak solution of (2.2.1) with initial data u_{in} given by Theorem 2.1.2 and $\{u_n\}_{n\in\mathbb{N}}$ is the sequence given by Proposition 2.2.4. Then, if H is the entropy defined in (2.1.36) and $D^{(n)}$ are the functionals defined in (2.2.20), the same calculations as in Section 2 and Section 6 of [29] yield

$$\int_T^\infty D^{(n)}(u_n(t))dt \le H(U_M) + C_1 \int_{[0,\infty)} (1+x)u_{in}(x)dx, \ \forall n \in \mathbb{N}$$

where U_M is the unique equilibrium with the same mass than u_{in}, $M = M_0(u_{in})$. Since the sequence of functions $\{b_n\}_{n\in\mathbb{N}}$ is increasing,

$$\int_T^\infty D^{(m)}(u_n(t))dt \le H(U_M) + C_1 \int_{[0,\infty)} (1+x)u_{in}(x)dx, \forall n > m.$$

Therefore, by the weak lower semi continuity of the function D_m (cf. Theorem 4.6 in [29]), and the weak convergence of u_n to u:

$$\int_T^\infty D^{(m)}(u(t))dt \le H(U_M) + C_1 \int_{[0,\infty)} (1+x)u_{in}(x)dx, \ \forall m \in \mathbb{N}$$

However, it is only possible to obtain a partial characterization of the measures $u \in \mathscr{M}_+([0,\infty))$ with total mass M and such that $D^{(m)}(u) = 0$ for all $m \in \mathbb{N}$.

Proposition 2.4.1. *A measure $u \in \mathscr{M}_+([0,\infty))$ with total mass $M > 0$, satisfies $D^{(m)}(u) = 0$ for all $m \in \mathbb{N}$, if and only if, there exists $\mu \le 0$ and $\alpha \ge 0$ such that $u = g_\mu + \alpha\delta_0$ and $\int_0^\infty g_\mu(x)dx + \alpha = M$, where*

$$g_\mu(x) = \frac{x^2}{e^{x-\mu} - 1}, \qquad x > 0. \tag{2.4.1}$$

Proof. It is straightforward to check that if $u = g_\mu + \alpha\delta_0$ for some $\mu \le 0$ and $\alpha \ge 0$, such that $\int_0^\infty g_\mu(x)dx + \alpha = M$, then $D^{(m)}(u) = 0$. On the other hand, if $u = g + G$ is the Lebesgue decomposition of u and $D^{(m)}(u) = 0$, then $D_1^{(m)}(g) = D_2^{(m)}(g, G) = D_3^{(m)}(G) = 0$. From $D_1^{(m)}(g) = 0$ it follows that, for a.e. $(x, y) \in [0,\infty)^2$,

$$b_m(x,y) j\Big(g'(x^2 + g)e^{-x}, g(y^2 + g')e^{-y}\Big) = 0. \tag{2.4.2}$$

Since $b_m(x, y) > 0$ for $(x, y) \in \Gamma_{\varepsilon,m}$ for all $\varepsilon > 0$ and all $m \in \mathbb{N}$, where

$$\Gamma_{\varepsilon,m} = \Big\{(x,y) \in \Gamma : d((x,y), \partial\Gamma) > \varepsilon,\ (x,y) \in \Big(\frac{1}{m}, m\Big) \times \Big(\frac{1}{m}, m\Big)\Big\},$$

we deduce from (2.4.2),

$$\frac{g(x)e^x}{x^2 + g(x)} = \frac{g(y)e^y}{y^2 + g(y)} \quad a.e.\ (x,y) \in \Gamma_{\varepsilon,m},$$

and both terms must then be equal to a nonnegative constant, say γ. If $\gamma = 0$, then $g = 0$ for a.e. $x > \varepsilon$. If $\gamma > 0$, then $\gamma = e^\mu$ for some $\mu \in \mathbb{R}$ and $g = g_\mu$ for a.e. $x > \varepsilon$. Letting $\varepsilon \to 0$ we obtain that either $g = 0$ or $g = g_\mu$ a.e. in $(0,\infty)$ and, since $g \ge 0$, then $\mu \le 0$.

From $D_3^{(m)}(G) = 0$ for all $m \in \mathbb{N}$, we obtain that $j(e^{-x}, e^{-y}) = (e^{-x} - e^{-y})(x - y) = 0$ for $G \times G$ a.e. $(x, y) \in \Gamma_{\varepsilon,m}$. Letting $\varepsilon \to 0$, we deduce that

$$G = \sum_{i=0}^{\infty} \alpha_i \delta_{x_i}, \tag{2.4.3}$$

for some $\alpha_i \geq 0$, $x_i \geq 0$ with $b_m(x_i, x_j) = 0$ for all $i \neq j$, and all $m \in \mathbb{N}$.

From $D_2^{(m)}(g, G) = 0$, $g = g_\mu$ and G as in (2.4.3), we deduce that, for all $m \in \mathbb{N}$,

$$D_2^{(m)}(g, G) = \sum_{i=0}^{\infty} \alpha_i (x_i - \mu)\big(e^{-\mu} - e^{-x_i}\big) \int_0^\infty b_m(x, x_i) g_\mu(x) dx = 0,$$

and therefore, each of the terms in the sum above is zero. If $\alpha_i > 0$ and $x_i > 0$ for some $i \in \mathbb{N}$, it then follows that $\mu = x_i$, which is a contradiction since $\mu \leq 0$. Hence $G = \alpha\delta_0$ for some $\alpha \geq 0$. □

Remark 2.4.2. The measure u in the statement of Proposition 2.4.1 is not uniquely determined because, since $b_m(x, 0) = 0$ for all $x > 0$, it is possible to have $\mu < 0$ and $\alpha > 0$.

2.5 A simplified equation.

There may be several reasons to consider the following simplified version of equation (2.1.15), (2.1.16):

$$\frac{\partial u}{\partial t}(t, x) = u(t, x) \int_0^\infty R(x, y) u(t, y) dy, \tag{2.5.1}$$

$$R(x, y) = b(x, y)(e^{-x} - e^{-y}). \tag{2.5.2}$$

Although the integral collision operator in (2.5.1) only contains the nonlinear terms of the integral collision operator in (2.1.15), it may supposed to be the dominant term when u is large. This was the underlying idea in [76] and [77], when such approximation was suggested. Let us also recall that, as shown in Section B.1 in Appendix B, if the variables k, t and f are suitably scaled with the parameter β to obtain the new variables x, τ and u (cf. (B.1.2) and (B.1.6)), the equation (2.1.15), (2.1.16) yields equation (B.1.7), where the dependence on the parameter $\beta > 0$ has been kept. Then, the reduced equation (2.5.1) appears as the lower order approximation as $\beta \to \infty$.

Due to its simpler form, the study of (2.5.1) is slightly easier. The existence of solutions $u \in C([0, \infty), L^1([0, \infty)))$, that do not form a Dirac mass at the origin in finite time, is proved (cf. Section 2.5.2) and it is also possible to describe the long time behaviour of the solutions. Both questions remain open for the equation (2.1.15).

2.5.1 Existence and properties of weak solutions.

In this Section we prove the following result on the existence of weak solutions of the equation (2.5.1), (2.5.2).

Theorem 2.5.1. *For any initial data* $u_0 \in \mathscr{M}_+([0,\infty))$ *satisfying*

$$X_\eta(u_0) < \infty \quad \textit{for some} \quad \eta > \frac{1-\theta}{2}, \tag{2.5.3}$$

there exists $u \in C([0,\infty), \mathscr{M}_+([0,\infty)))$ *such that:*

$$(i)\ \forall \varphi \in C_b([0,\infty)),\ \int_{[0,\infty)} u(\cdot,x)\varphi(x)dx \in C([0,\infty);\mathbb{R}) \tag{2.5.4}$$

$$\textit{and} \quad \int_{[0,\infty)} u(0,x)\varphi(x)dx = \int\limits_{[0,\infty)} u_0(x)\varphi(x)dx,$$

(ii) $\forall \varphi \in C_b^1([0,\infty))$, $\varphi'(0) = 0$,

$\int_{[0,\infty)} u(\cdot,x)\varphi(x)dx \in W^{1,\infty}_{loc}([0,\infty);\mathbb{R})$, *and for a.e.* $t > 0$,

$$\frac{d}{dt}\int\limits_{[0,\infty)} u(t,x)\varphi(x)dx = \frac{1}{2}\iint\limits_{[0,\infty)^2} R(x,y)u(t,x)u(t,y)(\varphi(x)-\varphi(y))dydx. \tag{2.5.5}$$

(We will say that u *is a weak solution of (2.5.1) with initial data* u_0*). The solution also satisfies,*

$$M_0(u(t)) = M_0(u_0) \quad \forall t > 0, \tag{2.5.6}$$

$$X_\eta(u(t)) \le X_\eta(u_0) \quad \forall t > 0. \tag{2.5.7}$$

This result is similar to Theorem 2.1.2 for the equation (2.2.1)–(2.2.11), and its proof uses similar arguments. The main difference is that Theorem 3 in [29] can not be used to obtain approximate solutions, and this must be done using a classical truncation argument. Let us then consider the following auxiliary problem:

$$\frac{\partial u_n}{\partial t}(t,x) = u_n(t,x)\int_0^\infty R_n(x,y)u_n(t,y)dy, \tag{2.5.8}$$

$$u_n(0,x) = u_{in}(x) \tag{2.5.9}$$

$$R_n(x,y) = b_n(x,y)(e^{-x} - e^{-y}) \tag{2.5.10}$$

where b_n is defined in (2.2.25).

Proposition 2.5.2. *For every* $n \in \mathbb{N}$ *and for every nonnegative initial data* $u_{in} \in L^1([0,\infty))$*, there exists a nonnegative function*

$$u_n \in C([0,\infty), L^1([0,\infty))) \cap C^1((0,\infty), L^1([0,\infty)))$$

that satisfies (2.5.8) and (2.5.9) in $C((0,\infty), L^1([0,\infty)))$ *and* $L^1([0,\infty))$ *respectively and is such that,*

$$M_0(u_n(t)) = M_0(u_{in}) \quad \forall t \ge 0. \tag{2.5.11}$$

Moreover, for all φ*, defined on* $[0,\infty)$*, measurable, non negative and non decreasing function,*

$$\int_{[0,\infty)} u_n(t,x)\varphi(x)dx \le \int_{[0,\infty)} u_{in}(x)\varphi(x)dx, \ \ \forall n \in \mathbb{N},\ \forall t > 0. \tag{2.5.12}$$

Proof. The proof uses a simple Banach fixed point argument. For any nonnegative $f \in C([0,\infty), L^1([0,\infty)))$ we consider the solution u to the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = u(t,x)\displaystyle\int_0^\infty R_n(x,y)f(t,y)dy \quad x>0,\ t>0, \\ u(0,x) = u_{in}(x), \quad x>0, \end{cases}$$

given by:

$$A_n(f) \equiv u(t,x) = u_{in}(x)e^{\int_0^t\int_0^\infty R_n(x,y)f(s,y)dyds}.$$

Our goal is then to prove first that A_n is a contraction on $\mathcal{X}_{\rho,T}$ for some $\rho>0$ and $T>0$ where,

$$\mathcal{X}_{\rho,T} = \left\{ f \in C([0,T); L^1([0,\infty))); \sup_{0\le t\le T} \|f(t)\|_1 \le \rho \right\}.$$

For all $T>0$, $t\in[0,T)$ and $f\in\mathcal{X}_{\rho,T}$,

$$\|A_n(f)(t)\|_1 \le \|u_{in}\|_1 e^{T\rho\|R_n\|_\infty}; \tag{2.5.13}$$

and for all t_1, t_2 such that $0\le t_1\le t_2\le T$:

$$\begin{aligned} &|A_n(f)(t_1,x) - A_n(f)(t_2,x)| = \\ &= u_{in}(x)\left| e^{\int_0^{t_1}\int_0^\infty R_n(x,y)f(s,y)dyds} - e^{\int_0^{t_2}\int_0^\infty R_n(x,y)f(s,y)dyds}\right| \\ &\le u_{in}(x)\left|\int_{t_1}^{t_2}\int_0^\infty R_n(x,y)f(s,y)dyds\right| \times \\ &\quad \times e^{\theta\int_0^{t_1}\int_0^\infty R_n(x,y)f(s,y)dyds + (1-\theta)\int_0^{t_2}\int_0^\infty R_n(x,y)f(s,y)dyds} \\ &\le u_{in}(x)\rho\|R_n\|_\infty |t_1-t_2| e^{T\rho\|R_n\|_\infty}. \end{aligned}$$

It then follows that

$$\|A_n(f)(t_1) - A_n(f)(t_2)\|_1 \le \|u_{in}\|_1\rho\|R_n\|_\infty e^{\rho T\|R_n\|_\infty}|t_1-t_2|. \tag{2.5.14}$$

Let now f and g be in $\mathcal{X}_{\rho,T}$ and denote $v=A_n(g)$ and $u=A_n(f)$. Arguing as before,

$$\|u(t)-v(t)\|_1 \le \|u_{in}\|_1\|R_n\|_\infty\|f-g\|_{C([0,T],\,L^1([0,\infty)))}Te^{\rho T\|R_n\|_\infty}. \tag{2.5.15}$$

By (2.5.13)–(2.5.15), if

$$\begin{aligned} &\|u_{in}\|_1 e^{T\rho\|R_n\|_\infty} < \rho, \text{ and} \\ &\|u_{in}\|_1\|R_n\|_\infty Te^{\rho T\|R_n\|_\infty} < 1, \end{aligned}$$

then A_n is a contraction on $C([0,T], L^1([0,\infty)))$, and has a fixed point u_n that satisfies

$$u_n(x,t) = u_{in}(x)e^{\int_0^t\int_0^\infty R_n(x,y)u_n(s,y)dyds}. \tag{2.5.16}$$

This solution may then be extended to $C([0,T_{\max}), L^1([0,\infty)))$. It immediately follows from (2.5.16) that $u_n \ge 0$.

Moreover, since $u_n \in C([0,T_{\max}), L^1([0,\infty)))$ and R_n is bounded, we deduce from (2.5.16) that $u_n \in C^1([0,T_{\max}), L^1([0,\infty)))$ and, for every $t \in (0,T_{\max})$, the equation (2.5.8) is satisfied in $L^1([0,\infty))$. For all $T < T_{\max}$ and all $t \in [0,T]$,

$$\left| u_{in}(\cdot)\frac{d}{dt}\left(e^{\int_0^t\int_0^\infty R_n(\cdot,y)u_n(s,y)dy}\right)\right|$$
$$\leq u_{in}(\cdot)\|R_n\|_\infty\|u_n\|_{C([0,T],L^1([0,\infty)))}e^{T\|R_n\|_\infty\|u_n\|_{C([0,T],L^1([0,\infty)))}} \in L^1([0,\infty)),$$

then if we multiply (2.5.16) by any $\varphi \in L^\infty([0,\infty))$, we deduce that for all $t < T_{\max}$,

$$\frac{d}{dt}\int_0^\infty u_n(t,x)\varphi(x)dx = \int_0^\infty\int_0^\infty R_n(x,y)u_n(t,x)u_n(t,y)\varphi(x)dydx.$$

Recalling the definition of R_n, then by the symmetry of b_n and Fubini's theorem,

$$\frac{d}{dt}\int_0^\infty u_n(t,x)\varphi(x)dx = \int_0^\infty\int_0^\infty k_{\psi,n}(x,y)u_n(s,x)u_n(s,y)dydx, \tag{2.5.17}$$

i.e. u_n is a weak solution of (2.5.8), (2.5.9) for $t \in [0,T)$. If we chose $\varphi = 1$ we deduce that (2.5.11) holds for all $t < T_{\max}$. Then, by a classical argument, $T_{\max} = \infty$.

In order to prove (2.5.12) let ψ be non negative and measurable function such that $\int_0^\infty u_0(x)\psi(x)dx < \infty$, and consider $\{\psi_k\}_{k\in\mathbb{N}}$ the sequence of simple functions that converges monotonically to ψ as $k \to \infty$. Since $\psi_k \in L^\infty([0,\infty))$, then (2.5.17) holds with $\varphi = \psi_k$ for all k, and by Lebesgue's and monotone convergence Theorems,

$$\int_0^\infty u_n(t,x)\psi(x)dx = \int_0^\infty u_{in}(x)\psi(x)dx$$
$$+\int_0^t\int_0^\infty\int_0^\infty k_{\psi,n}(x,y)u_n(s,x)u_n(s,y)dydxds.$$

Using that $u_n \in C([0,\infty), L^1([0,\infty)))$, (2.5.11) and

$$\int_0^\infty\int_0^\infty k_{\psi,n}(x,y)\big|u_n(t_1,x)u_n(t_1,y) - u_n(t_2,x)u_n(t_2,y)\big|dydx$$
$$\leq 2\|k_{\psi,n}\|_\infty M_0(u_{in})\|u_n(t_1) - u_n(t_2)\|_1,$$

so that $t \mapsto \int_0^\infty\int_0^\infty k_{\psi,n}(x,y)u_n(s,x)u_n(s,y)dydx$ is continuous, it follows by the fundamental theorem of calculus that (2.5.17) holds for ψ. If, in addition, φ is nondecreasing, then $(e^{-x} - e^{-y})(\varphi(x) - \varphi(y)) \leq 0$ for all $(x,y) \in [0,\infty)^2$, and then (2.5.12) follows. □

Proof of Theorem 2.5.1 . Consider first a initial data $u_0 \in L^1([0,\infty))$. Let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence of solutions to (2.5.8)–(2.5.9) constructed in Proposition 2.5.2 for $n \in \mathbb{N}$. As in the proof of Theorem 2.1.2, the result follows from the precompactness (given by the conservation of $M_0(u)$) and the equicontinuity of $\{u_n\}_{n\in\mathbb{N}}$. These properties follow as in the proof of Proposition 2.2.8 and Proposition 2.2.9 respectively. The existence of the solution u follows using the same arguments as in Corollary 2.2.10 and the end of the Proof of Theorem 2.1.2.

Property (2.5.7) for u follows from (2.5.12), the lower semicontinuity of the non negative function $e^{\eta x}$, and the weak convergence to u of u_n.

For a general initial data $u_0 \in \mathscr{M}_+([0,\infty))$, by Corollary 8.6 in [24] there exists a sequence $\{u_{0,n}\}_{n\in\mathbb{N}} \subset L^1([0,\infty))$ such that

$$\lim_{n\to\infty} \int_0^\infty \varphi(x) u_{0,n}(x)dx = \int_{[0,\infty)} \varphi(x)u_0(x)dx \quad \forall \varphi \in C_b([0,\infty)). \tag{2.5.18}$$

Since $u_{0,n} \in L^1([0,\infty))$, using the previous step there exists a weak solution u_n that satisfies (2.5.4)–(2.5.7). By (2.5.6) and (2.5.7), the sequence $\{u_n\}_{n\in\mathbb{N}}$ is precompact in $C([0,\infty),\mathscr{M}_+([0,\infty)))$. Arguing as in Proposition 2.2.9, we deduce that it is also equicontinuous. Therefore, using the same arguments as in the end of the Proof of Theorem 2.1.2, we deduce the existence of a subsequence, still denoted $\{u_n\}_{n\in\mathbb{N}}$, and a weak solution of (2.5.1), $u \in C([0,\infty),\mathscr{M}_+([0,\infty)))$, satisfying (2.5.4)–(2.5.7).

The property (2.5.7) is obtained using first in the weak formulation (2.5.5) a sequence of monotone non decreasing test functions $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_b^1([0,\infty)$ such that $\varphi_k'(0) = 0$ and $\varphi_k(x) \to e^{\eta x}$ for all $x \geq 0$, to obtain:

$$\int_{[0,\infty)} u(t,x)\varphi_k(x)dx \leq \int_{[0,\infty)} u_0(x)\varphi_k(x)dx, \tag{2.5.19}$$

and then pass to the limit as $k \to \infty$. □

Remark 2.5.3. In Theorem 2.1.2, the initial data is required to satisfy $X_\eta(u_0) < \infty$ for some $\eta \in \left(\frac{1-\theta}{2}, \frac{1}{2}\right)$. On the one hand, the condition $\eta > \frac{1-\theta}{2}$ is sufficient in order to have boundedness of the operators $K_\varphi(u,u)$ and $L_\varphi(u)$. On the other hand, the condition $\eta < 1/2$ comes from the estimate (2.2.40). In Theorem 2.5.1, however, that last condition is not needed, thanks to the estimate (2.5.12).

We show now that the support of $u(t)$ is constant in time.

Proposition 2.5.4. *Let u be a weak solution of (2.5.1) constructed in Theorem 2.5.1 for an initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (2.5.3). The following statements hold:*

(i) For all $r > 0$, t_0 and t with $0 \leq t_0 \leq t$, and $\varphi \in C_c^1((0,\infty))$ nonnegative such that $\operatorname{supp}(\varphi) \subset [r,L]$ *for some $L > r$,*

$$\int_{[0,\infty)} \varphi(x)u(t,x)dx \geq e^{-(t-t_0)C_1} \int_{[0,\infty)} \varphi(x)u(t_0,x)dx, \tag{2.5.20}$$

$$\int_{[0,\infty)} \varphi(x)u(t,x)dx \leq e^{(t-t_0)C_2} \int_{[0,\infty)} \varphi(x)u(t_0,x)dx, \tag{2.5.21}$$

where

$$C_1 = \frac{C_*\rho_* M_0(u_0)}{\sqrt{\theta(1+\theta)}} \frac{e^{\frac{(1-\theta)L}{2}}}{r^{3/2}}, \quad C_2 = \frac{C_*\rho_* M_0(u_0)}{\sqrt{2}} \frac{e^{\frac{(1-\theta)L}{2\theta}}}{r^{3/2}}. \tag{2.5.22}$$

(ii) For all $r > 0$, t_0 and t with $0 \leq t_0 \leq t$,

$$\int_{[0,r)} u(t_0,x)dx \leq \int_{[0,r)} u(t,x)dx \leq e^{(t-t_0)C_r} \int_{[0,r)} u(t_0,x)dx, \tag{2.5.23}$$

where
$$C_r = \frac{C_*\rho_* M_0(u_0)}{\sqrt{\theta(1+\theta)}} \frac{e^{\frac{(1-\theta)r}{2\theta}}}{r^{3/2}}. \tag{2.5.24}$$

(iii) $\mathrm{supp}(u(t)) = \mathrm{supp}(u_0)$ *for all* $t > 0$.

Proof. Proof of (i). Since there are no integrability issues near the origin because $\mathrm{supp}(\varphi) \subset [r, L]$, then by Fubini's theorem

$$\frac{1}{2}\iint_{[0,\infty)^2} R(x,y)(\varphi(x) - \varphi(y))u(t,x)u(t,y)dydx$$
$$= \int_{[0,\infty)} \varphi(x)u(t,x)\int_{[0,\infty)} R(x,y)u(t,y)dydx.$$

Let us prove the lower bound (2.5.20). Using (2.2.8)–(2.2.11), for all $(x,y) \in \Gamma$, $y \le x$,

$$|R(x,y)| \le \frac{C_* e^{\frac{x-y}{2}}(x-y)}{xy(x+y)} \le C^* \frac{e^{\frac{(1-\theta)x}{2}}}{x^{3/2}}, \quad C^* = \frac{C_*\rho_*}{\sqrt{\theta(1+\theta)}}, \tag{2.5.25}$$

and taking into account the support of φ, we deduce that

$$\int_{[0,\infty)} \varphi(x)u(t,x)\int_{[0,\infty)} R(x,y)u(t,y)dydx$$
$$\ge \int_r^L \varphi(x)u(t,x)\int_0^x R(x,y)u(t,y)dydx$$
$$\ge -\frac{C^* e^{\frac{(1-\theta)L}{2}}}{r^{3/2}}\int_r^L \varphi(x)u(t,x)\int_0^x u(t,y)dydx$$
$$\ge -C_1\int_r^L \varphi(x)u(t,x)dx,$$

and then, from the weak formulation, we obtain that for all $t > 0$,

$$\frac{d}{dt}\int_r^L \varphi(x)u(t,x)dx \ge -C_1\int_r^L \varphi(x)u(t,x)dx,$$

and (2.5.20) follows by Gronwall's Lemma.

We now prove the upper bound (2.5.21) by similar arguments. Since $R(x,y) \le 0$ for $y \le x$, then

$$\int_{[0,\infty)} \varphi(x)u(t,x)\int_{[0,\infty)} R(x,y)u(t,y)dydx$$
$$\le \int_r^L \varphi(x)u(t,x)\int_x^\infty R(x,y)u(t,y)dydx,$$

and since for all $(x,y) \in \Gamma$, $x \le y$,

$$R(x,y) \le \frac{C_* e^{\frac{y-x}{2}}(y-x)}{xy(x+y)} \le C' \frac{e^{\frac{(1-\theta)x}{2\theta}}}{x^{3/2}}, \qquad C' = \frac{C_*\rho_*}{\sqrt{2}},$$

we deduce from the weak formulation that for all $t > 0$,

$$\frac{d}{dt}\int_r^L \varphi(x)u(t,x)dx \le \frac{C' e^{\frac{(1-\theta)L}{2\theta}}}{r^{3/2}}\int_r^L \varphi(x)u(t,x)\int_x^\infty u(t,y)dydx$$
$$\le C_2\int_r^L \varphi(x)u(t,x)dx,$$

and then (2.5.21) follows by Gronwall's Lemma.

Proof of (ii). We first prove the lower bound in (2.5.23). Given $r > 0$, let $0 < r_* < r$, and $\varphi \in C_c^1([0,\infty))$ be nonnegative, nonincreasing, and such that $\varphi(x) = 1$ for all $x \in [0, r_*]$ and $\varphi(x) = 0$ for all $x \geq r$. Since $(e^{-x} - e^{-y})(\varphi(x) - \varphi(y)) \geq 0$ for all $0 \leq y \leq x$, it follows from the weak formulation

$$\frac{d}{dt}\int_{[0,r)} \varphi(x)u(t,x)dx \geq 0 \qquad \forall t > 0,$$

hence

$$\int_{[0,r)} \varphi(x)u(t,x)dx \geq \int_{[0,r)} \varphi(x)u(t_0,x)dx \qquad \forall t \geq t_0 \geq 0,$$

and then the lower bound in (2.5.23) follows by taking the supremum over all φ as above, i.e., letting $r_* \to r$.

Let us prove now the upper bound in (2.5.23). Given $r > 0$, let r_* and φ be as before. Keeping only the positive terms in the weak formulation and taking Γ into account, we deduce

$$\frac{d}{dt}\int_{[0,r)} \varphi(x)u(t,x)dx \leq \int_{r_*}^{\frac{r}{\theta}} \int_{\theta x}^{\min\{x,r\}} |R(x,y)|\varphi(y)u(t,x)u(t,y)dydx,$$

and by (2.5.25) we obtain

$$\begin{aligned}\frac{d}{dt}\int_{[0,r)} \varphi(x)u(t,x)dx &\leq \frac{C^* e^{\frac{(1-\theta)r}{2\theta}}}{r_*^{3/2}} \int_{r_*}^{\frac{r}{\theta}} u(t,x) \int_{\theta x}^{\min\{x,r\}} \varphi(y)u(t,y)dydx \\ &\leq C_{r_*}\int_{[0,r)} \varphi(y)u(t,y)dy,\end{aligned}$$

where

$$C_{r_*} = \frac{C^* e^{\frac{(1-\theta)r}{2\theta}}}{r_*^{3/2}} M_0(u_0),$$

and then it follows from Gronwall's Lemma

$$\int_{[0,r)} \varphi(x)u(t,x)dx \leq e^{(t-t_0)C_{r_*}} \int_{[0,r)} \varphi(x)u(t_0,x)dx \qquad \forall t \geq t_0 \geq 0.$$

The upper bound in (2.5.23) then follows by letting r_* tend to r..

Proof of (iii). We recall the following characterization of the support of a Radon measure μ (see [35], Chapter 7): $x \in \operatorname{supp}(\mu)$ if and only if $\int_{[0,\infty)} \varphi d\mu > 0$ for all $\varphi \in C_c([0,\infty))$ with $0 \leq \varphi \leq 1$ such that $\varphi(x) > 0$.

Then, from (2.5.20) and (2.5.21) for $t_0 = 0$, we deduce that

$$(0,\infty) \cap \operatorname{supp}(u_0) = (0,\infty) \cap \operatorname{supp}(u(t)) \qquad \forall t > 0,$$

and from (2.5.23) for $t_0 = 0$, we deduce that for all $t > 0$,

$$0 \in \operatorname{supp}(u_0) \quad \text{if and only if} \quad 0 \in \operatorname{supp}(u(t)),$$

which completes the proof. □

The queues of the weak solutions are decreasing in time, as proved in the following Proposition.

Proposition 2.5.5. *Let u be the weak solution of (2.5.1) constructed in Theorem 2.5.1 for an initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (2.5.3). Then*

(i) For all $r \geq 0$, the map $t \mapsto \int_{[r,\infty)} u(t,x)dx$ is nonincreasing on $[0,\infty)$.

(ii) For all $r > 0$, if

$$\exists x_0 \in [r, \gamma_2(r)) \cap \operatorname{supp}(u_0),\ \exists y_0 \in (\gamma_1(r), r) \cap \operatorname{supp}(u_0), \quad \textit{such that} \quad B(x_0,y_0) > 0, \tag{2.5.26}$$

then the map $t \mapsto \int_{[r,\infty)} u(t,x)dx$ is strictly decreasing on $[0,\infty)$.

Remark 2.5.6. Condition (2.5.26) holds, for instance, if r is an interior point of the support of u_0.

Proof. Proof of (i). For $r = 0$, the result follows from the conservation of mass (2.5.6). For $r > 0$, let $\varepsilon \in (0,r)$ and $\varphi_\varepsilon \in C_b^1([0,\infty))$ be an increasing function such that $\varphi_\varepsilon(x) = 1$ for all $x \geq r$, $\varphi_\varepsilon(x) = 0$ for all $x \in [0, r-\varepsilon]$. Using the monotonicity of φ_ε, we deduce from the weak formulation (2.5.5) that for all $t \geq 0$,

$$\frac{d}{dt}\int_{[0,\infty)} \varphi_\varepsilon(x) u(t,x)dx \leq 0,$$

and then the map $t \mapsto \int_{[0,\infty)} \varphi_\varepsilon(x)u(t,x)dx$ is nonincreasing. The result then follows by letting $\varepsilon \to 0$.

Proof of (ii). Since (2.5.5) is invariant under time translations, it suffices to prove that for all $r > 0$,

$$\int_{[r,\infty)} u(t,x)dx < \int_{[r,\infty)} u_0(x)dx \qquad \forall t > 0,$$

provided (2.5.26) holds. To this end, consider φ_ε as in part (i). By (2.5.5)

$$\begin{aligned}\int_{[0,\infty)} \varphi_\varepsilon(x)u(t,x)dx = &\int_{[0,\infty)} \varphi_\varepsilon(x)u_0(x)dx \\ &+ \int_0^t \int_0^\infty \int_0^x k_{\varphi_\varepsilon}(x,y)u(s,x)u(s,y)dydxds.\end{aligned}$$

Then, since $\lim_{\varepsilon\to 0} k_{\varphi_\varepsilon}(x,y) = k_\varphi(x,y)$ for all $(x,y) \in [0,\infty)^2$, where $\varphi(x) = \mathbb{1}_{[r,\infty)}(x)$, and for all ε small enough,

$$\begin{aligned}\int_0^\infty \int_0^x &|k_{\varphi_\varepsilon}(x,y)|u(s,x)u(s,y)dydx \\ &= \int_{r-\varepsilon}^\infty \int_{\theta x}^x |k_{\varphi_\varepsilon}(x,y)|u(s,x)u(s,y)dydx \\ &\leq 2C_*\rho_* \int_{r-\varepsilon}^\infty \int_{\theta x}^x \frac{e^{\frac{x-y}{2}}}{\sqrt{xy(x+y)}} u(s,x)u(s,y)dydx \\ &\leq \frac{2C_*\rho_*}{\sqrt{\theta(1+\theta)}(r-\varepsilon)^{3/2}} \int_{r-\varepsilon}^\infty e^{\frac{(1-\theta)x}{2}} u(s,x)\int_{\theta x}^x u(s,y)dydx \\ &\leq \frac{4C_*\rho_* M_0(u_0)}{\sqrt{\theta(1+\theta)}r^{3/2}} \int_0^\infty e^{\frac{(1-\theta)x}{2}} u_0(x)dx < \infty,\end{aligned}$$

we deduce from dominated convergence Theorem

$$\int_{[r,\infty)} u(t,x)dx = \int_{[r,\infty)} u_0(x)dx + \int_0^t \int_{[r,\infty)} \int_{[0,r)} R(x,y)u(s,x)u(s,y)dydxds. \tag{2.5.27}$$

Taking Γ into account, we observe that

$$\int_{[r,\infty)} \int_{[0,r)} R(x,y)u(s,x)u(s,y)dydxds = \int_{[r,\gamma_2(r))} \int_{(\gamma_1(x),r)} R(x,y)u(s,x)u(s,y)dydxds \leq 0. \tag{2.5.28}$$

The goal is to show that the integral above is, indeed, strictly negative for all $s \in [0,t]$. By (2.5.26) and Proposition 2.5.4 (iii), there exists an open rectangle $G = G_1 \times G_2$ centered around (x_0, y_0) and contained in $\{(x,y) \in [0,\infty)^2 : x \in [r, \gamma_2(r)),\ y \in (\gamma_1(x), r)\}$ such that

$$\int_{G_i} u(t,x)dx > 0 \qquad \forall t > 0,\ i = 1,2.$$

We then obtain

$$\int_{[r,\gamma_2(r))} \int_{(\gamma_1(x),r)} R(x,y)u(s,x)u(s,y)dydxds \leq \max_{(x,y)\in G} R(x,y) \int_{G_1} u(t,x)dx \int_{G_2} u(t,y)dy < 0,$$

and the result then follows from (2.5.27) and (2.5.28). □

2.5.2 Global "regular" solutions.

We prove in this section Theorem 2.1.4 for initial data u_0 sufficiently flat around the origin. This condition on u_0 is sufficient to prevent the formation of a Dirac mass in finite time. We do not know if it is necessary. We prove first the following,

Proposition 2.5.7. *For all $v_0 \in L^1([0,\infty))$, $v_0 \geq 0$, satisfying (2.1.40) for some $\eta > (1-\theta)/2$, there exists a nonnegative global weak solution $v \in C([0,\infty), L^1([0,\infty)))$ of (2.5.1), (2.5.2) such that*

$$u(t,x) = u_0(x)e^{\int_0^t \int_0^\infty R(x,y)u(s,y)dyds} \quad \forall t > 0,\ a.e.\ x > 0, \tag{2.5.29}$$

and also satisfies $v(0) = v_0$, (2.1.42), (2.1.43).

Proof of Proposition 2.5.7. The proof has two steps.
Step 1. We consider first a compactly supported initial data, say $\operatorname{supp} u_0 \subset [0,L]$, $L > 0$. We first prove that the operator

$$A(f)(t,x) = u_0(x)e^{\int_0^t \int_0^\infty R(x,y)f(s,y)dyds} \tag{2.5.30}$$

is a contraction on $Y_{\rho,T}$ for some $\rho>0$ and $T>0$, where

$$\begin{gathered}Y_{\rho,T}=\left\{f\in C([0,T),L^1([0,\infty),\omega dx)):\ \|f\|_T\le\rho\right\},\\ \|f\|_T=\sup_{0\le t<T}\int_0^\infty\omega(x)|f(t,x)|dx=\sup_{0\le t<T}\|f(t)\|_\omega,\\ \omega(x)=(1+x^{-3/2}).\end{gathered}$$

Using (2.2.8)–(2.2.11), for all $(x,y)\in\Gamma$, $x\le L$,

$$|R(x,y)|\le\frac{C_*\rho_*e^{\frac{|x-y|}{2}}}{\sqrt{xy(x+y)}}\le\frac{C_*\rho_*}{\sqrt{\theta(1+\theta)}}\frac{e^{\frac{(1-\theta)x}{2\theta}}}{y^{3/2}}\le C_L\omega(y),$$

where $C_L=\frac{C_*\rho_*e^{\frac{(1-\theta)L}{2\theta}}}{\sqrt{\theta(1+\theta)}}$. Then, for all nonnegative $f\in Y_{\rho,T}$, $x\in[0,L]$, and $t\in[0,T)$,

$$\int_0^t\int_0^\infty|R(x,y)|f(s,y)dyds\le C_L\rho T,$$

and then

$$\begin{aligned}A(f)(t,x)&\le u_0(x)e^{C_L\rho T},\\ \|A(f)\|_T&\le\|u_0\|_\omega e^{C_L\rho T}.\end{aligned}\tag{2.5.31}$$

Notice that $\|u_0\|_\omega<\infty$ by the hypothesis (2.1.40). Let now t_1 and t_2 be such that $0\le t_1\le t_2<T$. Then, for all $x\in[0,L]$,

$$\begin{aligned}|A(f)(t_1,x)-A(f)(t_2,x)|&=\\ &=u_0(x)\left|e^{\int_0^{t_1}\int_0^\infty R(x,y)f(s,y)dyds}-e^{\int_0^{t_2}\int_0^\infty R(x,y)f(s,y)dyds}\right|\\ &\le u_0(x)e^{C_L\rho T}C_L\rho|t_1-t_2|,\end{aligned}$$

and therefore

$$\|A(f)(t_1)-A(f)(t_2)\|_\omega\le\|u_0\|_\omega e^{C_L\rho T}C_L\rho|t_1-t_2|,$$

from where it follows that $A\in C([0,T),L^1([0,\infty),\omega dx))$. On the other hand, if we chose $\rho=2\|u_0\|_\omega$ and $T>0$ such that $e^{C_L\rho T}\le 2$, we deduce from (2.5.31) that $\|A(f)\|_T\le\rho$, i.e., $A(f)\in Y_{\rho,T}$.

Let now f and g be in $Y_{\rho,T}$. By similar computations as before,

$$\|A(f)-A(g)\|_T\le\|u_0\|_\omega e^{C_L\rho T}C_LT\|f-g\|_T,$$

and if T is such that

$$\|u_0\|_\omega e^{C_L\rho T}C_LT<1,$$

then A is a contraction on $Y_{\rho,T}$, and has then a fixed point u that satisfies (2.5.29) for all $t\in(0,T)$ and $a.e.\,x>0$. It then follows in particular that $u\ge0$. Let us denote

$$T_{\max}=\sup\Big\{T>0;\exists\,\rho>0,\exists u\in Y_{\rho,T}\text{ satisfying (2.5.29)},\ \forall t\in[0,T)\Big\}.$$

We claim that if $T_{\max} < \infty$, then $\limsup_{t\to T_{\max}} \|u(t)\|_\omega = \infty$. Suppose that $T_{\max} < \infty$ and $\limsup_{t\to T_{\max}} \|u(t)\|_\omega = \ell < \infty$, and let $t_n \to T_{\max}$. For every $n \in \mathbb{N}$ we define $\rho_n = 2\|u(t_n)\|_\omega$, and the map

$$A_n(f)(t,x) = u(t_n,x)e^{\int_0^t\int_0^\infty R(x,y)f(s,y)dyds},$$

for $f \in C([0,T), L^1([0,\infty),\omega dx))$, $T > 0$. For every $T > 0$, $t \in [0,T)$, $x \in [0,L]$, and $f \in Y_{\rho_n,T}$,

$$A_n(f)(t,x) \le u(t_n,x)e^{C_L\rho_n T},$$
$$\|A_n(f)\|_T \le \|u(t_n)\|_\omega e^{C_L\rho_n T},$$

and for all T such that $T \le (\ln 2)/(C_L\rho_n) = T_n$, it follows that $A_n(f) \in Y_{\rho_n,T}$. Notice that by hypothesis, $\rho_n \le 2\ell$ for all n, and then

$$T_n \ge \frac{\ln 2}{C_L 2\ell} := \tau_1 \quad \forall n \in \mathbb{N}.$$

Now let f and g be in $Y_{\rho_n,T}$ for $T > 0$. Arguing as before

$$\|A_n(f) - A_n(g)\|_T \le \|u(t_n)\|_\omega e^{C_L 2\ell T} C_L T\|f - g\|_T,$$

and since

$$u(t_n,x) \le u_0(x)e^{C_L 2\ell T_{\max}}, \quad \forall n \in \mathbb{N},$$

then

$$\|A_n(f) - A_n(g)\|_T \le \|u_0\|_\omega e^{C_L 2\ell(T+T_{\max})} C_L T\|f - g\|_T.$$

If we chose $\tau_2 > 0$ such that

$$\|u_0\|_\omega e^{C_L 2\ell(\tau_2+T_{\max})} C_L\tau_2 < 1,$$

and we let $\tau_* = \min\{\tau_1,\tau_2\}$, then A_n is a contraction from Y_{ρ_n,τ_*} into itself, and has then a fixed point, say v_n. The function v_n satisfies

$$v_n(t,x) = u(t_n,x)e^{\int_0^t\int_0^\infty R(x,y)\, v_n(t,y)\, dy}, \quad \forall t \in [0,\tau_*), \text{ a.e. } x \in [0,\infty).$$

Therefore, the function w_n defined as

$$w_n(t,x) = \begin{cases} u(t,x) & \text{if } t \in [0,t_n) \\ v_n(t-t_n,x), & \text{if } t \in [t_n, t_n+\tau_*) \end{cases}$$

satisfies the integral equation:

$$w_n(t,x) = u_0(x)e^{\int_0^t\int_0^\infty R(x,y)w_n(s,y)dyds}, \quad \forall t \in [0, t_n+\tau_*).$$

Since $t_n \to T_{\max}$, then $t_n + \tau_* > T_{\max}$ for n large enough, and this contradicts the definition of $T_{\max}$. We deduce that, either $T_{\max} = \infty$, and the solution is said to be global, or $\limsup_{t\to T_{\max}} \|u(t)\|_\omega = \infty$ and the solution is said to blow up in finite time, at $T_{\max}$.

Since for all $T < T_{\max}$, $t \in [0,T]$ and $a.e.\, x \in [0,L]$,

$$\left|\frac{d}{dt}\left(u_0(x)\varphi(x)e^{\int_0^t\int_0^\infty R(x,y)u(s,y)dyds}\right)\right|$$
$$\leq u_0(x)|\varphi(x)|e^{C_LT\|u\|_T}C_LT\|u\|_T \quad \text{(integrable in } x\text{)},$$

we may then multiply both sides of the equation (2.5.29) by a function $\varphi \in C_b([0,\infty))$ and integrate on $[0,\infty)$:

$$\frac{d}{dt}\int_0^\infty u(t,x)\varphi(x)dx = \int_0^\infty u(t,x)\varphi(x)\left(\int_0^\infty R(x,y)u(t,y)dy\right)dx,$$

and since

$$|u(t,\cdot)u(t,\cdot)\varphi(t,\cdot)R(\cdot,\cdot)| \in L^1([0,\infty)\times[0,\infty)) \quad \forall t \in [0,T_{\max}),$$

by Fubini's Theorem and the antysimmetry of $R(x,y)$,

$$\begin{aligned}\frac{d}{dt}\int_0^\infty u(t,x)\varphi(x)dx &= \int_0^\infty\int_0^\infty \varphi(x)R(x,y)u(t,x)u(t,y)dxdy\\ &= \frac{1}{2}\int_0^\infty\int_0^\infty(\varphi(x)-\varphi(y))R(x,y)u(t,x)u(t,y)dxdy. \end{aligned} \tag{2.5.32}$$

This shows that u is a weak solution of (2.5.1), (2.5.2) . If $\varphi = 1$:

$$\frac{d}{dt}\int_0^\infty u(t,x)dx = 0,$$

and then $\|u(t)\|_1 = \|u_0\|_1$ for all $t > 0$. Then, since

$$\int_0^\infty R(x,y)u(s,y)dx \leq \frac{C_L\|u_0\|_1}{x^{3/2}},$$

we obtain from (2.5.29)

$$u(t,x) \leq u_0(x)e^{\frac{tC_L\|u_0\|_1}{x^{3/2}}},$$

and then

$$\|u(t)\|_\omega \leq \int_0^\infty \omega(x)e^{\frac{tC_L\|u_0\|_1}{x^{3/2}}}u_0(x)dx. \tag{2.5.33}$$

Notice that (2.1.40) implies

$$\forall r > 0, \qquad \int_0^1 u_0(x)\frac{e^{\frac{r}{x^{3/2}}}}{x^{3/2}}dx < \infty. \tag{2.5.34}$$

Indeed, if we write $x^{-3/2} = e^{-\frac{3}{2}\ln x}$, then for all $r > 0$,

$$\int_0^1 u_0(x)\frac{e^{\frac{r}{x^{3/2}}}}{x^{3/2}}dx = \int_0^1 u_0(x)e^{\frac{r}{x^{3/2}}-\frac{3}{2}\ln x}dx \leq \int_0^1 u_0(x)e^{\frac{r'}{x^{3/2}}}dx < \infty,$$

where

$$r' = r + e^{-1} = \max_{x\in[0,1]} \left(r - \frac{3}{2} x^{3/2} \ln x \right).$$

We then obtain from (2.5.33), (2.5.34) that

$$\|u(t)\|_\omega \le \int_0^\infty \omega(x) e^{\frac{tC_L\|u_0\|_1}{x^{3/2}}} u_0(x)dx < \infty \quad \forall t \in [0, T_{\max}), \tag{2.5.35}$$

therefore $\lim_{t\to T_{\max}} \|u(t)\|_\omega < \infty$ if $T_{\max} < \infty$, and then by the alternative, $T_{\max} = \infty$.

Step 2. For a general initial data u_0, let $u_{0,n}(x) = u_0(x)\mathbb{1}_{[0,n]}(x)$, and u_n be the weak solution constructed in Step 1 for the initial data $u_{0,n}$ that satisfies

$$u_n(t,x) = u_{0,n}(x) e^{\int_0^t \int_0^\infty R(x,y)u_n(s,y)dyds}, \tag{2.5.36}$$

and $\|u_n(t)\|_1 = \|u_{0,n}\|_1 \le \|u_0\|_1$ for all $t > 0$ and all $n \in \mathbb{N}$. Then, arguing as in the proof of Theroem 2.1.2, a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (not relabelled) converges to some $u \in C([0,\infty), \mathscr{M}_+([0,\infty)))$ in the space $C([0,\infty), \mathscr{M}_+([0,\infty)))$. On the other hand, since for all $n \in \mathbb{N}$,

$$\int_0^\infty R(x,y)u_n(s,y)dy \le \frac{C^*}{x^{3/2}} \int_x^\infty e^{\frac{(1-\theta)y}{2}} u_n(s,y)ds \le \frac{C_0}{x^{3/2}}, \tag{2.5.37}$$

where $C_0 = C^* \int_0^\infty e^{\eta y} u_0(y)dy$, it follows from (2.1.40) that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $E \subset [0,\infty)$ mesasurable with $|E| < \delta$,

$$\int_E u_n(t,x)dx \le \int_E u_0(x) e^{\frac{C_0 t}{x^{3/2}}} dx < \varepsilon \quad \forall n \in \mathbb{N},\ \forall t > 0. \tag{2.5.38}$$

Moreover, for all $\varepsilon > 0$ there exists $M > 0$ such that

$$\begin{aligned} \int_M^\infty u_n(t,x)dx &\le e^{-\eta M} \int_M^\infty e^{\eta x} u_n(t,x)dx \\ &\le e^{-\eta M} \int_0^\infty e^{\eta x} u_0(x)dx < \varepsilon \quad \forall n \in \mathbb{N},\ \forall t > 0. \end{aligned} \tag{2.5.39}$$

It then follows from (2.5.38)–(2.5.39) and Dunford-Pettis Theorem, that for all $t > 0$, a subsequence of $u_n(t)$ (not relabelled) converges to a function $U(t) \in L^1([0,\infty))$ in the weak topology $\sigma(L^1, L^\infty)$. Therefore we deduce that for all $t > 0$,

$$\int_0^\infty \varphi(x)U(t,x)dx = \int_{[0,\infty)} \varphi(x)u(t,x)dx \, \forall \varphi \in C_b([0,\infty)),$$

i.e., the measure $u(t)$ is absolutely continuous with respect to the Lebesgue measure, with density $U(t)$. With some abuse of notation we identify u and U. The goal now is to pass to the limit in (2.5.36) as $n \to \infty$. Since $R(x,\cdot) \in L^\infty([0,\infty))$ for a.e. $x > 0$ and all $t > 0$, and

$$\int_0^\infty |R(x,y)|u_n(s,y)dy \le \frac{C^*}{x^{3/2}} \left(\int_0^x e^{\frac{x-y}{2}} u_n(s,y)dy + \right. \tag{2.5.40}$$

$$\left. + \int_x^\infty e^{\frac{y-x}{2}} u_n(s,y)dy \right)$$

$$\le \frac{C^*}{x^{3/2}} \left(e^{\eta x}\|u_0\|_1 + \int_0^\infty e^{\eta y} u_0(y)dy \right),\ \forall n \in \mathbb{N}, \tag{2.5.41}$$

it follows by the weak convergence $u_n(t) \rightharpoonup u(t)$ and dominated convergence, that for all $t>0$, a.e. $x>0$,

$$\lim_{n\to\infty}\int_0^t\int_0^\infty R(x,y)u_n(s,y)dyds = \int_0^t\int_0^\infty R(x,y)u(s,y)dyds,$$

and then, using that $u_{0,n}\to u_0$ a.e., (2.5.37), and dominated convergence,

$$\begin{aligned}\lim_{n\to\infty}&\int_0^\infty u_{0,n}(x)e^{\int_0^t\int_0^\infty R(x,y)u_n(s,y)dyds}dx\\ &=\int_0^\infty u_0(x)e^{\int_0^t\int_0^\infty R(x,y)u(s,y)dyds}dx.\end{aligned}$$

Therefore, u satisfies (2.5.29) for all $t>0$ and $a.e.\, x>0$.

Arguing as in (2.5.37) we obtain (2.1.43), and arguing as in Step 1 we obtain (2.1.42).

We now claim that

$$u\in C([0,\infty),L^1((0,\infty))). \tag{2.5.42}$$

For all $T>0$, t_1 and t_2 with $0\le t_1\le t_2\le T$, we have by (2.5.40),

$$\begin{aligned}&\|u(t_1)-u(t_2)\|_1\\ &\le\int_0^\infty u_0(x)\left|e^{\int_0^{t_1}\int_0^\infty R(x,y)u(s,y)dyds}-e^{\int_0^{t_2}\int_0^\infty R(x,y)u(s,y)dyds}\right|dx\\ &\le\int_0^\infty u_0(x)e^{\frac{TC_0}{x^{3/2}}}\left(\int_{t_1}^{t_2}\int_0^\infty |R(x,y)|u(s,y)dyds\right)dx\\ &\le|t_1-t_2|\int_0^\infty u_0(x)e^{\frac{TC_0}{x^{3/2}}}\frac{C^*}{x^{3/2}}\left(e^{\eta x}\|u_0\|_1+\int_0^\infty e^{\eta y}u_0(y)dy\right)dx,\end{aligned}$$

and then (2.5.42) follows using (2.1.40). Arguing as in Step 1 we deduce that u is a weak solution of (2.5.1), (2.5.2). □

Proof of Theorem 2.1.4. Theorem 2.1.4 follows from Proposition 2.5.7 since the function $b(k,k')=\frac{\Phi\mathcal{B}_\beta}{kk'}$ satisfies (2.2.5)–(2.2.11). □

Remark 2.5.8. The same proof shows that Theorem 2.1.4 is still true for the equation

$$\frac{\partial v}{\partial t}(t,k)=v(t,k)\int_{[0,\infty)}v(t,k')\big(e^{-\beta k}-e^{-\beta k'}\big)\frac{\mathcal{B}_\beta(k,k')}{kk'}dk'. \tag{2.5.43}$$

where the redistribution function $\mathcal{B}_\beta$ is kept without truncation. This is possible because the property (2.1.40) is also propagated by the weak solutions of (2.5.43) such that

$$v(t,k)=v_0(k)e^{\int_0^t\int_0^\infty\left(e^{-\beta k}-e^{-\beta k'}\right)\frac{\mathcal{B}_\beta(k,k')}{kk'}v(s,k')\,dk'\,ds}. \tag{2.5.44}$$

Notice in particular that the integral term in the exponential is well defined when $v(t)$ satisfies (2.1.40).

Remark 2.5.9. Let u and v be two solutions of (2.1.39), with a compactly supported initial data $u_0 \in L^1([0,\infty))$ satisfying (2.1.40) and such that $\mathrm{supp}(u_0) \subset [0,L]$, $L>0$. It follows from the representation(2.5.29) that, for all $t>0$ and $a.e.\, x>0$,

$$|u(t,x)-v(t,x)| \leq u_0(x)e^{\frac{tC_L\|u_0\|_1}{x^{3/2}}}\frac{C_L}{x^{3/2}}\int_0^t\int_0^\infty |u(s,y-v(s,y)|dyds,$$

and then, by Gronwall's Lemma, $u=v$ for a.e. $t>0$ and a.e. $x>0$.

2.5.3 M_α as Lyapunov functional.

The goal of this Section is the study of the functionals $M_\alpha(u(t))$, defined in (2.1.24), and $D_\alpha(u(t))$, defined in (2.1.44), acting on the weak solutions of problem (2.5.1), and to prove, in particular, Theorem 2.1.6.

Let us start with the following simple lemma, that establishes a monotonicity property for the moments of a solution to (2.5.1).

Lemma 2.5.10. *Let u be the weak solution of (2.5.1) given by Theorem 2.5.1 for an initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (2.5.3). Then (2.5.5) holds for $\varphi(x)=x^\alpha$ for all $\alpha\geq 1$. Moreover, for all $t_0\geq 0$,*

$$M_\alpha(u(t)) \leq M_\alpha(u(t_0)) \qquad \forall t\geq t_0 \tag{2.5.45}$$

Proof. Let $\alpha\geq 1$ and $\varphi(x)=x^\alpha$. We first notice from (2.5.3) and (2.5.7) that $M_\alpha(u(t))<\infty$ for all $t\geq 0$. Then, consider an approximation $\{\varphi_k\}_{k\in\mathbb{N}}\subset C_b^1([0,\infty))$ such that φ_k is nondecreasing, $\varphi_k'(0)=0$, $\varphi_k'\leq\varphi'$ for all $k\in\mathbb{N}$, and $\varphi_k\to\varphi$ pointwise as $k\to\infty$. Using the definite sign of the right hand side of (2.5.5) for the test function φ_k, we obtain

$$\frac{d}{dt}\int_{[0,\infty)}\varphi_k(x)u(t,x)dx \leq 0 \qquad \forall t>0,\ \ \forall k\in\mathbb{N},$$

from where, for all $t_0\geq 0$, $\int_{[0,\infty)}\varphi_k(x)u(t,x)dx \leq \int_{[0,\infty)}\varphi_k(x)u(t_0,x)dx$ for all $t\geq t_0$ and all $k\in\mathbb{N}$, and then (2.5.45) follows from dominated convergence theorem, by letting $k\to\infty$.

Let us prove now that (2.5.5) holds for $\varphi(x)=x^\alpha$. From (2.5.5) for the test function φ_k,

$$\int_{[0,\infty)}\varphi_k(x)u(t,x)dx = \int_{[0,\infty)}\varphi_k(x)u_0(x)dx + \int_0^t\iint_{[0,\infty)^2}k_{\varphi_k}(x,y)u(s,x)u(s,y)dydxds. \tag{2.5.46}$$

Using that $\varphi_k'\leq\varphi'$ for all $k\in\mathbb{N}$, we obtain from (A.0.4) that for all $(x,y)\in\Gamma$,

$$|k_{\varphi_k}(x,y)| \leq C\alpha\max\{x,y\}^{\alpha-1}e^{\frac{|x-y|}{2}}, \quad C=\max\left\{\frac{(1-\theta)^2}{\theta\delta_*(1+\theta)},\rho_*\right\},$$

and, since $|x-y|\leq(1-\theta)\max\{x,y\}$ and $\max\{x,y\}\leq\theta^{-1}\min\{x,y\}$ for all $(x,y)\in$

Γ, we then deduce using also (2.5.7) and (2.5.45), that for all $t \geq 0$ and $k \in \mathbb{N}$,

$$\begin{aligned}
&\iint_{[0,\infty)^2} |k_{\varphi_k}(x,y)| u(t,x)u(t,y)dydx \\
&\leq \frac{C\alpha}{\theta^{\alpha-1}} \iint_{[0,\infty)^2} \min\{x,y\}^{\alpha-1} e^{\frac{(1-\theta)}{2}\max\{x,y\}} u(t,x)u(t,y)dydx \\
&\leq \frac{2C\alpha}{\theta^{\alpha-1}} \left(\int_{[0,\infty)} e^{\frac{(1-\theta)x}{2}} u(t,x)dx \right) \left(\int_{[0,\infty)} y^{\alpha-1} u(t,y)dy \right) \\
&\leq \frac{2C\alpha}{\theta^{\alpha-1}} \left(\int_{[0,\infty)} e^{\eta x} u_0(x)dx \right) \left(\int_{[0,\infty)} y^{\alpha-1} u_0(y)dy \right).
\end{aligned}$$

On the other hand, $k_{\varphi_k}(x,y) \to k_\varphi(x,y)$ for all $(x,y) \in [0,\infty)^2$ as $k \to \infty$. Passing to the limit as $k \to \infty$ in (2.5.46), it then follows from dominated convergence theorem that for all $t \geq 0$,

$$\begin{aligned}
\int_{[0,\infty)} \varphi(x)u(t,x)dx = &\int_{[0,\infty)} \varphi(x)u_0(x)dx \\
&+ \int_0^t \iint_{[0,\infty)^2} k_\varphi(x,y)u(s,x)u(s,y)dydxds,
\end{aligned} \tag{2.5.47}$$

and then (2.5.5) holds. □

If u is a weak solution to (2.5.1) given by Theorem 2.5.1, then by Lemma 2.5.10 the following identity holds,

$$\frac{d}{dt} M_\alpha(u(t)) = \frac{1}{2} D_\alpha(u(t)) \qquad \forall t > 0. \tag{2.5.48}$$

Since $D_\alpha(u(t)) \leq 0$ for all $t > 0$, this shows that M_α is a Lyapunov functional on these solutions. The identity (2.5.48) is reminiscent of the usual entropy - dissipation of entropy identity.

As already observed in the Introduction, since the support of the function B is contained in the region $\Gamma \subset [0,\infty)^2$, if $a > 0$ and $b > 0$ are such that $(a,b) \notin \Gamma$ (they do not see each other) then, for all $\varphi \in C_b^1([0,\infty))$ such that $\varphi'(0) = 0$,

$$\iint_{[0,\infty)^2} \delta(x-a)\delta(y-b)R(x,y)(\varphi(x) - \varphi(y))dxdy = 0.$$

Let us then see some of the consequences of this simple observation.

Definition 2.5.11. We say that two points a and c on $[0,\infty)$ are Γ-disjoint if $(a,c) \notin \Gamma$. We say that two sets A and C on $[0,\infty)$ are Γ-disjoint if for all $(a,c) \in A \times C$, $(a,c) \notin \Gamma$, i.e., if $A \times C \subset [0,\infty)^2 \setminus \mathring{\Gamma}$.

Since the support of any given measure $u \in \mathscr{M}_+([0,\infty))$ is, by definition, a closed subset of $[0,\infty)$, then

$$(\text{supp}(u))^c = \bigcup_{k=0}^{\infty} I_k, \;\; I_k \text{ open interval }, \; I_k \cap I_j = \emptyset \;\text{ if }\; k \neq j. \tag{2.5.49}$$

We may write $I_k = (a_k, b_k)$ for $0 \leq a_k < b_k$ for all $k \in \mathbb{N}$, except if $\operatorname{supp} u \subset [r, \infty)$, $r > 0$, for which $I_k = [0 = a_k, b_k)$ for some k. We now define

$$\mathcal{I} = \{I_k : \gamma_1(b_k) \geq a_k\},$$

and denote $\{C_k\}_{k\in\mathcal{J}}$ the connected components of $\left(\bigcup_{I\in\mathcal{I}} I\right)^c$. Notice that, in general, $\mathcal{J}$ could be uncountable. Finally define, for all $u \in \mathscr{M}_+([0,\infty))$

$$A_k(u) = C_k \cap \operatorname{supp}(u), \ \ \forall k \in \mathcal{J}. \tag{2.5.50}$$

Notice by (2.5.50) that $A_k(u)$ is a closed subset of $[0, \infty)$ for all $k \in \mathbb{N}$, since it is the intersection of two closed sets.

We write $A_k(u) = A_k$ when no confusion is possible.

Lemma 2.5.12. *$\mathcal{J}$ is a countable set.*

Proof. Given two elements of $\mathcal{I}$, there is at most a finite number of elements of $\mathcal{I}$ between them. More precisely, we claim that, for any given $I_i \in \mathcal{I}$, $I_j \in \mathcal{I}$, with $I_i = (a_i, b_i)$, $I_j = (a_j, b_j)$, $0 < b_i \leq a_j$, then: $\operatorname{card}(\{I_k = (a_k, b_k) \in \mathcal{I} : b_i \leq a_k < b_k \leq a_j\}) < \infty$. The proof of this fact start with this trivial remark: if $I_k \in \mathcal{I}$, then $|I_k| = b_k - a_k \geq b_k - \gamma_1(b_k)$. Using that, if we consider the decreasing sequence b_j, $\gamma_1(b_j)$, $\gamma_1^2(b_j) = \gamma_1(\gamma_1(b_j))$, $\gamma_1^3(b_j)$,..., then $\gamma_1^m(b_j) < b_i$ for some integer m, and therefore there could be only m elements of $\mathcal{I}$ between I_i and I_j.

For the sake of the argument, let us say that given two elements $I_i = (a_i, b_i)$ and $I_j = (a_j, b_j)$ of $\mathcal{I}$, there are 2 more elements $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ of $\mathcal{I}$ between them, i.e.,

$$a_i < b_i \leq a_1 < b_1 \leq a_2 < b_2 \leq a_j < bj.$$

Then, there are 3 connected components in $(a_i, b_j) \setminus \left(I_i \cup I_1 \cup I_2 \cup I_j\right)$, namely $[b_i, a_1]$, $[b_1, a_2]$ and $[b_2, a_j]$. With this idea, it can be proved that the number of connected components of $[0,\infty) \setminus \left(\bigcup_{I\in\mathcal{I}} I\right)$, i.e., the collection $\{C_k\}_{k\in\mathcal{J}}$, is at most countable. □

We prove now several useful properties of the collection $\{A_k\}_{k\in\mathbb{N}}$.

Lemma 2.5.13. *Let $u \in \mathscr{M}_+([0,\infty))$ and consider the collection $\{A_k\}_{k\in\mathbb{N}}$ constructed above. Then A_i and A_j are Γ-disjoint if and only if $i \neq j$, and*

$$\operatorname{supp}(u) = \bigcup_{k=0}^{\infty} A_k. \tag{2.5.51}$$

Proof. It is clear that A_i and A_i are not Γ-disjoint, since $A_i \times A_i$ contains points on the diagonal, and therefore on $\mathring{\Gamma}$. Now, if $i \neq j$, we first observe that A_i and A_j are disjoint. Indeed, by definition $A_i \subset C_i$ and $A_j \subset C_j$, where C_i and C_j are different connected components of $[0,\infty) \setminus \left(\bigcup_{I\in\mathcal{I}} I\right)$, therefore disjoint. We now prove that A_i and A_j are in fact Γ-disjoint. Let us assume that A_i is on the left of A_j, i.e., $\sup A_i < \inf A_j$. It follows from the construction that there exists at least one $I_k = (a_k, b_k) \in \mathcal{I}$ between A_i and A_j, i.e., such that

$$\sup A_i \leq a_k < b_k \leq \inf A_j.$$

By definition of $\mathcal{I}$, the points a_k and b_k are Γ-disjoint, and then, for all $(a_i, a_j) \in A_i \times A_j$,

$$\gamma_1(a_j) \geq \gamma_1(b_k) \geq a_k \geq a_i,$$

hence a_i and a_j are Γ-disjoint. Finally, (2.5.51) follows from the construction. Indeed, since by definition $A_k = C_k \cap \operatorname{supp}(u)$, then $\cup_{k\in\mathbb{N}} A_k \subset \operatorname{supp} u$. On the other hand, by definition $\cup_{k\in\mathbb{N}} C_k = [0,\infty) \setminus \big(\cup_{I\in\mathcal{I}} I\big)$, and then by (2.5.49)

$$\operatorname{supp}(u) = \bigcap_{k\in\mathbb{N}} I_k^c \subset \bigcup_{k\in\mathbb{N}} C_k,$$

from where the inclusion $\operatorname{supp}(u) \subset \cup_{k\in\mathbb{N}} A_k$ follows. □

In the remaining part of the section we will use several times the following simple remark.

Remark 2.5.14. Consider the function $z(x) = x - \gamma_1(x)$, $x \geq 0$, where γ_1 is given by (2.2.13) in Remark 2.2.1. Then, z is a continuous and strictly increasing function on $[0,\infty)$, with $z(0) = 0$.

In the next Lemma we prove that any two sets A_i and A_j of the collection $\{A_k\}_{k\in\mathbb{N}}$ are separated from each other by a positive distance, given by the function $z(x)$ of Remark 2.5.14 .

Lemma 2.5.15. *Let $u \in \mathscr{M}_+([0,\infty))$ and consider the collection $\mathcal{A} = \{A_k\}_{k\in\mathbb{N}}$ constructed above. Suppose that* $\operatorname{card}(\mathcal{A}) \geq 2$. *For any $k \in \mathbb{N}$, let us denote $x_k =$* $\min A_k$ *and $y_k =$* $\sup A_k$. *Given two elements A_i, A_j in $\mathcal{A}$, suppose that $y_i < x_j$. Then,*

$$\operatorname{dist}(A_i, A_j) \geq x_j - \gamma_1(x_j) > 0. \tag{2.5.52}$$

Moreover, for every $\varepsilon > 0$, let

$$\mathcal{A}_\varepsilon = \{A_k \in \mathcal{A} : A_k \subset (\varepsilon, \infty)\}. \tag{2.5.53}$$

If $\mathcal{A}_\varepsilon \neq \emptyset$ and $\operatorname{card}(\mathcal{A}_\varepsilon) \geq 2$, *then*

$$\operatorname{dist}(A_i, A_j) > \varepsilon - \gamma_1(\varepsilon) > 0 \qquad \forall A_i, A_j \in \mathcal{A}_\varepsilon,\ i \neq j. \tag{2.5.54}$$

Proof. Since A_i and A_j are closed sets and $y_i < x_j$, it follows that $\operatorname{dist}(A_i, A_j) = x_j - y_i$. By Lemma 2.5.13, the closed sets A_i and A_j are Γ-disjoint and then, by Definition 2.5.11,

$$y_i \leq \gamma_1(x_j).$$

Therefore $\operatorname{dist}(A_i, A_j) \geq x_j - \gamma_1(x_j)$ and, since $x_j > 0$, (2.5.52) follows from Remark 2.5.14.

Let now $\varepsilon > 0$ be fixed and consider A_i and A_j in $\mathcal{A}_\varepsilon$. Without loss of generality, we may assume that $y_i < x_j$. Using Remark 2.5.14, it then follows from (2.5.52) and (2.5.53) that

$$\operatorname{dist}(A_i, A_j) \geq z(x_j) > z(\varepsilon) > 0.$$

□

Lemma 2.5.16. *Let u be the weak solution of (2.5.1) constructed in Theorem 2.5.1 for an initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (2.5.3), and consider the collection $\mathcal{A} = \{A_k(u_0)\}_{k\in\mathbb{N}}$ constructed above. Then*

$$\int_{A_k} u(t,x)dx = \int_{A_k} u_0(x)dx \qquad \forall t>0,\ \forall k\in\mathbb{N}. \tag{2.5.55}$$

Proof. In the trivial case $A_k = \mathrm{supp}(u_0)$ for all $k\in\mathbb{N}$, then (2.5.55) is just the conservation of mass (2.5.6). Suppose then that $\mathrm{card}(\mathcal{A}) \geq 2$. We consider separately two different cases.

(i) Suppose that there exists $\varepsilon > 0$ such that $[0,\varepsilon] \subset \mathrm{supp}(u_0)$. Then, since $[0,\varepsilon]$ can not intersect A_k for two different values of k, there exists $k_0 \in \mathbb{N}$ such that $[0,\varepsilon] \subset A_{k_0}$. In particular $A_{k_0} \notin \mathcal{A}_\varepsilon$. Let us see that

$$\mathcal{A} = \mathcal{A}_\varepsilon \cup \{A_{k_0}\}. \tag{2.5.56}$$

Arguing by contradiction, suppose that for some $\ell \neq k_0$ we have $A_\ell \in \mathcal{A}\setminus\mathcal{A}_\varepsilon$. Since $[0,\varepsilon] \subset A_{k_0}$ and $A_{k_0}\cap A_\ell = \emptyset$, then $x_\ell = \min A_\ell > \varepsilon$. Therefore $A_\ell \in \mathcal{A}_\varepsilon$, which is a contradiction.

We wish now to estimate from below the distances $\mathrm{dist}(A_i, A_j)$ for all $A_i \in \mathcal{A}$, $A_j \in \mathcal{A}$, $i \neq j$. By (2.5.54) and (2.5.56),

$$\mathrm{dist}(A_i, A_j) > \varepsilon - \gamma_1(\varepsilon) > 0 \qquad \forall i\neq k_0, \forall j \neq k_0, i\neq j. \tag{2.5.57}$$

On the other hand, for all $i \neq k_0$, $x_i = \min A_i > \varepsilon$ by (2.5.56) and then, by (2.5.52) and Remark 2.5.14,

$$\mathrm{dist}(A_i, A_{k_0}) \geq x_i - \gamma_1(x_i) = z(x_i) > z(\varepsilon). \tag{2.5.58}$$

By (2.5.56), (2.5.57) and (2.5.58) we have then:

$$\mathrm{dist}(A_i, A_j) > z(\varepsilon) > 0,\ \forall A_j\in\mathcal{A}, \forall A_i \in \mathcal{A}. \tag{2.5.59}$$

For any fixed $k\in\mathbb{N}$, we now claim that, since A_i is closed for every $i\in\mathbb{N}$, by (2.5.59) the set

$$D_k = \bigcup_{i\in\mathbb{N}, i\neq k} A_i \tag{2.5.60}$$

is a closed subset of $[0,\infty)$. In order to prove that property, let us assume, by contradiction, that there exists a point $x_* \in \overline{D}_k \setminus D_k$. Let $\{x_n\}_{n\in\mathbb{N}} \subset D_k$ be a sequence such that converges to x_*. In particular $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore, by (2.5.59), there exists $k_* \in \mathbb{N}\setminus\{k\}$ such that, for some n_* sufficiently large:

$$x_n \in A_{k_*},\ \forall n \geq n_*.$$

Since A_{k_*} is a closed set, it follows that $x_* \in A_{k_*} \subset D_k$, and this is a contradiction.

By (2.5.59), D_k and A_k are disjoint subsets of $[0,\infty)$. Therefore, by Urysohn's lemma, there exists a function $\varphi \in C_b([0,\infty))$ such that

$$\varphi(x) = 1 \quad \forall x\in A_k \quad \text{and} \quad \varphi(x) = 0 \quad \forall x \in D_k.$$

Using (2.5.59) and a density argument, we may assume that $\varphi \in C_b^1([0,\infty))$. Then, since $\mathrm{supp}(u(t)) = \mathrm{supp}(u_0)$ (cf. Proposition 2.5.4 (iii)), it follows from (2.5.51)

$$\int_{[0,\infty)} \varphi(x)u(t,x)dx = \int_{A_k} u(t,x)dx,$$

and since A_i and A_j are Γ-disjoint for $i \neq j$ (cf. Lemma 2.5.13), then by construction of φ,

$$\begin{aligned}\iint_{[0,\infty)^2} & R(x,y)(\varphi(x)-\varphi(y))u(t,x)u(t,y)dydx \\ &= \sum_{i=0}^{\infty} \iint_{A_i\times A_i} R(x,y)(\varphi(x)-\varphi(y))u(t,x)u(t,y)dydx \\ &= \iint_{A_k\times A_k} R(x,y)(\varphi(x)-\varphi(y))u(t,x)u(t,y)dydx = 0,\end{aligned}$$

from where (2.5.55) follows by the weak formulation.

(ii) Suppose that the assumption of part (i) does not hold. In this case, there exists a strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to 0$ as $n \to \infty$ such that $x_n \notin \mathrm{supp}(u_0)$ for all $n \in \mathbb{N}$. Moreover, since $\mathrm{supp}(u_0)$ is a closed set, for each $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that

$$(x_n - \delta_n, x_n + \delta_n) \subset (\mathrm{supp}(u_0))^c.$$

For every $n \in \mathbb{N}$ and $k \in \mathbb{N}$ fixed such that

$$A_k \in \mathcal{A}_{x_n}, \tag{2.5.61}$$

where $\mathcal{A}_{x_n}$ is defined in (2.5.53), we consider the set:

$$D_{k,n} = \bigcup_{\substack{A_i\in\mathcal{A}_{x_n} \\ A_i\neq A_k}} A_i$$

Using now (2.5.54) for $\varepsilon = x_n$ we deduce that $D_{k,n}$ is a closed set by the same argument as for D_k in (2.5.60). By Urysohn's lemma again, we can then construct a test function $\varphi \in C_b^1([0,\infty))$ such that

$$\varphi(x) = 1 \quad \forall x \in A_k \qquad \text{and} \qquad \varphi(x) = 0 \quad \forall x \in [0,x_n] \cup D_{k,n}.$$

Arguing as in part (i), we then deduce that

$$\int_{A_k} u(t,x)dx = \int_{A_k} u_0(x)dx \qquad \forall t > 0,\ \forall A_k \in \mathcal{A}_{x_n}. \tag{2.5.62}$$

We use now that

$$\mathcal{A} = \left(\bigcup_{n\in\mathbb{N}} \mathcal{A}_{x_n}\right) \bigcup \left\{A_i \in \mathcal{A} : A_i \not\subset (0,\infty)\right\}$$

because

$$\bigcup_{n\in\mathbb{N}} \mathcal{A}_{x_n} = \{A_j \in \mathcal{A} : A_j \subset (0,\infty)\}.$$

But, if $A_i \not\subset (0,\infty)$, then $0 \in A_i$. Therefore, if $0 \notin \operatorname{supp}(u_0)$ there is no such A_i. If $0 \in \operatorname{supp}(u_0)$, since the sets A_k are pairwise disjoint, such subset A_i is unique. It follows that there exists at most a unique $k_0 \in \mathbb{N}$ such that:

$$\mathcal{A} = \Big(\bigcup_{n\in\mathbb{N}} \mathcal{A}_{x_n}\Big) \bigcup \Big\{A_{k_0}\Big\}. \tag{2.5.63}$$

The equality (2.5.55) then follows from (2.5.62), (2.5.63) and the conservation of mass (2.5.6). □

We may prove now the main result of this Section.

Proof of Theorem 2.1.6. Let us prove $(i) \implies (iii)$. Suppose that $D_\alpha(u) = 0$, and let, for $\varepsilon > 0$,

$$\Gamma_\varepsilon = \{(x,y) \in \Gamma : d((x,y), \partial\Gamma) > \varepsilon,\ |x-y| > \varepsilon\}.$$

Since $b(x,y)(e^{-x} - e^{-y})(x^\alpha - y^\alpha) < 0$ for all $(x,y) \in \Gamma_\varepsilon$, it follows from (i) that $\operatorname{supp}(u \times u) \subset \Gamma_\varepsilon^c$. Letting $\varepsilon \to 0$, we deduce that

$$\operatorname{supp}(u \times u) \subset \Delta \cup (\mathring{\Gamma})^c, \qquad \Delta = \{(x,x) : x \geq 0\}. \tag{2.5.64}$$

Notice that any two points $y < x$ in the support of u have to be at distance, namely, $x - y \geq x - \gamma_1(x)$. Otherwise $\gamma_1(x) < y$ and then $(x,y) \in \mathring{\Gamma} \setminus \Delta$, in contradiction with (2.5.64). Moreover, since the map $z(x) = x - \gamma_1(x)$ is continuous and strictly increasing on $[0,\infty)$, with $z(0) = 0$, it follows that the support of u consists, at most, on a countable number of points, where the only possible accumulation point is $x = 0$. Therefore $A_k = \{x_k\}$ for all $k \in \mathbb{N}$, and then (iii) holds.

Let us prove $(iii) \implies (i)$. If u is as in (iii), then $\operatorname{supp}(u \times u) = \{(x_i, x_j) : i,j \in \mathbb{N}\}$, and then

$$D_\alpha(u) = \sum_{i \leq j} \chi(i,j)\alpha_i\alpha_j b(x_i,x_j)(e^{-x_i} - e^{-x_j})(x_i^\alpha - x_j^\alpha) = 0,$$

where $\chi(i,j) = 2$ if $i \neq j$ and $\chi(i,j) = 1$ if $i = j$. Indeed, the terms with $i = j$ vanish due to the factor $(e^{-x_i} - e^{-x_j})(x_i^\alpha - x_j^\alpha)$, and for those terms with $i \neq j$, then $b(x_i, x_j) = 0$ since $(x_i, x_j) \notin \Gamma$.

We now prove $(iii) \implies (ii)$. Using (2.5.51) in Lemma 2.5.13 and the definition of x_k, for any $v \in \mathcal{F}$,

$$M_\alpha(v) = \sum_{k=0}^{\infty} \int_{A_k} x^\alpha v(x)dx \geq \sum_{k=0}^{\infty} x_k^\alpha m_k = M_\alpha(u),$$

and since $u \in \mathcal{F}$, u is indeed the minimizer of M_α.

We finally prove $(ii) \implies (iii)$. Let u be a minimizer of M_α and let $v = \sum_{k=0}^{\infty} m_k \delta_{x_k}$. We already know by the previous case that v is also a minimizer of M_α, hence $M_\alpha(u) = M_\alpha(v)$. Since moreover

$$M_\alpha(v) = \sum_{k=0}^{\infty} x_k^\alpha m_k = \sum_{k=0}^{\infty} x_k^\alpha \int_{A_k} u(x)dx,$$

it follows that

$$\sum_{k=0}^{\infty} \int_{A_k} (x^\alpha - x_k^\alpha) u(x) dx = 0.$$

By definition of x_k, all the terms in the sum above are nonnegative, and therefore

$$\int_{A_k} (x^\alpha - x_k^\alpha) u(x) dx = 0 \qquad \forall k \in \mathbb{N},$$

which implies that $A_k = \{x_k\}$ for all $k \in \mathbb{N}$, and therefore $u = v$. □

2.5.4 Long time behavior.

This Section is devoted to he proof of Theorem 2.1.8, that we have divided in several steps. For a given increasing sequence $t_n \to \infty$ as $n \to \infty$, let us define

$$u_n(t) = u(t + t_n), \qquad t \geq 0,\ n \in \mathbb{N}, \tag{2.5.65}$$

where u is the weak solution of (2.5.1) constructed in Theorem 2.5.1 for an initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (2.5.3). We first notice by (2.5.48) and Lemma 2.5.5 that for all $\alpha > 1$ and $t > 0$,

$$\frac{1}{2} \int_0^t |D_\alpha(u(s))| ds = M_\alpha(u_0) - M_\alpha(u(t)) \leq M_\alpha(u_0),$$

so by letting $t \to \infty$ we deduce $D_\alpha(u) \in L^1([0,\infty))$. Since moreover

$$\int_0^t D_\alpha(u_n(s)) ds = \int_{t_n}^{t_n + t} D_\alpha(u(s)) ds, \qquad \forall t \geq 0,$$

it follows that

$$\lim_{n \to \infty} \int_0^t D_\alpha(u_n(s)) ds = 0 \qquad \forall t \geq 0. \tag{2.5.66}$$

Proposition 2.5.17. *Let u be the weak solution of (2.5.1) constructed in Theorem 2.5.1 for an initial data $u_0 \in \mathscr{M}_+([0,\infty))$ satisfying (2.5.3). For every sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$, there exist a subsequence, still denoted $\{t_n\}_{n\in\mathbb{N}}$, and*

$$U \in C([0,\infty), \mathscr{M}_+([0,\infty))) \tag{2.5.67}$$

such that for all $\varphi \in C([0,\infty))$ satisfying (2.2.43), and all $t > 0$,

$$\lim_{n \to \infty} \int_{[0,\infty)} \varphi(x) u(t + t_n, x) dx = \int_{[0,\infty)} \varphi(x) U(t,x) dx. \tag{2.5.68}$$

Moreover, U is a weak solution of (2.5.1) such that $M_0(U(t)) = M_0(u_0)$ for all $t > 0$.

Proof. The proof is the same as the first part of the proof of Theorem 2.1.2 for equation (2.2.1). □

Lemma 2.5.18. *Let u, u_0 and U be as in Proposition 2.5.17. Then*

$$\operatorname{supp}(U(t)) = \operatorname{supp}(U(0)) \subset \operatorname{supp}(u_0) \qquad \forall t \geq 0. \tag{2.5.69}$$

Proof. On the one hand, since U is a weak solution of (2.5.1), then by Proposition 2.5.4 it follows that $\operatorname{supp}(U(t)) = \operatorname{supp}(U(0))$ for all $t > 0$, where $U(0)$ is given by (2.5.68) for $t = 0$. On the other hand, again by Proposition 2.5.4 we have, in particular, that $\operatorname{supp}(u_n(0)) = \operatorname{supp}(u_0)$ for all $n \in \mathbb{N}$. The result then follows from the convergence of $u_n(0)$ towards $U(0)$ in the sense of (2.5.68). Indeed, let $x_0 \in \operatorname{supp} U(0)$. We use the characterization of the support of a measure given in the proof of part (iii) of Proposition 2.5.4. Then

$$\rho_\varphi = \int_{[0,\infty)} \varphi(x) U(0,x) dx > 0,$$

for all $\varphi \in C_c([0,\infty))$ such that $0 \le \varphi \le 1$ and $\varphi(x_0) > 0$. Using then (2.5.68) for $t = 0$, we deduce that for all φ as before, there exists $n_* \in \mathbb{N}$, such that

$$\int_{[0,\infty)} \varphi(x) u_n(0,x) dx \ge \frac{\rho_\varphi}{2} > 0 \qquad \forall n \ge n_*,$$

and then $x_0 \in \operatorname{supp}(u_n(0)) = \operatorname{supp}(u_0)$. □

A partial identification of the limit U is given in our next Proposition.

Proposition 2.5.19. *Let u, u_0 and U be as in Proposition 2.5.17. Then*

$$U(t) = \mu \qquad \forall t \ge 0, \tag{2.5.70}$$

where μ is the measure defined in (2.1.46).

Proof. We first prove that $D_\alpha(U(t)) = 0$ for a.e. $t > 0$ and for all $\alpha > 1$. Indeed, if we define as in proof of Theorem 2.1.2, $u_n(t) = u(t + t_n)$, we deduce by the same arguments

$$\lim_{n\to\infty} \int_0^t D_\alpha(u_n(s)) ds = \int_0^t D_\alpha(U(s)) ds = 0 \qquad \forall t \ge 0,$$

hence $D_\alpha(U(t)) = 0$ for a.e. $t \ge 0$.
Then by Theorem 2.1.6 , there exist $m_j(t) \ge 0$, $x_j(t) \ge 0$ such that,

$$U(t) = \sum_{j=0}^{\infty} m_j(t) \delta_{x_j(t)}, \tag{2.5.71}$$

$$x_i(t),\ x_j(t) \quad \text{are } \Gamma\text{-disjoint } \forall i \ne j. \tag{2.5.72}$$

By (2.5.69) in Proposition 2.5.18,

$$x_j(t) = x_j(0) := x'_j \in \operatorname{supp}(u_0) \qquad \forall t \ge 0,\ \forall j \in \mathbb{N}. \tag{2.5.73}$$

Furthermore, since by Proposition 2.5.17, U is a weak solution of (2.5.1), it follows from Lemma 2.5.16 that for all $t \ge 0$, $j \in \mathbb{N}$,

$$m_j(t) = \int_{\{x_j(t)\}} U(t,x) dx = \int_{\{x_j(0)\}} U(0,x) dx = m_j(0) := m'_j,$$

and then by (2.5.71) we conclude that U is independent of t

Let us prove now that U satisfies Properties 1-4. Properties 1 and 2 are already proved in (2.5.72) and (2.5.73). In order to prove 3, let $k \in \mathbb{N}$ and $\varphi \in C_c^1([0,\infty))$ be such that $\varphi(x) = 1$ for all $x \in A_k$ and $\varphi(x) = 0$ for all $x \in \cup_{i \neq k} A_i$. This construction is possible by Uryshon's Lemma. Then, by (2.5.55) in Lemma 2.5.16,

$$\int_{[0,\infty)} \varphi(x) u_n(t,x)dx = \int_{A_k} u_n(t,x)dx = m_k, \qquad \forall n \in \mathbb{N},$$

and then by (2.5.68) in Proposition 2.5.17,

$$\int_{[0,\infty)} \varphi(x) U(x)dx = m_k.$$

Since $\mathrm{supp}(U) \subset \mathrm{supp}(u_0)$ by Lemma 2.5.18, we then deduce

$$\int_{[0,\infty)} \varphi(x) U(x)dx = \int_{A_k} U(x)dx = \sum_{j \in \mathcal{J}_k} m'_j,$$

and thus Property 3 holds.

Let us prove Property 4. Let $k \in \mathbb{N}$ and suppose that $x_k = \min\{x \in A_k\} > 0$. By (2.1.47) in Property 3, the set $\mathcal{J}_k$ in non empty. Let then $x'_j \in \mathcal{J}_k$. If $x'_j = x_k$, there is nothing left to prove. Suppose then $x'_j \neq x_k$, which by definition of x_k implies $x'_j > x_k$. We first notice that between x_k and x'_j, there can only be a finite number of elements in $\mathcal{J}_k$. This is because $x_k > 0$ and the points in $\mathcal{J}_k$ are pairwise Γ-disjoint, thus, the only possible accumulation point for any sequence in $\mathcal{J}_k$ is $x = 0$. Consequently, the point $x'_{j_0} = \min\{x \in \mathcal{J}_k\}$ is well define. Again, if $x'_{j_0} = x_k$, there is nothing left to prove. Suppose then $x'_{j_0} > x_k$, and let $0 < \varepsilon < (x'_{j_0} - x_k)/2$. On the one hand,

$$\int_{[x_k, x'_{j_0} - \varepsilon]} U(x)dx = 0. \tag{2.5.74}$$

On the other hand, let us show that the integral in (2.5.74) is strictly positive, which will be a contradiction. Since $A_k \subset \mathrm{supp}(u_0)$, in particular

$$\delta = \int_{[x_k, x'_{j_0} - \varepsilon)} u_0(x)dx > 0, \quad \text{and} \quad \int_{A_k \cap [x'_{j_0} - \varepsilon, \infty)} u_0(x)dx > 0,$$

and then by Proposition 2.5.5,

$$\int_{[0, x'_{j_0} - \varepsilon)} u(t,x)dx > \int_{[0, x'_{j_0} - \varepsilon)} u_0(x)dx \qquad \forall t > 0.$$

We now deduce from Lemma 2.5.16 that

$$\int_{\{x < x_k\}} u(t,x)dx = \int_{\{x < x_k\}} u_0(x)dx \qquad \forall t > 0,$$

and then we obtain

$$\int_{[x_k, x'_{j_0} - \varepsilon)} u(t,x)dx > \int_{[x_k, x'_{j_0} - \varepsilon)} u_0(x)dx = \delta \qquad \forall t > 0.$$

It then follows from (2.5.68) that

$$\int_{[x_k, x'_{j_0}-\varepsilon]} U(x)dx \geq \limsup_{n\to\infty} \int_{[x_k, x'_{j_0}-\varepsilon]} u_n(t,x)dx > \delta > 0,$$

in contradiction with (2.5.74). □

Proof of Theorem 2.1.8. By Proposition 2.5.17, Lemma 2.5.18, and Proposition 2.5.19, there exists a sequence, $\{t_n\}_{n\in\mathbb{N}}$ such that, if $u_n(t) = u(t+t_n)$ for all $t > 0$ and $n \in \mathbb{N}$, then u_n converges in $C([0,\infty), \mathscr{M}_+([0,\infty)))$ to the measure μ defined in (2.1.46).

Let us assume that for some other sequence $\{s_m\}_{m\in\mathbb{N}}$, the sequence $\omega_m(t) = u(t+s_m)$ is such that ω_m converges in $C([0,\infty), \mathscr{M}_+([0,\infty)))$ to a measure $W \in C([0,\infty), \mathscr{M}_+([0,\infty)))$.

Arguing as before, there exists a subsequence of $\{\omega_m\}_{m\in\mathbb{N}}$, still denoted $\{\omega_m\}_{m\in\mathbb{N}}$, such that, $\{\omega_m(t)\}_{m\in\mathbb{N}}$ converges narrowly to a measure $W \in \mathscr{M}_+([0,\infty))$ for every $t \geq 0$ as $m \to \infty$. Moreover, the limit W is of the form

$$W = \sum_{j=0}^{\infty} c_j \delta_{y_j},$$

and satisfies the properties 1-4 in Theorem 2.1.8. We claim that $W = U$.

By Point (i) of Proposition 2.5.5, for any $x \geq 0$, the map $t \mapsto \int_{[x,\infty)} u(t,y)dy$ is monotone nonincreasing on $[0,\infty)$. Therefore the following limit exists:

$$F(x) = \lim_{t\to\infty} \int_{[x,\infty)} u(t,y)dy, \quad x \geq 0.$$

From,

$$\int_{[x,\infty)} u(t,y)dy \geq \int_{[x,\infty)} \omega_m(t,y)dy, \quad \forall m \in \mathbb{N},$$

we first deduce that,

$$\int_{[x,\infty)} u(t,y)dy \geq \int_{[x,\infty)} W(y)dy.$$

On the other hand, it follows from the narrow convergence,

$$\int_{[x,\infty)} W(y)dy \geq \limsup_{m\to\infty} \int_{[x,\infty)} w_m(t,y)dy,$$

and then

$$F(x) = \int_{[x,\infty)} W(y)dy.$$

The same argument yields

$$F(x) = \int_{[x,\infty)} \mu(y)dy,$$

and then, using that $M_0(W) = M_0(u_0) = M_0(\mu)$, it follows that W and μ have the same (cumulative) distribution function, and therefore $W = \mu$ (cf. [48], Example 1.44, pg.20). □

We describe in the following example the behavior of a particularly simple solution of the reduced equation for which, although the sequence $\{A_k(u_0)\}_{k\in\mathbb{N}}$ has only one element, the asymptotic limit μ has two Dirac measures.

Example 1. Let $0 < a < b < c$ be such that $B(a,b) > 0$, $B(b,c) > 0$, and $B(a,c) = 0$, and let $x_0 > 0$, $y_0 > 0$ and $z_0 > 0$ be such that $x_0 + y_0 + z_0 = 1$. If we define

$$u_0 = x_0\delta_a + y_0\delta_b + z_0\delta_c,$$

it follows from the choice of the constant a, b, c that $A_0(u_0) = \{a, b, c\}$ and $A_k(u_0) = \emptyset$ for all $k > 0$. On the other hand, by Proposition 2.5.4, (iii) the weak solution u of (2.5.1) given by Theorem 2.5.1 is of the form,

$$u(t) = x(t)\delta_a + y(t)\delta_b + z(t)\delta_c, \ \ \forall t > 0,$$

where, in addition, $x(t) + y(t) + z(t) = 1$ for all $t > 0$. Using the weak formulation (2.5.5) for the test functions $\mathbb{1}_{[b,\infty)}$, $\mathbb{1}_{[c,\infty)}$, and the conservation of mass, we obtain the following system of equations:

$$\begin{aligned} &x'(t) = R(a,b)x(t)y(t), \ \ x(0) = x_0 \\ &y'(t) = -R(a,b)x(t)y(t) + R(b,c)y(t)z(t), \ \ y(0) = y_0 \\ &z'(t) = -R(b,c)y(t)z(t), \ \ z(0) = z_0. \end{aligned}$$

Since $x'(t) \geq 0$ for all t and $x(t) \in (x_0, 1)$,

$$\lim_{t\to\infty} x(t) = x_\infty \in [x_0, 1].$$

Moreover, for all $t > 0$,

$$\begin{aligned} y(t) &= y_0 e^{\int_0^t \big(R(b,c)z(s) - R(a,b)x(s)\big)ds} \\ z(t) &= z_0 e^{-R(b,c)\int_0^t y(s)ds}, \end{aligned}$$

and, by the conservation of mass,

$$\begin{aligned} \frac{y(t)}{z(t)} &= \frac{y_0}{z_0} e^{\int_0^t \big(R(b,c) - x(s)(R(a,b)+R(b,c))\big)ds} \leq \frac{y_0}{z_0}e^{Ct}, \\ C &= \big(R(b,c) - x_0(R(a,b) + R(b,c))\big). \end{aligned} \tag{2.5.75}$$

If we suppose that

$$x_0 > \frac{R(b,c)}{R(a,b) + R(b,c)},$$

then $C < 0$ and, by (2.5.75),

$$\lim_{t\to\infty} \frac{y(t)}{z(t)} = 0.$$

Using that for all $t > 0$, $z(t) \leq z_0$, we also have, using again (2.5.75), $y(t) \leq y_0 e^{Ct}$, and then $\lim_{t\to\infty} y(t) = 0$. However, since for all $t > 0$,

$$z'(t) = -R(b,c)z(t)y(t) \geq -R(b,c)z(t)y_0e^{Ct},$$

we have,

$$z(t) \geq z_0 \exp\left(\frac{R(b,c)y_0(1-e^{Ct})}{C}\right).$$

Then, since $z'(t) < 0$,

$$z_\infty = \lim_{t\to\infty} z(t) \geq z_0 \exp\left(\frac{R(b,c)y_0}{C}\right) > 0,$$

and the measure μ is,

$$\mu = x_\infty \delta_a + z_\infty \delta_c.$$

Chapter 3

On a system of two coupled equations for the normal fluid - condensate interaction in a Bose gas

3.1 Introduction

We consider the existence and properties of radially symmetric weak solutions to the following system of differential equations:

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,p) = n(t) I_3(F(t))(p) \qquad t>0,\ p\in\mathbb{R}^3, & (3.1.1)\\ n'(t) = -n(t)\displaystyle\int_{\mathbb{R}^3} I_3(F(t))(p)dp \qquad t>0, & (3.1.2)\end{cases}$$

where

$$I_3(F(t))(p) = \iint_{(\mathbb{R}^3)^2}\big[R(p,p_1,p_2) - R(p_1,p,p_2) - R(p_2,p_1,p)\big]\,dp_1dp_2, \tag{3.1.3}$$

$$\begin{aligned} R(p,p_1,p_2) &= \big[\delta(|p|^2-|p_1|^2-|p_2|^2)\delta(p-p_1-p_2)\big] \times \\ &\quad \times \left[F_1F_2(1+F) - (1+F_1)(1+F_2)F\right], \end{aligned} \tag{3.1.4}$$

and we denote $F = F(t,p)$ and $F_\ell = F(t,p_\ell)$ for $\ell = 1,2$.

The system (3.1.1), (3.1.2) is motivated by the mathematical description of a weakly interacting dilute gas of bosons. Given such a gas at equilibrium, if its temperature is below the so-called critical temperature T_c, a macroscopic density of bosons, called a condensate, appears at the lowest quantum state (cf.[52]). A description of the system of particles out of equilibrium at zero temperature has also been rigorously obtained ([27]). The system (3.1.1), (3.1.2) is more directly related to a gas out of equilibrium and at non zero temperature. The equations that, in the physic's literature, describe a gas in such a situation have not been the object of a mathematical proof; they have rather been deduced on the basis of physical arguments (cf. [37], [40, 73], [69] for example). We are particularly interested in the kinetic description of the interaction between the condensate and the particles in the dilute gas, when most of the particles are still in the gas, and so when the system is at a temperature close to T_c.

3.1.1 The Nordheim equation

The kinetic equation consistently used to describe the evolution of the distribution function for a spatially homogeneous, weakly interacting dilute gas of bosons of momentum p_1 is

$$\frac{\partial F}{\partial t}(t,p_1) = I_4(F(t))(p_1) \qquad t>0,\ p_1\in\mathbb{R}^3, \tag{3.1.5}$$

where

$$I_4(F(t))(p_1) = \iiint_{(\mathbb{R}^3)^3} q(F)d\nu(p_2,p_3,p_4), \tag{3.1.6}$$

$$q(F) = F_3F_4(1+F_1)(1+F_2) - F_1F_2(1+F_3)(1+F_4), \tag{3.1.7}$$

$$d\nu(p_1,p_2,p_3) = 2a^2\pi^{-3}\delta\left(p_1+p_2-p_3-p_4\right)\times \\ \delta\left(E(p_1)+E(p_2)-E(p_3)-E(p_4)\right)dp_2dp_3dp_4. \tag{3.1.8}$$

sometimes called Nordheim equation ([58]), (cf. for example [37], [40], [69]). We are assuming that the particles have mass $m = 1/2$ and $E(p)$ denotes the energy of a particle of momentum p. The constant a is the scattering length that parametrizes the Fermi pseudopotential of scattering. In the absence of condensate, the energy of the particles is taken to be $E(p) = |p|^2$.

For a condensed Bose gas, it is necessary to include the collisions involving the condensate. A kinetic equation is derived in [26] and [47] describing such processes. More recently, [73] extended the treatment to a trapped Bose gas by including Hartree-Fock corrections to the energy of the excitations, and have derived coupled kinetic equations for the distribution functions of the normal and superfluid components. Later on the results where generalized to low temperatures in [41] using the Bogoliubov-Popov approximation to describe the energy particle. The system is as follows

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,p) = I_4(F(t))(p) + 32a^2n(t)\widetilde{I}_3(F(t))(p) \quad t>0,\ p\in\mathbb{R}^3, & (3.1.9)\\ n'(t) = -n(t)\displaystyle\int_{\mathbb{R}^3}\widetilde{I}_3(F(t))(p)dp \qquad t>0. & (3.1.10)\end{cases}$$

(cf. [26], [40], [47] for a deduction based on physic's arguments). The term $I_4(F)$ is exactly as in (3.1.6) and the constant $32a^2$ comes from the approximation of the transition probability: $|\mathcal{M}(p,p_1,p_2)|^2 \approx 32a^2n(t)$. The integral collision $\widetilde{I}_3$ is given by an expression similar to (3.1.3), (3.1.4) but where the corresponding terms $\widetilde{R}(p,p_1,p_2)$ are as follows,

$$\widetilde{R}(p,p_1,p_2) = \left[\delta(E(p)-E(p_1)-E(p_2))\delta(p-p_1-p_2)\right]\times \\ \times\left[F(p_1)F(p_2)(1+F(p)) - (1+F(p_1))(1+F(p_2))F(p)\right]. \tag{3.1.11}$$

In presence of a condensate, the energy $E(t,p)$ of the particles at time t is now taken as $E(t,p) = \sqrt{|p|^4 + 16a\,n(t)|p|^2}$, where $n(t)$ is the condensate density ([14], [40]). Once equation (3.1.9) has been obtained, the equation (3.1.10) is just what is needed in order to ensure that the total number of particles $n(t) + \int_{\mathbb{R}^3} F(t,p)dp$ in the system is constant in time.

We are particularly interested in a situation where most of the particles are in the gas, and the condensate density n is very small. The energy of the particles is then usually approximated as $E(t,p) \approx |p|^2 + 4a\pi n(t)$ (cf.[40]). In all what follows we need the strongest simplification $E(t,p) \approx |p|^2$ to have the collision integral I_3 in (3.1.3).

Moreover, in the problem (3.1.1), (3.1.2) only the term that in the equation (3.1.9) describes the interactions involving one particle of the condensate has been kept. The term I_4, the same as in equation (3.1.5), that only considers interactions between particles in the gas, has been dropped. The term I_4 has been studied with detail to prove the existence of solutions to the Nordheim equation (3.1.5) and describe some of their properties. The problem (3.1.1), (3.1.2) only takes into account the collision processes involving a particle of the condensate.

Since we are only concerned with radial solutions (F, n) of (3.1.1), (3.1.2), a very natural independent variable is $x = |p|^2$. But this introduces a jacobian and then, the most suitable quantity is not always $f(x) = F(p)$ but may be sometimes $\sqrt{x}f(x)$.

3.1.2 The term I_4 and the Nordheim equation

The local existence of bounded solutions for Nordheim equation (3.1.5) was proved in [16]. Global existence of bounded solutions has been proved in [51] for bounded and suitably small initial data. The existence of radially symmetric weak solutions was first proved in [53] for all initial data f_0 in the space of nonnegative radially symmetric measures on $[0, \infty)$.

For radially symmetric solutions $F(p) = f(x)$, $x = |p|^2$, the expression of the Nordheim equation simplifies because it is possible to perform the angular variables in the collision integral. After rescaling the time variable t (in order to absorb some constants), the Nordheim equation reads:

$$\frac{\partial f}{\partial t}(t, x_1) = J_4(f(t))(x_1), \qquad t > 0,\ x_1 \geq 0, \tag{3.1.12}$$

where

$$J_4(f)(x_1) = \iint_{[0,\infty)^2} \frac{w(x_1, x_2, x_3)}{\sqrt{x_1}} q(f)(x_1, x_2, x_3) dx_2 dx_3, \tag{3.1.13}$$

$$q(f) = (1 + f_1)(1 + f_2)f_3 f_4 - (1 + f_3)(1 + f_4)f_1 f_2, \tag{3.1.14}$$

$$w(x_1, x_2, x_3) = \min\left\{\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, \sqrt{x_4}\right\},\ x_4 = (x_1 + x_2 - x_3)_+. \tag{3.1.15}$$

The factor $\frac{w}{\sqrt{x_1}}$ in the collision integral comes from the angular integration of the Dirac's delta of the energies $|p_\ell|^2$.

If we denote $\mathscr{M}_+([0, \infty))$ the space of positive and finite Radon measures on $[0, \infty)$, and define for all $\alpha \in \mathbb{R}$

$$\mathscr{M}_+^\alpha([0, \infty)) = \left\{G \in \mathscr{M}_+([0, \infty)) :\ M_\alpha(G) < \infty\right\}, \tag{3.1.16}$$

$$M_\alpha(G) = \int_{[0,\infty)} x^\alpha G(x) dx \qquad \text{(moment of order } \alpha), \tag{3.1.17}$$

the definition of weak solution introduced in [53] is the following.

Definition 3.1.1 (Weak radial solutions of (3.1.5))**.** Let G be a map from $[0,\infty)$ into $\mathscr{M}_+^1([0,\infty))$ and consider f defined as $\sqrt{x}f(t) = G(t)$. We say that f is a weak radial solution of (3.1.5) if G satisfies:

$$\forall t > 0: \quad G(t) \in \mathscr{M}_+^1([0,\infty)), \tag{3.1.18}$$

$$\forall T > 0: \quad \sup_{0\le t<T} \int_{[0,\infty)} (1+x)G(t,x)dx < \infty, \tag{3.1.19}$$

$$\forall\, \varphi \in C_b^{1,1}([0,\infty)): \quad \int_{[0,\infty)} \varphi(x)G(t,x)dx \in C^1([0,\infty)), \tag{3.1.20}$$

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(t,x)G(t,x)dx = \mathcal{Q}_4(\varphi, G(t)), \tag{3.1.21}$$

$$\begin{aligned}\mathcal{Q}_4(\varphi, G) = &\iiint_{[0,\infty)^3} \frac{G_1G_2G_3}{\sqrt{x_1x_2x_3}} w\Delta\varphi\; dx_1dx_2dx_3 + \\ &+ \frac{1}{2}\iiint_{[0,\infty)^3} \frac{G_1G_2}{\sqrt{x_1x_2}} w\Delta\varphi\; dx_1dx_2dx_3\end{aligned} \tag{3.1.22}$$

$$\Delta\varphi(x_1,x_2,x_3) = \varphi(x_4)+\varphi(x_3)-\varphi(x_2)-\varphi(x_1), \tag{3.1.23}$$

$$w(x_1,x_2,x_3) = \min\{\sqrt{x_1},\sqrt{x_2},\sqrt{x_3},\sqrt{x_4}\}, \;\; x_4 = (x_1+x_2-x_3)_+. \tag{3.1.24}$$

For all initial data f_0 such that $G_0 = \sqrt{x}f_0 \in \mathscr{M}_+^1([0,\infty))$, the existence of a weak solution was proved in [53]. The moments of order zero and one of G where shown to be constant in time. It was shown in [55] that a definition equivalent to Definition 3.1.1 would be to impose $\varphi(0) = 0$ to the test functions in Definition 3.1.1 and impose the conservation of mass on $G(t)$ for all $t > 0$. Further properties of the solutions, such as the gain of moments, asymptotic behavior, where obtained in a series of articles [53, 54, 55, 56]

It is proved in Proposition 3.2.1 below that if the measure G is written as $G(t) = n(t)\delta_0 + g(t)$, where $n(t) = G(t,\{0\})$, then for all $\varphi \in C_b^{1,1}([0,\infty))$ the term $\mathcal{Q}_4(\varphi, G)$ may be decomposed as follows:

$$\mathcal{Q}_4(\varphi, G(t)) = \mathscr{Q}_4(\varphi, g(t)) + n(t)\mathscr{Q}_3(\varphi, g(t)), \tag{3.1.25}$$

where

$$\begin{aligned}\mathscr{Q}_4(\varphi, g) = &\iiint_{(0,\infty)^3} \frac{g_1g_2g_3}{\sqrt{x_1x_2x_3}} w\Delta\varphi\; dx_1dx_2dx_3 \\ &+ \frac{1}{2}\iiint_{(0,\infty)^3} \frac{g_1g_2}{\sqrt{x_1x_2}} w\Delta\varphi\; dx_1dx_2dx_3,\end{aligned} \tag{3.1.26}$$

$$\mathscr{Q}_3(\varphi, g) = \mathscr{Q}_3^{(2)}(\varphi, g) - \mathscr{Q}_3^{(1)}(\varphi, g), \tag{3.1.27}$$

$$\mathscr{Q}_3^{(2)}(\varphi, g) = \iint_{(0,\infty)^2} \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}} g(x)g(y)dxdy, \tag{3.1.28}$$

$$\mathscr{Q}_3^{(1)}(\varphi, g) = \int_{(0,\infty)} \frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}} g(x)dx, \tag{3.1.29}$$

$$\Lambda(\varphi)(x,y) = \varphi(x+y) + \varphi(|x-y|) - 2\varphi(\max\{x,y\}), \tag{3.1.30}$$

$$\mathcal{L}_0(\varphi)(x) = x\big(\varphi(0)+\varphi(x)\big) - 2\int_0^x \varphi(y)dy. \tag{3.1.31}$$

It was also proved in [53] that as $t \to \infty$, the measure G converges in the weak sense of measures to one of the measures:

$$G_{\beta,\mu,C} = \frac{\sqrt{x}}{e^{\beta x - \mu} - 1} + C\delta_0, \;\; \beta > 0, \;\; \mu \leq 0, \;\; C \geq 0 \tag{3.1.32}$$

where the constants C and μ are such that $C\mu = 0$.

When $C = 0$ and $\mu \leq 0$, the function $F_{\beta,\mu,0}(p) = |p|^{-1} G_{\beta,\mu,0}(|p|^2)$ is an equilibrium of the Nordheim equation (3.1.5) because $q(F_{\beta,\mu,0})d\nu \equiv 0$. When $C > 0$ and $\mu = 0$, then $F_{\beta,0,C}(p) = |p|^{-1} G_{\beta,0,C}(|p|^2)$ is an equilibria of (3.1.9) because $q(f_{\beta,0,0}) \equiv 0$ and $R(p,p',p'') \equiv 0$ for all $(p,p',p'') \in (\mathbb{R}^3)^3$ for $f_{\beta,0,0}$, where $R(p,p',p'')$ is defined in (3.1.4). It was proved in [53] that $F_{\beta,\mu,C}$ is a weak solution of the Nordheim equation (3.1.12) if and only if $\mu C = 0$.

On the other hand, it was proved in [33] that, given any $N > 0$, $E > 0$ there exists initial data $f_0 \in L^\infty(\mathbb{R}_+; (1+x)^\gamma)$ with $\gamma > 3$, satisfying

$$\int_{\mathbb{R}^+} f_0(x)\sqrt{x}dx = N, \qquad \int_{\mathbb{R}^+} f_0(x)\sqrt{x^3}dx = E,$$

and such that there exists a global weak solution f and positive times $0 < T_* < T^*$ such that:

$$\sup_{0<t\leq T_*} \|f(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} < \infty, \qquad \sup_{T_*<t\leq T^*} \int_{\{0\}} \sqrt{x} f(t,x)\, dx > 0. \tag{3.1.33}$$

Property (3.1.33) shows that the solution $G = \sqrt{x} f$ of (3.1.18)–(3.1.24) is a bounded function on the time interval $[0, T^*)$ and a Dirac mass is formed at the origin at some time T_0 between T_* and T^*. After that time T_0, the solution G is such that $G(t, \{0\}) > 0$.

In the simplified description of the physical system of particles that we are using, where only the radial density G of particles of momentum p is considered, the description of the physical Bose-Einstein condensate can just be given by a Dirac measure at the origin.

Notwithstanding the similarity of these two phenomena, the extent to which the first one is a truthful mathematical description of the second is not clear. Nevertheless, we refer to the term $n(t)\delta_0$ that appears in finite time in some of the weak solutions of the Nordheim equation as "condensate", with some abuse of language.

3.1.3 The term I_3 in radial variables.

The results briefly presented in the previous sub Section describe some of the properties of the weak solutions to the Nordheim equation in terms of the measure G. In particular, the weak convergence of G to the measures defined in (3.1.32) shows what is the limit of $G(t, \{0\})$ as $t \to \infty$. To understand better the dynamics of $G(t, \{0\})\delta_0$ and its interaction with $G(t) - G(t, \{0\})\delta_0$ it seems suitable to write $G(t) = G(t, \{0\})\delta_0 + g(t)$ and consider the system (3.1.9), (3.1.10).

For radially symmetric functions $F(p) = f(x)$, $x = |p^2|$, the system (3.1.1), (3.1.2) reads, after a suitable time rescaling to absorb some constants:

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,x) = \dfrac{n(t)}{\sqrt{x}} J_3(f(t))(x) \qquad t > 0, \; x > 0, & (3.1.34) \\ n'(t) = -n(t) \displaystyle\int_0^\infty J_3(f(t))(x) dx \qquad t > 0, & (3.1.35) \end{cases}$$

where

$$J_3(f)(x) = \int_0^x \Big(f(x-y)f(y) - f(x)\big[1 + f(x-y) + f(y)\big]\Big)dy +$$
$$+ 2\int_x^\infty \Big(f(y)\big[1 + f(y-x) + f(x)\big] - f(y-x)f(x)\Big)dy. \tag{3.1.36}$$

(cf. [65] and [70] for the isotropic system that also contains the term $J_4(f)$, that comes from I_4 in (3.1.9)). Notice that

$$\int_0^\infty J_3(f(t))(x)dx$$
$$= \int_0^\infty \int_0^\infty \Big(f(t,x)f(t,y) - f(t,x+y)\big[1 + f(t,x) + f(t,y)\big]\Big)dxdy \tag{3.1.37}$$

whenever the integral in the right hand side is finite, for example, if $f \in L^1\big(\mathbb{R}_+, (1+x)dx\big)$. In that case we also have,

$$\int_0^\infty J_3(f(t))(x)dx = M_1(f(t)). \tag{3.1.38}$$

The factor $x^{-1/2}$ in the right hand side of (3.1.34) comes from the angular integration of the Dirac's measure of energies of I_3, just as the $\frac{w}{\sqrt{x_1}}$ term of (3.1.13) in I_4. But since $\frac{w}{\sqrt{x_1}}$ is a bounded function, it appears that the operator I_3 is more singular than I_4 for small values of x.

If we denote $F(t,p) = f(t,|p|^2) = |p|^{-1}g(t,|p|^2)$ and $x = |p^2|$, from the original motivation of the Nordheim equation it is very natural to expect

$$\int_{\mathbb{R}^3} F(t,p)dp = 2\pi \int_0^\infty f(t,x)\sqrt{x}dx = 2\pi \int_0^\infty g(t,x)dx < \infty,$$

(that corresponds to the number of particles in the normal fluid), and

$$\int_{\mathbb{R}^3} F(t,p)|p|^2 dp = 2\pi \int_0^\infty f(t,x)x^{3/2}dx = 2\pi \int_0^\infty g(t,x)xdx < \infty,$$

(corresponding to the total energy in the system). But there is no particular reason to expect

$$\int_{\mathbb{R}^3} F(t,p)\frac{dp}{|p|} = 2\pi \int_0^\infty f(t,x)dx = 2\pi \int_0^\infty g(t,x)\frac{dx}{\sqrt{x}} < \infty.$$

Without that last condition, the convergence of the integrals in the term $I_3(F(t))$ (cf. (3.1.3), (3.1.4)), or in (3.1.34), (3.1.36), is delicate. That difficulty is usually avoided using a suitable weak formulation.

If we suppose that $f = x^{-1/2}g \in L^1\big(\mathbb{R}_+, (1+x)dx\big)$, and multiply the equation (3.1.34) by $\sqrt{x}\,\varphi$, we obtain by Fubini's Theorem,

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(x)g(t,x)dx = n(t)\widetilde{\mathscr{Q}}_3(\varphi, g(t)) \quad \forall \varphi \in C_b^1([0,\infty)), \tag{3.1.39}$$

where

$$\widetilde{\mathscr{Q}}_3(\varphi, g) = \mathscr{Q}_3^{(2)}(\varphi, g) - \widetilde{\mathscr{Q}}_3^{(1)}(\varphi, g), \tag{3.1.40}$$

$$\widetilde{\mathscr{Q}}_3^{(1)}(\varphi, g) = \int_{(0,\infty)} \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} g(x)dx, \tag{3.1.41}$$

$$\mathcal{L}(\varphi)(x) = x\varphi(x) - 2\int_0^x \varphi(y)dy. \tag{3.1.42}$$

Notice that, by (3.1.27),

$$\mathscr{Q}_3(\varphi, g) = \widetilde{\mathscr{Q}}_3(\varphi, g) - \varphi(0)M_{1/2}(g). \tag{3.1.43}$$

A natural weak formulation for $G = n(t)\delta_0 + g$ is then obtained by adding (3.1.35) to (3.1.39). We then define a weak radially symmetric solution of the Problem (3.1.1), (3.1.2) as follows.

Definition 3.1.2 (Weak radial solution of (3.1.1), (3.1.2))**.** Consider a map $G : [0,T) \to \mathscr{M}_+^1([0,\infty))$ for some $T \in (0,\infty]$, that we decompose as follows:

$$\forall t \in [0,T): \quad G(t) = n(t)\delta_0 + g(t), \quad \text{where} \;\; n(t) = G(t, \{0\});$$

and define $F(t,p) = |p|^{-1}g(t,|p|^2)$ for all $t > 0$ and $p \in \mathbb{R}^3$. We say that (F, n) is a weak radial solution of (3.1.1), (3.1.2) on $(0,T)$ if:

$$\forall T' \in (0,T]: \qquad \sup_{0\le t<T'} \int_{[0,\infty)} (1+x)G(t,x)dx < \infty, \tag{3.1.44}$$

$$\forall \varphi \in C_b^1([0,\infty)): \quad t \mapsto \int_{[0,\infty)} \varphi(x)G(t,x)dx \in W_{loc}^{1,\infty}([0,T)), \tag{3.1.45}$$

and for a.e. $t \in (0,T)$

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(x)G(t,x)dx = n(t)\mathscr{Q}_3(\varphi, g(t)) \quad \forall \varphi \in C_b^1([0,\infty)), \tag{3.1.46}$$

where $\mathscr{Q}_3(\varphi, g)$ is defined by (3.1.27)-(3.1.29).

We show in Proposition 3.2.1 that the Definition 3.1.2 substantially coincides with the Definition 3.1.1 of radial weak solution of (3.1.5) when the term $\mathscr{Q}_4(\varphi, g)$ in (3.2.1) is dropped (cf. Remark 3.2.2). As a consequence, the measures $f_{\beta,0,C}(p)$ defined above are weak radial solutions of (3.1.1), (3.1.2) (cf. Proposition 3.2.3).

3.1.4 Main results

The existence of weak radial solutions for the Cauchy problem associated with the system (3.1.1), (3.1.2) is given in the following Theorem.

Theorem 3.1.3 (Existence result)**.** *Suppose that $G_0 \in \mathscr{M}_+^1([0,\infty))$ satisfies $G_0(\{0\}) > 0$, and define $F_0(p) = |p|^{-1}g_0(|p|^2)$, where $g_0 = G_0 - G_0(\{0\})\delta_0$. Then, there exists a weak radial solution (F, n) of (3.1.1), (3.1.2) on $(0,\infty)$ such that $F(t,p) = |p|^{-1}g(t,|p|^2)$, where $G = n\delta_0 + g$ satisfies:*

$$G \in C\big([0,\infty), \mathscr{M}_+^1([0,\infty))\big), \quad G(0) = G_0 \tag{3.1.47}$$

and:

(i) G conserves the total number of particles N and energy E:

$$M_0(G(t)) = M_0(G_0) = N \qquad \forall t \geq 0, \tag{3.1.48}$$
$$M_1(G(t)) = M_1(G_0) = E \qquad \forall t \geq 0. \tag{3.1.49}$$

(ii) For all $\alpha \geq 3$, if $M_\alpha(G_0) < \infty$, then $G \in C\big((0,\infty), \mathscr{M}_+^\alpha([0,\infty))\big)$ and

$$M_\alpha(G(t)) \leq \Big(M_\alpha(G_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau(t)\Big)^{\frac{\alpha-1}{2}} \qquad \forall t > 0, \tag{3.1.50}$$

$$\text{where} \quad \tau(t) = \int_0^t G(s,\{0\}) ds. \tag{3.1.51}$$

(iii) For all $\alpha \geq 3$,

$$M_\alpha(G(t)) \leq C(\alpha, E) \left(\frac{1}{1 - e^{-\gamma(\alpha,E)\tau(t)}}\right)^{2(\alpha-1)} \qquad \forall t > 0, \tag{3.1.52}$$

where $\tau(t)$ is given by (3.1.51), and the constants $C(\alpha, E)$ and $\gamma(\alpha, E)$ are defined in Theorem 3.3.1.

(iv) If $\alpha \in (1, 3]$ and

$$E > C(\alpha) N^{5/3}, \tag{3.1.53}$$

$$\text{where} \qquad C(\alpha) = \begin{cases} \left(\frac{(2^\alpha - 2)(\alpha+1)}{(\alpha-1)}\right)^{\frac{2}{3}} & \text{if} \quad \alpha \in (1,2], \\ \big(\alpha(\alpha+1)\big)^{\frac{2}{3}} & \text{if} \quad \alpha \in (2,3], \end{cases} \tag{3.1.54}$$

then $M_\alpha(G(t))$ is a decreasing function on $(0, \infty)$.

The next result is a property satisfied by all the weak radial solutions of (3.1.1), (3.1.2).

Theorem 3.1.4. *Let G_0 be as in Theorem 3.1.3, and G a weak radial solution of (3.1.1), (3.1.2). Then for all $T > 0$, $R > 0$ and $\alpha \in \big(-\frac{1}{2}, \infty\big)$,*

$$\int_0^T G(t,\{0\}) \int_{(0,R]} x^\alpha G(t,x) dx dt \leq$$
$$\leq \frac{2R^{\frac{1}{2}+\alpha}}{1 - \left(\frac{2}{3}\right)^{\frac{1}{2}+\alpha}} \left(\int_0^T G(t,\{0\}) dt\right)^{\frac{1}{2}} \left(\frac{\sqrt{E}}{2} \int_0^T G(t,\{0\}) dt + \sqrt{N}\right). \tag{3.1.55}$$

The only possible algebraic behavior for such a measure G near the origin is then $x^{-1/2}$.

Remark 3.1.5. The functions $F_{\beta,0,C}$ defined above are weak radial solutions of (3.1.1), (3.1.2) for all $\beta > 0$ and $C \geq 0$ (cf. Proposition 3.2.3). Since

$$\int_{(0,\infty)} x^\alpha G_{\beta,0,C}\, dx < \infty \quad \Longleftrightarrow \quad \alpha > -1/2,$$

the estimate (3.1.55) can not hold for all radial weak solutions if $\alpha \leq -1/2$.

In the next two results we describe the evolution of the measure at the origin $n(t) = G(t, \{0\})$ by taking the limit $\varepsilon \to 0$ in the weak formulation (3.1.46) for test functions φ_ε as follows:

Remark 3.1.6. Given $\varphi \in C_b^1([0,\infty))$ nonnegative, convex, with $\varphi(0) = 1$ and $\lim_{x\to\infty} \sqrt{x}\varphi(x) = 0$, denote $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$ for $\varepsilon > 0$. Notice that for any $G \in \mathscr{M}_+([0,\infty))$,

$$G(\{0\}) = \lim_{\varepsilon\to 0} \int_{[0,\infty)} \varphi_\varepsilon(x) dG(x). \tag{3.1.56}$$

The standard example is $\varphi_\varepsilon(x) = (1 - x/\varepsilon)_+^2$.

Theorem 3.1.7. *Let G be the solution of (3.1.46) obtained in Theorem 3.1.3, with initial data $G_0 \in \mathscr{M}_+^1([0,\infty))$ such that $N = M_0(G_0) > 0$, $E = M_1(G_0) > 0$ and $G_0(\{0\}) > 0$. Denote $G(t) = n(t)\delta_0 + g(t)$, with $n(t) = G(t, \{0\})$. Then n is right continuous and a.e. differentiable on $[0,\infty)$. Moreover, there exists a positive measure μ on $[0,\infty)$ whose cumulative distribution function is given by*

$$\mu((0,t]) = \lim_{\varepsilon\to 0} \int_0^t n(s)\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, g(s))ds \tag{3.1.57}$$

for any φ_ε as in Remark 3.1.6, and such that:

$$n(t) - n(0) + \int_0^t n(s)M_{1/2}(g(s))ds = \mu((0,t]) \qquad \forall t > 0. \tag{3.1.58}$$

Theorem 3.1.8. *Let G and μ be as in Theorem 3.1.7. Then*

$$0 < \mu((0,t])) < \infty \qquad \forall t > 0. \tag{3.1.59}$$

The measure μ in (3.1.57) depends on the atomic part of g, and on the behaviour of g at the origin (it seems to be actually related with its moment of order $-1/2$ c.f. Proposition 3.6.4 and Remark 3.6.5). This measure μ appears as a source term in the equation (3.1.58) for n. Given the function n, the equation (3.1.34) satisfied by g on $(0,\infty)$ has also a natural weak formulation by itself. In terms of $g(t)$, where $g(t) = G(t) - G(t, \{0\})\delta_0$ and $\sqrt{x}f(t,x) = g(t,x)$ it reads

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(x)g(t,x)dx = n(t)\mathscr{Q}_3(\varphi, g(t)),\ \forall \varphi \in C_b^1([0,\infty)),\ \varphi(0) = 0. \tag{3.1.60}$$

In the next result we describe the relation between a weak solution (F, n) of (3.1.1), (3.1.2), where $F(t,p) = |p|^{-1}g(t,|p|^2)$, $G(t) = n(t)\delta_0 + g(t)$, $n(t) = G(t, \{0\})$, and a pair (g, n) where g is a weak radial solution of the equation (3.1.1) and n satisfies (3.1.2).

Theorem 3.1.9. *Suppose that $G \in C\big([0,\infty), \mathscr{M}_+([0,\infty))\big)$ is such that $G(0) = G_0 \in \mathscr{M}_+^1([0,\infty))$ with $G_0(\{0\}) > 0$, and denote $G(t) = n(t)\delta_0 + g(t)$ with $n(t) = G(t, \{0\})$.*
(i) If (F, n) is a weak radial solution of (3.1.1), (3.1.2) and $F(t,p) = |p|^{-1}g(t,|p|^2)$, then n is given by (3.1.58), (3.1.57), and g satisfies (3.1.60) for a.e. $t > 0$.

(ii) On the other hand, if g satisfies (3.1.44), (3.1.45) and (3.1.60) for some non-negative bounded function n, then the limit in (3.1.57) exists. If n also satisfies

$$n(t) = n(0) + \lim_{\varepsilon\to 0} \int_0^t n(s)\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, g(s))ds - \int_0^t n(s)M_{1/2}(g(s))ds \tag{3.1.61}$$

and $F(t,p) = |p|^{-1}g(t,|p|^2)$, then (F,n) is a weak radial solution of (3.1.1), (3.1.2).

If in the Definition 3.1.2 only test functions satisfying $\varphi(0) = 0$ are taken, it becomes necessary to introduce some other condition to the system. Otherwise the system would be reduced to find g satisfying (3.1.45)–(3.1.46) for a given function $n(t)$ and for test functions such that $\varphi(0) = 0$. If we impose just the conservation of mass, we prove below (Corollary 3.1.10) that we recover a solution that satisfies the Definition 3.1.2.

Corollary 3.1.10. *If g satisfies (3.1.44), (3.1.45) and (3.1.60) for some nonnegative bounded function $n = n(t)$ such that*

$$n(t) + \int_{(0,\infty)} g(t,x)dx = constant \tag{3.1.62}$$

and $F(t,p) = |p|^{-1}g(t,|p|^2)$, then (F,n) is a weak radial solution of (3.1.1), (3.1.2).

In our last result we show that, under some sufficient conditions, the condensate density $n(t)$ tends to zero as $t \to \infty$, fast enough to be integrable.

Theorem 3.1.11. *Suppose that $G_0 \in \mathscr{M}_+^1([0,\infty))$ satisfies $G_0(\{0\}) > 0$ and let (F,n) be the weak radial solution of (3.1.1), (3.1.2) obtained in Theorem 3.1.3. Let us call $N = M_0(G_0)$ and $E = M_1(G_0)$. If condition (3.1.53), (3.1.54) hold for some $\alpha \in (1,3]$, then, for all $t_0 > 0$,*

$$\int_{t_0}^\infty n(t)dt \le M_\alpha(G(t_0))C(N,E,\alpha) \tag{3.1.63}$$

for some explicit constant $C(N,E,\alpha)$ given in (3.7.1), and

$$\lim_{t\to\infty} n(t) = 0. \tag{3.1.64}$$

Remark 3.1.12. The quantity $E/N^{5/3}$ has a very precise interpretation in physical terms. Suppose that T is the temperature of a system of particles at equilibrium with total number of particles N and total energy E. And denote T_c the critical temperature, that is the temperature at which the ground state of the system becomes macroscopically occupied. Then:

$$\frac{E}{N^{5/3}} = b\,\frac{T}{T_c}, \qquad \text{where} \qquad b = \frac{3}{(2\pi)^{\frac{1}{3}}}\frac{\zeta(5/2)}{\zeta(3/2)^{5/3}}.$$

and condition (3.1.53) implies

$$\frac{T}{T_c} = \frac{1}{b}\,\frac{E}{N^{5/3}} > \frac{C(\alpha)}{b}.$$

The function $C(\alpha)/b$ is continuous and strictly increasing on $[1,3]$ and its limit as $\alpha \to 1^+$ is $\log(16)^{2/3}/b \approx 4.48403$. Condition (3.1.53) means that, when at equilibrium, the system of particles would be at a temperature clearly above the critical temperature. Anyway, the solution F of the problem (3.1.1), (3.1.2) may be far from any real distribution of particles of the original system of particles.

3.1.5 Some arguments of the proofs.

It is very natural to make the following change of variables in problem (3.1.46). Given $G(t) = n(t)\delta_0 + g(t)$, where $n(t) = G(t, \{0\})$, we define

$$H(\tau) = G(t), \qquad \text{where} \qquad \tau = \int_0^t n(s)ds. \tag{3.1.65}$$

In terms of H, (3.1.46) reads

$$\frac{d}{d\tau}\int_{[0,\infty)} \varphi(x)H(\tau, x)dx = \mathscr{Q}_3(\varphi, H(\tau)) \quad \forall \varphi \in C_b^1([0,\infty)). \tag{3.1.66}$$

To obtain a measure H that satisfies (3.1.66), we first find h satisfying

$$\frac{d}{d\tau}\int_{[0,\infty)} \varphi(x)h(\tau, x)dx = \widetilde{\mathscr{Q}}_3(\varphi, h(\tau)) \quad \forall \varphi \in C_b^1([0,\infty)), \tag{3.1.67}$$

where $\widetilde{\mathscr{Q}}_3$ is given in (3.1.40)–(3.1.42). Then we define H as

$$H(\tau) = h(\tau) - \left(\int_0^\tau M_{1/2}(h(\sigma))d\sigma\right)\delta_0. \tag{3.1.68}$$

By (3.1.43), the measure H will satisfy (3.1.66).

As it will be seen in Section 3.3, all the arguments are much simpler and clear in the equation for H than in the equation for G. In particular, the measure λ, that corresponds to the measure μ of Theorem 3.1.7, appears as the Lebesgue-Stieltjes measure associated to $m(\tau) = h(\tau, \{0\})$.

The proofs of Theorem 3.1.3 and Theorem 3.1.4 make great use of the change of variables (3.1.65). Several of our arguments will need the measure $h(\tau)$ to satisfy only one inequality in (3.1.67). This requires the following:

Definition 3.1.13. A function $h : [0,\infty) \to \mathscr{M}_+([0,\infty))$ is said to be a super solution of (3.1.67) if

$$\begin{cases} \forall \varphi \in C_b^1([0,\infty)) \text{ nonnegative, convex and decreasing :} \\ \dfrac{d}{d\tau}\displaystyle\int_{[0,\infty)} \varphi(x)h(\tau, x)dx \geq \mathscr{Q}_3^{(2)}(\varphi, h(\tau)) \qquad a.e.\ \tau > 0. \end{cases} \tag{3.1.69}$$

The operator $\mathscr{Q}_3^{(2)}$ is considered in [44] and [45], where a problem similar to (3.1.67) is studied, with $\widetilde{\mathscr{Q}}_3$ replaced by $\mathscr{Q}_3^{(2)}$ and for which, the property of instantaneous condensation is proved. We extend this result to the solutions h of the problem (3.1.67) with the whole $\widetilde{\mathscr{Q}}_3$, using similar arguments (monotonicity, convexity of test functions) and taking care of the linear term.

Theorem 3.1.3 is deduced from the corresponding existence result of h, that is proved using very classical arguments: regularization of the problem, fixed point, a priori estimates and passage to the limit. Then, the delicate point is to invert the change of variables (3.1.65) in order to obtain a global in time nonnegative solution G.

The Plan of the article is the following. In Section 3.2 we prove Proposition 3.2.1. Section 3.3 is devoted to the proof of the existence of the measure H. In Section

3.4 we obtain several properties of $h(\tau,\{0\})$. In Section 3.5 we prove Theorem 3.1.3 (existence for the measure G) and Theorem 3.1.4. The contents of Section 3.6 are the proofs of Theorem 3.1.7, Theorem 3.1.8, Theorem 3.1.9 and Corollary 3.1.10. Finally in Section 3.7 we prove Theorem 3.1.11. Several technical results are presented in an Appendix.

3.2 On weak formulations.

We deduce first a detailed expression of the weak formulation of (3.1.12) for a radial measure G.

Proposition 3.2.1. *Let G satisfy (3.1.18)–(3.1.24) for some $T>0$, and write $G(t)=n(t)\delta_0+g(t)$, where $n(t)=G(t,\{0\})$. Then, for all $\varphi\in C_b^{1,1}([0,\infty))$ and for all $t\in(0,T)$:*

$$\frac{d}{dt}\int_{[0,\infty)}\varphi(x)G(t,x)dx=\mathscr{Q}_4(\varphi,g(t))+n(t)\mathscr{Q}_3(\varphi,g(t)),\tag{3.2.1}$$

where $\mathscr{Q}_4(\varphi,g)$ and $\mathscr{Q}_3(\varphi,g)$ are defined in (3.1.26)–(3.1.31).

Remark 3.2.2. If the term $\mathscr{Q}_4(\varphi,g)$ in (3.2.1) is dropped, we recover the equation (3.1.46) that defines a radial weak solution of (3.1.1), (3.1.2).

Proof of Proposition 3.2.1. We may rewrite $\mathcal{Q}_4(\varphi,G)$ in (3.1.22) as

$$\mathcal{Q}_4(\varphi,G)=\iiint_{[0,\infty)^3}\Phi_\varphi\, dG_1dG_2dG_3+\frac{1}{2}\iiint_{[0,\infty)^3}\sqrt{x_3}\Phi_\varphi\, dG_1dG_2dx_3,$$

where Φ_φ is as in Lemma C.2.23, and we have used notation dG instead of Gdx. Then we decompose $[0,\infty)^3=(0,\infty)^3\cup A\cup P$, where, for $\{i,j,k\}=\{1,2,3\}$,

$$\begin{aligned}A&=\{(x_1,x_2,x_3)\in\partial[0,\infty)^3\ :\ x_i=x_j=0,\ x_k>0\}\cup\{(0,0,0)\},\\ P&=\{(x_1,x_2,x_3)\in\partial[0,\infty)^3\ :\ x_i=0,\ (x_j,x_k)\in(0,\infty)^2\}.\end{aligned}$$

Let $\varphi\in C_b^{1.1}([0,\infty)$. By (C.2.34) in Lemma C.2.22 and the definition (C.2.35) of W, it follows that $\Phi_\varphi\equiv 0$ on A. Hence, recalling the definition (3.1.26) of $\mathscr{Q}_4(\varphi,g)$ and the definition of Φ_φ in Lemma C.2.22, we have

$$\mathcal{Q}_4(\varphi,G)=\mathscr{Q}_4(\varphi,g)+\iiint_P\Phi_\varphi\, dG_1dG_2dG_3+\frac{1}{2}\iiint_P\sqrt{x_3}\Phi_\varphi\, dG_1dG_2dx_3.\tag{3.2.2}$$

We now study the integral over P for the cubic and the quadratic terms in (3.2.2). (a) The cubic term. Since Φ_φ is symmetric in the x_1, x_2 variables, and Φ_φ is uniformly continuous on $[0,\infty)^3$ by Lemma C.2.23, then

$$\begin{aligned}\iiint_P\Phi_\varphi\, dG_1dG_2dG_3=&2\iiint_{\{x_2=0,\ x_1>0,\, x_3>0\}}\Phi_\varphi\, dG_1dG_2dG_3\\&+\iiint_{\{x_3=0,\ x_1>0,\, x_2>0\}}\Phi_\varphi\, dG_1dG_2dG_3\\=&2G(t,\{0\})\iint_{(0,\infty)^2}\Phi_\varphi(x_1,0,x_3)\, dG_1dG_3\\&+G(t,\{0\})\iint_{(0,\infty)^2}\Phi_\varphi(x_1,x_2,0)\, dG_1dG_2.\end{aligned}\tag{3.2.3}$$

Using now the definition of Φ_φ, we have

$$
\begin{aligned}
&2\iint_{(0,\infty)^2} \Phi_\varphi(x_1,0,x_3)\, dG_1dG_3 \qquad (3.2.4)\\
&= 2\iint_{\{x_1>x_3>0\}} \big[\varphi(x_1-x_3)+\varphi(x_3)-\varphi(0)-\varphi(x_1)\big]\frac{dG_1dG_3}{\sqrt{x_1x_3}}\\
&= \iint\limits_{(0,\infty)^2} \big[\varphi(|x_1-x_3|)+\varphi(\min\{x_1,x_3\})-\varphi(0)-\varphi(\max\{x_1,x_3\})\big]\frac{dG_1dG_3}{\sqrt{x_1x_3}}.
\end{aligned}
$$

and

$$
\begin{aligned}
&\iint_{(0,\infty)^2} \Phi_\varphi(x_1,x_2,0)\, dG_1dG_2 \qquad (3.2.5)\\
&= \iint\limits_{(0,\infty)^2} \big[\varphi(x_1+x_2)+\varphi(0)-\varphi(\min\{x_1,x_2\})-\varphi(\max\{x_1,x_2\})\big]\frac{dG_1dG_2}{\sqrt{x_1x_2}}.
\end{aligned}
$$

Notice in (3.2.4) that $\varphi(|x_1-x_3|)+\varphi(\min\{x_1,x_3\})-\varphi(0)-\varphi(\max\{x_1,x_3\})=0$ on the diagonal $\{x_1=x_3>0\}$. Then, using (3.2.4) (changing the labels x_3 by x_2) and (3.2.5) in (3.2.3), and recalling the definition (3.1.30) of $\Lambda(\varphi)$, we obtain

$$
\iiint_P \Phi_\varphi\, dG_1dG_2dG_3 = G(t,\{0\})\iint_{(0,\infty)^2}\frac{\Lambda(\varphi)(x_1,x_2)}{\sqrt{x_1x_2}}dG_1dG_2. \qquad (3.2.6)
$$

(b) The quadratic term. Again, by the symmetry of Φ_φ in x_1, x_2, and the continuity of Φ_φ on $[0,\infty)^3$, we obtain

$$
\begin{aligned}
\frac{1}{2}\iiint_P \sqrt{x_3}\,\Phi_\varphi\, dG_1dG_2dx_3 &= \iiint_{\{x_2=0,\ x_1>0,\, x_3>0\}} \sqrt{x_3}\,\Phi_\varphi\, dG_1dG_2dx_3\\
&= G(t,\{0\})\iint_{(0,\infty)^2} \sqrt{x_3}\,\Phi_\varphi(x_1,0,x_3)\, dG_1dx_3\\
&= G(t,\{0\})\iint_{\{x_1>x_3>0\}} \frac{\Delta\varphi(x_1,0,x_3)}{\sqrt{x_1}}dG_1dx_3\\
&= -G(t,\{0\})\int_{(0,\infty)}\frac{\mathcal{L}_0(\varphi)(x_1)}{\sqrt{x_1}}dG_1, \qquad (3.2.7)
\end{aligned}
$$

where $\mathcal{L}_0(\varphi)$ is given in (3.1.31). Using (3.2.6) and (3.2.7) in (3.2.2), the result follows. □

Proposition 3.2.3. *For all $C>0$ and all $\beta>0$, the measure $f_{\beta,0,C}$ is a radial weak solutions of (3.1.1),(3.1.2).*

Proof. By Proposition 3.2.1,

$$
\mathcal{Q}_4(\varphi,G_{\beta,0,C}) = \mathscr{Q}_4(\varphi,G_{\beta,0,0}) + C\mathscr{Q}_3(\varphi,G_{\beta,0,0}).
$$

We already know by Theorem 5 of [53] that $\mathcal{Q}_4(\varphi,G_{\beta,0,C})=0$ for all $\varphi\in C^{1,1}([0,\infty))$. Since $\mathscr{Q}_4(\varphi,G_{\beta,0,0})\equiv\mathcal{Q}_4(\varphi,G_{\beta,0,0})$, we deduce $\mathscr{Q}_4(\varphi,G_{\beta,0,0})=0$ for all $\varphi\in C^{1,1}([0,\infty))$. Then, since $C>0$,

$$
\mathscr{Q}_3(\varphi,G_{\beta,0,0}) = 0 \quad \forall\varphi\in C^{1,1}([0,\infty)).
$$

□

3.3 Existence of solutions H to (3.1.66)

The main result of this Section is the following,

Theorem 3.3.1. *Let $h_0 \in \mathscr{M}_+^1([0,\infty))$ with $N = M_0(h_0) > 0$ and $E = M_1(h_0) > 0$. Then, there exists $h \in C\big((0,\infty), \mathscr{M}_+^\alpha([0,\infty))\big)$ for any $\alpha \geq 1$, that satisfies the following properties: for all $\varphi \in C_b^1([0,\infty)$*

$$(i) \quad \tau \mapsto \int_{[0,\infty)} \varphi(x) h(\tau,x) dx \in W^{1,\infty}_{loc}([0,\infty)), \tag{3.3.1}$$

$$(ii) \quad \frac{d}{d\tau}\int_{[0,\infty)} \varphi(x) h(\tau,x)\, dx = \widetilde{\mathscr{Q}}_3(\varphi, h(\tau)) \quad a.e.\, \tau > 0, \tag{3.3.2}$$

$$(iii) \quad h(0) = h_0, \tag{3.3.3}$$

$$(iv) \quad M_0(h(\tau)) \leq \left(\frac{\sqrt{E}}{2}\tau + \sqrt{N}\right)^2 \quad \forall \tau \geq 0, \tag{3.3.4}$$

$$(v) \quad M_1(h(\tau)) = E \quad \forall \tau \geq 0, \tag{3.3.5}$$

(vi) *For all $\alpha \geq 3$, if $M_\alpha(h_0) < \infty$, then*

$$M_\alpha(h(\tau)) \leq \left(M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau\right)^{\frac{\alpha-1}{2}} \quad \forall \tau \geq 0, \tag{3.3.6}$$

$$(vii) \quad M_\alpha(h(\tau)) \leq C(\alpha,E)\left(\frac{1}{1-e^{-\gamma(\alpha,E)\tau}}\right)^{2(\alpha-1)} \quad \forall \alpha \geq 3,\ \forall \tau > 0, \tag{3.3.7}$$

where $C = C(\alpha, E)$ is the unique positive root of the algebraic equation

$$2^{\alpha-2}(\alpha+1)E^{\frac{2\alpha+3}{2(\alpha-1)}}(1+C) = C^{\frac{2\alpha-1}{2(\alpha-1)}}, \tag{3.3.8}$$

and $\gamma = \gamma(\alpha, E)$:

$$\gamma = \frac{1}{2(\alpha+1)}\left(\frac{C}{E}\right)^{\frac{1}{2(\alpha-1)}}. \tag{3.3.9}$$

The proof of Theorem 3.3.1 is in three steps. In the first, a regularized problem is solved (Theorem 3.3.6). Then, using an approximation argument, a solution is obtained that satisfies (3.3.1)–(3.3.6) but not yet (3.3.7) (Theorem 3.3.4). The Theorem 3.3.1 is proved with a second approximation argument on the initial data.

As a Corollary, we obtain the measure H (not necessarily positive).

Corollary 3.3.2. *Suppose that $h_0 \in \mathscr{M}_+^1([0,\infty))$ with $N = M_0(h_0) > 0$ and $E = M_1(h_0) > 0$, consider h given by Theorem 3.3.1, and define, for $\tau \geq 0$*

$$H(\tau) = h(\tau) - \left(\int_0^\tau M_{1/2}(h(\sigma))d\sigma\right)\delta_0. \tag{3.3.10}$$

Then $H \in C\big([0,\infty), \mathscr{M}^1([0,\infty))\big)$ *and for all* $\tau \in [0,\infty)$ *and* $\varphi \in C_b^1([0,\infty))$*:*

$$(i) \quad \tau \mapsto \int_{[0,\infty)} \varphi(x) H(\tau,x) dx \in W^{1,\infty}_{loc}([0,\infty)), \tag{3.3.11}$$

$$(ii) \quad \frac{d}{d\tau}\int_{[0,\infty)} \varphi(x) H(\tau,x)\, dx = \mathscr{Q}_3(\varphi, H(\tau)) \quad a.e.\, \tau > 0, \tag{3.3.12}$$

$$(iii) \quad H(0) = h_0, \tag{3.3.13}$$

$$(iv) \quad M_0(H(\tau)) = N \quad \forall \tau \geq 0, \tag{3.3.14}$$

$$(v) \quad M_1(H(\tau)) = E \quad \forall \tau \geq 0, \tag{3.3.15}$$

(vi) $\forall \alpha \geq 3$, *if* $M_\alpha(h_0) < \infty$ *then, for all* $\tau > 0$,

$$M_\alpha(H(\tau)) \leq \Big(M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau\Big)^{\frac{\alpha-1}{2}}, \tag{3.3.16}$$

$$(vii) \quad M_\alpha(H(\tau)) \leq C(\alpha, E)\left(\frac{1}{1-e^{-\gamma(\alpha,E)\tau}}\right)^{2(\alpha-1)}, \ \forall \alpha \geq 3, \tag{3.3.17}$$

where the constants $C(\alpha,E)$ *and* $\gamma(\alpha,E)$ *are defined in Theorem 3.3.1.*

Remark 3.3.3. Under the hypothesis that all the moments of the initial data h_0 are bounded it is easy to obtain the estimate (3.3.7) using the weak formulation (3.3.2). However, it is not so easy using the regularized weak formulation (3.3.21) below. For that reason, we first want to obtain a solution h satisfying (3.3.2) with an initial data with bounded moments of all order.

3.3.1 A first result.

Theorem 3.3.4. *For any* $h_0 \in \mathscr{M}_+^1([0,\infty))$ *with* $N = M_0(h_0)$ *and* $E = M_1(h_0)$*, there exists* $h \in C\big([0,\infty), \mathscr{M}_+^1([0,\infty))\big)$ *that satisfies (3.3.1)–(3.3.6).*

The proof of Theorem 3.3.4 is made in two steps. We first solve a regularised version of (3.3.2). Then, in a second step, we use an approximation argument. More precisely, we consider the following cutoff:

Cutoff 3.3.5. For every $n \in \mathbb{N}$ let $\phi_n \in C_c([0,\infty))$ be such that supp $\phi_n = [0, n+1]$, $\phi_n(x) \leq x^{-1/2}$ for all $x > 0$ and $\phi_n(x) = x^{-1/2}$ for all $x \in \big(\frac{1}{n}, n\big)$, in such a way that:

$$\forall x > 0 \quad \lim_{n\to\infty} \phi_n(x) = \frac{1}{\sqrt{x}}. \tag{3.3.18}$$

3.3.2 Regularised problem

We now solve in Theorem 3.3.6 a regularised version of (3.3.2) with the operator $\widetilde{\mathscr{Q}}_{3,n}$ defined in (C.1.14)–(C.1.16). The solution h_n is obtained as a mild solution to the equation

$$\frac{\partial h_n}{\partial \tau}(\tau, x) = J_{3,n}(h_n(\tau))(x), \tag{3.3.19}$$

where $J_{3,n}$ is defined in (C.1.17)-(C.1.20), and corresponds to a regularised version of the term J_3 defined in (3.1.36). Namely, $J_{3,n}(h) = J_3(h\phi_n)$, where ϕ_n is as in Cutoff 3.3.5.

Theorem 3.3.6. *For any $n \in \mathbb{N}$ and any nonnegative function $h_0 \in C_c([0,\infty))$, there exists a unique nonnegative function $h_n \in C\big([0,\infty), L^\infty(\mathbb{R}_+) \cap L^1_x(\mathbb{R}_+)\big)$ such that for all $\tau \in [0,\infty)$ and all $\varphi \in L^1_{loc}(\mathbb{R}_+)$:*

$$\tau \mapsto \int_{[0,\infty)} \varphi(x) h(\tau,x)dx \in W^{1,\infty}_{loc}([0,\infty)) \tag{3.3.20}$$

$$\frac{d}{d\tau}\int_0^\infty \varphi(x)h_n(\tau,x)dx = \widetilde{\mathscr{Q}}_{3,n}(\varphi, h_n(\tau)). \tag{3.3.21}$$

$$h_n(0,x) = h_0(x) \tag{3.3.22}$$

Moreover, if we denote by $N = M_0(h_0)$ and $E = M_1(h_0)$, then for every $\tau \in [0,\infty)$ and $\alpha \geq 3$:

$$M_0(h_n(\tau)) \leq \left(\frac{E}{2}\tau + \sqrt{N}\right)^2, \tag{3.3.23}$$

$$M_1(h_n(\tau)) = E, \tag{3.3.24}$$

$$M_\alpha(h_n(\tau)) \leq \left(M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}}\tau\right)^{\frac{\alpha-1}{2}}. \tag{3.3.25}$$

Furthermore, there exist two positive constants $C_{1,n}$ and $C_{2,n}$ depending on n and $\|h_0\|_{L^\infty \cap L^1_x}$ such that for all $\tau > 0$:

$$\|h_n(\tau)\|_\infty \leq C_{1,n} e^{C_{2,n}(\tau^2+\tau)}. \tag{3.3.26}$$

Proof. Using (C.1.17) we write equation (3.3.19) as

$$\frac{\partial h_n}{\partial \tau} + h_n A_n(h_n) = K_n(h_n) + L_n(h_n), \tag{3.3.27}$$

and the solution h_n is obtained as a fixed point of the operator:

$$\begin{aligned} R_n(h_n)(\tau,x) =& h_0(x)S_n(0,\tau;x) \\ &+ \int_0^\tau S_n(\sigma,\tau;x)\big(K_n(h_n)(\sigma,x) + L_n(h_n)(\sigma,x)\big)d\sigma, \end{aligned} \tag{3.3.28}$$

$$S_n(\sigma,\tau;x) = e^{-\int_\sigma^\tau A_n(h_n)(\sigma,x)d\sigma} \tag{3.3.29}$$

on

$$\begin{aligned} B(T) := \Big\{ h \in C\big([0,T], L^\infty(\mathbb{R}_+)\cap L^1_x(\mathbb{R}_+)\big) : h \geq 0 \quad \text{and} \\ \sup_{\tau\in[0,T]} \|h(\tau)\|_{L^\infty\cap L^1_x} \leq 2\|h_0\|_{L^\infty \cap L^1_x} \Big\}. \end{aligned} \tag{3.3.30}$$

Let us show first that R_n sends $B(T)$ into itself. Let $r_0 := \|h_0\|_{L^\infty\cap L^1_x}$ and for an arbitrary $T > 0$, let $h \in B(T)$. By Proposition C.1.21 with $\rho(x) = x$,

$$R_n(h)(\tau,x) \geq 0 \qquad \forall \tau \in [0,T],\ \forall x \in \mathbb{R}_+,$$
$$R_n(h) \in C\big([0,T], L^\infty(\mathbb{R}_+)\cap L^1_x(\mathbb{R}_+)\big).$$

Moreover, using (C.1.28) and (C.1.29):

$$\sup_{\tau\in[0,T]} \|R_n(h)(\tau)\|_{L^\infty\cap L^1_x} \leq r_0 + T\,C(n)(4r_0^2 + 2r_0).$$

If T satisfies:

$$T \le \frac{1}{C(n)(4r_0+2)} \tag{3.3.31}$$

then $R_n(h) \in B(T)$.

To prove that R_n is a contraction, let $h_1 \in B(T)$, $h_2 \in B(T)$ and write:

$$\begin{aligned}
\big|R_n(h_1)(\tau,x) - R_n(h_2)(\tau,x)\big| &\le h_0(x)\,|S_1(0,\tau;x) - S_2(0,\tau;x)| + \\
&+ \int_0^\tau |S_1(\sigma,\tau;x) - S_2(\sigma,\tau;x)|\,\big(K_n(h_1)(\sigma,x) + L_n(h_1)(\sigma,x)\big)d\sigma \\
&+ \int_0^\tau \big|K_n(h_1)(\sigma,x) - K_n(h_2)(\sigma,x)\big|d\sigma \\
&+ \int_0^\tau \big|L_n(h_1)(\sigma,x) - L_n(h_2)(\sigma,x)\big|d\sigma.
\end{aligned}$$

By (C.1.31), for all $\sigma \ge 0$ and $\tau \ge 0$

$$\begin{aligned}
|S_1(\sigma,\tau;x) - S_2(\sigma,\tau;x)| &\le \int_0^\tau |A_n(h_1)(\sigma,x) - A_n(h_2)(\sigma,x)|d\sigma \\
&\le C(n)\,\tau \sup_{\tau\in[0,T]} \|h_1(\tau) - h_2(\tau)\|_\infty.
\end{aligned} \tag{3.3.32}$$

Using now (3.3.32) and (C.1.28)–(C.1.31), we deduce:

$$\begin{aligned}
&\|R_n(h_1)(\tau) - R_n(h_2)(\tau)\|_{L^\infty\cap L^1_x} \le C_1 \sup_{\tau\in[0,T]} \|h_1(\tau) - h_2(\tau)\|_\infty, \\
&C_1 \equiv C_1(n,T,r_0) = C(n)T\left(1 + 3r_0 + 2Tr_0(1+2r_0)\right).
\end{aligned}$$

If (3.3.31) holds and

$$C(n)T\left(1 + 3r_0 + 2Tr_0(1+2r_0)\right) < 1,$$

R_n will be a contraction from $B(T)$ into itself. This is achieved, for example, as soon as:

$$T < \min\left\{\frac{1}{2r_0(1+2r_0)}, \frac{1}{2C(n)(1+2r_0)}\right\} = \kappa_{r_0}.$$

The fixed point h_n of R_n in $B(T)$ is then a mild solution of (3.3.19), that can be extended to a maximal interval of existence $[0, T_{n,\max})$.

We claim now that h_n satisfies (3.3.20), (3.3.21). Since h_n is a mild solution of (3.3.19):

$$h_n(\tau,x) = h_0(x)S_n(0,\tau;x) + \int_0^\tau S_n(\sigma,\tau;x)\big(K_n(h_n)(\sigma,x) + L_n(h_n)(\sigma,x)\big)d\sigma \tag{3.3.33}$$

We multiply this equation by $\varphi \in L^1_{loc}(\mathbb{R}_+)$ and integrate on $(0,\infty)$:

$$\begin{aligned}
\int_0^\infty h_n(\tau,x)\varphi(x)dx &= \int_0^\infty h_0(x)S_n(0,\tau;x)\varphi(x)dx + \\
&+ \int_0^\tau\int_0^\infty S_n(\sigma,\tau;x)\big(K_n(h_n)(\sigma,x) + L_n(h_n)(\sigma,x)\big)\varphi(x)dxd\sigma.
\end{aligned}$$

Using Lemma C.1.21 and $h_0 \in C_c([0,\infty))$, it follows that the integrals above are well define. It also follows from Lemma C.1.21 and (3.3.29) that $\tau \mapsto \int_0^\infty h_n(\tau,x)\varphi(x)dx$ is locally Lipschitz on $(0, T_{n,\max})$, and:

$$\begin{aligned}\frac{d}{dt}\int_0^\infty h_n(\tau,x)\varphi(x)dx &= \int_0^\infty h_0(x)(S_n(0,\tau;x))_\tau\varphi(x)dx+\\ &+\int_0^\infty \big(K_n(h_n)(\tau,x)+L_n(h_n)(\tau,x)\big)\varphi(x)dx+\\ &+\int_0^\tau\int_0^\infty (S_n(\sigma,\tau;x))_\tau\big(K_n(h_n)(\sigma,x)+L_n(h_n)(\sigma,x)\big)\varphi(x)dxd\sigma.\end{aligned}$$

We use now that $(S_n(\sigma,\tau;x))_\tau = -A_n(h_n)(\tau,x)S_n(\sigma,\tau;x)$ and the identity (3.3.33) to deduce:

$$\begin{aligned}\frac{d}{dt}\int_0^\infty h_n(\tau,x)\varphi(x)dx = &\int_0^\infty \big(K_n(h_n)(\tau,x)+L_n(h_n)(\tau,x)\big)\varphi(x)dx-\\ &-\int_0^\infty A_n(h_n)h_n(\tau,x)\varphi(\tau,x)dx,\end{aligned}$$

that is (3.3.21).

Suppose now that $T_{n,\max}<\infty$ and

$$\sup_{\tau\in[0,T_{n,\max})}\|h_n(\tau)\|_{L^\infty\cap L^1_x}<\infty.$$

Then there is an increasing sequence $\tau_j \to T_{n,\max}$ as $j\to\infty$ and $L>0$ such that

$$\sup_j \|h_n(\tau_j)\|_{L^\infty\cap L^1_x}\le L<\infty.$$

Fix $\delta>0$ such that $\delta<\kappa_{r_0+1}$. Starting with the initial value $h(\tau_j)$ we have a mild solution h_j defined on $[0,\delta]$. Gluing together h with h_j we obtain a mild solution on $[0, t_j+\delta]$. For j large enough, $t_j+\delta>T_{n,\max}$, and this is a contradiction. Therefore, either $T_{n,\max}=\infty$ or, if $T_{n,\max}=\infty$, then $\limsup\|h_n(\tau)\|_{L^\infty\cap L^1_x}=\infty$ as $\tau\to T_{n,\max}$.

Let us prove now the estimates (3.3.23), (3.3.24) and (3.3.26), first for all $\tau\in(0,T_{n,\max})$. Then, the property $T_{n,\max}=\infty$ will follow. We start proving (3.3.24). To this end we use (3.3.21) with $\varphi=x$. Since in that case $\Lambda(\varphi)(x,y)=0$ and $\mathcal{L}(\varphi)(x)=0$, (3.3.24) is immediate. To prove (3.3.23), we use (3.3.21) with $\varphi=1$. Then, $\Lambda(\varphi)(x,y)=0$ and $\mathcal{L}(\varphi)(x)=-x$ and then, using $\phi_n\le x^{-1/2}$, Hölder inequality and (3.3.24):

$$\frac{d}{d\tau}\left(\int_0^\infty h_n(\tau,x)dx\right)^{1/2}\le\frac{\sqrt{E}}{2},$$

from where (3.3.23) follows.

In order to prove (3.3.26) we use (3.3.23):

$$\begin{aligned}\|K_n(h_n)(\sigma)\|_\infty &\le \|\phi_n\|_\infty^2\|h_n(\sigma)\|_1\|h_n(\sigma)\|_\infty\\ &\le\|\phi_n\|_\infty^2\left(\frac{\sqrt{E}}{2}\sigma+\sqrt{N}\right)^2\|h_n(\sigma)\|_\infty,\end{aligned}$$

which combined with the estimate $\|L_n(h_n)(\sigma)\|_\infty \le 2\|\phi_n\|_1\|h_n(\sigma)\|_\infty$, gives

$$\begin{aligned}\|h_n(\tau)\|_\infty &\le \|h_0\|_\infty + \int_0^\tau \big(\|K_n(h_n)(\sigma)\|_\infty + \|L_n(h_n)(\sigma)\|_\infty\big)d\sigma\\ &\le \|h_0\|_\infty + C(n,h_0)\int_0^\tau (\sigma^2+1)\|h_n(\sigma)\|_\infty d\sigma.\end{aligned}$$

where

$$C(n,h_0) = \max\left\{\|\phi_n\|_1\|\phi_n\|_\infty^2\|h_0\|_1,\ \frac{\|\phi_n\|_\infty^2}{4}\|h_0\|_{L^1_x}\right\}.$$

Then (3.3.26) follows from Gronwall's inequality.

For the proof of (3.3.25) we use (3.3.21) with $\varphi(x) = x^\alpha$ for $\alpha \ge 3$:

$$\frac{d}{d\tau}M_\alpha(h_n(\tau)) = \widetilde{\mathscr{Q}}_{3,n}(\varphi, h_n(\tau)). \tag{3.3.34}$$

Since:

$$\mathcal{L}(\varphi)(x) = \left(\frac{\alpha-1}{\alpha+1}\right)x^{\alpha+1} \ge 0, \tag{3.3.35}$$

we have,

$$\frac{d}{d\tau}M_\alpha(h_n(\tau)) \le 2\int_0^\infty\int_0^x \Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y)h_n(\tau,x)h_n(\tau,x)dydx.$$

Then, we write $\Lambda(\varphi)(x,y) = x^\alpha\big((1+z)^\alpha + (1-z)^\alpha - 2\big)$, where $z = y/x$, and by Taylor's expansion around $z = 0$:

$$u(z) \le \frac{\|u''\|_\infty}{2}z^2 \le \alpha(\alpha-1)2^{\alpha-3}z^2.$$

Hence for all $0 \le y \le x$:

$$\Lambda(\varphi)(x,y) \le C_\alpha x^{\alpha-2}y^2, \qquad \text{where} \qquad C_\alpha = \alpha(\alpha-1)2^{\alpha-3}, \tag{3.3.36}$$

and then, using $\phi_n(x)\phi_n(y) \le y^{-1}$ and (3.3.24),

$$\frac{d}{d\tau}M_\alpha(h_n(\tau) \le 2C_\alpha M_{\alpha-2}(h_n(\tau))E.$$

Since by Holder's inequality and (3.3.24)

$$M_{\alpha-2}(h_n(\tau)) \le E^{\frac{2}{\alpha-1}}M_\alpha(h_n(\tau))^{\frac{\alpha-3}{\alpha-1}},$$

we deduce

$$\frac{d}{d\tau}\left(M_\alpha(h_n(\tau))^{\frac{2}{\alpha-1}}\right) \le \frac{4C_\alpha}{\alpha-1}E^{\frac{\alpha+1}{\alpha-1}},$$

and (3.3.25) follows. □

3.3.3 Proof of Theorem 3.3.4.

The solution h whose existence is claimed in Theorem 3.3.4 is obtained as the limit of a subsequence of solutions $(h_n)_{n\in\mathbb{N}}$ to the regularized problems obtained in Theorem 3.3.6. We first prove the following Lemma.

Lemma 3.3.7. *Let $h_0 \in C_c([0,\infty))$ be nonnegative with $N = M_0(h_0) > 0$ and $E = M_1(h_0) > 0$, and consider $(h_n)_{n\in\mathbb{N}}$ the sequence of functions given by Theorem 3.3.6. Then for every $\tau \in [0,\infty)$ there exists a subsequence, still denoted $(h_n(\tau))_{n\in\mathbb{N}}$, and a measure $h(\tau) \in \mathscr{M}_+^1([0,\infty))$ such that, as $n\to\infty$, $h_n(\tau)$ converges to $h(\tau)$ in the following sense:*

$$\forall\varphi \in C([0,\infty));\ \exists\theta\in[0,1):\quad \sup_{x\geq 0}\frac{\varphi(x)}{1+x^\theta} < \infty, \tag{3.3.37}$$

$$\lim_{n\to\infty}\int_{[0,\infty)}\varphi(x)h_n(\tau,x)dx = \int_{[0,\infty)}\varphi(x)h(\tau,x)dx. \tag{3.3.38}$$

Moreover, for every $\tau\in[0,\infty)$:

$$M_0(h(\tau)) \leq \left(\frac{\sqrt{E}}{2}\tau + \sqrt{N}\right)^2, \tag{3.3.39}$$

$$M_1(h(\tau)) \leq E. \tag{3.3.40}$$

Proof. Let us prove first the convergence for a subsequence of $(h_n(\tau))_{n\in\mathbb{N}}$. For every $\tau\geq 0$ we have by (3.3.23) that

$$\sup_{n\in\mathbb{N}}\int_0^\infty h_n(\tau,x)dx \leq \left(\frac{\sqrt{E}}{2}\tau+\sqrt{N}\right)^2.$$

Therefore, there exists a subsequence, still denoted $(h_n(\tau))_{n\in\mathbb{N}}$, and a measure $h(\tau)$ such that $(h_n(\tau))_{n\in\mathbb{N}}$ converges to $h(\tau)$ in the weak* topology of $\mathscr{M}([0,\infty))$, as $n\to\infty$:

$$\lim_{n\to\infty}\int_{[0,\infty)}\varphi(x)h_n(\tau,x)dx = \int_{[0,\infty)}\varphi(x)h(\tau,x)dx,\ \forall\varphi\in C_0([0,\infty)). \tag{3.3.41}$$

Since for all $n\in\mathbb{N}$, $h_n(\tau)$ is nonnegative, then $h(\tau)$ is a positive measure. Also by weak* convergence and (3.3.23) we deduce that $h(\tau)$ is a finite measure:

$$\int_{[0,\infty)}h(\tau,x)dx \leq \liminf_{n\to\infty}\int_0^\infty h_n(\tau,x)dx \leq \left(\frac{\sqrt{E}}{2}\tau+\sqrt{N}\right)^2. \tag{3.3.42}$$

Moreover, by (3.3.24) we also have that the sequence $(h_n(\tau))_{n\in\mathbb{N}}$ is bounded in $L^1_x(\mathbb{R}_+)$. Hence there exists a subsequence (not relabelled) that converges to a measure $\nu(\tau)$ in the weak* topology of $\mathscr{M}([0,\infty))$, i.e., such that

$$\lim_{n\to\infty}\int_0^\infty\varphi(x)\,x\,h_n(\tau,x)dx = \int_{[0,\infty)}\varphi(x)\nu(\tau,x)dx,\ \forall\varphi\in C_0([0,\infty)). \tag{3.3.43}$$

Again, since $h_n(\tau)$ is nonnegative for all $n\in\mathbb{N}$ then $\nu(\tau)$ is a positive measure. Also by weak* convergence and (3.3.24) we have

$$\int_{[0,\infty)}\nu(\tau,x)dx \leq \liminf_{n\to\infty}\int_0^\infty x\,h_n(\tau,x)dx = E. \tag{3.3.44}$$

Let us show now that $\nu(\tau) = x\,h(\tau)$. This will follow from

$$\forall\varphi \in C_0([0,\infty)) : \int_{[0,\infty)} \varphi(x)\nu(\tau,x)dx = \int_{[0,\infty)} \varphi(x)\,x\,h(\tau,x)dx \tag{3.3.45}$$

In a first step we show that (3.3.45) holds for $\varphi \in C_c([0,\infty))$ and then we use a density argument. Let $\varepsilon > 0$ and $\varphi \in C_c([0,\infty))$. Using (3.3.43) with test function φ, and (3.3.41) with test function $x\varphi(x)$, we deduce that

$$\begin{aligned}
&\left|\int_{[0,\infty)} \varphi(x)\nu(\tau,x)dx - \int_{[0,\infty)} \varphi(x)\,x\,h(\tau,x)dx\right| \\
&\quad\le \left|\int_0^\infty \varphi(x)\nu(\tau,x)dx - \int_{[0,\infty)} \varphi(x)\,x\,h_n(\tau,x)dx\right| \\
&\quad+ \left|\int_0^\infty \varphi(x)\,x\,h_n(\tau,x)dx - \int_{[0,\infty)} \varphi(x)\,x\,h(\tau,x)dx\right| < \varepsilon
\end{aligned}$$

for n large enough. Hence (3.3.45) holds for all $\varphi \in C_c([0,\infty))$. Now let $\varphi \in C_0([0,\infty))$ and consider a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_c([0,\infty))$ such that $\|\varphi_k - \varphi\|_\infty \to 0$ as $k \to \infty$. Using (3.3.45) with φ_k and the bounds (3.3.42) and (3.3.44), we deduce that

$$\begin{aligned}
&\left|\int_{[0,\infty)} \varphi(x)\nu(\tau,x)dx - \int_{[0,\infty)} \varphi(x)\,x\,h(\tau,x)dx\right| \\
&\quad\le \int_{[0,\infty)} \big|\varphi(x) - \varphi_k(x)\big|\nu(\tau,x)dx \\
&\quad+ \left|\int_{[0,\infty)} \varphi_k(x)\nu(\tau,x)dx - \int_{[0,\infty)} \varphi_k(x)\,x\,h(\tau,x)dx\right| \\
&\quad+ \int_{[0,\infty)} \big|\varphi_k(x) - \varphi(x)\big|\,x\,h(\tau,x)dx < \varepsilon
\end{aligned}$$

for k large enough. Therefore (3.3.45) holds for all $\varphi \in C_0([0,\infty))$, i.e., $\nu(\tau) = x\,h(\tau)$. Hence we rewrite (3.3.43) as

$$\lim_{n\to\infty} \int_0^\infty \varphi(x)x\,h_n(\tau,x)dx = \int_{[0,\infty)} \varphi(x)x\,h(\tau,x)dx,\ \forall\varphi \in C_0([0,\infty)). \tag{3.3.46}$$

Let us show now (3.3.37), (3.3.38). Let then $\varphi \in C([0,\infty))$ be any nonnegative test function that satisfies (3.3.37). We denote $(\zeta_j)_{j\in\mathbb{N}}$ a sequence of nonnegative and nonincreasing functions of $C_c^\infty([0,\infty))$ such that:

$$\zeta_j(x) = 1 \text{ if } x \in [0,j), \qquad \zeta_j(x) = 0 \text{ if } x > j+1,$$

and define $\varphi_j = \varphi\,\zeta_j$. Then for every n, $j \in \mathbb{N}$:

$$\begin{aligned}
&\left|\int_0^\infty \varphi(x)h_n(\tau,x)dx - \int_{[0,\infty)} \varphi(x)h(\tau,x)dx\right| \\
&\quad\le \int_0^\infty \big|\varphi(x) - \varphi_j(x)\big|h_n(\tau,x)dx \\
&\quad+ \left|\int_0^\infty \varphi_j(x)h_n(\tau,x)dx - \int_{[0,\infty)} \varphi_j(x)h(\tau,x)dx\right| \\
&\quad+ \int_{[0,\infty)} \big|\varphi_j(x) - \varphi(x)\big|h(\tau,x)dx
\end{aligned} \tag{3.3.47}$$

Since $\varphi_j \in C_0([0,\infty))$, using (3.3.43), the second term in the right hand side of (3.3.47) converges to zero as $n \to \infty$ for every $j \in \mathbb{N}$. The first and the third term in the right hand side of (3.3.47) are treated in the same way, using that $\varphi_j(x) = \varphi(x)$ for all $x \in [0,j)$. For instance, in the first term:

$$\int_0^\infty \big|\varphi(x) - \varphi_j(x)\big| h_n(\tau,x)dx = \int_j^\infty \big|\varphi(x) - \varphi_j(x)\big| h_n(\tau,x)dx$$
$$\leq 2\int_j^\infty |\varphi(x)| h_n(\tau,x)dx \leq 2C\int_j^\infty (1+x^\theta) h_n(\tau,x)dx$$
$$\leq 2C\left(\frac{1+j^\theta}{j}\right)\int_j^\infty x\, h_n(\tau,x)dx \leq 2C\left(\frac{1+j^\theta}{j}\right)E.$$

Therefore this term is small provided j is large enough. In conclusion, the difference in (3.3.47) is less than ε for n sufficiently large, i.e., (3.3.38) holds. □

Remark 3.3.8. The so-called narrow topology $\sigma(\mathscr{M}([0,\infty)), C_b([0,\infty)))$ on $\mathscr{M}_+([0,\infty))$ is generated by the metric $d(\mu,\nu) = \|\mu-\nu\|_0$, where

$$\|\mu\|_0 = \sup\left\{\int_{[0,\infty)} \varphi d\mu : \varphi \in \mathrm{Lip}_1([0,\infty)),\ \|\varphi\|_\infty \leq 1\right\},$$

(cf. [15] Theorem 8.3.2).

Using this Remark, Lemma 3.3.7 and the Arzelà-Ascoli's Theorem we prove now the following:

Proposition 3.3.9. *Let h_0 and $(h_n)_{n\in\mathbb{N}}$ be as in Lemma 3.3.7. Then there exist a subsequence (not relabelled) and $h \in C\big([0,\infty), \mathscr{M}_+([0,\infty))\big)$ such that*

$$h_n \xrightarrow[n\to\infty]{} h \quad in \quad C\big([0,\infty), \mathscr{M}_+([0,\infty))\big). \tag{3.3.48}$$

Moreover, if we denote by $N = M_0(h_0)$ and $E = M_1(h_0)$, then for all $\tau \geq 0$

$$M_0(h(\tau)) \leq \left(\frac{\sqrt{E}}{2}\tau + \sqrt{N}\right)^2, \tag{3.3.49}$$

$$M_1(h(\tau)) \leq E, \tag{3.3.50}$$

and for all $\varphi \in C([0,\infty))$ satisfying the growth condition (3.3.37):

$$\lim_{n\to\infty}\int_0^\infty \varphi(x)h_n(\tau,x)dx = \int_{[0,\infty)} \varphi(x)h(\tau,x)dx. \tag{3.3.51}$$

Proof of Proposition 3.3.9. By Lemma 3.3.7 the sequence $(h_n(\tau))_{n\in\mathbb{N}}$ is relatively compact in $\mathscr{M}([0,\infty))$ for every $\tau \in [0,\infty)$. Let us show now that $(h_n)_{n\in\mathbb{N}}$ is also equicontinuous. To this end let $\tau_2 \geq \tau_1 \geq 0$, and consider φ as in Remark 3.3.8, i.e., $\varphi \in \mathrm{Lip}([0,\infty))$ with Lipschitz constant $\mathrm{Lip}(\varphi) \leq 1$, and $\|\varphi\|_\infty \leq 1$. Then, using $\phi_n(x) \leq x^{-1/2}$, (C.1.2) and (C.1.4) in Lemma C.1.15, we have

$$\left|\int_0^\infty \varphi(x)h_n(\tau_1,x)dx - \int_0^\infty \varphi(x)h_n(\tau_2,x)dx\right|$$
$$\leq \int_{\tau_1}^{\tau_2} \big|\widetilde{\mathscr{Q}}_{3,n}(\varphi, h_n(\sigma))\big| d\sigma \leq 2\int_{\tau_1}^{\tau_2}\left(\int_0^\infty h_n(\sigma,x)dx\right)^2 d\sigma$$
$$+ 4\int_{\tau_1}^{\tau_2}\int_0^\infty \sqrt{x}\, h_n(\sigma,x)dxd\sigma. \tag{3.3.52}$$

Using Hölder's inequality and the estimates (3.3.23) and (3.3.24) in (3.3.52), it follows that

$$\begin{aligned}&\left|\int_0^\infty \varphi(x)h_n(\tau_1,x)dx-\int_0^\infty \varphi(x)h_n(\tau_2,x)dx\right|\\&\leq 2\int_{\tau_1}^{\tau_2}\left(\frac{\sqrt{E}}{2}\sigma+\sqrt{N}\right)^4 d\sigma+4\sqrt{E}\int_{\tau_1}^{\tau_2}\left(\frac{\sqrt{E}}{2}\sigma+\sqrt{N}\right)d\sigma\quad \forall n\in\mathbb{N}.\end{aligned}$$

We then deduce using Remark 3.3.8 that $(h_n)_{n\in\mathbb{N}}$ is equicontinuous. It then follows from Arzelà-Ascoli's Theorem (cf. for example [63]) that there exists $h\in C\big([0,\infty),\mathscr{M}_+([0,\infty))\big)$ such that $h_n\to h$ in $C\big([0,T],\mathscr{M}_+([0,\infty))\big)$, for every $T>0$, as $n\to\infty$.

The estimates (3.3.49), (3.3.50) and the convergence (3.3.51) are deduced in the same way as in the Proof of Lemma 3.3.7. □

Proof of Theorem 3.3.4. By Corollary C.1.18, there exists a sequence of nonnegative function $(h_{0,n})_{n\in\mathbb{N}}\in C_c([0,\infty))$ that approximate h_0 in the weak* topology of the space $C_b([0,\infty))^*$. Let then $(h_n)_{n\in\mathbb{N}}\subset C\big([0,\infty),\mathscr{M}_+([0,\infty))\big)$ be the sequence of solutions to (3.3.20), (3.3.21) obtained by Theorem 3.3.6 with the initial data $h_{0,n}$. By Proposition 3.3.9 there exists a subsequence, still denoted $(h_n)_{n\in\mathbb{N}}$, and $h\in C\big([0,\infty),\mathscr{M}_+([0,\infty))\big)$ such that h_n converges to h in the topology of $C\big([0,\infty),\mathscr{M}_+([0,\infty))\big)$.

By (3.3.21) and (3.3.22), for all $\varphi\in C_b^1([0,\infty))$ and $\tau>0$:

$$\int_0^\infty \varphi(x)h_n(\tau,x)dx-\int_0^\infty \varphi(x)h_{0,n}(x)dx=\int_0^\tau \widetilde{\mathscr{Q}}_{3,n}(\varphi,h_n(\sigma))d\sigma. \tag{3.3.53}$$

By construction, for every $\varphi\in C_b^1([0,\infty))$ and every $\tau\in[0,\infty)$:

$$\lim_{n\to\infty}\int_0^\infty \varphi(x)h_n(\tau,x)dx=\int_{[0,\infty)}\varphi(x)h(\tau,x)dx. \tag{3.3.54}$$

We prove now the convergence of the linear term: for all $\varphi\in C_b^1([0,\infty))$ and $\tau\in[0,\infty)$

$$\lim_{n\to\infty}\widetilde{\mathscr{Q}}_{3,n}^{(1)}(\varphi,h_n(\tau))=\widetilde{\mathscr{Q}}_3^{(1)}(\varphi,h_n(\tau)). \tag{3.3.55}$$

By definition:

$$\begin{aligned}&\left|\widetilde{\mathscr{Q}}_3^{(1)}(\varphi,h(\tau))-\widetilde{\mathscr{Q}}_{3,n}^{(1)}(\varphi,h_n(\tau))\right|\\&\leq\left|\int_0^\infty\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}h(\tau,x)dx-\int_0^\infty\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}h_n(\tau,x)dx\right|\\&+\int_0^\infty\left|\mathcal{L}(\varphi)(x)\phi_n(x)-\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}\right|h_n(\tau,x)dx.\end{aligned} \tag{3.3.56}$$

From Lemma C.1.15 (iii) and (3.3.51):

$$\lim_{n\to\infty}\left|\int_0^\infty\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}h(\tau,x)dx-\int_0^\infty\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}h_n(\tau,x)dx\right|=0 \tag{3.3.57}$$

For the second term in the right hand side of (3.3.56) we split the integral $\int_0^\infty$ in two: $\int_0^R$ and $\int_R^\infty$ for $R > 0$, and apply (C.1.4). We obtain:

$$\begin{aligned}\int_0^\infty \left|\mathcal{L}(\varphi)(x)\phi_n(x) - \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}\right| h_n(\tau,x)dx \\ \leq \left\|\mathcal{L}(\varphi)(x)\phi_n(x) - \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}\right\|_{C([0,R])} \int_0^R h_n(\tau,x)dx + \\ + 4\|\varphi\|_\infty \int_R^\infty \sqrt{x}\, h_n(\tau,x)dx.\end{aligned} \tag{3.3.58}$$

By (3.3.24), for any $\varepsilon > 0$ and $R > (E/\varepsilon)^2$:

$$\int_R^\infty \sqrt{x}\, h_n(\tau,x)dx \leq \frac{E}{\sqrt{R}} < \varepsilon \qquad \forall n \in \mathbb{N}.$$

Then by Lemma C.1.16 and (3.3.23), the part on $[0,R]$ converges to zero as $n \to \infty$. Since $R > 0$ is arbitrary we finally deduce that (3.3.58) converges to zero as $n \to \infty$. Therefore (3.3.55) holds.

Let us prove now the convergence of the quadratic term: for all $\varphi \in C_b^1([0,\infty))$ and all $\tau \in [0,\infty)$:

$$\lim_{n\to\infty} \mathscr{Q}_{3,n}^{(2)}(\varphi, h_n(\tau)) = \mathscr{Q}_3^{(2)}(\varphi, h_n(\tau)). \tag{3.3.59}$$

As before

$$\begin{aligned}&\left|\mathscr{Q}_3^{(2)}(\varphi, h(\tau)) - \mathscr{Q}_{3,n}^{(2)}(\varphi, h_n(\tau))\right| \\ &\leq \left|\mathscr{Q}_3^{(2)}(\varphi, h(\tau)) - \int_0^\infty\int_0^\infty \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}} h_n(\tau,x)h_n(\tau,y)dxdy\right| \\ &+ \int_0^\infty\int_0^\infty \left|\Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y) - \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}\right| h_n(\tau,x)h_n(\tau,y)dxdy.\end{aligned} \tag{3.3.60}$$

It follows from Lemma C.1.15 (ii) and (3.3.51) that the first term in the right hand side above converges to zero as $n \to \infty$. For the second term we proceed as before. For any $R > 0$ we split the double integral:

$$\begin{aligned}&\int_0^\infty\int_0^\infty \left|\Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y) - \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}\right| h_n(\tau,x)h_n(\tau,y)dxdy \\ &\leq \left\|\Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y) - \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}\right\|_{C([0,R]^2)} \left(\int_0^R h_n(\tau,x)dx\right)^2 \\ &+ \iint_{(0,\infty)^2\setminus(0,R)^2} \left|\Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y) - \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}\right| h_n(\tau,x)h_n(\tau,y)dxdy \\ &= I_1 + I_2.\end{aligned}$$

By Lemma C.1.16 and (3.3.23), I_1 converges to zero as $n \to \infty$. For the term I_2 we use (C.1.2) in Lemma C.1.15 and the estimates (3.3.24) and (3.3.23):

$$\begin{aligned}&\int_R^\infty\int_R^\infty \left|\Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y) - \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}\right| h_n(\tau,x)h_n(\tau,y)dxdy \\ &\leq 4\|\varphi'\|_\infty \left(\int_R^\infty h_n(\tau,x)dx\right)^2 \leq \frac{4\|\varphi'\|_\infty E^2}{R^2} \qquad \forall n \in \mathbb{N}\end{aligned}$$

and

$$2\int_R^\infty\int_0^R\left|\Lambda(\varphi)(x)\phi_n(x)\phi_n(y)-\frac{\Lambda(\varphi)(x)}{\sqrt{xy}}\right|h_n(\tau,x)h_n(\tau,y)dxdy$$
$$\leq 4\|\varphi'\|_\infty\int_R^\infty\int_0^R h_n(\tau,x)h_n(\tau,y)dxdy\leq\frac{4\|\varphi'\|_\infty E}{R}\left(\frac{\sqrt{E}}{2}\tau+\sqrt{N}\right)^2.$$

Since $R>0$ is arbitrary we deduce that I_2 also converges to zero as $n\to\infty$. We then conclude that (3.3.59) holds.

Combining (3.3.55) and (3.3.59) it follows that for all $\varphi\in C_b^1([0,\infty))$ and all $\tau\in[0,\infty)$:

$$\lim_{n\to\infty}\widetilde{\mathscr{Q}}_{3,n}(\varphi,h_n(\tau))=\widetilde{\mathscr{Q}}_3(\varphi,h(\tau)). \tag{3.3.61}$$

Moreover, using $\phi_n(x)\leq x^{-1/2}$, (C.1.2) and (C.1.4) in Lemma C.1.15, and the estimates (3.3.23) and (3.3.24), we have for all $\varphi\in C_b^1([0,\infty))$, all $\tau\in[0,\infty)$ and all $n\in\mathbb{N}$:

$$\left|\widetilde{\mathscr{Q}}_{3,n}(\varphi,h_n(\tau))\right|\leq$$
$$\leq 2\|\varphi'\|_\infty\left(\int_0^\infty h_n(\tau,x)dx\right)^2+4\|\varphi\|_\infty\int_0^\infty\sqrt{x}\,h_n(\tau,x)dx$$
$$\leq 2\|\varphi'\|_\infty\left(\frac{\sqrt{E}}{2}\tau+\sqrt{N}\right)^4+4\|\varphi\|_\infty\sqrt{E}\left(\frac{\sqrt{E}}{2}\tau+\sqrt{N}\right).$$

By (3.3.61) and the dominated convergence Theorem:

$$\lim_{n\to\infty}\int_0^\tau\widetilde{\mathscr{Q}}_{3,n}(\varphi,h_n(\sigma))d\sigma=\int_0^\tau\widetilde{\mathscr{Q}}_3(\varphi,h(\sigma))d\sigma. \tag{3.3.62}$$

Using now (3.3.54) and (3.3.62), we may pass to the limit as $n\to\infty$ in (3.3.53) for all $\varphi\in C_b^1([0,\infty))$ and all $\tau\in[0,\infty)$ to obtain:

$$\int_{[0,\infty)}\varphi(x)h(\tau,x)dx=\int_{[0,\infty)}\varphi(x)h_0(x)dx+\int_0^\tau\widetilde{\mathscr{Q}}_3(\varphi,h(\sigma))d\sigma. \tag{3.3.63}$$

The map $\tau\mapsto\int_{[0,\infty)}\varphi(x)h(\tau,x)dx$ is then locally Lipschitz on $[0,\infty)$, and h satisfies (3.3.1), (3.3.2) for all $\varphi\in C_b^1([0,\infty))$ and for a.e. $\tau\in[0,\infty)$. It also follows from (3.3.63) that $h(0)=h_0$ in $\mathscr{M}_+$.

The property (3.3.4) follows from (3.3.49). The conservation of energy (3.3.5) is obtained as follows. We already know by (3.3.50) that $M_1(h(\tau))\leq E$. On the other hand, let $\varphi_k\in C_b^1([0,\infty))$ be a concave test function such that $\varphi_k(x)=x$ for $x\in[0,k)$ and $\varphi_k(x)=k+1$ for $x\geq k+2$. Notice that there exists a positive constant C such that

$$\sup_{k\in\mathbb{N}}\|\varphi_k'\|_\infty\leq C. \tag{3.3.64}$$

By Remark C.1.14, $\widetilde{\mathscr{Q}}_3^{(1)}(\varphi_k,h)\leq 0$ and $\mathscr{Q}_3^{(2)}(\varphi_k,h)\leq 0$ for all $k\in\mathbb{N}$, and then, from (3.3.63):

$$\int_{[0,\infty)}\varphi_k(x)h(\tau,x)dx\geq\int_{[0,\infty)}\varphi_k(x)h_0(x)dx+\int_0^\tau\mathscr{Q}_3^{(2)}(\varphi_k,h(\sigma))d\sigma. \tag{3.3.65}$$

We now prove that for all $\tau \in [0, \infty)$:

$$\lim_{k\to\infty} \int_0^\tau \mathscr{Q}_3^{(2)}(\varphi_k, h(\sigma))d\sigma = 0. \tag{3.3.66}$$

Notice that $\Lambda(\varphi_k)(x, y) \to 0$ as $k \to \infty$, since $\varphi_k(x) \to x$. Then, using (C.1.2) in Lemma C.1.15, (3.3.64) and (3.3.23), we deduce for all $\tau \in [0, \infty)$ and $\sigma \in (0, \tau)$:

$$\lim_{k\to\infty} \mathscr{Q}_3^{(2)}(\varphi_k, h(\sigma)) = 0 \tag{3.3.67}$$

$$\left|\mathscr{Q}_3^{(2)}(\varphi_k, h(\tau))\right| \le 2C\left(\frac{\sqrt{E}}{2}\tau + \sqrt{N}\right)^4 \qquad \forall k \in \mathbb{N}. \tag{3.3.68}$$

and (3.3.66) follows from the dominated convergence Theorem. We take now limits in (3.3.65) as $k \to \infty$. By (3.3.66) and the monotone convergence Theorem we obtain that $M_1(h(\tau)) \ge E$ and then $M_1(h(\tau)) = E$ for all $\tau > 0$.

We assume now that $M_\alpha(h_0) < \infty$ for some $\alpha \ge 3$ and prove (3.3.6). By (3.3.25) and (C.1.10) in Corollary C.1.18:

$$\begin{aligned} M_\alpha(h(\tau)) &\le \liminf_{n\to\infty} \left(M_\alpha(h_{0,n})^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} M_1(h_{0,n})^{\frac{\alpha+1}{\alpha-1}}\tau\right)^{\frac{\alpha-1}{2}} \\ &\le \left(M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}}\tau\right)^{\frac{\alpha-1}{2}}. \end{aligned}$$

□

3.3.4 Proof of Theorem 3.3.1

Proof of Theorem 3.3.1. Consider again the sequence of initial data $h_{0,n}$ used in the proof of Theorem 3.3.4 and the sequence of solutions h_n obtained by Theorem 3.3.4. Using (3.3.25) we know that $M_\alpha(h_n(\tau)) < \infty$ for $\tau > 0$ and $n \in \mathbb{N}$.

Our first step is to prove that (3.3.2) holds also true for $\varphi(x) = x^\alpha$. Notice that h_n solves now the equation (3.3.2), with the operator $\widetilde{\mathscr{Q}}_3$ in the right-hand side, whose kernel is not compactly supported and the argument in the proof of (3.3.25) must be slightly modified.

In order to use (3.3.2) we consider a sequence $(\varphi_k) \subset C_b^1([0, \infty))$ such that:

$$\varphi_k \to \varphi \quad \text{as} \quad k \to \infty \tag{3.3.69}$$

$$\varphi_k \le \varphi_{k+1} \le \varphi \tag{3.3.70}$$

$$\varphi' \ge \varphi_k' \ge 0. \tag{3.3.71}$$

Let us prove by the dominated convergence Theorem that for all $\tau \ge 0$:

$$(i) \quad \widetilde{\mathscr{Q}}_3(\varphi, h_n) \in L^1(0, \tau), \tag{3.3.72}$$

$$(ii) \quad \lim_{k\to\infty} \int_0^\tau \widetilde{\mathscr{Q}}_3(\varphi_k, h_n(\sigma))d\sigma = \int_0^\tau \widetilde{\mathscr{Q}}_3(\varphi, h_n(\sigma))d\sigma. \tag{3.3.73}$$

To this end we first observe that, for $x \ge y > 0$:

$$\lim_{k\to\infty} \Lambda(\varphi_k)(x, y) = \Lambda(\varphi) \qquad \text{and} \qquad \lim_{n\to\infty} \mathcal{L}(\varphi_k) = \mathcal{L}(\varphi)(x) \tag{3.3.74}$$

and, by the mean value Theorem:

$$\frac{\Lambda(\varphi_k)(x,y)}{\sqrt{xy}} \le \varphi_k'(\xi_1) - \varphi_k'(\xi_2)$$

for some $\xi_1 \in (x, x+y)$ and $\xi_2 \in (x-y, x)$. Using then (3.3.71):

$$\frac{|\Lambda(\varphi_k)(x,y)|}{\sqrt{xy}} \le \alpha\big(2^{\alpha-1}+1\big)x^{\alpha-1} \qquad \forall k \in \mathbb{N}, \tag{3.3.75}$$

and by (3.3.70):

$$\frac{|\mathcal{L}(\varphi_k)(x)|}{\sqrt{x}} \le \left(\frac{\alpha+3}{\alpha+1}\right) x^{\alpha+\frac{1}{2}} \qquad \forall k \in \mathbb{N}.$$

Since by Theorem 3.3.4: $M_{\alpha-1}(h_n(\tau)) < \infty$ and $M_{\alpha+1/2}(h_n(\tau)) < \infty$, for every fixed n we may apply the Lebesgue's convergence Theorem to the sequences $\left\{\frac{\Lambda(\varphi_k)(x,y)}{\sqrt{xy}} h_n(\sigma,x)h_n(\sigma,y)\right\}_{k\in\mathbb{N}}$ and $\left\{\frac{\mathcal{L}(\varphi_k)(x)}{\sqrt{x}} h_n(\sigma,x)\right\}_{k\in\mathbb{N}}$ and obtain (3.3.72), (3.3.73).

We use now φ_k in (3.3.2) and take the limit $k \to \infty$. We obtain from (3.3.69), (3.3.70), (3.3.73) and monotone convergence:

$$M_\alpha(h_n(\tau)) = M_\alpha(h_{0,n}) + \int_0^\tau \widetilde{\mathscr{Q}}_3(\varphi, h_n(\sigma))d\sigma \qquad \forall \tau \ge 0, \tag{3.3.76}$$

and then, using (3.3.72):

$$\frac{d}{d\tau} M_\alpha(h_n(\tau)) = \widetilde{\mathscr{Q}}_3(\varphi, h_n(\tau)) \qquad a.e.\ \tau > 0. \tag{3.3.77}$$

If we use (3.3.35) and (3.3.36) in the right hand side of (3.3.77), we obtain

$$\frac{d}{d\tau} M_\alpha(h_n) \le 2^{\alpha-2}\alpha(\alpha-1)E_n M_{\alpha-2}(h_n) - \left(\frac{\alpha-1}{\alpha+1}\right) M_{\alpha+\frac{1}{2}}(h_n),$$

where $E_n = M_1(h_{0,n})$. Now by Hölder's inequality:

$$\begin{aligned} M_{\alpha-2}(h_n) &\le E_n^{2/(\alpha-1)} M_\alpha(h_n)^{(\alpha-3)/(\alpha-1)} \\ M_\alpha(h_n) &\le E_n^{1/(2\alpha-1)} M_{\alpha+\frac{1}{2}}(h_n)^{2(\alpha-1)/(2\alpha-1)}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \frac{d}{d\tau} M_\alpha(h_n) \le\ & 2^{\alpha-2}\alpha(\alpha-1)E_n^{1+2/(\alpha-1)} M_\alpha(h_n)^{(\alpha-3)/(\alpha-1)} \\ & - \left(\frac{\alpha-1}{\alpha+1}\right) E_n^{-1/(2(\alpha-1))} M_\alpha(h_n)^{(2\alpha-1)/(2(\alpha-1))}. \end{aligned}$$

Since $(\alpha-3)/(\alpha-1) \in [0,1)$ then

$$M_\alpha(h_n)^{(\alpha-3)/(\alpha-1)} \le 1 + M_\alpha(h_n),$$

and :

$$\begin{aligned} \frac{d}{d\tau} M_\alpha(h_n) \le\ & 2^{\alpha-2}\alpha(\alpha-1)E_n^{1+2/(\alpha-1)} \big(1 + M_\alpha(h_n)\big) \\ & - \left(\frac{\alpha-1}{\alpha+1}\right) E_n^{-1/(2(\alpha-1))} M_\alpha(h_n)^{(2\alpha-1)/(2(\alpha-1))}, \end{aligned} \tag{3.3.78}$$

where $(2\alpha-1)/(2(\alpha-1))>1$. If we define:

$$\begin{aligned}&u(\sigma)=M_\alpha(h_n(\tau)),\quad \sigma=C_1\tau,\quad q=2(\alpha-1),\\&C_1=2^{\alpha-2}\alpha(\alpha-1)E^{1+2/(\alpha-1)}\\&C_2=\left(\frac{\alpha-1}{\alpha+1}\right)E^{-1/(2(\alpha-1))},\quad C=\frac{C_2}{C_1}.\end{aligned}$$

We deduce from (3.3.78) that

$$u'\leq 1+u-Cu^{1+1/q}, \tag{3.3.79}$$

and then by Lemma 6.3 in [13], for every $n\in\mathbb{N}$:

$$M_\alpha(h_n(\tau))\leq C(\alpha,E_n)\left(\frac{1}{1-e^{-\gamma(\alpha,E_n)\tau}}\right)^{2(\alpha-1)}, \tag{3.3.80}$$

where the constants $C(\alpha,E_n)$ and $\gamma(\alpha,E_n)$ are defined as in Theorem 3.3.1. We may argue now as in the proof of Theorem 3.3.4 and pass to the limit along a subsequence to obtain a limit $h\in C\big([0,\infty),\mathscr{M}_+([0,\infty))\big)$ satisfying (3.3.1)–(3.3.5) and (3.3.7). Using (3.3.7) and $h\in C\big([0,\infty),\mathscr{M}_+([0,\infty))\big)$ we deduce as in the proof of Theorem 3.3.4 that $h\in C\big((0,\infty),\mathscr{M}_+^\alpha([0,\infty))\big)$ for all $\alpha\geq 1$. □

Proof of Corollary 3.3.2. We first observe that the map $\tau\mapsto M_{1/2}(h(\tau))$ is locally bounded. Indeed by Hölder's inequality, (3.3.4) and (3.3.5):

$$M_{1/2}(h(\tau))\leq\sqrt{M_1(h(\tau))M_0(h(\tau))}\leq\sqrt{E}\left(\frac{\sqrt{E}}{2}\tau+\sqrt{N}\right).$$

Then by (3.3.1) it follows (3.3.11). Now for all $\varphi\in C_b^1([0,\infty))$ and for a.e. $\tau>0$, we deduce from (3.3.2):

$$\begin{aligned}\frac{d}{d\tau}\int_{[0,\infty)}\varphi(x)H(\tau,x)dx&=\widetilde{\mathscr{Q}}_3(\varphi,h(\tau))-\varphi(0)M_{1/2}(h(\tau))\\&=\mathscr{Q}_3(\varphi,h(\tau)).\end{aligned}$$

Since $H=h$ on $(0,\infty)$ then $\mathscr{Q}_3(\varphi,H)\equiv\mathscr{Q}_3(\varphi,h)$, and therefore (3.3.12) holds.

Now for the initial data: $H(0)=h(0)=h_0$. The conservation of mass (3.3.14) follows from (3.3.12) for $\varphi=1$, since $\Lambda(\varphi)=0=\mathcal{L}_0(\varphi)$. The conservation of energy (3.3.15) follows directly from (3.3.5) since $H=h$ on $(0,\infty)$. □

3.4 Properties of $h(\tau,\{0\})$.

In all this Section we denote

$$m(\tau)=h(\tau,\{0\}). \tag{3.4.1}$$

The main result of this Section is the following.

Theorem 3.4.1. *Suppose that $h\in C([0,\infty);\mathscr{M}_+^1([0,\infty))$ is a solution of (3.1.67) with $h(0)=h_0\in\mathscr{M}_+^1([0,\infty))$, $N=M_0(h_0)>0$ and $E=M_1(h_0)>0$. Then m is right continuous, a.e. differentiable and strictly increasing on $[0,\infty)$.*

We begin with the following properties of the function m defined in (3.4.1).

Lemma 3.4.2. *The function m is nondecreasing, a.e. differentiable and right continuous on $[0, \infty)$.*

Proof. Given any φ_ε as in Remark 3.1.6, then for all $\tau \geq 0$

$$m(\tau) = \lim_{\varepsilon \to 0} \int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau, x) dx, \tag{3.4.2}$$

and by (3.1.67)-(3.1.40)

$$\frac{d}{d\tau} \int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau, x) dx = \mathscr{Q}_3^{(2)}(\varphi_\varepsilon, h(\tau)) - \widetilde{\mathscr{Q}}_3^{(1)}(\varphi_\varepsilon, h(\tau)). \tag{3.4.3}$$

Since φ_ε is convex, nonnegative and decreasing, it follows from Lemma C.1.13 that $\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, h) \geq 0$ and $\widetilde{\mathscr{Q}}_3^{(1)}(\varphi_\varepsilon, h) \leq 0$ for all $\varepsilon > 0$. Then by (3.4.3)

$$\int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau_2, x) dx \geq \int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau_1, x) dx \qquad \forall \tau_2 \geq \tau_1 \geq 0.$$

Letting $\varepsilon \to 0$ it follows from (3.4.2) that m is nondecreasing on $[0, \infty)$ and, for any $\tau \geq 0$ and $\delta > 0$,

$$\liminf_{\delta \to 0^+} m(\tau + \delta) \geq m(\tau). \tag{3.4.4}$$

Using Lebesgue's Theorem (cf. [63]), m is a.e. differentiable on $[0, \infty)$. On the other hand, if in (3.4.3) the term $\widetilde{\mathscr{Q}}_3^{(1)}(\varphi_\varepsilon, h)$ is dropped,

$$\int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau + \delta, x) dx \leq \int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau, x) dx + \int_\tau^{\tau+\delta} \mathscr{Q}_3^{(2)}(\varphi_\varepsilon, h(\sigma)) d\sigma.$$

Using $\mathbb{1}_{\{0\}} \leq \varphi_\varepsilon$ for all $\varepsilon > 0$, and the bound (C.1.2), we deduce

$$m(\tau + \delta) \leq \int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau, x) dx + \frac{2\delta}{\varepsilon} (M_0(h(\tau)))^2.$$

If we take now superior limits as $\delta \to 0^+$ at $\varepsilon > 0$ fixed,

$$\limsup_{\delta \to 0^+} m(\tau + \delta) \leq \int_{[0,\infty)} \varphi_\varepsilon(x) h(\tau, x) dx \qquad \forall \varepsilon > 0.$$

We may pass now to the limit as $\varepsilon \to 0$ in the right hand side above and use (3.4.2) to get,

$$\limsup_{\delta \to 0^+} m(\tau + \delta) \leq m(\tau). \tag{3.4.5}$$

The right continuity then follows from (3.4.4) and (3.4.5). □

Corollary 3.4.3. *The map $\tau \mapsto H(\tau, \{0\})$, defined for all $\tau \geq 0$, is right continuous on $[0, \infty)$ and*

$$\limsup_{\delta \to 0^+} H(\tau - \delta, \{0\}) \leq H(\tau, \{0\}) \qquad \forall \tau > 0. \tag{3.4.6}$$

Proof. By construction (cf.(3.1.68)) it follows

$$H(\tau, \{0\}) = m(\tau) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma.$$

Since $M_{1/2}(h) \in L^1_{loc}(\mathbb{R}_+)$ then $\tau \mapsto \int_0^\tau M_{1/2}(h(\sigma))d\sigma$ is absolutely continuous, and since m is right continuous by Lemma 3.4.2, it follows that $\tau \mapsto H(\tau, \{0\})$ is also right continuous. To prove (3.4.6) we use the continuity of $\tau \mapsto \int_0^\tau M_{1/2}(h(\sigma))d\sigma$ and the monotonicity of $h(\tau, \{0\})$: for all $\tau > 0$ and $\delta \in (0, \tau)$,

$$\begin{aligned}\limsup_{\delta\to 0^+} H(\tau - \delta, \{0\}) &= \limsup_{\delta\to 0^+} m(\tau - \delta) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma \\ &\le m(\tau) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma = H(\tau, \{0\}).\end{aligned}$$

□

Remark 3.4.4. We do not know if the map $\tau \mapsto H(\tau, \{0\})$ is continuous. By property (3.4.6) however, $H(\tau, \{0\})$ does not decrease through the possible discontinuities.

The proof of Theorem 3.4.1 closely follows the proof of Proposition 1.21 in [45] (see also [44], Ch. 3), where the authors proved the same result for the equation without the linear term $\widetilde{\mathscr{Q}}_3^{(1)}$. The main arguments in the proof are, on the one hand, the invariance of the problem (3.1.67) with respect to time translation and under a suitable scaling transformation. On the other hand, the fact that $\Lambda(\varphi) \ge 0$ on $\mathbb{R}^2_+$ for convex test functions φ, and that the map $\tau \mapsto \mathscr{Q}_3^{(2)}(\varphi, h(\tau))$ is locally integrable on $[0, T)$. When the linear term $\widetilde{\mathscr{Q}}_3^{(1)}$ is added, a slight modification of these argument still leads to the proof. Since by Lemma C.1.13, for all nonnegative, convex decreasing test function $\varphi \in C^1_b([0, \infty))$, we have $\widetilde{\mathscr{Q}}_3^{(1)}(\varphi, h) \le 0$, then solutions h to (3.1.67) are also super solutions (cf. Definition 3.1.13).

Proposition 3.4.5. *Let h be a super solution of (3.1.67). Then for any $R > 0$ and $\theta \in (0, 1)$*

$$\int_{[0,R]} h(\tau, x)dx \ge (1 - \theta) \int_{[0,\theta R]} h(\tau_0, x)dx \qquad \forall \tau \ge \tau_0 \ge 0. \tag{3.4.7}$$

Proof. Chose $\varphi_R(x) = (1 - x/R)_+$ for $R > 0$, and consider a sequence $(\varphi_{R,n})_{n\in\mathbb{N}} \subset C^1_b([0, \infty))$ such that $\varphi_{R,n} \to \varphi_R$, $\varphi_{R,n} \le \varphi_R$ and $\varphi_{R,n}(0) = 1$ for all $n \in \mathbb{N}$. Since by convexity $\mathscr{Q}_3^{(2)}(\varphi_{R,n}, h) \ge 0$, then for all τ and τ_0 with $\tau \ge \tau_0 \ge 0$,

$$\begin{aligned}\int_{[0,\infty)} \varphi_{R,n}(x)h(\tau, x)dx &\ge \int_{[0,\infty)} \varphi_{R,n}(x)h(\tau_0, x)dx \\ &\ge \int_{[0,\theta R]} \varphi_{R,n}(x)h(\tau_0, x)dx \ge \varphi_{R,n}(\theta R) \int_{[0,\theta R]} h(\tau_0, x)dx,\end{aligned}$$

and (3.4.7) follows since, if we let $n \to \infty$,

$$\int_{[0,R]} h(\tau, x)dx \ge \int_{[0,\infty)} \varphi_R(x)h(\tau, x)dx \ge \varphi_R(\theta R) \int_{[0,\theta R]} h(\tau_0, x)dx.$$

□

Lemma 3.4.6. *Let h be a super solution of (3.1.67). Let $R>0$ and consider a sequence $R:=a_0<a_1<a_2<...<a_n<...$ such that $|a_i-a_{i-1}|\le \frac{R}{2}$ for all $i\in\{1,2,3,...\}$. Then for all $\tau\ge\tau_0\ge 0$ there holds*

$$\int_{[0,R]} h(\tau,x)dx \ge \sum_{i=1}^{\infty}\frac{1}{2a_i}\int_{\tau_0}^{\tau}\left(\int_{(a_{i-1},a_i]} h(\sigma,x)dx\right)^2 d\sigma. \tag{3.4.8}$$

Proof. We chose φ_R and $\varphi_{R,n}$ as in the proof of Proposition 3.4.5 above. Since h is a super solution of (3.1.67), then for all $n\in\mathbb{N}$,

$$\frac{d}{d\tau}\int_{[0,\infty)} h(\tau,x)\varphi_{R,n}(x)dx \ge \mathscr{Q}_3^{(2)}(\varphi_{R,n},h(\tau)).$$

We have now:

$$\begin{aligned}\mathscr{Q}_3^{(2)}(\varphi_{R,n},h(\tau)) &\ge \iint_{(R,\infty)^2} h(\tau,x)h(\tau,y)\frac{\varphi_{R,n}(|x-y|)}{\sqrt{xy}}dxdy\\ &\ge \sum_{i=1}^{\infty}\frac{\varphi_{R,n}(R/2)}{a_i}\iint_{(a_{i-1},a_i]^2} h(\tau,x)h(\tau,y)dxdy\\ &= \sum_{i=1}^{\infty}\frac{\varphi_{R,n}(R/2)}{a_i}\left(\int_{(a_{i-1},a_i]} h(\tau,x)dx\right)^2.\end{aligned}$$

Estimate (3.4.8) follows in the limit $n\to\infty$, since $\varphi_{R,n}(R/2)\to 1/2$. □

Proposition 3.4.7. *Let h be a super solution of (3.1.67) with initial data $h_0\in\mathscr{M}_+^1([0,\infty))$, and denote $N=M_0(h_0)$ and $E=M_1(h_0)$. Then for all $R>0$, $\alpha\in\left(-\frac{1}{2},\infty\right)$, and τ_1 and τ_2 with $0\le\tau_1\le\tau_2$:*

$$\int_{\tau_1}^{\tau_2}\int_{(0,R]} x^\alpha h(\tau,x)dxd\tau \le \frac{2R^{\frac{1}{2}+\alpha}\sqrt{\tau_2-\tau_1}}{1-\left(\frac{2}{3}\right)^{\frac{1}{2}+\alpha}}\left(\frac{\sqrt{E}}{2}\tau_2+\sqrt{N}\right). \tag{3.4.9}$$

Proof. Since h is a super solution of (3.1.67), if we chose $\varphi(x)=(1-x/r)_+^2$ for $r>0$, then

$$\int_{[0,\infty)}\varphi(x)h(\tau_2,x)dx \ge \int_{\tau_1}^{\tau_2}\mathscr{Q}_3^{(2)}(\varphi,h(\tau))d\tau. \tag{3.4.10}$$

Since $\operatorname{supp}\Lambda(\varphi)=\{(x,y)\in[0,\infty)^2:|x-y|\le r\}$ and $\Lambda(\varphi)(x,y)=\varphi(|x-y|)$ for all $(x,y)\in[r,\infty)^2$, then for all $\tau\ge 0$:

$$\begin{aligned}\mathscr{Q}_3^{(2)}(\varphi,h(\tau)) &\ge \iint_{\left(r,\frac{3r}{2}\right]^2}\frac{\varphi(|x-y|)}{\sqrt{xy}}h(\tau,x)h(\tau,y)dxdy\\ &\ge \frac{1}{4}\left(\int_{\left(r,\frac{3r}{2}\right]}\frac{h(\tau,x)}{\sqrt{x}}dx\right)^2.\end{aligned}$$

If we use that $\varphi\le 1$ in the left hand side of (3.4.10), and the estimate above in the right hand side, then

$$\int_{\tau_1}^{\tau_2}\left(\int_{\left(r,\frac{3r}{2}\right]}\frac{h(\tau,x)}{\sqrt{x}}dx\right)^2 d\tau \le 4M_0(h(\tau_2)).$$

Since for any $\alpha \in (-1/2, \infty)$

$$\int_{\left(r, \frac{3r}{2}\right]} \frac{h(\tau,x)}{\sqrt{x}} dx \geq \left(\frac{3r}{2}\right)^{-\alpha-\frac{1}{2}} \int_{\left(r, \frac{3r}{2}\right]} x^{\alpha} h(\tau,x) dx,$$

we then obtain

$$\int_{\tau_1}^{\tau_2} \left(\int_{\left(r, \frac{3r}{2}\right]} x^{\alpha} h(\tau,x) dx\right)^2 d\tau \leq 4M_0(h(\tau_2)) \left(\frac{3r}{2}\right)^{1+2\alpha}. \tag{3.4.11}$$

For any given $R > 0$, using the decomposition

$$(0, R] = \bigcup_{k=0}^{\infty} (a_{k+1}, a_k], \qquad a_k = \left(\frac{2}{3}\right)^k R,$$

and Cauchy-Schwarz inequality we obtain

$$\int_{\tau_1}^{\tau_2} \int_{(0,R]} x^{\alpha} h(\tau,x) dx d\tau \leq \sqrt{\tau_2 - \tau_1} \sum_{k=0}^{\infty} \left(\int_{\tau_1}^{\tau_2} \left(\int_{(a_{k+1}, a_k]} x^{\alpha} h(\tau,x) dx\right)^2 d\tau\right)^{\frac{1}{2}}.$$

If we chose $r = a_{k+1}$ so that $(a_{k+1}, a_k] = (r, (3/2)r]$ for every $k \in \mathbb{N}$, then by (3.4.11) we deduce

$$\int_{\tau_1}^{\tau_2} \int_{(0,R]} x^{\alpha} h(\tau,x) dx d\tau \leq 2\sqrt{(\tau_2 - \tau_1) M_0(h(\tau_2))} \sum_{k=0}^{\infty} a_k^{\frac{1}{2}+\alpha}.$$

Using the estimate (3.3.4) for $M_0(h(\tau_2))$ and

$$\sum_{k=0}^{\infty} a_k^{\frac{1}{2}+\alpha} = \frac{R^{\frac{1}{2}+\alpha}}{1 - \left(\frac{2}{3}\right)^{\frac{1}{2}+\alpha}},$$

we finally obtain (3.4.9). □

Lemma 3.4.8. *Let h be a super solution of (3.1.67). Then for all $r > 0$, $\tau \geq \tau_0 \geq 0$ and $n \in \mathbb{N}$:*

$$\int_{[0,r]} h(\tau,x) dx \geq \frac{1}{4^{n+1} r} \int_{\tau_0}^{\tau} \left(\int_{(r, r2^n]} h(\sigma,x) dx\right)^2 d\sigma. \tag{3.4.12}$$

Proof. Consider the decomposition

$$(r, 2^n r] = \bigcup_{i=3}^{2^{n+1}} \left(\frac{r}{2}(i-1), \frac{r}{2} i\right].$$

Then by Lemma 3.4.6, and Lemma 3.12 in [45], we have

$$\begin{aligned}
\int_{[0,r]} h(\tau,x) dx &\geq \int_{\tau_0}^{\tau} \sum_{i=3}^{2^{n+1}} \frac{1}{ri} \left(\int_{\left(\frac{r}{2}(i-1), \frac{r}{2} i\right]} h(\sigma,x) dx\right)^2 d\sigma \\
&\geq \int_{\tau_0}^{\tau} \frac{1}{r} \left(\sum_{i=3}^{2^{n+1}} i\right)^{-1} \left(\int_{(r, r2^n]} h(\sigma,x) dx\right)^2 d\sigma \\
&\geq \frac{1}{(2^n - 1)(2^{n+1} + 3) r} \int_{\tau_0}^{\tau} \left(\int_{(r, r2^n]} h(\sigma,x) dx\right)^2 d\sigma.
\end{aligned}$$

Notice that $(2^n - 1)(2^{n+1} + 3) \leq 4^{n+1}$. □

The next Lemma takes into account the linear term $\widetilde{\mathscr{Q}}_3^{(1)}$.

Lemma 3.4.9. *Let h be a solution of (3.1.67) with initial data $h_0 \in \mathscr{M}_+^1([0,\infty))$ satisfying*

$$m_0 = \int_{(0,\infty)} h_0(x)dx > 0. \tag{3.4.13}$$

Then, for any $\tau_0 \geq 0$ there exist $R_1 > 0$, $C_1 > 0$ such that

$$\int_{[0,r]} h(\tau,x)dx \geq C_1\, r \qquad \forall r \in [0,R_1], \quad \forall \tau \geq \tau_0. \tag{3.4.14}$$

Proof. By (3.4.13), there exist $0 < a \leq b < \infty$ such that

$$\int_{(a,b]} h_0(x)dx > \frac{m_0}{2}. \tag{3.4.15}$$

We prove now

$$\exists T' > 0; \forall \tau \in [0,T') : \int_{\left(\frac{a}{2},2b\right]} h(\tau,x)dx \geq \frac{m_0}{4}. \tag{3.4.16}$$

To this end we use (3.1.67) with a test function $\varphi \in C_c^1([0,\infty))$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $(a,b]$ and $\varphi = 0$ on $[0,\infty) \setminus \left(\frac{a}{2},2b\right]$ and (3.4.15) to obtain:

$$\int_{\left(\frac{a}{2},2b\right]} h(\tau,x)dx \geq \frac{m_0}{2} + \int_0^\tau \widetilde{\mathscr{Q}}_3(\varphi,h(\sigma))d\sigma. \tag{3.4.17}$$

Now using (C.1.2) and (3.3.4) we deduce

$$\left|\mathscr{Q}_3^{(2)}(\varphi,h(\sigma))\right| \leq 2\|\varphi'\|_\infty \left(\frac{\sqrt{M_1(h_0)}}{2}\sigma + \sqrt{M_0(h_0)}\right)^4.$$

Using now $\frac{|\mathcal{L}(\varphi)(x)|}{\sqrt{x}} \leq 3\|\varphi\|_\infty\sqrt{x}$ and $M_{1/2}(h) \leq \sqrt{M_0(h)M_1(h)}$, we have by the conservation of energy and the mass inequality

$$\left|\widetilde{\mathscr{Q}}_3^{(1)}(\varphi,h(\sigma))\right| \leq 2\|\varphi\|_\infty \sqrt{M_1(h_0)}\left(\frac{M_1(h_0)}{2}\sigma + \sqrt{M_0(h_0)}\right).$$

It follows that $\widetilde{\mathscr{Q}}_3(\varphi,h) \in L^1_{loc}(\mathbb{R}_+)$ and we deduce (3.4.16) from (3.4.17).

By Lemma 3.4.8 and (3.4.16), for any $r \in \left(0,\frac{a}{2}\right]$ and $n \in \mathbb{N}$ such that $r2^n \in (2b,3b]$ we have

$$\begin{aligned}\int_{[0,r]} h(\tau,x)dx &\geq \int_0^\tau \frac{1}{4^{n+1}r}\left(\int_{\left(\frac{a}{2},2b\right]} h(\sigma,x)dx\right)^2 d\sigma \\ &\geq \frac{\tau}{4^{n+1}r}\left(\frac{m_0}{4}\right)^2 \geq \frac{m_0^2}{4^3(3b)^2}\tau\, r \qquad \forall \tau \in [0,T'].\end{aligned} \tag{3.4.18}$$

where $\left(\frac{a}{2},2b\right] \subset (r,r2^n]$ has been used.

For any given $\tau_0 \geq 0$ define $\tau' = \min\{\tau_0,T'\}$. Then by (3.4.7) in Proposition 3.4.5 with $\theta = \frac{1}{2}$ and $R = 2r$, we deduce from (3.4.18):

$$\int_{[0,2r]} h(\tau,x)dx \geq \frac{C\tau'}{2}r \qquad \forall \tau \geq \tau'. \tag{3.4.19}$$

and this proves the Lemma, where $R_1 = a/2$ and $C_1 = C\tau'/4$. □

Proposition 3.4.10. *Let h and h_0 be as in Lemma 3.4.9. For all $L > 0$ and every $\tau_1 > 0$ there exists $R_0 = R_0(h, L, \tau_1) > 0$ such that*

$$\int_{[0,R_0]} h(\tau, x)dx \geq LR_0 \qquad \forall \tau \geq \tau_1. \tag{3.4.20}$$

Proof. By Lemma 3.4.9 for $\tau_0 = \frac{\tau_1}{2}$

$$\exists C_1 > 0,\ \exists R_1 > 0; \int_{[0,r]} h(\tau, x)dx \geq C_1 r, \quad \forall r \in [0, R_1],\ \forall \tau \geq \frac{\tau_1}{2}. \tag{3.4.21}$$

Now fix an integer $p \geq 2$ such that $C_1 p \geq 8L$. We divide the proof in two parts. Assume first :

$$\exists r' \in (0, R_1],\ \exists \tau' \in \left[\frac{\tau_1}{2}, \tau_1\right] : \int_{\left[0, \frac{r'}{p}\right]} h(\tau', x)dx \geq \frac{C_1 r'}{2}. \tag{3.4.22}$$

It follows from lemma 3.4.5 with $\theta = \frac{1}{2}$ and $R = \frac{2r'}{p}$ that

$$\int_{\left[0, \frac{2r'}{p}\right]} h(\tau, x)dx \geq \frac{C_1 r'}{4} \qquad \forall \tau \geq \tau',$$

If we take $R_0 := \frac{2r'}{p}$, we have, by our choice of p,

$$\int_{[0,R_0]} h(\tau, x)dx \geq \frac{C_1 p}{8} R_0 \geq LR_0 \qquad \forall \tau \geq \tau',$$

so (3.4.20) holds.

Assume now that (3.4.22) does not hold, then, by (3.4.21):

$$\int_{\left(\frac{r}{p}, r\right]} h(\tau, x)dx \geq \frac{C_1 r}{2} \qquad \forall r \in (0, R_1], \quad \forall \tau \in \left[\frac{\tau_1}{2}, \tau_1\right]. \tag{3.4.23}$$

Take now any $r \in \left(0, \frac{R_1}{p}\right]$, let $n \in \mathbb{N}$ be the largest integer such that $rp^n \in \left(\frac{R_1}{p}, R_1\right]$, and consider now the following decomposition

$$(r, rp^n] = \bigcup_{i=p+1}^{p^{n+1}} \left(\frac{r}{p}(i-1), \frac{r}{p}i\right] = \bigcup_{k=1}^{n} \bigcup_{i=p^k+1}^{p^{k+1}} \left(\frac{r}{p}(i-1), \frac{r}{p}i\right].$$

By lemma 3.4.6 on $(\tau_1/2, \tau_1)$ with $a_i = ri/p$, $i = p+1, \cdots, p^{n+1}$:

$$\begin{aligned}
\int_{[0,r]} h(\tau_1, x)dx &\geq \int_{\frac{\tau_1}{2}}^{\tau_1} \left[\frac{p}{2r} \sum_{i=p+1}^{p^{n+1}} \frac{1}{i} \left(\int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x)dx\right)^2\right] d\sigma \\
&= \int_{\frac{\tau_1}{2}}^{\tau_1} \left[\frac{p}{2r} \sum_{k=1}^{n} \sum_{i=p^k+1}^{p^{k+1}} \frac{1}{i} \left(\int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x)dx\right)^2\right] d\sigma \\
&\geq \int_{\frac{\tau_1}{2}}^{\tau_1} \left[\frac{1}{2r} \sum_{k=1}^{n} \frac{1}{p^k} \sum_{i=p^k+1}^{p^{k+1}} \left(\int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x)dx\right)^2\right] d\sigma.
\end{aligned} \tag{3.4.24}$$

We use now Lemma 3.12 in [45]

$$\sum_{i=p^k+1}^{p^{k+1}}\left(\int_{\left(\frac{r}{p}(i-1),\frac{r}{p}i\right]} h(\sigma,x)dx\right)^2 \geq \frac{1}{p^k(p-1)}\times$$
$$\times\left(\sum_{i=p^k+1}^{p^{k+1}}\int_{\left(\frac{r}{p}(i-1),\frac{r}{p}i\right]} h(\sigma,x)dx\right)^2 \geq \frac{1}{p^{k+1}}\left(\int_{(rp^{k-1},rp^k]} h(\sigma,x)dx\right)^2$$

and deduce

$$\int_{[0,r]} h(\tau_1,x)dx \geq \int_{\frac{\tau_1}{2}}^{\tau_1}\left[\frac{1}{2r}\sum_{k=1}^{n}\frac{1}{p^{2k+1}}\left(\int_{(rp^{k-1},rp^k]} h(\sigma,x)dx\right)^2\right]d\sigma.$$

Due to the choice of the integer n, $rp^k \in (0,R_1]$ for all $k=1,\cdots,n$, and we can use (3.4.23) on each interval $(rp^{k-1},rp^k]$ to obtain:

$$\int_{[0,r]} h(\tau_1,x)dx \geq \int_{\frac{\tau_1}{2}}^{\tau_1}\left[\frac{1}{2r}\sum_{k=1}^{n}\frac{1}{p^{2k+1}}\left(\frac{C_1rp^k}{2}\right)^2\right]d\sigma = \frac{\tau_1C_1^2n}{16p}\,r.$$

It then follows from lemma 3.4.5 with $\theta=\frac{1}{2}$ and $R=2r$ that

$$\int_{[0,2r]} h(\tau,x)dx \geq \frac{\tau_1C_1^2n}{32p}\,r \qquad \forall\tau\geq\tau_1. \tag{3.4.25}$$

Since $rp^n \in \left(\frac{R_1}{p},R_1\right]$, then $n \geq \frac{\log\left(\frac{R_1}{rp}\right)}{\log(p)}$, and we chose $r>0$ small enough in order to have $r\in(0,R_1/p)$ and

$$\frac{\tau_1C_1^2}{64p}\frac{\log\left(\frac{R_1}{rp}\right)}{\log p} \geq L;$$

and set $R_0 := 2r$. The result then follows from (3.4.25). □

Lemma 3.4.11. *Let h be a solution of (3.1.67) and, for any $\kappa>0$ and $\lambda>0$, consider the rescaled measure $h_{\kappa,\lambda}$ defined as:*

$$\int_{[0,\infty)} h_{\kappa,\lambda}(\tau,x)\varphi(x)dx = \kappa\int_{[0,\infty)} h(\kappa\lambda\tau,x)\varphi\left(\frac{x}{\lambda}\right)dx,\ \forall\varphi\in C_b([0,\infty)). \tag{3.4.26}$$

Then $h_{\kappa,\lambda}$ is a super solution of (3.1.67).

Proof. Let $\varphi\in C_b^1([0,\infty))$ be nonnegative, convex and decreasing, $\psi(x)=\varphi(x/\lambda)$, and $\eta=\kappa\lambda\tau$. By Lemma C.1.13, $\widetilde{\mathscr{Q}}_3^{(1)}(\psi,h)\leq 0$, and by (3.1.67)

$$\frac{d}{d\eta}\int_{[0,\infty)}\psi(x)h(\eta,x)dx \geq \mathscr{Q}_3^{(2)}(\psi,h(\eta)).$$

Since $\mathscr{Q}_3^{(2)}(\psi,h(\eta)) = \kappa^{-2}\lambda^{-1}\mathscr{Q}_3^{(2)}(\varphi,h_{\kappa,\lambda}(\tau))$, then

$$\frac{d}{d\tau}\int_{[0,\infty)}\varphi(x)h_{\kappa,\lambda}(\tau,x)dx = \kappa^2\lambda\frac{d}{d\eta}\int_{[0,\infty)}\psi(x)h(\eta,x)dx \geq \mathscr{Q}_3^{(2)}(\varphi,h_{\kappa,\lambda}(\tau)).$$

□

Lemma 3.4.12. *Let h be a super solution of (3.1.67). Suppose that there exists $\tau'>0$ such that*

$$\int_{[0,1]} h(\tau,x)dx \geq 1 \qquad \forall \tau \geq \tau'. \tag{3.4.27}$$

Then for any given $\delta>0$ there exist τ_0 such that

$$\tau' \leq \tau_0 \leq \tau' + T_0(\delta), \qquad T_0(\delta) = \frac{64}{\delta^3}\left(1-\frac{\delta}{2}\right) \tag{3.4.28}$$

$$\textit{and} \quad \int_{[0,\frac{\delta}{4}]} h(\tau_0,x)dx \geq 1-\frac{\delta}{2}. \tag{3.4.29}$$

Proof. The statement of the Lemma is equivalent to show that the following set

$$A := \left\{\tau \in [\tau', \tau'+T_0(\delta)] : \int_{[0,\frac{\delta}{4}]} h(\tau,x)dx \geq 1-\frac{\delta}{2}\right\}.$$

is non empty, where $T_0(\delta)$ is defined in (3.4.28). To this end we first apply Lemma 3.4.6 with $a_0=\frac{\delta}{4}$, $a_i=\frac{\delta}{4}\left(1+\frac{i}{2}\right)$ for $i\in\{1,...,n-1\}$ and $a_n=1$. The number n is chosen to be the largest integer such that $a_{n-1}<1$, which implies

$$\frac{1}{n+1} > \frac{\delta}{8}. \tag{3.4.30}$$

Then, using ${a_i}^{-1} \geq 1$ for all $i\in\{1,...,n\}$:

$$\int_{[0,\frac{\delta}{4}]} h(\tau,x)dx \geq \frac{1}{2}\int_{\tau'}^{\tau}\sum_{i=1}^{n}\left(\int_{(a_{i-1},a_i]} h(\sigma,x)dx\right)^2 d\sigma, \ \forall \tau>\tau'.$$

Since by Lemma 3.12 in [45] and (3.4.30):

$$\sum_{i=1}^{n}\left(\int_{(a_{i-1},a_i]} h(\sigma,x)dx\right)^2 \geq \frac{\delta}{8}\left(\int_{(\frac{\delta}{4},1]} h(\sigma,x)dx\right)^2,$$

we obtain, for all $\tau>\tau'$

$$\int_{[0,\frac{\delta}{4}]} h(\tau,x)dx \geq \frac{\delta}{16}\int_{\tau'}^{\tau}\left(\int_{(\frac{\delta}{4},1]} h(\sigma,x)dx\right)^2 d\sigma. \tag{3.4.31}$$

Arguing by contradiction suppose that $A=\emptyset$:

$$\int_{(0,\frac{\delta}{4}]} h(\tau,x)dx < 1-\frac{\delta}{2} \qquad \forall \tau \in [\tau', \tau'+T_0(\delta)]$$

and by (3.4.27):

$$\int_{(\frac{\delta}{4},1]} h(\tau,x)dx \geq \frac{\delta}{2} \qquad \forall \tau \in [\tau', \tau'+T_0(\delta)].$$

It follows from (3.4.31) that $1-\frac{\delta}{2} > \frac{\delta^3}{64}(\tau-\tau')$ for all $\tau\in[\tau',\tau'+T_0(\delta)]$ which is a contradiction for $\tau=\tau'+T_0(\delta)$. □

Proposition 3.4.13. *Let h be a solution of (3.1.67). Suppose that there exist m, $R>0$ such that*

$$\int_{[0,R]} h(\tau,x)dx \geq m \qquad \forall \tau \in [0,\infty). \tag{3.4.32}$$

Then given any $\alpha\in(0,1)$ there exists $T_* = T_*(\alpha)>0$ such that

$$\int_{[0,r]} h(\tau,x)dx \geq \frac{m}{(2R)^\alpha}\, r^\alpha \qquad \forall r\in[0,R], \quad \forall \tau \in \left[\frac{RT_*}{m},\infty\right). \tag{3.4.33}$$

Proof. We argue by induction and define first the scaled measure $h_1 = h_{\kappa_1,\lambda_1}$, defined as in (3.4.26), that satisfies condition (3.4.27) for $\kappa_1 = \frac{1}{m},\ \lambda_1 = R$. From Lemma 3.4.11, and Lemma 3.4.12 with $\tau'=0$, we deduce that for all $\delta\in(0,1)$ there exists $\tau_1>0$ such that:

$$0\leq \tau_1 \leq T_0(\delta), \qquad \int_{[0,\frac{\delta}{4}]} h_1(\tau_1,x)dx \geq 1-\frac{\delta}{2}.$$

Then from Lemma 3.4.11, and Proposition 3.4.5 with $\theta=\delta/2$ and $R=1/2$,

$$\int_{[0,\frac{1}{2}]} h_1(\tau,x)dx \geq \left(1-\frac{\delta}{2}\right)^2, \ \ \forall \tau \geq T_0(\delta),$$

$$\int_{[0,\frac{R}{2}]} h\left(\tau,x\right)dx \geq m\left(1-\delta\right), \ \ \forall \tau \geq \frac{R}{m} T_0(\delta). \tag{3.4.34}$$

Exactly as before we now define $h_2 = h_{\kappa_2,\lambda_2}$ as in (3.4.26), that satisfies condition (3.4.27) for $\kappa_2 = \frac{1}{m(1-\delta)^2},\ \lambda_2 = \frac{R}{2},\ \tau' = 2(1-\delta)T_0(\delta)$. The same argument gives then:

$$\int_{[0,\frac{R}{4}]} h\left(\tau,x\right)dx \geq m\left(1-\delta\right)^2, \quad \forall \tau \geq \frac{RT_0(\delta)}{m}\left(1+\frac{1}{2(1-\delta)}\right). \tag{3.4.35}$$

We deduce after n iterations

$$\int_{[0,\frac{R}{2^n}]} h\left(\tau,x\right)dx \geq m\left(1-\delta\right)^n, \quad \forall \tau \geq \frac{RT_0(\delta)}{m}\sum_{k=0}^{n-1}\frac{1}{2^k(1-\delta)^k} \tag{3.4.36}$$

If we chose $\delta = 1-2^{-\alpha}$, for any $0<\alpha<1$, we may define

$$T_* = T_0(\delta)\sum_{k=0}^{\infty} 2^{-(1-\alpha)k} = \frac{T_0(\delta)}{1-2^{-(1-\alpha)}}. \tag{3.4.37}$$

Since for any $r\in(0,R)$ there exists $n\in\mathbb{N}$ such that $r\in\left(\frac{R}{2^n},\frac{R}{2^{n-1}}\right]$,

$$\int_{[0,r]} h\left(\tau,x\right)dx \geq m2^{-\alpha n}, \ \ \forall \tau > \frac{RT_*}{m}$$

and using $2^{-n} > r/2R$, (3.4.33) follows. □

Proposition 3.4.14. *Let h be a solution of (3.1.67). Then, for all $\tau_0>0$ and for any $\alpha\in(0,1)$ there exists $R_* = R_*(h,\tau_0,\alpha)>0$ such that*

$$\int_{[0,r]} h(\tau,x)dx \geq C\, r^\alpha \qquad \forall r\in[0,R_*] \quad \forall \tau\in[\tau_0,\infty), \tag{3.4.38}$$

where $C = \frac{T_*(\alpha)}{\tau_0}(2R_*)^{1-\alpha}$, and $T_*(\alpha)$ is given by Proposition 3.4.13.

Proof. By Proposition 3.4.10 with $L > 0$ and for $\tau_1 = \tau_0/2$, there exists $R_0(h, L, \tau_1) > 0$ such that

$$\int_{[0,R_0]} h(\tau, x)dx \geq LR_0 \qquad \forall \tau \geq \frac{\tau_0}{2}.$$

Then by Proposition 3.4.13, with $m = LR_0$ and $R = R_0$, we obtain that for any given $\alpha \in (0, 1)$ there exists $T_* = T_*(\alpha) > 0$ such that

$$\int_{[0,r]} h(\tau, x)dx \geq \frac{LR_0}{(2R_0)^\alpha} r^\alpha \qquad \forall r \in [0, R_0], \quad \forall \tau \in \left[\frac{\tau_0}{2} + \frac{T_*}{L}, \infty\right).$$

If we chose $L = 2T_*/\tau_0$, then the Proposition follows with $R_* = R_0$. □

Proof of Theorem 3.4.1. By Lemma 3.4.2 the map $\tau \mapsto h(\tau, \{0\})$ is right continuous, nondecreasing and a.e. differentiable on $[0, \infty)$. It remains to prove that it is actually strictly increasing. We first suppose that h_0 is such that

$$\int_{\{0\}} h_0(x)dx = 0, \qquad \int_{(0,\infty)} h_0(x)dx > 0, \tag{3.4.39}$$

and prove

$$h(\tau, \{0\}) > 0 \qquad \forall \tau > 0. \tag{3.4.40}$$

Arguing by contradiction, if we suppose that there exists $\tau_0 > 0$ such that $h(\tau_0, \{0\}) = 0$, by monotonicity $h(\tau, \{0\}) = 0$ for all $\tau \in [0, \tau_0]$. In particular

$$\int_{\frac{\tau_0}{2}}^{\tau_0} \int_{[0,r]} h(\sigma, x)dxd\sigma = \int_{\frac{\tau_0}{2}}^{\tau_0} \int_{(0,r]} h(\sigma, x)dxd\sigma \tag{3.4.41}$$

for all $r > 0$. Now using Proposition 3.4.7 with $\alpha = 0$, and Proposition 3.4.14, we deduce that, for any $\alpha \in (0, 1/2)$, there exists $R_* = R_*(h, \tau_0/2, \alpha)$ such that

$$C_2\, r^\alpha \leq \int_{\frac{\tau_0}{2}}^{\tau_0} \int_{(0,r]} h(\sigma, x)dxd\sigma \leq C_1\sqrt{r}, \qquad \forall r \in [0, R_*];$$

$$C_1 = 8\sqrt{\frac{\tau_0}{2}} \left(\frac{\sqrt{M_1(h_0)}}{2}\tau_0 + \sqrt{M_0(h_0)}\right), \quad C_2 = \frac{T_*(\alpha)}{2}(2R_*)^{1-\alpha},$$

and that leads to a contradiction for r small enough.

Consider now a general initial data h_0 such that $\int_{\{0\}} h_0(x)dx > 0$. Let h be a solution of (3.1.67) with initial data h_0 and define

$$\tilde{h}(\tau) = h(\tau) - h_0(\{0\})\delta_0.$$

Then, on the one hand, the initial data of $\tilde{h}$ satisfies $\tilde{h}(0, \{0\}) = 0$. On the other hand we claim that $\tilde{h}$ is still a solution of (3.1.67). Notice indeed that $\tilde{h}_\tau \equiv h_\tau$ and, moreover, $\widetilde{\mathscr{Q}}_3(\varphi, h(\tau)) = \widetilde{\mathscr{Q}}_3(\varphi, \tilde{h}(\tau))$. Using the previous case

$$\int_{\{0\}} \tilde{h}(\tau, x)dx > 0, \quad \forall \tau > 0,$$

and then

$$\int_{\{0\}} h(\tau, x)dx > \int_{\{0\}} h_0(x)dx, \quad \forall \tau > 0.$$

The Theorem follows using now the time translation invariance of the equation. □

The last result of this section describes the relation between the Lebesgue-Stieltjes measure associated to the (right continuous and strictly increasing) function $m(\tau)=h(\tau,\{0\})$, and the equation for h (3.1.67).

Proposition 3.4.15. *Let h be a solution of (3.1.67) for a initial data $h_0\in\mathscr{M}_+^1([0,\infty))$ with $N=M_0(h_0)>0$ and $E=M_1(h_0)>0$. If we denote $m(\tau)=h(\tau,\{0\})$ and λ is the Lebesgue-Stieltjes measure associated to m, then for all φ_ε as in Remark 3.1.6 and for all τ_1 and τ_2 with $0\le\tau_1<\tau_2$:*

$$m(\tau_2)-m(\tau_1)=\lambda((\tau_1,\tau_2]), \tag{3.4.42}$$

$$\lambda((\tau_1,\tau_2])=\lim_{\varepsilon\to 0}\int_{\tau_1}^{\tau_2}\mathscr{Q}_3^{(2)}(\varphi_\varepsilon,h(\tau))d\tau, \tag{3.4.43}$$

and

$$0<\lambda((\tau_1,\tau_2]))<\infty. \tag{3.4.44}$$

Furthermore, for all φ_ε as in Remark 3.1.6

$$\lim_{\varepsilon\to 0}\mathscr{Q}_3^{(2)}(\varphi_\varepsilon,h)\in\mathscr{D}'(0,\infty), \tag{3.4.45}$$

and if we denote m' the derivative in the sense of Distributions of m, then

$$m'=\lambda=\lim_{\varepsilon\to 0}\mathscr{Q}_3^{(2)}(\varphi_\varepsilon,h)\quad in\quad \mathscr{D}'(0,\infty). \tag{3.4.46}$$

Proof. By Lemma 3.4.2, m is right continuous and nondecreasing on $[0,\infty)$. Then it has a Lebesgue-Stieltjes measure associated to it, λ, that satisfies (3.4.42) (c.f. [**?**] Ch.1).

On the other hand, since h is a solution of (3.1.67), using φ_ε as in Remark 3.1.6 and taking the limit $\varepsilon\to 0$, it follows from (C.1.25) in Lemma C.1.20 that for all τ_1 and τ_2 with $0\le\tau_1<\tau_2$:

$$m(\tau_2)-m(\tau_1)=\lim_{\varepsilon\to 0}\int_{\tau_1}^{\tau_2}\mathscr{Q}_3^{(2)}(\varphi_\varepsilon,h(\sigma))d\sigma, \tag{3.4.47}$$

and then (3.4.43) follows from (3.4.42). Moreover, since by Theorem 3.4.1 m is strictly increasing, then (3.4.44) holds.

Notice that the limit in (3.4.47) is independent of the choice of the test function φ_ε. Indeed, if ψ_ε is another test function as in Remark 3.1.6, since for all $\tau\ge 0$

$$\lim_{\varepsilon\to 0}\int_{[0,\infty)}\psi_\varepsilon(x)h(\tau,x)dx=m(\tau)=\lim_{\varepsilon\to 0}\int_{[0,\infty)}\varphi_\varepsilon(x)h(\tau,x)dx,$$

it follows from (3.4.47) that for all $0\le\tau_1\le\tau_2$

$$\lim_{\varepsilon\to 0}\int_{\tau_1}^{\tau_2}\mathscr{Q}_3^{(2)}(\psi_\varepsilon,h(\sigma))d\sigma=\lim_{\varepsilon\to 0}\int_{\tau_1}^{\tau_2}\mathscr{Q}_3^{(2)}(\varphi_\varepsilon,h(\sigma))d\sigma.$$

Now, for all φ_ε as in Remark 3.1.6, consider the absolutely continuous function

$$\theta_\varepsilon(\tau)=\int_{[0,\infty)}\varphi_\varepsilon(x)h(\tau,x)dx.$$

Then the equation in (3.3.2) reads $\theta_\varepsilon'(\tau) = \widetilde{\mathscr{Q}}_3(\varphi_\varepsilon, h(\tau))$. Using integration by parts we deduce that for all $\varepsilon > 0$:

$$-\int_0^\infty \phi'(\tau)\theta_\varepsilon(\tau)d\tau = \int_0^\infty \phi(\tau)\widetilde{\mathscr{Q}}_3(\varphi_\varepsilon, h(\tau))d\tau \quad \forall \phi \in C_c^\infty(0,\infty).$$

Taking the limit $\varepsilon \to 0$ it then follows from Lemma C.1.20 that

$$-\int_0^\infty \phi'(\tau)m(\tau)d\tau = \lim_{\varepsilon\to 0}\int_0^\infty \phi(\tau)\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, h(\tau))d\tau,$$

hence, $m' = \lim_{\varepsilon\to 0}\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, h)$. On the other hand, by Fubini's theorem

$$\int_0^\infty \phi(\tau)d\lambda(\tau) = \int_0^\infty\int_0^\tau \phi'(\sigma)d\sigma d\lambda(\tau) = -\int_0^\infty \phi'(\sigma)m(\sigma)d\sigma$$

for all $\phi \in C_c^\infty(0,\infty)$ (cf. [64], Example 6.14), thus $m' = \lambda$. □

3.5 Existence of solutions G, proof of Theorem 3.1.3.

Given a initial data $G_0 \in \mathscr{M}_+^1$ as in Theorem 3.1.3, let $h \in C\big([0,\infty), \mathscr{M}_+([0,\infty))\big)$ satisfy (3.3.1)–(3.3.5), (3.3.7), given by (3.3.2) and H defined by (3.3.10) and satisfying (3.3.11)–(3.3.15), (3.3.17) by Corollary 3.3.2. It is natural, in view of the change of variables (3.1.65) to define now,

$$G(t) = H(\tau), \quad \tau = \int_0^t G(s,\{0\})ds. \tag{3.5.1}$$

Notice nevertheless that since $G(s,\{0\})$ is still unknown, (3.5.1) does not define $G(t)$ actually. What we know is rather, given $\tau > 0$, what would be the value of t such that

$$t = \int_0^\tau \frac{d\sigma}{H(\sigma,\{0\})}, \tag{3.5.2}$$

since we expect to have $G(s,\{0\}) = H(\sigma,\{0\})$ for s and σ such that

$$\sigma = \int_0^s G(r,\{0\})dr, \quad \text{or} \quad s = \int_0^\sigma \frac{d\rho}{H(\rho,\{0\})}.$$

If G is going to be defined in that way it is then necessary first to check that the range of values taken by the variable t in (3.5.2) is all of $[0,\infty)$. By definition (3.3.10),

$$H(\tau,\{0\}) = h(\tau,\{0\}) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma. \tag{3.5.3}$$

Since both terms in the right hand side are nonnegative, $H(\tau,\{0\})$ has no a priori definite sign. We must then consider that question in some detail. Our first step is to prove the following

Lemma 3.5.1. *If $G_0(\{0\}) > 0$, then*

$$\tau_* = \inf\{\tau > 0 : H(\tau,\{0\}) = 0\} > 0, \tag{3.5.4}$$

$$H(\tau_*,\{0\}) = 0, \tag{3.5.5}$$

$$H(\tau,\{0\}) > 0 \qquad \forall \tau \in [0,\tau_*). \tag{3.5.6}$$

Proof. $H(0) = G_0$ by (3.3.13), and then, using φ_ε as in Remark 3.1.6, we deduce $H(0, \{0\}) = G_0(\{0\})$, which is strictly positive by hypothesis. Then (3.5.4) follows from the right continuity of $H(\tau, \{0\})$ (cf. Corollary 3.4.3).

In order to prove (3.5.5) we use a minimizing sequence $(\tau_n)_{n\in\mathbb{N}}$, i.e., $\tau_n \geq \tau_*$, $H(\tau_n, \{0\}) = 0$ for every $n \in \mathbb{N}$, and $\tau_n \to \tau_*$ as $n \to \infty$. Then from the right continuity (3.5.5) holds.

Let us prove now (3.5.6). If $H(\tau_0, \{0\}) < 0$ for some $\tau_0 \in (0, \tau_*)$, then τ_0 must be a left discontinuity point of $H(\tau, \{0\})$ and

$$\limsup_{\delta\to 0^+} H(\tau_0 - \delta, \{0\}) > H(\tau_0, \{0\}),$$

and this would contradict (3.4.6). That proves (3.5.6). □

It follows from Lemma 3.5.1 that the function:

$$t = \xi(\tau) = \int_0^\tau \frac{d\sigma}{H(\sigma, \{0\})} \tag{3.5.7}$$

introduced in (3.5.2) is well defined, monotone nondecreasing and continuous on the interval $[0, \tau_*)$. We then define,

$$\forall t \in [0, \xi(\tau_*)) : \quad G(t) = H(\xi^{-1}(t)). \tag{3.5.8}$$

By (3.5.8) and (3.5.3), if $G(t) = G(t, \{0\})\delta_0 + g(t)$ and $H(\tau) = H(\tau, \{0\})\delta_0 + \tilde{h}(\tau)$, then

$$G(t, \{0\}) = H(\tau, \{0\}), \tag{3.5.9}$$

$$\tilde{h}(\tau) = h(\tau) - h(\tau, \{0\})\delta_0, \tag{3.5.10}$$

$$g(t) = \tilde{h}(\tau). \tag{3.5.11}$$

Remark 3.5.2. Formula (3.5.8) defines the function G at time $t \in (0, \xi(\tau_*))$ from the knowledge of the function $H(\tau)$ for $\tau > 0$ such that $\tau = \xi^{-1}(t)$. Moreover,

$$\forall t \in (0, \xi(\tau_*)) : \quad \xi^{-1}(t) = \int_0^t G(s, \{0\}) ds. \tag{3.5.12}$$

We have now,

Proposition 3.5.3. *The function G defined by (3.5.8) is such that*

$$G \in C\big([0, \xi(\tau_*)), \mathscr{M}_+^1([0, \infty))\big), \ G(0) = G_0 \tag{3.5.13}$$

and satisfies (3.1.45), (3.1.46), (3.1.48) and (3.1.49) on the time interval $[0, \xi(\tau_*))$.

Proof. We first prove that $G(t)$ is a positive measure for all $t \in [0, \xi(\tau_*))$. By (3.5.6) and (3.5.9) we have $G(t, \{0\}) > 0$ for all $t \in [0, \xi(\tau_*))$. Then, since $h(\tau)$ is a positive measure for all $\tau \in [0, \infty)$, we deduce from (3.5.11) and (3.5.10) that $g(t)$ is a positive measure for all $t \in [0, \xi(\tau_*))$. Hence $G(t) = G(t, \{0\})\delta_0 + g(t)$ is also positive.

All the properties of $G(t)$ at $t \in [0, \xi(\tau_*))$ fixed follow from the corresponding property of $H(\tau)$ with $t = \xi(\tau)$. The only property where t is not fixed are (3.1.44) and (3.1.45). Since

$$\left|\frac{\partial G(t)}{\partial t}\right| = \left|\frac{\partial \tau}{\partial t}\frac{\partial H(\tau)}{\partial \tau}\right| \leq |H(\tau, \{0\})| \left|\frac{\partial H(\tau)}{\partial \tau}\right|$$

By definition,

$$|H(\tau,\{0\})| \leq |h(\tau,\{0\})| + \int_0^\tau M_{1/2}(h(\sigma))d\sigma.$$

Since $h \in C([0,\infty), \mathscr{M}_+^1)$ it follows using also (3.3.4), (3.3.5) and Hölder inequality that $H(\tau,\{0\}) \in L^\infty_{loc}([0,\infty))$.Then, by (3.3.1), $G(t)$ is locally Lipschitz on $[0,\xi(\tau_*))$ and satisfies (3.1.45). Since H satisfies (3.3.2) the change of variables ensures that G satisfies (3.1.46). □

We prove now the following property of the function G defined in (3.5.8).

Proposition 3.5.4. *Let G be the function defined in (3.5.8) for $t \in (0,\xi(\tau_*))$. Then the map $t \mapsto G(t,\{0\})$ is right continuous and differentiable for almost every $t \in [0,\xi(\tau_*))$ and, for all $t_0 \in (0,\xi(\tau_*))$*

$$G(t,\{0\}) \geq G(t_0,\{0\})e^{-\int_{t_0}^t M_{1/2}(g(s))ds} \quad \forall t \in (t_0,\xi(\tau_*)). \tag{3.5.14}$$

In particular, if $G(0,\{0\}) > 0$, then $G(t,\{0\}) > 0$ for all $t \in (0,\xi(\tau_*))$.

Proof. Using (3.1.46) and (3.1.43) with φ_ε as in Remark 3.1.6, we have

$$\frac{d}{dt}\int_{[0,\infty)} \varphi_\varepsilon(x)G(t,x)dx + G(t,\{0\})M_{1/2}(G(t)) = G(t,\{0\})\widetilde{\mathscr{Q}}_3(\varphi_\varepsilon, G(t)). \tag{3.5.15}$$

We use now that for all $\varepsilon > 0$:

$$G(t,\{0\}) \leq \int_{[0,\infty)} \varphi_\varepsilon(x)G(t,x)dx, \tag{3.5.16}$$

and we deduce from (3.5.15), using $J(t) = \exp\left(\int_0^t M_{1/2}(G(s))ds\right)$,

$$\frac{d}{dt}\left(J(t)\int_{[0,\infty)} \varphi_\varepsilon(x)G(t,x)dx\right) \geq G(t,\{0\})J(t)\widetilde{\mathscr{Q}}_3(\varphi_\varepsilon, G(t)). \tag{3.5.17}$$

By Lemma C.1.13 the right hand side of (3.5.17) is nonnegative, and we deduce

$$J(t)\int_{[0,\infty)} \varphi_\varepsilon(x)G(t,x)dx \geq J(t_0)\int_{[0,\infty)} \varphi_\varepsilon(x)G(t_0,x)dx$$

for all $t \in (t_0,\xi(\tau_*))$ and all $\varepsilon > 0$. If we pass now to the limit as $\varepsilon \to 0$:

$$J(t)G(t,\{0\}) \geq J(t_0)G(t_0,\{0\}), \tag{3.5.18}$$

and this proves the estimate (3.5.14). It also follows from Lebesgue's Theorem that $J(t)G(t,\{0\})$ is differentiable for almost every $t \in (0,\xi(\tau_*))$ (cf. [63], Theorem 2). On the other hand, since $J(t)$ is a.e differentiable and $J(t) > 0$ for all $t \in [0,\xi(\tau_*))$, we deduce that $G(t,\{0\})$ is also differentiable for almost every $t \in [0,\xi(\tau_*))$.

We prove now the right continuity of $G(t,\{0\})$. It follows from (3.5.18),

$$J(t+\delta)G(t+\delta,\{0\}) \geq J(t)G(t,\{0\}), \quad \forall\delta > 0\ \forall t > 0.$$

If we take inferior limits and use that J is continuous and strictly positive we obtain,

$$\liminf_{\delta\to 0} G(t+\delta,\{0\}) \geq G(t,\{0\}), \quad \forall t>0. \tag{3.5.19}$$

Since $\mathcal{L}_0(\varphi_\varepsilon)\geq 0$ by convexity (cf. Lemma C.1.13), we deduce

$$\frac{d}{dt}\int_{[0,\infty)}\varphi_\varepsilon(x)G(t,x)dx \leq G(t,\{0\})\iint_{(0,\infty)^2}\frac{\Lambda(\varphi_\varepsilon)(x,y)}{\sqrt{xy}}G(t,x)G(t,y)dxdy,$$

and the argument follows now as in the proof of the right continuity of H. From the inequality (3.5.16), the bound (C.1.2) and the conservation of mass, we deduce for all $t\in[0,\xi(\tau_*))$ fixed and $\delta\in[0,\xi(\tau_*)-t)$,

$$G(t+\delta,\{0\}) \leq \int_{[0,\infty)}\varphi_\varepsilon(x)G(t,x)dx + \frac{2N^2\delta}{\varepsilon}\int_0^t G(s,\{0\})ds.$$

If we take superior limits as $\delta\to 0$, and then let $\varepsilon\to 0$ we obtain, using (3.4.2) with G instead of H:

$$\limsup_{\delta\to 0} G(t+\delta,\{0\}) \leq G(t,\{0\}).$$

and this combined with (3.5.19) proves that $G(t,\{0\})$ is right continuous on $[0,\xi(\tau_*))$.

□

In the next Lemma we prove that the function G defined by (3.5.8) is actually well defined for all $t>0$.

Lemma 3.5.5.

$$\lim_{\tau\to\tau_*^-}\xi(\tau)=\infty. \tag{3.5.20}$$

Proof. Since the function $\xi(\tau)$ is monotone nondecreasing and continuous on $[0,\tau_*)$, its limit as $\tau\to\tau_*^-$ exists in $\overline{\mathbb{R}}_+$. Let us call it ℓ and suppose $\ell\in\mathbb{R}_+$. Now, from (3.5.14) and the fact that G satisfies : $0\leq M_{1/2}(G(s))\leq\sqrt{NE}$, we deduce

$$\limsup_{t\to\ell^-} G(t,\{0\}) \geq e^{-\sqrt{NE}\ell}G(0,\{0\})>0, \tag{3.5.21}$$

and by (3.4.6)

$$H(\tau_*,\{0\}) \geq \limsup_{\tau\to\tau_*^-} H(\tau,\{0\}) = \limsup_{t\to\ell^-} G(t,\{0\})>0,$$

and this contradicts (3.5.5). This proves that $\ell=\infty$. □

Proof of Theorem 3.1.3. By Lemma 3.5.5 the function G is defined for all $t>0$. As we have seen in the proof of Lemma 3.5.5, $G(t)\in\mathscr{M}_+([0,\infty))$ for all $t>0$. It then follows from Proposition 3.5.3 that G satisfies now all the conditions (3.1.44)–(3.1.46) and (3.1.47)–(3.1.49). Property (3.1.50) follows from the corresponding estimate (3.3.6) for h. Similarly, property (3.1.52) follows from the property (3.3.7) of h. We prove now the point (iv). Suppose then $\alpha\in(1,3]$ and condition (3.1.53). For $\varphi(x)=x^\alpha$ we have,

$$\mathscr{Q}_3^{(1)}(\varphi,G(t)) = \left(\frac{\alpha-1}{\alpha+1}\right)M_{\alpha+\frac{1}{2}}(G(t)).$$

On the other hand, for $0 \le y \le x$,

$$\Lambda(\varphi)(x,y) = x^\alpha \left(\left(1+z\right)^\alpha + \left(1-z\right)^\alpha - 2 \right), \quad z = \frac{y}{x} \in [0,1],$$

If $\alpha \in (1,2]$, for all $x \ge y > 0$,

$$\frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}} \le (2^\alpha - 2) x^{\alpha - \frac{3}{2}} y^{\frac{1}{2}} \le (2^\alpha - 2)(xy)^{\frac{\alpha-1}{2}}.$$

We deduce

$$\mathscr{Q}_3^{(2)}(\varphi, G(t)) \le (2^\alpha - 2)\Big(M_{\frac{\alpha-1}{2}}(G(t)) \Big)^2.$$

and obtain

$$\frac{d}{dt} M_\alpha(G(t)) \le G(t,\{0\}) \left[C_{1,1} \Big(M_{\frac{\alpha-1}{2}}(G(t)) \Big)^2 - C_2 M_{\alpha + \frac{1}{2}}(G(t)) \right],$$

where $C_{1,1} = 2^\alpha - 2$ and $C_2 = (\alpha-1)/(\alpha+1)$. Using Hölder's inequality

$$\frac{d}{dt} M_\alpha(G(t)) \le G(t,\{0\}) \Big[C_{1,1} N^{3-\alpha} E^{\alpha-1} - C_2 E^{(2\alpha+1)/2} N^{(1-2\alpha)/2} \Big]. \tag{3.5.22}$$

By (3.1.53), the right hand side of (3.5.22) is negative, and then $M_\alpha(G(t))$ is decreasing on $(0,\infty)$.

For $\alpha \in [2,3]$ we use the estimate (3.3.36) with $C_{1,2} = \alpha(\alpha-1)$ instead of C_α. Then we proceed as in the previous case to obtain

$$\frac{d}{dt} M_\alpha(G(t)) \le G(t,\{0\}) \Big[C_{1,2} N^{3-\alpha} E^{\alpha-1} - C_2 E^{(2\alpha+1)/2} N^{(1-2\alpha)/2} \Big]. \tag{3.5.23}$$

As before, (3.1.53) implies that the right hand side of (3.5.23) is negative, and then $M_\alpha(G(t))$ is decreasing. □

Proof of Theorem 3.1.4. By construction

$$G(t) = H(\tau) = h(\tau) - \left(\int_0^\tau M_{1/2}(h(\sigma)) d\sigma \right) \delta_0,$$

where τ and t are related by

$$t = \xi(\tau) = \int_0^\tau \frac{d\sigma}{H(\sigma,\{0\})}; \qquad \tau = \xi^{-1}(t) = \int_0^t G(s,\{0\}) ds. \tag{3.5.24}$$

Therefore $G(t,x) = h(\tau,x)$ for $x \in (0,\infty)$, and

$$\int_0^T G(t,\{0\}) \int_{(0,\infty)} x^\alpha G(t,x) dx dt = \int_0^{\xi^{-1}(T)} \int_{(0,\infty)} x^\alpha h(\tau,x) dx d\tau.$$

The result then follows from Proposition 3.4.7. □

Remark 3.5.6. One could try to directly solve the system (3.1.34), (3.1.35), written in (g,n) variables. First, to obtain a sequence of solutions (g_k, n_k) of an approximated system where the factor $x^{-1/2}$ is modified by truncation and regularization, and then pass to the limit. However, the limit obtained in that way, say (g,n) is not a solution of (3.1.34), (3.1.35). The reason is that all the solutions g_k of the approximated system will be functions with a bounded moment of order $-1/2$. Then, the right hand side of the equation (3.1.37) is equal to $M_{1/2}(g_k)$ and by passage to the limit the equation for n will be $n'(t) = -n(t) M_{1/2}(g(t))$, and the total mass will not be conserved.

3.6 Proofs of Theorems 3.1.7, 3.1.8 and 3.1.9.

We first prove Theorem 3.1.7.

Proof of Theorem 3.1.7 . We already know by Proposition 3.5.4 and Lemma 3.5.5 that n is right continuous and a.e. differentiable on $[0,\infty)$. Then, by construction

$$G(t) = H(\tau) = h(\tau) - \left(\int_0^\tau M_{1/2}(h(\sigma))d\sigma \right) \delta_0,$$

where $\tau \in [0,\tau^*)$ and $t \in [0,\infty)$ are related by (3.5.24). Hence

$$n(t) = m(\tau) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma = m(\tau) - \int_0^t n(s)M_{1/2}(g(s))ds. \tag{3.6.1}$$

Since $n(0) = m(0)$, it then follows from Proposition 3.4.15 that for all $t > 0$:

$$n(t) - n(0) + \int_0^t n(s)M_{1/2}(g(s))ds = \lambda((0,\tau]), \tag{3.6.2}$$

and using (3.5.24)

$$\lambda((0,\tau]) = \lim_{\varepsilon\to 0} \int_0^t n(s)\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, g(s))ds. \tag{3.6.3}$$

If we denote $\mu = \xi_\# \lambda$ (c.f. [2], Ch. 5), i.e., the push-forward of λ through the function $\xi : [0,\tau^*) \to [0,\infty)$ in (3.5.24), then from the definition of μ we obtain

$$\mu((0,t]) = \lambda((0,\tau]) \qquad \forall t > 0. \tag{3.6.4}$$

Then (3.1.58) and (3.1.57) follows from (3.6.2), (3.6.3) and (3.6.4). Moreover, (3.1.59) follows from (3.4.44) in Proposition 3.4.15. □

The following properties of $n(t)$ follows by the same arguments used in the proofs of properties (3.4.45) and (3.4.46) of Proposition 3.4.15

Proposition 3.6.1. *Let G, g, and $n(t)$ be as in Theorem 3.1.7. Then, for all φ_ε as in Remark 3.1.6, the following limit exists in $\mathscr{D}'(0,\infty)$:*

$$\lim_{\varepsilon\to 0} n\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, g) = T(G), \tag{3.6.5}$$

and

$$n' + nM_{1/2}(g) = T(G) \quad \text{in} \quad \mathscr{D}'(0,\infty). \tag{3.6.6}$$

Proof. Consider, for all φ_ε as in Remark 3.1.6, the absolutely continuous functions

$$\eta_\varepsilon(t) = \int_{[0,\infty)} \varphi_\varepsilon(x)G(t,x)dx. \tag{3.6.7}$$

Then equation (3.1.46) becomes $\eta'_\varepsilon = n\mathscr{Q}_3(\varphi_\varepsilon, g)$. Using integration by parts,

$$-\int_0^\infty \phi'(t)\eta_\varepsilon(t)dt = \int_0^\infty \phi(t)n(t)\mathscr{Q}_3(\varphi_\varepsilon, g(t))dt \quad \forall \phi \in C_c^\infty(0,\infty).$$

Taking the limit $\varepsilon \to 0$ we deduce, using Lemma C.1.20, that

$$-\int_0^\infty \phi'(t)n(t)dt = \lim_{\varepsilon\to 0}\int_0^\infty \phi(t)n(t)\mathscr{Q}_3^{(2)}(\varphi_\varepsilon, g(t))dt \\ - \int_0^\infty \phi(t)n(t)M_{1/2}(g(t))dt,$$

and then (3.6.5), (3.6.6) follows. □

Remark 3.6.2. If we take distributional derivatives in both sides of (3.1.58) we obtain:

$$n' + nM_{1/2}(g) = \mu \quad \text{in} \quad \mathscr{D}'(0,\infty),$$

and by (3.6.6), $\mu = T(G)$.

Proof of Theorem 3.1.8. The statement of the Theorem follows from (3.4.44) in Proposition 3.4.15 and (3.6.4). □

Proof of Theorem 3.1.9. Proof of part (i). By Theorem 3.1.7, n is given by (3.1.58) and (3.1.57). On the other hand, since G satisfies (3.1.46), and for all $\varphi \in C_b^1([0,\infty))$ such that $\varphi(0) = 0$:

$$\int_{[0,\infty)} \varphi(x)G(t,x)dx = \int_{[0,\infty)} \varphi(x)g(t,x)dx, \tag{3.6.8}$$

then g satisfies (3.1.60). In order to prove part (ii) we first show the existence of the limit in (3.1.57). To this end we write $\varphi_\varepsilon = (1-\psi_\varepsilon)$, where ψ_ε is as in Remark 3.1.6. Then $\varphi_\varepsilon(0) = 0$, and by (3.1.60) and (3.1.43), using that $\mathscr{Q}_3(1-\psi_\varepsilon, g) = \mathscr{Q}_3(1,g) - \mathscr{Q}_3(\psi_\varepsilon, g)$, and $\mathscr{Q}_3(1,g) = 0$, we deduce

$$\int_0^t n(s)\widetilde{\mathscr{Q}}_3(\psi_\varepsilon, g(s))ds = \int_{(0,\infty)} \varphi_\varepsilon(x)\left(g(0,x) - g(t,x)\right)dx \\ + \int_0^t n(s)M_{1/2}(g(s))ds. \tag{3.6.9}$$

The existence of the limit in (3.1.57) follows and, if we pass to the limit,

$$\lim_{\varepsilon\to 0}\int_0^t n(s)\mathscr{Q}_3^{(2)}(\psi_\varepsilon, g(s))ds = \int_{(0,\infty)} (g(0,x) - g(t,x))dx \\ + \int_0^t n(s)M_{1/2}(g(s))ds. \tag{3.6.10}$$

We now check that, if n satisfies the equation (3.1.61) then G satisfies equation (3.1.46) for a.e. $t > 0$ and for every $\varphi \in C_b^1([0,\infty))$. If $\varphi(0) = 0$ this follows from (3.1.60) and (3.6.8).

For $\varphi(0) \neq 0$ we may assume without loss of generality that $\varphi(0) = 1$, and write $\varphi = (\varphi - \psi_\varepsilon) + \psi_\varepsilon$, where ψ_ε is as in Remark 3.1.6. Since $(\varphi - \psi_\varepsilon)(0) = 0$, using (3.1.60) and (3.1.45)

$$\int_{[0,\infty)} (\varphi - \psi_\varepsilon)(x)g(t,x)dx = \int_{[0,\infty)} (\varphi - \psi_\varepsilon)(x)g(0,x)dx \\ + \int_0^t n(s)\widetilde{\mathscr{Q}}_3((\varphi - \psi_\varepsilon), g(s))ds. \tag{3.6.11}$$

In order to pas to the limit as $\varepsilon \to 0$, we first use $\widetilde{\mathscr{Q}}_3((\varphi - \psi_\varepsilon), g) = \widetilde{\mathscr{Q}}_3(\varphi, g) - \widetilde{\mathscr{Q}}_3(\psi_\varepsilon, g)$. Then, since for all $t \geq 0$

$$\lim_{\varepsilon \to 0} \int_{[0,\infty)} \psi_\varepsilon(x) g(t,x) dx = 0, \tag{3.6.12}$$

and n satisfies (3.1.61), we deduce from (3.6.11) and Lemma C.1.20:

$$\begin{aligned}\int_{[0,\infty)} \varphi(x) g(t,x) dx = \int_{[0,\infty)} \varphi(x) g(0,x) dx + \int_0^t n(s) \widetilde{\mathscr{Q}}_3(\varphi, g(s)) ds \\ + n(0) - n(t) - \int_0^t n(s) M_{1/2}(g(s)) ds.\end{aligned}$$

Since $\widetilde{\mathscr{Q}}_3(\varphi, G) - M_{1/2}(g) = \mathscr{Q}_3(\varphi, G)$, it follows that G satisfies

$$\int_{[0,\infty)} \varphi(x) G(t,x) dx = \int_{[0,\infty)} \varphi(x) G(0,x) dx + \int_0^t n(s) \mathscr{Q}_3(\varphi, g(s)) ds,$$

thus (3.1.46) holds for a.e. $t > 0$.

In order to check that G satisfies (3.1.44) we first use (3.1.46) with $\varphi = 1 \in C_b^1([0,\infty))$. For that choice of φ we have $\Lambda(\varphi) = \mathcal{L}_0(\varphi) \equiv 0$ and then:

$$\int_{[0,\infty)} G(t,x) dx = \int_{[0,\infty)} G_0(x) dx.$$

Because:

$$\int_{[0,\infty)} x\, G(t,x) dx = \int_{[0,\infty)} x\, g(t,x) dx,$$

G satisfies (3.1.44) since by hypothesis so does g. □

Remark 3.6.3. If G is a weak radial solution of (3.1.1), (3.1.2), we know by Theorem 3.1.9 that g satisfies (3.1.60). It is straightforward to check that it also satisfies,

$$\frac{d}{dt} \int_{(0,\infty)} \varphi(x) g(t,x) dx = n(t) \widetilde{\mathscr{Q}}_3(\varphi, g(t)) - \varphi(0) \frac{d}{dt} \mu((0,t]),$$

where μ is as in Theorem 3.1.7, and $\widetilde{\mathscr{Q}}_3$ is defined in (3.1.40)–(3.1.42).

Proof of Corollary 3.1.10. If we prove that n satisfies (3.1.61), the conclusion of the Corollary will follow from part (ii) of Theorem 3.1.9. By the hypothesis and part (ii) of Theorem 3.1.9, the limit in (3.1.57) exists, and (3.6.10) holds, that we write:

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_0^t n(s) \mathscr{Q}_3^{(2)}(\psi_\varepsilon, g(s)) ds - \int_0^t n(s) M_{1/2}(g(s)) ds = \\ = \int_{[0,\infty)} (G(0,x) - G(t,x)) dx + n(t) - n(0).\end{aligned}$$

Using the conservation of mass (3.1.62) it follows that n satisfies equation (3.1.61). □

Proposition 3.6.4. *(i) Let $G \in \mathscr{M}_+([0,\infty))$. If G has no atoms on $(0,\infty)$ and $\int_{(0,\infty)} \frac{G(x)}{\sqrt{x}} dx < \infty$, then, for all φ_ε as in Remarrk 3.1.6,*

$$\mathscr{T}(G) = \lim_{\varepsilon \to 0} \mathscr{Q}_3^{(2)}(\varphi_\varepsilon, G) = 0. \tag{3.6.13}$$

(ii) Let $T > 0$ and $F : [0,T] \to \mathscr{M}_+([0,\infty))$ be such that $F(t)$ has no atoms on $(0,\infty)$ for all $t \in [0,T]$ and

$$\rho(T) = \sup_{t \in [0,T]} \int_{(0,\infty)} \frac{F(t,x)}{\sqrt{x}} dx < \infty. \tag{3.6.14}$$

Then, for any bounded measurable function $\eta : [0,T] \to [0,\infty)$,

$$\lim_{\varepsilon \to 0} \int_0^t \eta(s) \mathscr{Q}_3^{(2)}(\varphi_\varepsilon, F(s)) ds = 0 \qquad \forall t \in [0,T]. \tag{3.6.15}$$

Proof. Let us prove (i). By definition

$$\mathscr{T}(G) = \lim_{\varepsilon \to 0} \iint_{(0,\infty)^2} \frac{\Lambda(\varphi_\varepsilon)(x,y)}{\sqrt{xy}} G(x)G(y) dxdy,$$

Since $\Lambda(\varphi_\varepsilon) \le 1$ for all $\varepsilon > 0$ and

$$\lim_{\varepsilon \to 0} \Lambda(\varphi_\varepsilon)(x,y) = \mathbb{1}_{\{x=y>0\}}(x,y) \qquad \forall (x,y) \in (0,\infty)^2,$$

and $\int_{(0,\infty)} \frac{G(x)}{\sqrt{x}} dx < \infty$, then by dominated convergence

$$\mathscr{T}(G) = \iint_{\{x=y>0\}} \frac{G(x)G(y)}{\sqrt{xy}} dxdy.$$

Since G has no atoms on $(0,\infty)$, i.e., $G(\{x\}) = 0$ for all $x > 0$, by Fubini's theorem

$$\iint_{\{x=y>0\}} \frac{G(x)G(y)}{\sqrt{xy}} dxdy = \int_{(0,\infty)} \frac{G(x)}{x} G(\{x\}) dx = 0.$$

That proves (3.6.13).

Let us prove now (ii) using part (i) and dominated convergence Theorem. We first consider a bounded measurable function $\eta \ge 0$ defined on $[0,T]$. By part (i),

$$\lim_{\varepsilon \to 0} \eta(t) \mathscr{Q}_3^{(2)}(\varphi_\varepsilon, F(t)) = 0 \qquad \forall t \in [0,T]. \tag{3.6.16}$$

On the other hand, using again that $\Lambda(\varphi_\varepsilon) \le 1$ for all $\varepsilon > 0$, we deduce

$$\left| \eta(t) \mathscr{Q}_3^{(2)}(\varphi_\varepsilon, F(t)) \right| \le \|\eta\|_\infty \rho(T) < \infty \qquad \forall \varepsilon > 0, \quad \forall t \in [0,T]. \tag{3.6.17}$$

Identity (3.6.15) then follows from (3.6.16), (3.6.17) and dominated convergence Theorem. □

Remark 3.6.5. From Proposition 3.6.4, if $M_{-1/2}(g) < \infty$ and g has no atoms, then $\mu((0,t]) = 0$ for all $t > 0$. If $g \in L^1(0,\infty)$ and $x = 0$ is a Lebesgue point of g then $\mathscr{T}(g) = 0$ (cf. [60]) and again $\mu((0,t]) = 0$ for all $t > 0$. If $g(x) = x^{-1/2}$, then $\mathscr{T}(g) = \pi^2/6$, (cf. [55]), and a similar result holds if $\lim_{x \to 0} \sqrt{x} g(x) = C > 0$ (cf. [67]). In that case, $\mu((0,t]) = \pi^2/6 \int_0^t n(s) ds$.

3.7 Proof of Theorem 3.1.11

Proof. By (3.5.22) and (3.5.23), we deduce that for all $t > t_0 > 0$:

$$\int_{t_0}^{t} G(s,\{0\})ds \le \big(M_\alpha(G(t_0)) - M_\alpha(G(t))\big)C(N,E,\alpha)$$
$$C(N,E,\alpha) = \left[\left(\frac{\alpha-1}{\alpha+1}\right)E^{(2\alpha+1)/2}N^{(1-2\alpha)/2} - C_1N^{3-\alpha}E^{\alpha-1}\right]^{-1}, \tag{3.7.1}$$

where $C_1 = 2^\alpha - 2$ for $\alpha \in (1,2]$ and $C_1 = \alpha(\alpha-1)$ for $\alpha \in [2,3]$. Since by part (i), $0 \le M_\alpha(G(t_0)) - M\alpha(G(t)) \le M_\alpha(G(t_0))$ for every $t > t_0$, we immediately deduce (3.1.63).

We prove now (3.1.64). Since, as we have seen in (3.5.18), the function $n(t)J(t)$ is monotone nondecreasing, from where, for all $t > 0$ and $s \in (0,t)$:

$$n(t) \ge e^{-\int_s^t M_{1/2}(g(r))dr}n(s).$$

As we have $M_{1/2}(g(r)) \le \sqrt{NE}$ for all $r \ge 0$,

$$n(t) \ge e^{-\sqrt{NE}(t-s)}n(s). \tag{3.7.2}$$

By (3.1.63) we already have a sequence of times θ_k such that $\theta_k \to \infty$ and $n(\theta_k) \to 0$ as $k \to \infty$. Suppose that there exists, for some $\rho > 0$, an increasing sequence of times $(s_k)_{k\in\mathbb{N}}$ such that $s_k \to \infty$ as $k \to \infty$ and :

$$\forall k,\ n(s_k) \ge \rho \quad \text{and} \quad s_{k+1} - s_k > \frac{\log 2}{\sqrt{NE}}.$$

Then, if we denote $t_k = s_k + \frac{\log 2}{\sqrt{NE}}$, we deduce from (3.7.2) that for all $t \in (s_k, t_k)$:

$$n(t) \ge e^{-\sqrt{NE}(t-s_k)}n(s_k) \ge e^{-\sqrt{NE}(t_k-s_k)}\rho = \frac{\rho}{2}.$$

This would imply

$$\int_0^\infty n(t)dt \ge \sum_{k=0}^{\infty}\int_{s_k}^{t_k} n(t)dt = \infty,$$

and this contradiction proves (3.1.64). $\square$

Appendix A

Some useful estimates

Lemma A.0.1. *If B satisfies (2.2.5)–(2.2.10), and φ is L-Lipschitz on $[0,\infty)$, then for all $(x,y) \in \Gamma$:*

$$|k_\varphi(x,y)| \le LC_* A e^{\frac{|x-y|}{2}}, \quad A = \max\left\{\frac{(1-\theta)^2}{\theta\delta(1+\theta)}, \rho_*\right\}, \tag{A.0.1}$$

$$|\ell_\varphi(x,y)| \le \frac{LC_*(1-\theta)}{\theta^2(1+\theta)} e^{\frac{x-y}{2}}. \tag{A.0.2}$$

Moreover, the function $\mathcal{L}_\varphi$ given in (2.2.31) is continuous on $[0,\infty)$ and for all $x \in [0,\infty)$,

$$|\mathcal{L}_\varphi(x)| \le \frac{LC_*(1-\theta)}{\theta^2(1+\theta)}\big(e^{\frac{x-\gamma_1(x)}{2}} - e^{\frac{x-\gamma_2(x)}{2}}\big). \tag{A.0.3}$$

In particular, $\mathcal{L}_\varphi(0) = 0$.

Proof. We first prove (A.0.1). Let $(x,y) \in \operatorname{supp}(B) = \Gamma$, and assume, by the symmetry of k_φ, that $0 \le y \le x$. By the mean value theorem, $|e^{-x} - e^{-y}| \le e^{-y}(x-y)$, and from (2.2.11) and the Lipschitz condition,

$$|k_\varphi(x,y)| \le LC_* e^{\frac{x-y}{2}} \frac{(x-y)^2}{(x+y)xy}.$$

Then by (2.2.8)–(2.2.10)

$$\frac{(x-y)^2}{(x+y)xy} \le \begin{cases} \frac{(1-\theta)^2}{\theta\delta_*(1+\theta)} & \text{if } (x,y) \in \Gamma_1, \\ \rho_* & \text{if } (x,y) \in \Gamma_2, \end{cases}$$

and (A.0.1) follows.

In order to prove (A.0.2) we use (2.2.11) and the Lipschitz condition to have, for all $(x,y) \in \Gamma$,

$$|\ell_\varphi(x,y)| \le LC_* e^{\frac{x-y}{2}} \frac{y|x-y|}{x(x+y)}.$$

Using that $\Gamma \subset \{(x,y) \in [0,\infty)^2 : \theta x \le y \le \theta^{-1}x\}$, then

$$\frac{y|x-y|}{x(x+y)} \le \frac{(1-\theta)}{\theta^2(1+\theta)},$$

and (A.0.2) follows. We obtain (A.0.3) directly from (A.0.2) and Remark 2.2.1.

We finally prove the continuity of $\mathcal{L}_\varphi$ on $[0,\infty)$. By (2.2.7)–(2.2.10), $\mathcal{L}_\varphi(x)$ is continuous for all $x>0$, so we only need to prove $\mathcal{L}_\varphi(x)\to 0$ as $x\to 0$. This follows from (A.0.3) and the mean value theorem, using $\gamma_2(x)-\gamma_1(x)\le(\theta^{-1}-\theta)x$. □

Remark A.0.2. Under the hypothesis of Lemma A.0.1, the function k_φ could not be continuous at the origin $(x,y)=(0,0)$, since we do not know if $\lim_{(x,y)\to(0,0)} k_\varphi(x,y)=0$. However we have the following.

Lemma A.0.3. *If B satisfies (2.2.5)–(2.2.10), then $k_\varphi\in C([0,\infty)^2)$ for all $\varphi\in C^1([0,\infty))$ with $\varphi'(0)=0$, and $k_\varphi(0,0)=0$.*

Proof. By definition and (2.2.7), it is clear that $k_\varphi\in C([0,\infty)^2\setminus\{0\})$. If we prove that $\lim_{(x,y)\to(0,0)} k_\varphi(x,y)=0$, the continuity at the origin follows. To this end we mimic the proof of (A.0.1) using $\varphi(x)-\varphi(y)=\varphi'(\xi)(x-y)$ for some $\xi\in(\min\{x,y\},\max\{x,y\})$ instead of the Lipschitz condition, and we obtain

$$|k_\varphi(x,y)|\le\max\left\{\frac{(1-\theta)^2}{\theta\delta(1+\theta)},\rho\right\}|\varphi'(\xi)|e^{\frac{|x-y|}{2}} \tag{A.0.4}$$

for all $(x,y)\in\Gamma$ and all $\varphi\in C^1([0,\infty))$. If $\varphi'(0)=0$, it follows from (A.0.4) that $\lim_{(x,y)\to(0,0)} k_\varphi(x,y)=0$. □

Proposition A.0.4. *Suppose that B satisfies (2.2.5)–(2.2.10), φ is L-Lipschitz on $[0,\infty)$, and $u\in\mathscr{M}_+([0,\infty))$. Then*

$$|K_\varphi(u,u)|\le LC_*A\left(\int_{[0,\infty)} e^{\frac{x-\gamma_1(x)}{2}}u(x)dx\right)\left(\int_{[0,\infty)} u(y)dy\right), \tag{A.0.5}$$

$$|L_\varphi(u)|\le\frac{LC_*(1-\theta)}{2\theta^2(1+\theta)}\int_{[0,\infty)}\big(e^{\frac{x-\gamma_1(x)}{2}}-e^{\frac{x-\gamma_2(x)}{2}}\big)u(x)dx, \tag{A.0.6}$$

where A is given in (A.0.1).

Proof. In order to prove (A.0.5), we use Remark 2.2.1, Remark 2.2.3 and (A.0.1):

$$\begin{aligned}|K_\varphi(u,u)|&\le LC_*A\int_0^\infty e^{\frac{x}{2}}u(x)\int_{\gamma_1(x)}^x e^{-\frac{y}{2}}u(y)dydx\\&\le LC_*A\int_0^\infty e^{\frac{x-\gamma_1(x)}{2}}u(x)\int_{\gamma_1(x)}^x u(y)dydx,\end{aligned}$$

from where (A.0.5) follows. The estimate (A.0.6) follows directly from (A.0.3). □

Let us define now

$$K_{\varphi,n}(u,u)=\frac{1}{2}\iint_{[0,\infty)^2} k_{\varphi,n}(x,y)u(t,x)u(t,y)dydx, \tag{A.0.7}$$

$$k_{\varphi,n}(x,y)=b_n(x,y)(e^{-x}-e^{-y})(\varphi(x)-\varphi(y)), \tag{A.0.8}$$

$$L_{\varphi,n}(u_n)=\frac{1}{2}\int_{[0,\infty)}\mathcal{L}_{\varphi,n}(x)u(t,x)dx, \tag{A.0.9}$$

$$\mathcal{L}_{\varphi,n}(x)=\int_0^\infty \ell_{\varphi,n}(x,y)dy \tag{A.0.10}$$

$$\ell_{\varphi,n}(x,y)=b_n(x,y)y^2e^{-y}(\varphi(x)-\varphi(y)). \tag{A.0.11}$$

Remark A.0.5. Since $\phi_n \leq x^{-1}$, the estimates (A.0.1), (A.0.2) and (A.0.3) in Lemma A.0.1 hold for $k_{\varphi,n}$, $\ell_{\varphi,n}$ and $\mathcal{L}_{\varphi,n}$ respectively, and estimates (A.0.5) and (A.0.6) in Lemma A.0.4 hold for $K_{\varphi,n}(u,u)$ and $L_{\varphi,n}(u)$ respectively, for all $n \in \mathbb{N}$.

Lemma A.0.6. *$\mathcal{L}_{\varphi,n} \to \mathcal{L}_\varphi$ as $n \to \infty$ uniformly on the compact sets of $[0,\infty)$ for all φ L-Lipschitz on $[0,\infty)$.*

Proof. Let $R > 0$ and $x \in [0,R]$. On the one hand, if $x \in [0,1/n]$, we have $|\mathcal{L}_\varphi(x) - \mathcal{L}_{\varphi,n}(x)| \leq 2|\mathcal{L}_\varphi(x)| \to 0$ as $n \to \infty$, since $\mathcal{L}_\varphi(0) = 0$ (cf. Lemma A.0.1). On the other hand, if $x \in [1/n, R]$ and $y \in [1/n, n]$, by definition $\phi_n(x)\phi_n(y) = (xy)^{-1}$, and then

$$\begin{aligned}|\mathcal{L}_\varphi(x) - \mathcal{L}_{\varphi,n}(x)| \leq &\int_0^{\frac{1}{n}} |\ell_\varphi(x,y) - \ell_{\varphi,n}(x,y)|dy \\ &+ \int_n^\infty |\ell_\varphi(x,y) - \ell_{\varphi,n}(x,y)|dy.\end{aligned}$$

The two integrals in the right hand side above are treated in the same way. Using $|\ell_\varphi(x,y) - \ell_{\varphi,n}(x,y)| \leq |\ell_\varphi(x,y)|$ and (A.0.2),

$$\int_0^{\frac{1}{n}} |\ell_\varphi(x,y)|dy \leq \frac{LC_*(1-\theta)}{\theta^2(1+\theta)} e^{\frac{R}{2}} \int_0^{\frac{1}{n}} e^{-\frac{y}{2}} dy \xrightarrow[n\to\infty]{} 0,$$
$$\int_n^\infty |\ell_\varphi(x,y)|dy \leq \frac{LC_*(1-\theta)}{\theta^2(1+\theta)} e^{\frac{R}{2}} \int_n^\infty e^{-\frac{y}{2}} dy \xrightarrow[n\to\infty]{} 0,$$

and the result follows. □

Appendix B

The function $\mathcal{B}_\beta$, properties and scalings

In this Section, we describe several properties of the function $\mathcal{B}_\beta$. First, the parameter β is used to scale the variables, in such a way that the total mass of the solution is conserved. In the scaled equation, the linear term appears as a lower order term for β large. Then, for each $\beta > 0$ fixed, the behavior of $\mathcal{B}_\beta(k, k')$ is studied when k and k' are varying on $(0, \infty)$.

B.1 β-scalings of $\mathcal{B}_\beta$.

It looks natural from (2.1.2) to introduce the scaled variable

$$\mathbf{x} = \beta\,\mathbf{k}, \tag{B.1.1}$$

and define

$$F(\tau, x) = \beta^{-3} f(t, k), \qquad \tau = \beta^3 t, \quad x = \beta k. \tag{B.1.2}$$

The scaling (B.1.2) preserves the total number of particles:

$$\int_0^\infty x^2 F(\tau, x) dx = \int_0^\infty k^2 f(t, k) dk = \int_0^\infty k^2 f(0, k) dk \quad \forall \tau > 0.$$

In terms of F,

$$k^2 \frac{\partial f}{\partial t}(t, k) = \beta^4 x^2 \frac{\partial F}{\partial \tau}(\tau, x),$$
$$\tilde{q}(f, f') = \beta^6 F F' \big(e^{-x} - e^{-x'}\big) + \beta^3 \big(F' e^{-x} - F e^{-x'}\big),$$

and if we define

$$B_\beta(x, x') = \beta^{-1} \mathcal{B}_\beta(k, k'), \tag{B.1.3}$$

that is,

$$B_\beta(x, x') = \sqrt{\beta} e^{\frac{(x'+x)}{2}} \int_0^\pi \frac{(1 + \cos^2\theta)}{|\mathbf{x}' - \mathbf{x}|} e^{-\beta \frac{m(x-x')^2 + \frac{|\mathbf{x}'-\mathbf{x}|^4}{4m\beta^2}}{2|\mathbf{x}'-\mathbf{x}|^2}} d\cos\theta, \tag{B.1.4}$$

the equation (2.1.1)–(2.1.2) then reads

$$x^2\frac{\partial F}{\partial \tau}(\tau,x)=\int_0^\infty B_\beta(x,x')FF'\big(e^{-x}-e^{-x'}\big)xx'dx'+ \\ +\beta^{-3}\int_0^\infty B_\beta(x,x')\big(F'e^{-x}-Fe^{-x'}\big)xx'dx'. \qquad \text{(B.1.5)}$$

If we now define

$$u(\tau,x)=x^2F(\tau,x) \qquad \text{(B.1.6)}$$

then from (B.1.5) we finally obtain

$$\frac{\partial u}{\partial \tau}(\tau,x)=\int_0^\infty \frac{B_\beta(x,x')}{xx'}\big(e^{-x}-e^{-x'}\big)uu'dx'+ \\ +\beta^{-3}\int_0^\infty \frac{B_\beta(x,x')}{xx'}\big(u'x^2e^{-x}-ux'^2e^{-x'}\big)dx', \qquad \text{(B.1.7)}$$

The second term in the right hand side of (B.1.7) seems then negligible when $\beta\to\infty$, but no rigorous result on that direction is known.

B.2 The function $B_\beta(x,x')$ for β fixed.

In this Section we show some properties of the kernel B_β defined in (B.1.4).

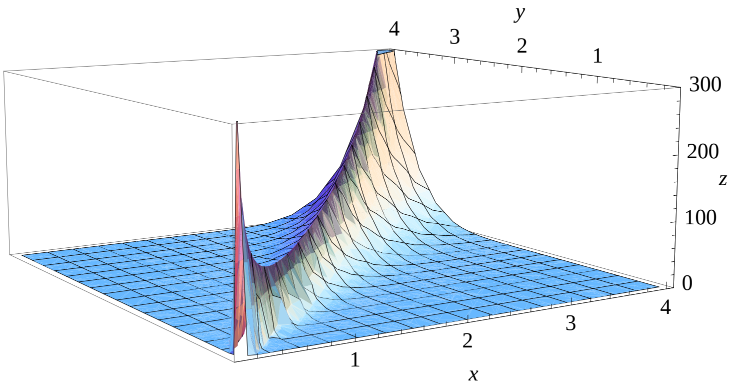

Figure B.1: The kernel $B_\beta(x,y)$ for $\beta = 100$, $m = 1$, $(x,y) \in [0.1, 4]^2$.

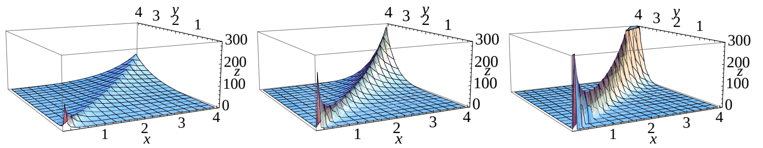

Figure B.2: From left to right, the kernel $B_\beta(x,y)$ for $m = 1$, $(x,y) \in [0.1, 4]^2$ and $\beta = 10$, $\beta = 50$ and $\beta = 200$.

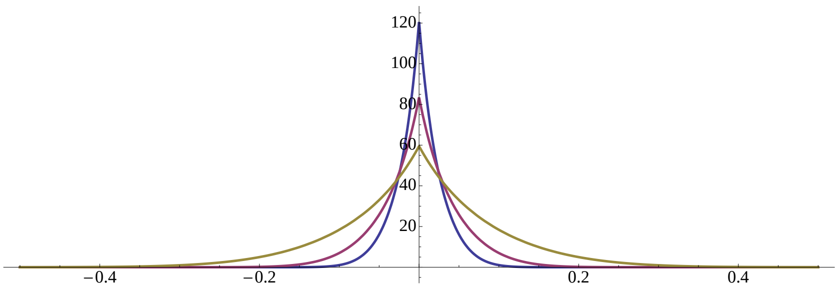

Figure B.3: Sections of B_β for $\beta = 50$ and $m = 1$. The horizontal axis corresponds to the variable $\xi = (x-y)/\sqrt{2}$. The vertical axis corresponds to $B_\beta(x,y)$ for $x+y =$ constant. In blue, $x+y = 0.3$, in red $x+y = 0.5$ and in yellow, $x+y = 1$.

Proposition B.2.7. *For all $\beta > 0$, $x > 0$ and $x' > 0$,*

$$B_\beta(x,x') \le \sqrt{\beta}\, \frac{4\big(10\max^2\{x,x'\} + \min^2\{x,x'\}\big)}{15\max^3\{x,x'\}} e^{\frac{(x'+x)}{2}}, \tag{B.2.8}$$

and for all $x > 0$, $x' > 0$ with $x \neq x'$,

$$\lim_{\beta\to\infty} B_\beta(x,x') = 0. \tag{B.2.9}$$

Proof. For all $x>0$ and $x'>0$,

$$\frac{e^{-\frac{(x+x')}{2}}}{\sqrt{\beta}}B_\beta(x,x') \le \int_0^\pi \frac{(1+\cos^2\theta)}{|\mathbf{x}'-\mathbf{x}|}d\cos\theta = \int_{-1}^1 \frac{(1+t^2)}{\sqrt{x^2+x'^2-2xx't}}dt$$
$$= \frac{4\big(10\max^2\{x,x'\}+\min^2\{x,x'\}\big)}{15\max^3\{x,x'\}}, \tag{B.2.10}$$

and then (B.2.8) holds. If $x'\neq x$, we have first

$$\lim_{\beta\to\infty} e^{-\beta\frac{m(x-x')^2+\frac{|\mathbf{x}'-\mathbf{x}|^4}{4m\beta^2}}{2|\mathbf{x}'-\mathbf{x}|^2}} = 0 \qquad \forall\theta\in[0,\pi],$$

and since

$$\frac{(1+\cos^2\theta)}{|\mathbf{x}'-\mathbf{x}|}e^{-\beta\frac{(x-x')^2+\frac{|\mathbf{x}'-\mathbf{x}|^4}{\beta^2}}{2m|\mathbf{x}'-\mathbf{x}|^2}} \le \frac{(1+\cos^2\theta)}{|\mathbf{x}'-\mathbf{x}|} \in L^1(d\cos\theta) \quad \forall\beta>0,$$

then (B.2.9) follows from Lebesgue's convergence Theorem. □

Proposition B.2.8.

$$B_\beta(x,x) = \sqrt{\beta}\left(\frac{2\sqrt{2\pi m\beta}}{x^2}+\mathcal{O}\left(\frac{1}{x}\right)^3\right)e^x \quad as \quad x\to\infty, \tag{B.2.11}$$

$$B_\beta(x,x) = \sqrt{\beta}\,\frac{44}{15}\left(\frac{1}{x}+1\right)+\mathcal{O}(x) \quad as \quad x\to 0. \tag{B.2.12}$$

Proof. By definition, for all $x>0$,

$$B_\beta(x,x) = \sqrt{\frac{\beta}{2}}\frac{e^x}{x}\int_0^\pi \frac{(1+\cos^2\theta)}{\sqrt{1-\cos\theta}}e^{-\frac{x^2(1-\cos\theta)}{4m\beta}}d\cos\theta$$
$$= \sqrt{\beta}\frac{2e^x\sqrt{\beta m}}{x^6}\Big(\sqrt{2\pi}\big(6\beta^2m^2-2\beta m x^2+x^4\big)Erf\left(\frac{x}{\sqrt{2\beta m}}\right)-$$
$$-12e^{-\frac{x^2}{2\beta m}}(\beta m)^{3/2}x\Big),$$

and the result follows. □

The function B_β is exponentially decreasing in the direction orthogonal to the first diagonal, as shown in the next two Propositions.

Proposition B.2.9. *For all $\beta>0$,*

$$\nabla B_\beta(x,x')\cdot(1,-1)>0 \quad if \quad x'>x>0, \tag{B.2.13}$$
$$\nabla B_\beta(x,x')\cdot(1,-1)<0 \quad if \quad x>x'>0. \tag{B.2.14}$$

Proof. It is only a straightforward calculation. With the help of Mathematica, using the change of variables $t=\cos\theta$,

$$\frac{\partial B_\beta}{\partial x}(x,x') = \frac{e^{\frac{(x'+x)}{2}}}{4m\sqrt{\beta}}\int_{-1}^1 \frac{(1+t^2)}{|\mathbf{x}'-\mathbf{x}|^5}e^{-\beta\frac{m(x-x')^2+\frac{|\mathbf{x}'-\mathbf{x}|^4}{4m\beta^2}}{2|\mathbf{x}'-\mathbf{x}|^2}}\Theta(x,x',t)dt, \tag{B.2.15}$$

$$\Theta(x,x',t) = 4(\beta m)^2(t-1)x'(x-x')(x+x')-(x-tx')|\mathbf{x}'-\mathbf{x}|^4+$$
$$+2\beta m\left(x'^2-2tx'(x-1)+(x-2)x\right)|\mathbf{x}'-\mathbf{x}|^2. \tag{B.2.16}$$

The expression of $\frac{\partial B_\beta}{\partial x'}$ is obtained from (B.2.15) and (B.2.16) using the permutation $x \leftrightarrow x'$. Then,

$$\nabla B_\beta(x,x')\cdot(1,-1) = \frac{e^{\frac{(x'+x)}{2}}}{4m\sqrt{\beta}}\int_{-1}^{1}\frac{(1+t^2)}{|\mathbf{x}'-\mathbf{x}|^5}e^{-\beta\frac{m(x-x')^2+\frac{|\mathbf{x}'-\mathbf{x}|^4}{4m\beta^2}}{2|\mathbf{x}'-\mathbf{x}|^2}}\times \\ \times\big(\Theta(x,x',t)-\Theta(x',x,t)\big)dt, \qquad \text{(B.2.17)}$$

$$\Theta(x,x',t)-\Theta(x',x,t) = (x'-x)\Big[4(\beta m)^2(1-t)(x+x')^2 \\ + 4\beta m(1+t)|\mathbf{x}'-\mathbf{x}|^2+(1+t)|\mathbf{x}'-\mathbf{x}|^4\Big],$$

and the result follows. □

Proposition B.2.10. *For all $\beta>0$, $x>0$ and $x'>0$,*

$$B_\beta(x,x') \le \mathscr{B}_\beta(x,x'), \qquad \text{(B.2.18)}$$

where

$$\mathscr{B}_\beta(x,x') = \sqrt{\beta}e^{-\beta\frac{m(x-x')^2+\frac{(x-x')^4}{4m\beta^2}}{2(x+x')^2}}N(x+x',|x-x'|), \qquad \text{(B.2.19)}$$

$$N(p,q) = \frac{8\,e^{\frac{p}{2}}\big(10(p+q)^2+(p-q)^2\big)}{15(p+q)^3}, \quad \forall p>0,\ \forall q>0. \qquad \text{(B.2.20)}$$

Proof. For all $\mathbf{x}\in\mathbb{R}^3$ and $\mathbf{x}'\in\mathbb{R}^3$ such that $|\mathbf{x}|=x$, $|\mathbf{x}'|=x'$,

$$|x-x'| \le |\mathbf{x}-\mathbf{x}'| \le x+x'.$$

Therefore,

$$B_\beta(x,x') \le \sqrt{\beta}e^{\frac{(x'+x)}{2}}e^{-\beta\frac{m(x-x')^2+\frac{(x-x')^4}{4m\beta^2}}{2(x+x')^2}}\int_0^\pi\frac{(1+\cos^2\theta)}{|\mathbf{x}'-\mathbf{x}|}d\cos\theta,$$

and the result follows using (B.2.10). □

Corollary B.2.11.

$$\forall x>0, x'>0:\ B_\beta(x,x') \le B_\beta\left(\frac{x+x'}{2},\frac{x+x'}{2}\right), \qquad \text{(B.2.21)}$$

$$B_\beta\left(\frac{x+x'}{2},\frac{x+x'}{2}\right) = \sqrt{\beta}\left(\frac{2\sqrt{2\pi m\beta}}{(x+x')^2}+\mathcal{O}\left(\frac{1}{x+x'}\right)^3\right)e^{\frac{x+x'}{2}},\ \ x+x'\to\infty,$$

$$B_\beta\left(\frac{x+x'}{2},\frac{x+x'}{2}\right) = \frac{44\sqrt{\beta}}{15}\left(\frac{1}{x+x'}+1\right)+\mathcal{O}(x+x'),\ \ x+x'\to 0.$$

If $x+x'\to\infty$, and $|x-x'|\le\theta x$:

$$|e^{-x}-e^{-x'}|B_\beta(x,x') \le 2\sqrt{\beta}\left(\frac{2\sqrt{2\pi m\beta}}{(x+x')^2}+\mathcal{O}\left(\frac{1}{x+x'}\right)^3\right)\left|\sinh\left(\frac{\theta x}{2}\right)\right|. \qquad \text{(B.2.22)}$$

For all $\rho>0$ fixed and $x>0$, $x'>0$ such that $x+x'=\rho$,

$$B_\beta(x,x') \le \sqrt{\beta}\,e^{-\beta\frac{(x-x')^2}{2m\rho^2}}\,\Phi(\rho,|x-x'|). \qquad \text{(B.2.23)}$$

Proof. By Proposition B.2.9, the function B_β is strictly decreasing in the direction orthogonal to the first diagonal, and then property (B.2.21) follows. In order to prove (B.2.22) we have first, when $x + x' \to \infty$,

$$|e^{-x} - e^{-x'}|B_\beta(x, x') \le 2\left(\frac{2\sqrt{2\pi m\beta}}{(x+x')^2} + \mathcal{O}\left(\frac{1}{x+x'}\right)^3\right)\left|\sinh\left(\frac{x'-x}{2}\right)\right|$$

If moreover, $0 \le x' - x \le \theta x$ then

$$0 \le (e^{-x} - e^{-x'})B_\beta(x, x') \le 2\left(\frac{2\sqrt{2\pi m\beta}}{(x+x')^2} + \mathcal{O}\left(\frac{1}{x+x'}\right)^3\right)\sinh\left(\frac{\theta x}{2}\right)$$

If $-\theta x \le x' - x \le 0$ then,

$$0 \le -\sinh\left(\frac{x'-x}{2}\right) = \sinh\left(\frac{x-x'}{2}\right) \le \sinh\left(\frac{\theta x}{2}\right),$$

and (B.2.22) follows. □

Proposition B.2.12. *For all $\varphi \in C_c((0,\infty)\times(0,\infty))$:*

$$\lim_{\beta\to\infty}\iint_{(0,\infty)^2}\varphi(x,y)\Phi_\beta(x,y)\mathscr{B}_\beta(x,y)dxdy = \\ = \frac{88}{15}\sqrt{\frac{m\pi}{2}}erf(1)\int_{(0,\infty)}\varphi\left(\frac{z}{2},\frac{z}{2}\right)e^{\frac{z}{2}}dz \qquad \text{(B.2.24)}$$

Proof. Define the new variables

$$\xi = x - y,\ \zeta = x + y,\ \psi(\xi,\zeta) = \varphi\left(\frac{\xi+\zeta}{2},\frac{\zeta-\xi}{2}\right)$$

and denote $\Psi_\beta(\xi,\zeta) = \Phi_\beta(x,y)$. Then,

$$I = \iint_{(0,\infty)^2}\varphi(x,y)\Phi_\beta(x,y)\mathscr{B}_\beta(x,y)dxdy = \\ = \iint_D e^{-\frac{\beta^2\xi^2+\xi^4}{2m\beta\zeta^2}}\Psi_\beta(\xi,\zeta)\mathscr{B}_\beta\left(\frac{\xi+\zeta}{2},\frac{\zeta-\xi}{2}\right)\psi(\xi,\zeta)d\xi d\zeta$$

where $D = \{(\zeta,\xi)\in\mathbb{R}^2 : \zeta > 0,\ -\zeta < \xi < \zeta\}$. We write now,

$$\frac{\beta^2\xi^2+\xi^4}{2m\beta\zeta^2} = \frac{\beta\xi^2}{2m\zeta^2}\left(1+\frac{\xi^2}{\beta^2}\right)$$

and the change of variables:

$$\sqrt{\frac{\beta}{2m}}\frac{\xi}{\zeta} = z_1,\ \zeta = z_2;\ \ \xi = \sqrt{\frac{2m}{\beta}}z_1z_2,\ \zeta = z_2$$

whose Jacobian is $\sqrt{2m/\beta}\,z_2$ and,

$$I = \iint_\Omega e^{-z_1^2\left(1+\frac{2mz_1^2z_2^2}{\beta}\right)}\mathscr{B}_\beta(Z_1,Z_2)\times \\ \times\Psi\left(\beta^{-1}\sqrt{\frac{2m}{\beta}}z_1z_2,\beta^{-1}z_2,\right)\psi\left(\sqrt{\frac{2m}{\beta}}z_1z_2,z_2\right)\sqrt{2m/\beta}\,z_2dz_1dz_2$$

$$Z_1 = \frac{1}{2}\left(z_2+\sqrt{\frac{2m}{\beta}}z_1z_2\right),\ Z_2 = \frac{1}{2}\left(z_2-\sqrt{\frac{2m}{\beta}}z_1z_2\right)$$

Due to the cut off function $\Phi_\beta(x,y)$, the actual domain of integration Ω_β is:

$$\Omega_\beta = \left\{(z_1,z_2) \in \mathbb{R}\times\mathbb{R}^+;\ \sqrt{2m}|\,z_1| \leq \theta z_2^{1/2}\left(1-\frac{2m}{\beta}z_1^2\right)^{1/2}\right\}$$

where Ω is the domain where $z_2 > 0$, $z_1 \in (-1,1)$. As $\beta \to \infty$,

$$\lim_{\beta\to\infty} e^{-z_1^2\left(1+\frac{2mz_1^2z_2^2}{\beta}\right)}\psi\left(\sqrt{\frac{2m}{\beta}}z_1z_2, z_2\right) = e^{-z_1^2}\psi\left(0,z_2\right).$$

On the other hand, using (B.2.12), for all z_1, z_2,

$$\lim_{\beta\to\infty}\frac{\mathscr{B}_\beta\left(Z_1,Z_2\right)}{\beta} = \frac{44\,e^{\frac{z_2}{2}}}{15z_2} \tag{B.2.25}$$

By definition of Ψ, for all $z_1 \in \mathbb{R}$ and $z_2 > 0$ fixed, if β is sufficiently large,

$$\Psi\left(\beta^{-1}\sqrt{\frac{2m}{\beta}}z_1z_2, \beta^{-1}z_2,\right) = 1$$

Then,

$$\begin{aligned}\lim_{\beta\to\infty} I =& \frac{44}{15}\sqrt{2m}\iint_\Omega e^{-z_1^2}e^{\frac{z_2}{2}}\psi\left(0,z_2\right)dz_1dz_2\\ =& \frac{44}{15}\sqrt{\frac{m\pi}{2}}erf(1)\int_{(0,\infty)}\varphi\left(\frac{z_2}{2},\frac{z_2}{2}\right)e^{\frac{z_2}{2}}dz_2\end{aligned}$$

□

The function $B_\beta(x,y) \geq 0$ coincides with $\mathscr{B}_\beta(x,y)$ for $x = y$ and is below that function, that tends to a Dirac measure along the first diagonal as $\beta \to \infty$. From properties (B.2.9) and (B.2.24), the truncation of $\mathcal{B}_\beta$ may then be seen as reasonable.

Appendix C

Appendix

We have gathered in this Appendix several results that are important and useful, but not directly related to the main results of Chapter 3. For the sake of clarity, we present them in two different sections. In the first one, we find results that are used all along Chapter 3, perhaps several times. In the second, we present results that are needed in Section 3.2.

C.1 A1

Lemma C.1.13 (Convex-positivity)**.** *Let $\varphi \in C([0,\infty))$. If φ is convex then $\Lambda(\varphi)(x,y) \geq 0$ for all $(x,y) \in [0,\infty)^2$ and $\mathcal{L}_0(\varphi)(x) \geq 0$ for all $x \in [0,\infty)$. If φ is nonnegative and nonincreasing, then $\mathcal{L}(\varphi)(x) \leq 0$ for all $x \in [0,\infty)$.*

Proof. Since $\Lambda(\varphi)(x,y)$ is symmetric we may reduce the proof to the case $0 \leq y \leq x$. Putting $x = \frac{x+y}{2} + \frac{x-y}{2}$, then by the very definition of convexity

$$\varphi(x) \leq \frac{\varphi(x+y)}{2} + \frac{\varphi(x-y)}{2},$$

therefore $\Lambda(\varphi)(x,y) \geq 0$.

The positivity of $\mathcal{L}_0(\varphi)$ is equivalent to prove

$$\frac{1}{x}\int_0^x \varphi(y)dy \leq \frac{\varphi(0)+\varphi(x)}{2} \qquad \forall x \in [0,\infty). \tag{C.1.1}$$

Since for any $0 \leq y \leq x$ we may trivially write $y = \left(1 - \frac{y}{x}\right) 0 + \frac{y}{x}\, x$, then by convexity $\varphi(y) \leq \left(1 - \frac{y}{x}\right)\varphi(0) + \frac{y}{x}\varphi(x)$, which implies (C.1.1).
If φ is nonnegative and nonincreasing, then $\mathcal{L}(\varphi)(x) \leq -x\,\varphi(x) \leq 0$ for all $x \in [0,\infty)$. □

Remark C.1.14. By linearity and Lemma C.1.13, it follows that for all $\varphi \in C([0,\infty))$ concave, $\Lambda(\varphi)(x,y) \leq 0$ for all $(x,y) \in [0,\infty)^2$ and $\mathcal{L}_0(\varphi)(x) \leq 0$ for all $x \in [0,\infty)$.

Lemma C.1.15. *Consider the operators $\Lambda(\cdot)$, $\mathcal{L}_0(\cdot)$ and $\mathcal{L}(\cdot)$ given in (3.1.30), (3.1.31) and (3.1.42) respectively. Then*

(i) If $\varphi \in \mathrm{Lip}([0,\infty))$ with Lipschitz constant L, then

$$\frac{|\Lambda(\varphi)(x,y)|}{\sqrt{xy}} \leq 2L \qquad \forall (x,y) \in [0,\infty)^2. \tag{C.1.2}$$

(ii) *If* $\varphi \in C^1([0,\infty))$*, then the map* $(x,y) \mapsto \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}$ *belongs to* $C([0,\infty)^2)$ *and*

$$\frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}} = 0 \qquad \forall (x,y) \in \partial[0,\infty)^2. \tag{C.1.3}$$

(iii) *If* $\varphi \in C([0,\infty))$ *then the maps* $x \mapsto \frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}}$ *and* $x \mapsto \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$ *belong to* $C([0,\infty))$ *and* $\frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}} = \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} = 0$ *at* $x = 0$*. If in addition* φ *is bounded, then*

$$\frac{|\mathcal{L}_0(\varphi)(x)|}{\sqrt{x}} \leq 4\|\varphi\|_\infty \sqrt{x} \qquad \forall x \in [0,\infty), \tag{C.1.4}$$

$$\frac{|\mathcal{L}(\varphi)(x)|}{\sqrt{x}} \leq 3\|\varphi\|_\infty \sqrt{x} \qquad \forall x \in [0,\infty). \tag{C.1.5}$$

Proof. (i) By the symmetry of $\Lambda(\varphi)$ we can assume that $0 \leq y \leq x$, and directly from the Lipschitz continuity

$$|\Lambda(\varphi)(x,y)| \leq |\varphi(x+y) - \varphi(x)| + |\varphi(x-y) - \varphi(x)| \leq 2L\, y,$$

which implies (C.1.2).
(ii) The only possible problem for the continuity is on the boundary of $[0,\infty)^2$. Again by the symmetry of $\Lambda(\varphi)$ we can assume $0 \leq y \leq x$. Then by the mean value theorem $\Lambda(\varphi)(x,y) = y\,(\varphi'(\xi_1) - \varphi'(\xi_2))$ for some $\xi_1 \in (x, x+y)$ and $\xi_2 \in (x-y, x)$. Hence

$$\frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}} \leq \varphi'(\xi_1) - \varphi'(\xi_2),$$

and the continuity of $\frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}$ on $[0,\infty)^2$ and (C.1.3) follow from the continuity of φ'.
(iii) The continuity of $\frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}}$ and $\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$ are clear for $x > 0$. Using that $\frac{1}{x}\int_0^x \varphi(y)dy \to \varphi(0)$ as $x \to 0$ by Lebesgue differentiation Theorem, it follows the continuity at $x = 0$ and that $\frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}} = \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} = 0$ for $x = 0$. The bounds (C.1.4) and (C.1.5) are straightforward for $\varphi \in C_b([0,\infty))$. □

Lemma C.1.16. *Consider the operators* $\Lambda(\cdot)$ *and* $\mathcal{L}_0(\cdot)$ *given in (3.1.30) and (3.1.31), and a sequence* $(\phi_n)_{n\in\mathbb{N}} \subset C_c([0,\infty))$ *as in Cutoff 3.3.5.*

(i) *If* $\varphi \in C^1([0,\infty))$ *then* $\Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y) \xrightarrow[n\to\infty]{} \frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}$ *uniformly on the compact sets of* $[0,\infty)^2$*.*

(ii) *If* $\varphi \in C([0,\infty))$ *then* $\mathcal{L}(\varphi)(x)\phi_n(x) \xrightarrow[n\to\infty]{} \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$ *uniformly on the compact sets of* $[0,\infty)$*.*

Proof. (i) The pointwise convergence on $[0,\infty)^2$ is trivial since $\phi_n(x) \to x^{-1/2}$ as $n \to \infty$. Then, let $\varepsilon > 0$ and $R > 0$. For $n \geq R$ there holds $\phi_n(x) = x^{-1/2}$ for all $x \in [1/n, R]$, so we only need to show the uniform convergence on the regions $(x,y) \in [0,R] \times [0,1/n]$ and $(x,y) \in [0,1/n] \times [0,R]$. By the symmetry of $\Lambda(\varphi)$, we may study only one region.

Using that $\frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}}$ is continuous (hence uniformly continuous on compacts) and vanishes when $(x,y) \in \partial[0,\infty)^2$ (c.f. Lemma C.1.15), there holds for all $(x,y) \in [0,R] \times [0,1/n]$ that, for n large enough,

$$\left|\frac{\Lambda(\varphi)(x,y)}{\sqrt{xy}} - \Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y)\right| \le \frac{|\Lambda(\varphi)(x,y)|}{\sqrt{xy}} \le \varepsilon$$

(ii) Let $\varepsilon > 0$ and $R > 0$. Since for $n \ge R$ there holds $\phi_n(x) = x^{-1/2}$ for all $x \in [1/n, R]$, we only need to prove the uniform convergence on the region $[0, 1/n]$. Using that $\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$ is continuous (hence uniformly continuous on compacts) and vanishes when $x \to 0$ (cf. Lemma C.1.15), we have

$$\left|\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} - \mathcal{L}(\varphi)(x)\phi_n(x)\right| \le \frac{|\mathcal{L}(\varphi)(x)|}{\sqrt{x}} \le \varepsilon \qquad \forall x \in [0, 1/n]$$

for n large enough. □

The following Lemma is about the approximation of a measure by continuous functions. It is a simplified version of Lemma 4 in [53].

Lemma C.1.17. *Let $\nu \in \mathscr{M}_+^\alpha([0,\infty))$ for some $\alpha \ge 0$. Then, there exists a sequence of functions $(\nu_n)_{n\in\mathbb{N}} \subset C([0,\infty)) \cap L^1\big(\mathbb{R}_+, (1+x^\alpha)dx\big)$ such that*

$$\forall \varphi \in C([0,\infty)) : \quad \sup_{x\ge 0} \frac{|\varphi(x)|}{1+x^\alpha} < \infty, \tag{C.1.6}$$

$$\lim_{n\to\infty} \int_0^\infty \varphi(x)\nu_n(x)dx = \int_{[0,\infty)} \varphi(x)d\nu(x). \tag{C.1.7}$$

Proof. Let $J(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$ for $x \ge 0$ and define, for $n \in \mathbb{N}$, $x \ge 0$,

$$\nu_n(x) = e^n \int_{[0,\infty)} J\left(e^n|x - y(1-e^{-n})|\right) d\nu(y).$$

In order to prove that ν_n is a continuous function on $[0,\infty)$, let $x \ge 0$ and $(x_k)_{k\in\mathbb{N}} \subset [0,\infty)$ be such that $x_k \to x$ as $k \to \infty$. Since J is a bounded continuous function on $[0,\infty)$ and $M_0(\nu) < \infty$, it is easily deduced using dominated convergence theorem that, for all $n \in \mathbb{N}$, $\nu_n(x_k) \to \nu_n(x)$ as $k \to \infty$, and therefore $\nu_n \in C([0,\infty))$.

Let us prove now that $\nu_n \in L^1\big(\mathbb{R}_+, (1+x^\alpha)dx\big)$. To this end, let $F_n(x,y) = (1+x^\alpha)e^n J\left(e^n|x - y(1-e^{-n})|\right)$. Using the change of variables $z = e^n(y(1-e^{-n})-x)$ we deduce that for all $y \ge 0$, $n \in \mathbb{N}$,

$$\begin{aligned}\int_0^\infty |F_n(x,y)|dx = &\int_0^{y(e^n-1)} \left(1 + [y(1-e^{-n}) - e^{-n}z]^\alpha\right) J(z)dz \\ &+ \int_0^\infty \left(1 + [y(1-e^{-n}) + e^{-n}z]^\alpha\right) J(z)dz.\end{aligned}$$

Since

$$\begin{aligned}&1 + [y(1-e^{-n}) - e^{-n}z]^\alpha \le 1 + [y(1-e^{-n}) + e^{-n}z]^\alpha \le 1 + 2^\alpha(y^\alpha + z^\alpha) \\ &\le 2^\alpha(1+y^\alpha)(1+z^\alpha),\end{aligned} \tag{C.1.8}$$

and $\nu \in \mathscr{M}_+^\alpha([0,\infty))$, then for all $n \in \mathbb{N}$,

$$\int_{[0,\infty)} \int_0^\infty |F_n(x,y)| dx d\nu(y) \leq 2^{\alpha+1} \int_{[0,\infty)} (1+y^\alpha) d\nu(y) \int_0^\infty (1+z^\alpha) J(z) dz < \infty,$$

which implies, by Fubini's theorem, that $\nu_n \in L^1\big(\mathbb{R}_+, (1+x^\alpha)dx\big)$.

Now, for any $\varphi \in C([0,\infty))$ satisfying (C.1.6), using Fubini's theorem and the change of variables $z = e^n\big(x - y(1-e^{-n})\big)$:

$$\int_0^\infty \varphi(x)\nu_n(x)dx = \int_{[0,\infty)} I_n(\varphi)(y) d\nu(y), \tag{C.1.9}$$

$$I_n(\varphi)(y) = \int_0^{y(e^n-1)} \varphi\left(y(1-e^{-n}) - ze^{-n}\right) J(z)dz$$
$$\int_0^\infty \varphi\big(y(1-e^{-n}) + ze^{-n}\big) J(z)dz.$$

By a similar estimate as in (C.1.8), using (C.1.6) we obtain that for some constant $C > 0$,

$$\max\left\{\left|\varphi\big(y(1-e^{-n}) - ze^{-n}\big)\right|, \left|\varphi\big(y(1-e^{-n}) + ze^{-n}\big)\right|\right\} \leq C\big(1+y^\alpha\big)\big(1+z^\alpha\big),$$

and $|I_n(\varphi)(y)| \leq C(1+y^\alpha)$. We then deduce, using dominated convergence, that

$$\lim_{n\to\infty} I_n(\varphi)(y) = 2\varphi(y) \int_0^\infty J(z)dz = \varphi(y), \quad \forall y \geq 0,$$

and

$$\lim_{n\to\infty} \int_{[0,\infty)} I_n(\varphi)(y) d\nu(y) = \int_{[0,\infty)} \varphi(y) d\nu(y),$$

which completes the proof, in view of (C.1.9). □

Corollary C.1.18. *Let $\nu \in \mathscr{M}_+^\alpha([0,\infty))$ for some $\alpha \geq 1$. Then, there exists a sequence of nonnegative functions $(f_n)_{n\in\mathbb{N}} \subset C_c([0,\infty))$ such that*

$$\limsup_{n\to\infty} M_\alpha(f_n) \leq M_\alpha(\nu), \tag{C.1.10}$$

and for all $\varphi \in C_b([0,\infty))$,

$$\lim_{n\to\infty} \int_0^\infty \varphi(x) f_n(x)dx = \int_{[0,\infty)} \varphi(x) d\nu(x). \tag{C.1.11}$$

Proof. We consider the sequence $(\nu_n)_{n\in\mathbb{N}}$ given by Lemma C.1.17 and a smooth cutoff $\zeta_n \in C([0,\infty))$ such that $0 \leq \zeta_n \leq 1$, $\zeta_n(x) = 1$ for $x \in [0,n]$ and $\zeta_n(x) = 0$ for $x \geq n+1$. Then we define for all $n \in \mathbb{N}$:

$$f_n(x) = \nu_n(x)\zeta_n(x). \tag{C.1.12}$$

It then follows that f_n is a nonnegative continuous function on $[0,\infty)$ with compact support. Since $f_n \leq \nu_n$, the property (C.1.10) directly follows from (C.1.7) in Lemma

C.1.17. Now let $\varphi \in C_b([0,\infty))$. Since ν_n satisfies (C.1.7), in order to prove (C.1.11) it is sufficient to prove

$$\lim_{n\to\infty} \left| \int_0^\infty \varphi(x) f_n(x) dx - \int_0^\infty \varphi(x)\nu_n(x)(x)dx \right| = 0, \tag{C.1.13}$$

and (C.1.13) follows from

$$\lim_{n\to\infty} \int_n^\infty \varphi(x)\nu_n(x)dx \leq \lim_{n\to\infty} \frac{\|\varphi\|_\infty M_1(\nu_n)}{n} = 0,$$

where we have used that $M_1(\nu_n) \to M_1(\nu) < \infty$ as $n \to \infty$. □

Definition C.1.19. Let h, ϕ_n and φ be real-valued functions with domain $\mathbb{R}_+$. Then, let

$$\widetilde{\mathscr{Q}}_{3,n}(\varphi, h) = \mathscr{Q}^{(2)}_{3,n}(\varphi, h) - \widetilde{\mathscr{Q}}^{(1)}_{3,n}(\varphi, h), \tag{C.1.14}$$

where

$$\mathscr{Q}^{(2)}_{3,n}(\varphi, h) = \int_0^\infty\int_0^\infty \Lambda(\varphi)(x,y)\phi_n(x)\phi_n(y)h(x)h(y)dxdy, \tag{C.1.15}$$

$$\widetilde{\mathscr{Q}}^{(1)}_{3,n}(\varphi, h) = \int_0^\infty \mathcal{L}(\varphi)(x)\phi_n(x)h(x)dx, \tag{C.1.16}$$

and let, for $x \in \mathbb{R}_+$:

$$J_{3,n}(h)(x) = K_n(h)(x) + L_n(h)(x) - h(x)A_n(h)(x), \tag{C.1.17}$$

where

$$\begin{aligned} K_n(h)(x) = &\int_0^x h(x-y)h(y)\phi_n(x-y)\phi_n(y)dy \\ &+ 2\int_x^\infty h(y)h(y-x)\phi_n(y)\phi_n(y-x)dy, \end{aligned} \tag{C.1.18}$$

$$L_n(h)(x) = 2\int_x^\infty h(y)\phi_n(y)dy, \tag{C.1.19}$$

$$A_n(h)(x) = \phi_n(x)\Big(x + 4\int_0^x h(y)\phi_n(y)dy\Big). \tag{C.1.20}$$

Lemma C.1.20. *Let $G \in \mathscr{M}_+([0,\infty))$, φ_ε as in Remark 3.1.6, and ϕ_n as in Cutoff 3.3.5. Then*

$$G(\{0\}) = \lim_{\varepsilon\to 0} \int_{[0,\infty)} \varphi_\varepsilon(x)G(x)dx, \tag{C.1.21}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathscr{Q}}^{(1)}_{3,n}(\varphi_\varepsilon, G) = 0 \qquad \forall n \in \mathbb{N}. \tag{C.1.22}$$

If in addition G has no singular part in $(0,\infty)$, then

$$\lim_{\varepsilon\to 0} \mathscr{Q}^{(2)}_{3,n}(\varphi_\varepsilon, G) = 0 \qquad \forall n \in \mathbb{N}. \tag{C.1.23}$$

Furthermore, if $G \in \mathscr{M}_+^{1/2}([0,\infty))$*, then*

$$\lim_{\varepsilon\to 0} \mathscr{Q}_3^{(1)}(\varphi_\varepsilon, G) = M_{1/2}(G), \tag{C.1.24}$$

$$\lim_{\varepsilon\to 0} \widetilde{\mathscr{Q}}_3^{(1)}(\varphi_\varepsilon, G) = 0, \tag{C.1.25}$$

where $\mathscr{Q}_3^{(1)}$ *and* $\widetilde{\mathscr{Q}}_3^{(1)}$ *are defined in (3.1.29) and (3.1.41) respectively.*

Proof. The proof only uses dominated convergence. Since $\varphi_\varepsilon \le 1$ for all $\varepsilon > 0$, and $M_0(G) < \infty$, and $\varphi_\varepsilon \to \mathbb{1}_{\{0\}}$ as $\varepsilon \to 0$, then (C.1.21) holds. Then, since for all $x \in [0,\infty)$ it follows from dominated convergence that

$$\lim_{\varepsilon\to 0} \mathcal{L}_0(\varphi_\varepsilon)(x) = x \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \mathcal{L}(\varphi_\varepsilon)(x) = 0, \tag{C.1.26}$$

and ϕ_n is compactly supported, then (C.1.22) follows. Also, since for all $(x,y) \in [0,\infty)^2$, $\Lambda(\varphi_\varepsilon)(x,y) \le 1$ for all $\varepsilon > 0$, and

$$\lim_{\varepsilon\to 0} \Lambda(\varphi_\varepsilon)(x,y) = \mathbb{1}_{\{x=y>0\}}(x,y),$$

then

$$\lim_{\varepsilon\to 0} \mathscr{Q}_{3,n}^{(2)}(\varphi_\varepsilon, G) = \iint_{\{x=y>0\}} \phi_n(x)\phi_n(y)G(x)G(y)dxdy,$$

Using that G has no singular part on $(0,\infty)$, (C.1.23) follows.

Lastly, since

$$\widetilde{\mathscr{Q}}_3^{(1)}(\varphi_\varepsilon, G) \le \mathscr{Q}_3^{(1)}(\varphi_\varepsilon, G) = \int_{(0,\infty)} \frac{\mathcal{L}_0(\varphi_\varepsilon)(x)}{\sqrt{x}} G(x)dx, \tag{C.1.27}$$

and by (C.1.4)

$$\int_{(0,\infty)} \frac{|\mathcal{L}_0(\varphi_\varepsilon)(x)|}{\sqrt{x}} G(x)dx \le 4M_{1/2}(G) \qquad \forall \varepsilon > 0.$$

then (C.1.24) and (C.1.25) follows from (C.1.26) and dominated convergence. □

Lemma C.1.21. *Consider* $n \in \mathbb{N}$, $\phi_n \in C_c([0,\infty))$ *nonnegative and* $\rho \in L^1_{loc}(\mathbb{R}_+)$ *nonnegative. Then for every nonnegative functions* h, h_1 *and* h_2 *in* $L^\infty(\mathbb{R}_+)$*, the functions* $K_n(h)$, $L_n(h)$, $A_n(h)$ *and* $hA_n(h)$ *are also nonnegative, belong to* $L^\infty(\mathbb{R}_+) \cap L^1_\rho(\mathbb{R}_+)$*, and there exists a positive constant* $C(n,\rho)$ *such that:*

$$\|K_n(h_1) - K_n(h_2)\|_{L^\infty \cap L^1_\rho} \le C(n,\rho)\|h_1\|_\infty \|h_1 - h_2\|_\infty \tag{C.1.28}$$

$$\|L_n(h)\|_{L^\infty \cap L^1_\rho} \le C(n,\rho)\|h\|_\infty \tag{C.1.29}$$

$$\|A_n(h)\|_{L^\infty \cap L^1_\rho} \le C(n,\rho)\big(1 + \|h\|_\infty\big) \tag{C.1.30}$$

$$\|A_n(h_1) - A_n(h_2)\|_{L^\infty \cap L^1_\rho} \le C(n,\rho)\|h_1 - h_2\|_\infty. \tag{C.1.31}$$

Moreover

$$J_{3,n}(h) \in L^\infty(\mathbb{R}_+) \cap L^1_\rho(\mathbb{R}_+). \tag{C.1.32}$$

Proof. The positivity of the operators is clear from their definitions. Notice that since ϕ_n is bounded and compactly supported on $\mathbb{R}_+$ and $\rho \in L^1_{loc}(\mathbb{R}_+)$, there exist two positive constants $C(n)$ and $C(n,\rho)$ such that

$$\sup_{x\geq 0} \int_0^\infty \phi_n(|x-y|)\phi_n(y)dy \leq C(n),$$
$$\int_0^\infty \int_0^\infty \rho(x)\phi_n(|x-y|)\phi_n(y)dydx \leq C(n,\rho).$$

1. Estimates for K_n. For all $x \geq 0$:

$$K_n(h)(x) \leq 3\|h\|_\infty^2 \int_0^\infty \phi_n(|x-y|)\phi_n(y)dy \leq 3\|h\|_\infty^2 C(n),$$

and

$$\|K_n(h)\|_{L^1_\rho} \leq 3\|h\|_\infty^2 \int_0^\infty \int_0^\infty \rho(x)\phi_n(|x-y|)\phi_n(y)dydx \leq 3\|h\|_\infty^2 C(n,\rho).$$

Then for all $x \geq 0$:

$$\begin{aligned} &\big|K_n(h_1)(x) - K_n(h_2)(x)\big| \\ &\leq 3\int_0^\infty \phi_n(|x-y|)\phi_n(y)\big|h_1(|x-y|)h_1(y) - h_2(|x-y|)h_2(y)\big|dy. \end{aligned} \tag{C.1.33}$$

Without loss of generality we assume that $\|h_1\|_\infty \geq \|h_2\|_\infty$. Using

$$\big|h_1(|x-y|)h_1(y) - h_2(|x-y|)h_2(y)\big| \leq 2\|h_1\|_\infty\|h_1 - h_2\|_\infty$$

in (C.1.33) then (C.1.28) follows.
2. Estimates for L_n. Since ϕ_n is bounded and compactly supported and $\rho \in L^1_{loc}(\mathbb{R}_+)$, there exist two positive constants $C(n)$ and $C(n,\rho)$ such that

$$\int_0^\infty \phi_n(x)dx \leq C(n) \quad \text{and} \quad \int_0^\infty \rho(x)\int_x^\infty \phi_n(y)dydx \leq C(n,\rho)$$

and (C.1.29) follows.
3. Estimates for A_n. The estimate (C.1.30) follows from

$$\|A_n(h)\|_\infty \leq \|x\,\phi_n(x)\|_\infty + 4\|\phi_n\|_\infty^2\|h\|_\infty|\operatorname{supp}(\phi_n)| \leq C(n)(1+\|h\|_\infty),$$

and

$$\begin{aligned} \|A_n(h)\|_{L^1_\rho} &\leq \int_0^\infty \rho(x)\,x\,\phi_n(x)dx + 4\,\|h\|_\infty \int_0^\infty \rho(x)\,\phi_n(x)\int_0^x \phi_n(y)dydx \\ &\leq C(n,\rho)(1+\|h\|_\infty). \end{aligned}$$

For all $x \geq 0$,

$$\begin{aligned} |A_n(h_1)(x) - A_n(h_2)(x)| &\leq 4\|h_1 - h_2\|_\infty \phi_n(x)\int_0^x \phi_n(y)dy \\ &\leq C(n)\|h_1 - h_2\|_\infty. \end{aligned}$$

We also have,

$$\begin{aligned}\|A_n(h_1)-A_n(h_2)\|_{L^1_\rho} &\leq 4\|h_1-h_2\|_\infty \int_0^\infty \rho(x)\phi_n(x)\int_0^x \phi_n(y)dydx\\ &\leq C(n,\rho)\,\|h_1-h_2\|_\infty,\end{aligned}$$

and then, (C.1.31) follows.
4. Since $h \in L^\infty(\mathbb{R}_+)$ and $A_n(h) \in L^\infty(\mathbb{R}_+) \cap L^1_\rho(\mathbb{R}_+)$, then $hA_n(h) \in L^\infty(\mathbb{R}_+) \cap L^1_\rho(\mathbb{R}_+)$.
5. It also follows from points 1 to 4 that $J_{3,n}(h)$ has the desired regularity. □

C.2 A2

Lemma C.2.22. *Let $\varphi \in C^{1.1}([0,\infty))$. Then, for all $(x_1,x_2,x_3) \in [0,\infty)^3$ such that $x_1+x_2 \geq x_3$:*

$$\begin{aligned}\Delta\varphi(x_1,x_2,x_3) &= (x_1-x_3)(x_2-x_3)\times\\ &\times \int_0^1\int_0^1 \varphi''\big(x_3+t(x_1-x_3)+s(x_2-x_3)\big)dsdt.\end{aligned}$$

Moreover, if $\varphi \in C^{1.1}_b([0,\infty))$, then for all $(x_1,x_2,x_3) \in [0,\infty)^3$

$$|\Delta\varphi(x_1,x_2,x_3)| \leq \min\{A,B,C,D\}\,. \tag{C.2.34}$$

where $A = 4\|\varphi\|_\infty,\quad B = 2\|\varphi'\|_\infty|x_1-x_3|,\quad C = 2\|\varphi'\|_\infty|x_2-x_3|,$
$D = \|\varphi''\|_\infty|x_1-x_3||x_2-x_3|.$

Proof. Let $(x_1,x_2,x_3) \in [0,\infty)^3$ be such that $x_1+x_2 \geq x_3$. By the fundamental Theorem of calculus

$$\begin{aligned}\Delta\varphi(x_1,x_2,x_3) &= \big[\varphi(x_4)-\varphi(x_2)\big]-\big[\varphi(x_1)-\varphi(x_3)\big]\\ &= \int_0^1 \frac{d}{dt}\varphi\big(x_2+t(x_1-x_3)\big)dt-\int_0^1 \frac{d}{dt}\varphi\big(x_3+t(x_1-x_3)\big)dt\\ &= (x_1-x_3)\int_0^1 \big[\varphi'\big(x_2+t(x_1-x_3)\big)-\varphi'\big(x_3+t(x_1-x_3)\big)\big]dt\\ &= (x_1-x_3)\int_0^1\int_0^1 \frac{d}{ds}\varphi'\big(x_3+t(x_1-x_3)+s(x_2-x_3)\big)dsdt\\ &= (x_1-x_3)(x_2-x_3)\int_0^1\int_0^1 \varphi''\big(x_3+t(x_1-x_3)+s(x_2-x_3)\big)dsdt.\end{aligned}$$

Assume now that $\varphi \in C^{1.1}_b([0,\infty))$. Using the first, the third, and the fifth line above, estimate (C.2.34) follows. □

We now consider the function w given in (3.1.24) and define

$$W(x_1,x_2,x_3) = \begin{cases} \frac{w(x_1,x_2,x_3)}{\sqrt{x_1x_2x_3}} & \text{if} \quad (x_1,x_2,x_3) \in (0,\infty)^3\\ \frac{1}{\sqrt{x_1x_2}} & \text{if} \quad x_3=0, \quad (x_1,x_2) \in (0,\infty)^2\\ \frac{1}{\sqrt{x_ix_3}} & \text{if } x_j=0,\ x_i>x_3>0;\ \{i,j\}=\{1,2\}\\ 0 & \text{otherwise.}\end{cases} \tag{C.2.35}$$

We then have:

Lemma C.2.23. *Consider the function $\Phi_\varphi = W\Delta\varphi$, where $\Delta\varphi$ and W are defined in (3.1.23) and (C.2.35) respectively.*

(i) If $\varphi \in C^{1.1}([0,\infty))$ then $\Phi_\varphi \in C([0,\infty)^3)$.

(ii) If $\varphi \in C_b^{1.1}([0,\infty))$ then $\Phi_\varphi \in C_0([0,\infty)^3)$. In particular Φ_φ is uniformly continuous on $[0,\infty)^3$.

Proof. **Proof of (i).** By definition $\Phi_\varphi \in C((0,\infty)^3)$. Therefore it only remains to study the behaviour of Φ_φ in a neighborhood of the boundary $\partial[0,\infty)^3$ of $[0,\infty)^3$. First we show that Φ_φ is continuous on $\partial[0,\infty)^3$.
Thanks to the symmetry of Φ_φ in the x_1, x_2 variables, we just need to prove:
(i)for all $(x_1,x_2) \in (0,\infty)^2$,

$$\Phi_\varphi(x_1,x_2,0) = \frac{\Delta\varphi(x_1,x_2,0)}{\sqrt{x_1x_2}} \longrightarrow 0 \tag{C.2.36}$$

whenever $x_1 \to 0$ or $x_2 \to 0$ or $(x_1,x_2) \to (0,0)$, and
(ii) for all $x_1 > x_3 > 0$,

$$\Phi_\varphi(x_1,0,x_3) = \frac{\Delta\varphi(x_1,0,x_3)}{\sqrt{x_1x_3}} \longrightarrow 0 \tag{C.2.37}$$

whenever $x_1 \to x_3$ or $x_3 \to 0$ or $(x_1,x_3) \to (0,0)$.

By (C.2.34) $|\Delta\varphi(x_1,x_2,0)| \le \|\varphi''\|_\infty x_1x_2$ for all $(x_1,x_2) \in (0,\infty)^2$, which implies (C.2.36). Also $|\Delta\varphi(x_1,0,x_3)| \le \|\varphi''\|_\infty x_3(x_1-x_3)$ for all $x_1 > x_3 > 0$. Hence

$$\frac{|\Delta\varphi(x_1,0,x_3)|}{\sqrt{x_1x_3}} \le \|\varphi''\|_\infty \sqrt{\frac{x_3}{x_1}}(x_1-x_3) \le \|\varphi''\|_\infty (x_1-x_3),$$

which implies (C.2.37).

Then we prove that for any $x \in \partial[0,\infty)^3$ and for any $(x_n)_{n\in\mathbb{N}} \subset (0,\infty)^3$ such that $x_n \to x$, then $\Phi_\varphi(x_n) \to \Phi_\varphi(x)$ as $n \to \infty$. Let us denote

$$\Omega = \{(x_1,x_2,x_3) \in (0,\infty)^3 : x_1+x_2 \le x_3\}.$$

Since x_4 is defined as $x_4 = (x_1+x_2-x_3)_+$, then for all $(x_1,x_2,x_3) \in (0,\infty)^3$,

$$(x_1,x_2,x_3) \in \Omega \quad \text{if and only if} \quad x_4 = 0.$$

It might happen that the sequence $(x_n)_{n\in\mathbb{N}}$ "jumps" from Ω to Ω^c. If in every neighbourhood of x the sequence has points in both regions, then we may consider two subsequences, each one contained in one region only. For the sequel, the main estimate is the following: if we denote $x_n = (x_1^n, x_2^n, x_3^n)$ and $w(x_n) = \min\{\sqrt{x_1^n}, \sqrt{x_2^n}, \sqrt{x_3^n}, \sqrt{x_4^n}\}$, then by (C.2.34)

$$|\Phi_\varphi(x_n)| \le \|\varphi''\|_\infty \frac{w(x_n)}{\sqrt{x_1^nx_2^nx_3^n}} |x_1^n - x_3^n||x_2^n - x_3^n|. \tag{C.2.38}$$

We study case by case depending on where x lies.

Case $x = (0,0,0)$. If $(x_n) \subset \Omega$ then $x_4^n = 0$, $w(x_n) = \sqrt{x_4^n} = 0$ and thus $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$.
If $\{x_n\} \subset \Omega^c$ then $x_4^n > 0$ and we study case by case depending on the relative order

of x_1^n, x_2^n, and x_3^n. Since Φ_φ is symmetric in the x_1, x_2 variables, we may assume without loss of generality that $x_1^n \le x_2^n$. Note by (C.2.38) that we also may assume $x_3^n \ne x_1^n$, $x_3^n \ne x_2^n$; otherwise the result follows directly.

If $x_1^n \le x_2^n < x_3^n$, then $w(x_n) = \sqrt{x_4^n}$ and by (C.2.38)

$$
\begin{aligned}
|\Phi_\varphi(x_n)| &\le \|\varphi''\|_\infty \frac{\sqrt{x_4^n}}{\sqrt{x_1^n x_2^n x_3^n}} \left(x_3^n - x_1^n\right)\left(x_3^n - x_2^n\right) \\
&\le \|\varphi''\|_\infty \left(\frac{\sqrt{x_4^n}\left(x_3^n\right)^{3/2}}{\sqrt{x_1^n x_2^n}} + \frac{\sqrt{x_4^n x_1^n x_2^n}}{\sqrt{x_3^n}} \right) \\
&\le \|\varphi''\|_\infty \left(\frac{\left(x_3^n\right)^{3/2}}{\sqrt{x_2^n}} + \sqrt{x_1^n x_2^n} \right).
\end{aligned}
$$

Since $x_n \to x = 0$, then $\sqrt{x_1^n x_2^n} \to 0$. Moreover, since $x_n \in \Omega^c$ and $x_1^n \le x_2^n$, then $x_3^n < 2x_2^n$, and so

$$
\frac{\left(x_3^n\right)^{3/2}}{\sqrt{x_2^n}} \le 2^{3/2} x_2^n \longrightarrow 0 \qquad \text{as} \quad n \to \infty.
$$

If $x_1^n < x_3^n < x_2^n$, then $w(x_n) = \sqrt{x_1^n}$ and by (C.2.38)

$$
\begin{aligned}
|\Phi_\varphi(x_n)| &\le \|\varphi''\|_\infty \frac{(x_2^n - x_3^n)(x_3^n - x_1^n)}{\sqrt{x_2^n x_3^n}} \\
&\le \|\varphi''\|_\infty \left(\sqrt{x_2^n x_3^n} + \frac{x_1^n \sqrt{x_3^n}}{\sqrt{x_2^n}} \right) \\
&\le \|\varphi''\|_\infty \left(\sqrt{x_2^n x_3^n} + \sqrt{x_1^n x_3^n} \right) \longrightarrow 0 \qquad \text{as} \quad n \to \infty.
\end{aligned}
$$

Lastly, if $x_3^n < x_1^n \le x_2^n$, then $w(x_n) = \sqrt{x_3^n}$ and by (C.2.38)

$$
\begin{aligned}
|\Phi_\varphi(x_n)| &\le \|\varphi''\|_\infty \frac{(x_1^n - x_3^n)(x_2^n - x_3^n)}{\sqrt{x_1^n x_2^n}} \\
&\le \|\varphi''\|_\infty \left(\sqrt{x_1^n x_2^n} + \frac{\left(x_3^n\right)^2}{\sqrt{x_1^n x_2^n}} \right) \\
&\le 2\|\varphi''\|_\infty \left(\sqrt{x_1^n x_2^n} + x_1 \right) \longrightarrow 0 \qquad \text{as} \quad n \to \infty.
\end{aligned}
$$

Hence, in the three cases above $\Phi_\varphi(x_n) \to 0 = \Phi_\varphi(x)$.

Case $x = (x_1, 0, 0)$ with $x_1 > 0$. Then $w(x_n) = \min\left\{\sqrt{x_2^n}, \sqrt{x_3^n}\right\}$ for n large enough. On the other hand

$$
\begin{aligned}
\left|x_2^n - x_3^n\right| &= \left(\sqrt{x_2^n} + \sqrt{x_3^n}\right)\left|\sqrt{x_2^n} - \sqrt{x_3^n}\right| \\
&\le 2\max\left\{\sqrt{x_2^n}, \sqrt{x_3^n}\right\} \left|\sqrt{x_2^n} - \sqrt{x_3^n}\right|.
\end{aligned}
$$

Since $\min\left\{\sqrt{x_2^n}, \sqrt{x_3^n}\right\} \max\left\{\sqrt{x_2^n}, \sqrt{x_3^n}\right\} = \sqrt{x_2^n x_3^n}$, then by (C.2.38)

$$
|\Phi_\varphi(x_n)| \le 2\|\varphi''\|_\infty \frac{\left|x_1^n - x_3^n\right|}{\sqrt{x_1^n}} \left|\sqrt{x_2^n} - \sqrt{x_3^n}\right|
$$

for n large enough. It then follows $\Phi_\varphi(x_n) \to 0 = \Phi_\varphi(x)$ as $n \to \infty$.

The case $x = (0, x_2, 0)$ with $x_2 > 0$ is analogous to the previous one thanks to the symmetry of Φ_φ in the x_1, x_2 variables.

Case $x = (0, 0, x_3)$ with $x_3 > 0$. Then $x_n \in \Omega$ for n large enough, $x_4^n = 0$ and $w(x_n) = \sqrt{x_4^n} = 0$. Thus $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$ for n large enough.

Case $x = (0, x_2, x_3)$ with $x_2 > 0$ and $x_3 > 0$. If $x_2 > x_3$ then $w(x_n) = \sqrt{x_1^n}$ for n large enough and

$$|\Phi_\varphi(x_n) - \Phi_\varphi(x)| = \left| \frac{1}{\sqrt{x_2^n x_3^n}} \Delta\varphi(x_1^n, x_2^n, x_3^n) - \frac{1}{\sqrt{x_2 x_3}} \Delta\varphi(0, x_2, x_3) \right|,$$

which clearly goes to zero as $n \to \infty$. If $x_2 < x_3$ then $x_4^n = 0$ for n large enough and $w(x_n) = \sqrt{x_4^n} = 0$, thus $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$. If $x_2 = x_3$ and $(x_n) \subset \Omega$ for n large enough, then $x_4^n = 0$, thus $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$.
If $x_2 = x_3$ and $(x_n) \subset \Omega^c$ for n large enough, then $w(x_n) = \min\left\{\sqrt{x_1^n}, \sqrt{x_4^n}\right\}$, and by (C.2.38)

$$|\Phi_\varphi(x_n)| \le \|\varphi''\|_\infty \frac{\min\left\{\sqrt{x_1^n}, \sqrt{x_4^n}\right\}}{\sqrt{x_1^n x_2^n x_3^n}} \left|x_1^n - x_3^n\right| \left|x_2^n - x_3^n\right|.$$

On the one hand

$$\left|x_1^n - x_3^n\right| \le 2 \max\left\{\sqrt{x_1^n}, \sqrt{x_3^n}\right\} \left|\sqrt{x_1^n} - \sqrt{x_3^n}\right|.$$

On the other hand $\min\left\{\sqrt{x_1^n}, \sqrt{x_4^n}\right\} \le \min\left\{\sqrt{x_1^n}, \sqrt{x_3^n}\right\}$ for n large enough. Since $\min\left\{\sqrt{x_1^n}, \sqrt{x_3^n}\right\} \max\left\{\sqrt{x_1^n}, \sqrt{x_3^n}\right\} = \sqrt{x_1^n x_3^n}$, then

$$|\Phi_\varphi(x_n)| \le 2\|\varphi''\|_\infty \frac{\left|x_2^n - x_3^n\right|}{\sqrt{x_2^n}} \left|\sqrt{x_1^n} - \sqrt{x_3^n}\right|,$$

which goes to zero as $n \to \infty$ since $x_2 = x_3$. Thus $\Phi_\varphi(x_n) \to 0 = \Phi_\varphi(x)$.

The case $x = (x_1, 0, x_3)$ with $x_1 > 0$ and $x_3 > 0$ is analogous to the previous one thanks to the symmetry of Φ_φ in the x_1, x_2 variables.

Case $x = (x_1, x_2, 0)$ with $(x_1, x_2) \in (0, \infty)^2$. Then $w(x_n) = \sqrt{x_3^n}$ for n large enough and

$$|\Phi_\varphi(x_n) - \Phi_\varphi(x)| = \left| \frac{1}{\sqrt{x_1^n x_2^n}} \Delta\varphi(x_1^n, x_2^n, x_3^n) - \frac{1}{\sqrt{x_1 x_2}} \Delta\varphi(x_1, x_2, 0) \right|,$$

which clearly goes to zero as $n \to \infty$.

Proof of (ii). By part (i) $\Phi_\varphi \in C([0, \infty)^3)$. Let us show now that for any given $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that $|\Phi_\varphi(x)| \le \varepsilon$ for all $x \in [0, \infty)^3 \setminus [0, R(\varepsilon)]^3$.

Given $R > 0$ and $\alpha > 0$, let $(x_1, x_2, x_3) \in [0, \infty)^3 \setminus [0, R]^3$ and denote $x_i = \min\{x_1, x_2, x_3\}$, $x_k = \max\{x_1, x_2, x_3\}$ and x_j neither x_i nor x_k. Notice that $x_k > R$ and the function W defined in (C.2.35) satisfies $W(x_1, x_2, x_3) \le \frac{1}{\sqrt{x_j x_k}}$. If $x_i > \alpha$ or $x_j > \alpha$ then by (C.2.34)

$$|\Phi_\varphi(x_1, x_2, x_3)| \le \frac{|\Delta\varphi(x_1, x_2, x_3)|}{\sqrt{x_j x_k}} \le \frac{4\|\varphi\|_\infty}{\sqrt{\alpha R}} \le \varepsilon,$$

provided $R \geq \frac{16\|\varphi\|_\infty^2}{\alpha\varepsilon^2}$. If $x_i \leq \alpha$ and $x_j \leq \alpha$ we study case by case depending on the relative position of x_1, x_2, x_3. Since Φ_φ is symmetric in variables x_1 and x_2, we may assume without loss of generality that $x_2 \leq x_1$. If $x_k = x_1$, using (C.2.34)

$$|\Phi_\varphi(x_1, x_2, x_3)| \leq \frac{2\|\varphi'\|_\infty (x_j - x_i)}{\sqrt{x_1 x_j}} \leq \frac{2\|\varphi'\|_\infty \sqrt{x_j}}{\sqrt{x_1}} \leq \frac{2\|\varphi'\|_\infty \sqrt{\alpha}}{\sqrt{R}} \leq \varepsilon,$$

provided $R \geq \frac{4\|\varphi'\|_\infty^2 \alpha}{\varepsilon^2}$. If $x_k = x_3$ and $x \in \Omega$ then $x_4 = 0$ and $\Phi_\varphi(x) = 0$. If $x_k = x_3$ and $x \in \Omega^c$, then $x_1 \geq R/2$ and

$$|\Phi_\varphi(x_1, x_2, x_3)| \leq \frac{4\|\varphi\|_\infty}{\sqrt{x_1 x_3}} \leq \frac{4\sqrt{2}\|\varphi\|_\infty}{R} \leq \varepsilon,$$

provided $R \geq \frac{4\sqrt{2}\|\varphi\|_\infty}{\varepsilon}$.

Finally, if we chose $R \geq \max\left\{\frac{16\|\varphi\|_\infty^2}{\alpha\varepsilon^2}, \frac{4\|\varphi'\|_\infty^2 \alpha}{\varepsilon^2}, \frac{4\sqrt{2}\|\varphi\|_\infty}{\varepsilon}\right\}$ then $\Phi_\varphi \in C_0([0,\infty)^3)$ and in particular, Φ_φ is uniformly continuous in $[0,\infty)^3$. $\square$

Bibliography

[1] R. Alonso, I. M. Gamba, and M.-B. Tran. The Cauchy problem and BEC stability for the quantum Boltzmann-Condensation system for bosons at very low temperature. *ArXiv e-prints*, May 2018.

[2] L. Ambrosio, N. Gigli, and G. Savare. *Gradient Flows: In Metric Spaces and in the Space of Probability Measures*. Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 2005.

[3] L. Arkeryd and A. Nouri. Bose condensates in interaction with excitations: A kinetic model. *Communications in Mathematical Physics*, 310(3):765–788, Mar 2012.

[4] L. Arkeryd and A. Nouri. Bose condensates in interaction with excitations: A two-component space-dependent model close to equilibrium. *Journal of Statistical Physics*, 160(1):209–238, Jul 2015.

[5] L. Arkeryd and A. Nouri. On the Cauchy problem with large data for a space-dependent Boltzmann-Nordheim boson equation. *ArXiv e-prints*, January 2016.

[6] L. Arkeryd and A. Nouri. On the Cauchy problem with large data for a space-dependent Boltzmann-Nordheim boson equation II. *ArXiv e-prints*, November 2016.

[7] L. Arkeryd and A. Nouri. On the Cauchy problem with large data for the space-dependent Boltzmann Nordheim equation III. *ArXiv e-prints*, January 2018.

[8] P. B. Arnold, G. D. Moore, and L. G. Yaffe. Effective kinetic theory for high temperature gauge theories. *Journal of High Energy Physics*, 2003(01):030, 2003.

[9] J. Ballew, G. Iyer, and R. L. Pego. Bose-Einstein condensation in a hyperbolic model for the Kompaneets equation. *SIAM J. Math. Analysis*, 48:3840–3859, 2016.

[10] P. K. Barik, A. K. Giri, and P. Laurençot. Mass-conserving solutions to the Smoluchowski coagulation equation with singular kernel. *ArXiv e-prints, arXiv:1804.00853*, April 2018.

[11] M. Birkinshaw. The SunyaevÄ-ZelÄôdovich effect. *Phys. Reports*, 310(2):97–195, 1999.

[12] J-P. Blaizot and E. Iancu. The Quark gluon plasma: Collective dynamics and hard thermal loops. *Phys. Rept.*, 359:355–528, 2002.

[13] A. V. Bobylev. Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems. *Journal of Statistical Physics*, 1997.

[14] C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein. Bogoliubov theory in the Gross-Pitaevskii limit. *arXiv:1801.01389 [math-ph]*, page 101, 2018.

[15] V. I. Bogachev. *Measure Theory*. Number vol. 2. Springer Berlin Heidelberg, 2007.

[16] M. Briant and A. Einav. On the Cauchy problem for the homogeneous Boltzmann–Nordheim equation for bosons: local existence, uniqueness and creation of moments. *Journal of Statistical Physics*, 2016.

[17] L. M. Brown and R. P. Feynman. Radiative corrections to Compton scattering. *Phys. Rev.*, 85:231–244, 1952.

[18] C. Buet and S. Cordier. *Numerical Method for the Compton Scattering Operator*, chapter 9, pages 223–258. Series on advances in mathematics for applied sciences. World Scientific, 2003.

[19] C. Buet, B. Després, and Th. Leroy. Anisotropic models and angular moments methods for the Compton scattering. *Hal e-prints, hal-01717173*, 2018.

[20] R. E. Caflisch and C. D. Levermore. Equilibrium for radiation in a homogeneous plasma. *The Physics of Fluids*, 29(3):748–752, 1986.

[21] C. Cueto Camejo, R. Gröpler, and G. Warnecke. Regular solutions to the coagulation equations with singular kernels. *Mathematical Methods in the Applied Sciences*, 38(11):2171–2184.

[22] M. Chane-Yook and A. Nouri. On a quantum kinetic equation linked to the Compton effect. *Transport Theory and Statistical Physics*, 33(5-7):403–427, 2004.

[23] W. R. Chappell and W. E. Brittin. Quantum kinetic equations for a multicomponent system of charged particles and photons. *Phys. Rev.*, 146:75–91, Jun 1966.

[24] E. Cortés and M. Escobedo. On a system of equations for the normal fluid-condensate interaction in a Bose gas. *ArXiv e-prints*, March 2018.

[25] H. Dreicer. Kinetic Theory of an Electron-Photon Gas. *Physics of Fluids*, 7:735–753, May 1964.

[26] U. Eckern. Relaxation processes in a condensed Bose gas. *Journal of Low Temperature Physics*, 54:333–359, 1984. 10.1007/BF00683281.

[27] L. Erdös, B. Schlein, and H.-T. Yau. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Annals of Mathematics*, 172(1):291–370, 2010.

[28] M. Escobedo, M. A. Herrero, and J. J. L. Velázquez. A nonlinear Fokker-Planck equation modelling the approach to thermal equilibrium in a homogeneous plasma. *Trans. Amer. Math. Soc.*, 350(10):3837–3901, 1998.

[29] M. Escobedo and S. Mischler. On a quantum Boltzmann equation for a gas of photons. *Journal de Mathématiques Pures et Appliquées*, 80(5):471 – 515, 2001.

[30] M. Escobedo, S. Mischler, and M. A. Valle. Homogeneous Boltzmann equation in quantum relativistic kinetic theory. *Electron. J. Diff. Eqns. Monograph*, 4(2):1–85, 2003.

[31] M. Escobedo, S. Mischler, and J. J. L. Velázquez. Asymptotic description of Dirac mass formation in kinetic equations for quantum particles. *Journal of Differential Equations*, 202(2):208 – 230, 2004.

[32] M. Escobedo, S. Mischler, and J. J. L. Velázquez. Singular solutions for the Uehling–Uhlenbeck equation. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 138(01):67–107, 2008.

[33] M. Escobedo and J. J. L. Velázquez. Finite time blow-up and condensation for the bosonic Nordheim equation. *Invent. Math.*, 200(3):761–847, 2015.

[34] E. Ferrari and A. Nouri. On the Cauchy problem for a quantum kinetic equation linked to the Compton effect. *Mathematical and Computer Modelling*, 43:838–853, 04 2006.

[35] G.B. Folland. *Real analysis: modern techniques and their applications.* Pure and Applied Mathematics. Wiley, 1999.

[36] N. Fournier and Ph. Laurençot. Well-posedness of smoluchowski's coagulation equation for a class of homogeneous kernels. *Journal of Functional Analysis*, 233(2):351 – 379, 2006.

[37] C. W. Gardiner and P. Zoller. Quantum kinetic theory V. Quantum kinetic master equation for mutual interaction of condensate and noncondensate. *Phys. Rev. A*, 61:033601, 2000.

[38] S. I. Grachev. Nonstationary radiative transfer: Evolution of a spectrum by multiple compton scattering. *Astrophysics*, 57(4):550–558, Dec 2014.

[39] S. I. Grachev. Nonstationary radiative transfer: Evolution of a spectrum by multiple Compton scattering. *Astrophysics*, 57(4):550–558, Dec 2014.

[40] A. Griffin, T. Nikuni, and E. Zaremba. *Bose-Condensed Gases at Finite Temperatures.* Cambridge University Press, 2009.

[41] M. Imamovic-Tomasovic and A. Griffin. Quasiparticle kinetic equation in a trapped Bose gas at low temperatures. *J. Low Temp. Phys.*, 122:617–655, 2001.

[42] J.I. Katz. *High energy astrophysics.* Frontiers in physics. Addison-Wesley Pub. Co., 1987.

[43] O. Kavian. Remarks on the Kompaneets equation, a simplified model of the Fokker-Planck equation. *Stud. Math. Appl. North-Holland, Amsterdam*, 2002.

[44] A. H. M Kierkels. *On a Kinetic Equation Arising in Weak Turbulence Theory for the Nonlinear Schrödinger Equation.* PhD thesis, Universität Bonn, 2016.

[45] A. H. M. Kierkels and J. J. L. Velázquez. On the Transfer of Energy Towards Infinity in the Theory of Weak Turbulence for the Nonlinear Schrödinger Equation. *Journal of Statistical Physics*, 2015.

[46] A. H. M. Kierkels and J. J. L. Velázquez. On self-similar solutions to a kinetic equation arising in weak turbulence theory for the nonlinear schrödinger equation. *Journal of Statistical Physics*, 163(6):1350–1393, Jun 2016.

[47] T. R. Kirkpatrick and J. R. Dorfman. Transport in a dilute but condensed nonideal Bose gas: Kinetic equations. *Journal of Low Temperature Physics*, 58(3):301–331, 02 1985.

[48] A. Klenke. *Probability Theory: A Comprehensive Course.* Universitext. Springer London, 2013.

[49] A. S. Kompaneets. The establishment of thermal equilibrium between quanta and electrons. *Soviet Journal of Experimental and Theoretical Physics*, 4:730–737, 1957.

[50] C. D. Levermore, H. Liu, and R. L. Pego. Global dynamics of Bose-Einstein condensation for a model of the Kompaneets equation. *SIAM J. Math. Anal.*, 48(4):2454–2494, 2016.

[51] W. Li and X. Lu. Global existence of solutions of the Boltzmann equation for Bose-Einstein particles with anisotropic initial data. *Preprint arXiv:1706.06235v3 [math.AP]*, page 48, 2017.

[52] E.H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason. *The mathematics of the Bose Gas and its condensation.* Oberwolfach Seminars. Springer Basel AG, 2005.

[53] X. Lu. On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles. *J. Statist. Phys.*, 116(5-6):1597–1649, 2004.

[54] X. Lu. The Boltzmann equation for Bose-Einstein particles: velocity concentration and convergence to equilibrium. *J. Stat. Phys.*, 119(5-6):1027–1067, 2005.

[55] X. Lu. The Boltzmann equation for Bose-Einstein particles: condensation in finite time. *J. Stat. Phys.*, 150(6):1138–1176, 2013.

[56] X. Lu. The Boltzmann equation for Bose-Einstein particles: regularity and condensation. *J. Stat. Phys.*, 156(3):493–545, 2014.

[57] P. Mészáros and R. W. Bussard. The angle-dependent Compton redistribution function in x-ray sources. *The Astrophysical Journal*, 306:238–247, 06 1986.

[58] L. W. Nordheim. On the kinetic method in the new statistics and its application in the electron theory of conductivity. *Proc. R. Soc. Lond. A*, 119:689–698, 1928.

[59] J. R. Norris. Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. *Ann. Appl. Probab.*, 9(1):78–109, 02 1999.

[60] A. Nouri. Bose-Einstein condensates at very low temperatures: a mathematical result in the isotropic case. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 2(2):649–666, 2007.

[61] G.C. Pomraning. *The Equations of Radiation Hydrodynamics*. Dover books on physics. Dover Publications, 2005.

[62] N. P. Proukakis, K. Burnett, and H. T. C. Stoof. Microscopic treatment of binary interactions in the nonequilibrium dynamics of partially bose-condensed trapped gases. *Phys. Rev. A*, 57:1230–1247, Feb 1998.

[63] H. L. Royden. *Real Analysis*. Macmillan, 2nd edition, 1968.

[64] W. Rudin. *Functional Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[65] D. V. Semikoz and I. I. Tkachev. Kinetics of Bose condensation. *Phys. Rev. Lett.*, 74(16):3093–3097, 1995.

[66] D. V. Semikoz and I. I. Tkachev. Condensation of bosons in the kinetic regime. *Phys. Rev. D*, 55(2):489–502, Jan 1997.

[67] H. Spohn. Kinetics of the Bose-Einstein condensation. *Physica D: Nonlinear Phenomena*, 239(10):627 – 634, 2010.

[68] H. T. C. Stoof. Coherent versus incoherent dynamics during bose-einstein condensation in atomic gases. *Journal of Low Temperature Physics*, 114(1):11–108, Jan 1999.

[69] H. T. C. Stoof. Field theory for trapped atomic gases. In R. Kaiser, C. Westbrook, and F. David, editors, *Coherent atomic matter waves*, pages 219–315, Berlin, Heidelberg, 2001. Springer Berlin Heidelberg.

[70] B. V. Svistunov. Highly nonequilibrium Bose condensation in a weakly interacting gas. *Journal of the Moscow Physical Society*, 1:373–390, 1991.

[71] R. Walser, J. Williams, J. Cooper, and M. Holland. Quantum kinetic theory for a condensed bosonic gas. *Phys. Rev. A*, 59:3878–3889, May 1999.

[72] R. Weyman. Diffusion Approximation for a Photon Gas Interacting with a Plasma via the Compton Effect. *Phys. Fluids*, 8:2112–2114, 1965.

[73] E. Zaremba, T. Nikuni, and A. Griffin. Dynamics of trapped Bose gases at finite temperatures. *Journal of Low Temperature Physics*, 116:277–345, 1999. 10.1023/A:1021846002995.

[74] Y. B. Zel'Dovich. Reviews of Topical Problems: Interaction of free electrons with electromagnetic radiation. *Soviet Physics Uspekhi*, 18:79–98, February 1975.

[75] Y. B. Zel'Dovich and E. V. Levich. Bose Condensation and Shock Waves in Photon Spectra. *Soviet Journal of Experimental and Theoretical Physics*, 28:1287–1290, June 1969.

[76] Y. B. Zel'Dovich, E. V. Levich, and R. A. Syunyaev. Stimulated Compton Interaction between Maxwellian Electrons and Spectrally Narrow Radiation. *Soviet Journal of Experimental and Theoretical Physics*, 35:733–740, 1972.

[77] Y. B. Zel'Dovich and R. A. Syunyaev. Shock Wave Structure in the Radiation Spectrum During Bose Condensation of Photons. *Soviet Journal of Experimental and Theoretical Physics*, 35:81–85, 1972.