Gaussian quadrature for C^1 cubic Clough-Tocher macro-triangles

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Abstract

A numerical integration rule for multivariate cubic polynomials over n-dimensional simplices was designed by Hammer and Stroud [14]. The quadrature rule requires $n+2$ quadrature points: the barycentre of the simplex and $n+1$ points that lie on the connecting lines between the barycentre and the vertices of the simplex. In the planar case, this particular rule belongs to a two-parameter family of quadrature rules that admit exact integration of bivariate polynomials of total degree three over triangles. We prove that this rule is exact for a larger space, namely the C^1 cubic Clough-Tocher spline space over macro-triangles if and only if the split-point is the barycentre. This results into a factor of three reduction in the number of quadrature points needed to integrate the Clough-Tocher spline space exactly.

Key words: Numerical integration, Clough-Tocher spline space, Gaussian quadrature rules

1. Introduction

Numerical quadrature provides an efficient way of numerically evaluating integrals of functions from a certain linear space over a suitable parametric domain [16]. Among various classes of functions, polynomials play an important role. Gauss-Legendre quadrature is exact for univariate polynomials up to a given degree and is indispensable in the context of finite element methods [18]. Multi-variate integration has attracted a lot of attention in the past decades, see e.g. [21, 24, 27], and the encyclopedia of cubature rules [10] and the references cited therein.

With the introduction of isogeometric analysis [11], univariate quadrature rules for polynomial spline spaces have become a topic of recent interest [3, 5–8, 15, 17, 19]. One can use Gauss-Legendre quadrature on each individual element (knot interval) to integrate a spline function. However, this is inefficient as the increased continuity between elements reduces the number of quadrature points needed for exact integration. Although theoretical results about the existence of Gaussian quadrature rules for univariate polynomial splines have been known since 1977 [25], it was shown only recently how to numerically find such rules [6, 7] over arbitrary knot vectors. This then naturally extends to tensor-product scenarios such as bivariate B-splines, where also the methods of [8] show great promise.

In contrast, quadrature rules for polynomial splines over triangulations have received, to the best of our knowledge, no treatment so far. The current state of the art in the case of spaces such as C^1 cubic Clough-Tocher macro-triangles [9] and various box-spline constructions [12] is to apply simplex quadrature [30] over each simplex separately. Again, this is, as we show below, inefficient. In the bivariate case, the situation is difficult even for polynomials because only bounds on the number of quadrature points are known [31], and algorithms that iteratively remove redundant quadrature points are used [26, 33].

We initiate the study of Gaussian quadrature rules for polynomial spline spaces over triangulations. By Gaussian we mean that the rule is optimal in terms of the number of quadrature points, that is, the rule uses

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the minimal number of quadrature points while guaranteeing exactness of integration for any function from the space under consideration. In this paper, we focus on the spline space of C^1 cubic Clough-Tocher macro-triangles. This space is frequently used, for example, to solve the C^1 Hermite interpolation problem over a general triangulation by splitting each triangle into three micro-triangles based on a split-point, typically the barycentre. Each original triangle thus becomes a macro-triangle.

It is known that total-degree cubic polynomials over triangles can be integrated using four quadrature points [14, 30] and this number of points is optimal. However, it turns out, as we show below in our main result, that this four-node quadrature is exact not only on the cubic space over a triangle, but also on the C^1 cubic Clough-Tocher spline space over the corresponding macro-triangle provided that the split-point in the construction is chosen to be the barycentre of the triangle. This result effectively reduces the number of quadrature points needed by a factor of three when compared to standard element-wise quadrature.

We first recall some basic concepts such as Bézier triangles, the Clouch-Tocher macro-triangle, and the Hammer-Stroud quadrature in Section 2. This is followed by Section 3, where we state and prove our main result regarding Gaussian quadrature for C^1 cubic Clough-Tocher macro-triangles. In Section 4 we outline possible generalisations of our result. Finally, we conclude the paper in Section 5.

2. Preliminaries

We start by introducing some basic concepts such as Bézier triangles, the C^1 cubic Clough-Tocher macro-element, and Hammer-Stroud quadrature.

2.1. Bézier triangles

Consider a non-degenerate triangle $\mathcal{T}$ given by three vertices V_0, V_1, and V_2 in $\mathbb{R}^2$. Without loss of generality, we assume that the area of $\mathcal{T}$ is equal to one. Any point P in $\mathcal{T}$ can be uniquely expressed in terms of its barycentric coordinates $\boldsymbol{\tau} = (\tau_0, \tau_1, \tau_2)$ as

$$P = \sum_{i=0}^{2} \tau_i V_i; \quad \tau_0 + \tau_1 + \tau_2 = 1,\ 0 \le \tau_i \le 1, \tag{1}$$

where τ_i, $i = 0, 1, 2$, is equal to the area of triangle (V_{i+1}, V_{i+2}, P) with i being treated cyclically modulo 3.

With $\mathbf{i} = (i, j, k)$, $|\mathbf{i}| = i + j + k$, and $i, j, k \ge 0$, let

$$B^d_{\mathbf{i}}(\boldsymbol{\tau}) := \frac{d!}{i!j!k!} \tau_0^i \tau_1^j \tau_2^k \tag{2}$$

be the *Berstein polynomials* of degree d on $\mathcal{T}$. Then any polynomial p of total degree at most d on $\mathcal{T}$ can be expressed in the Berstein-Bézier form

$$p(\boldsymbol{\tau}) = \sum_{|\mathbf{i}|=d} p_{\mathbf{i}} B^d_{\mathbf{i}}(\boldsymbol{\tau}). \tag{3}$$

These polynomials span the linear space $\Pi_d := \operatorname{span}\{B^d_{\mathbf{i}}\}_{|\mathbf{i}|=d}$. The Bézier *ordinates* $p_{\mathbf{i}}$, associated with their *abscissae* $\mathbf{i}/d$ expressed in barycentric coordinates with respect to $\mathcal{T}$, form a triangular *control net* of the *Bézier triangle* (3), for more details see [13].

2.2. Clough-Tocher macro-triangle and continuity conditions

The Clough-Tocher-Hsieh split [9] partitions $\mathcal{T}$ using a given inner *split-point* $S \in \mathcal{T}$ into three *micro-triangles* $\mathcal{T}_i$ given by (V_{i+1}, V_{i+2}, S), $i = 0, 1, 2$; see Figure 1, top left. This gives rise to three *micro-edges* from S to V_i. The triangulation $\mathcal{T}^\star$ of $\mathcal{T}$ consisting of $\mathcal{T}_i$ is called the *Clough-Tocher macro-triangle* (also known as *macro-element*) corresponding to $\mathcal{T}$. The *Clough-Tocher spline space* on $\mathcal{T}^\star$ is the C^1 cubic spline space on $\mathcal{T}^\star$, i.e.,

$$\mathcal{S}^1_3(\mathcal{T}^\star) := \{s \in C^1(\mathcal{T}^\star) : s|_{\mathcal{T}_i} \in \Pi_3, i = 0, 1, 2\}. \tag{4}$$

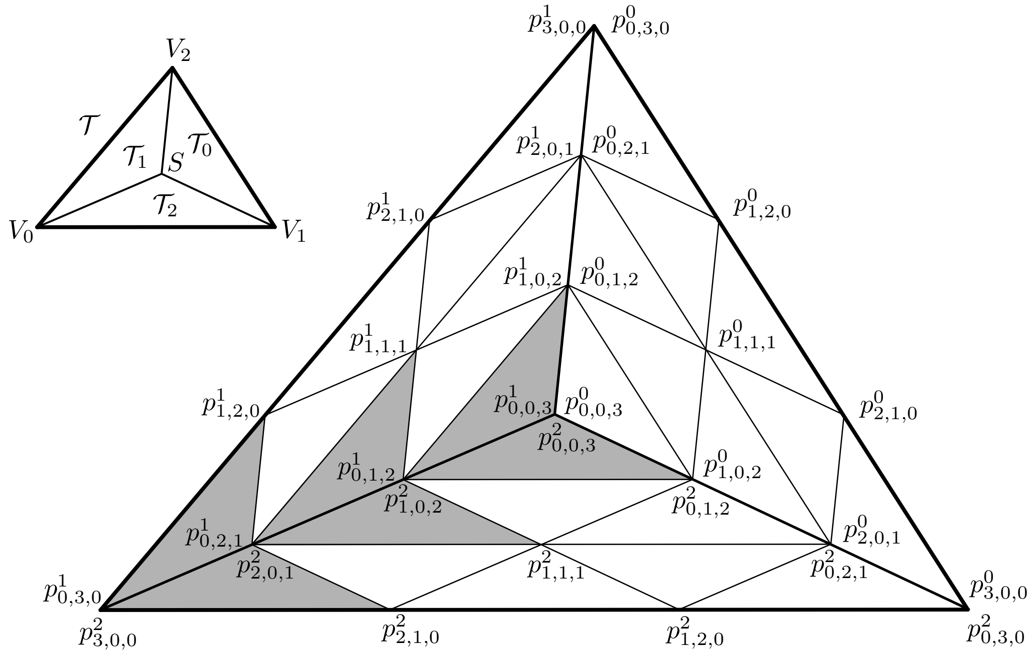

Figure 1: Top left: The Clough-Tocher-Hsieh split of $\mathcal{T}$ (thick edges) using the split point S into $\mathcal{T}^\star$ (thin edges are micro-edges). Right: Labelling of Bézier ordinates (see Section 2.1) over the macro-triangle. The control net triangles involved in the C^1 continuity conditions (6) between p^1 and p^2 are shown in grey.

In other words, any s in $\mathcal{S}^1_3(\mathcal{T}^\star)$ consists of three cubic Bézier triangles (one on each of the micro-triangles $\mathcal{T}_i$) joined with C^1 continuity across their pair-wise shared micro-edges. It is known that $\dim(\mathcal{S}^1_3(\mathcal{T}^\star)) = 12$.

A discussion on choosing the split point S can be found in [20, 28]. Although various options exist, the split point is typically placed at the barycentre of $\mathcal{T}$.

For later use, we now recall C^0, C^1, and C^2 continuity conditions between Bézier cubic triangles [20, 23, 29]. Let p^i be a cubic Bézier triangle defined on $\mathcal{T}_i$, $i = 0, 1, 2$. Using the notation from Figure 1, the C^0 continuity conditions between p^i and p^{i+1} at their shared micro-edge SV_{i+2} are simply

$$p^i_{0,j,3-j} = p^{i+1}_{j,0,3-j} \quad \text{for} \quad j = 0, 1, 2, 3. \tag{5}$$

Let (τ_0, τ_1, τ_2) be the barycentric coordinates of S with respect to $\mathcal{T}$ (of unit area), i.e., $S = \tau_0 V_0 + \tau_1 V_1 + \tau_2 V_2$. Note that this means that τ_i is the area of $\mathcal{T}_i$. Then the C^1 continuity conditions between p^i and p^{i+1} read

$$p^i_{0,j,3-j} = \tau_i p^{i+1}_{j,1,2-j} + \tau_{i+1} p^i_{1,j,2-j} + \tau_{i+2} p^i_{0,j+1,2-j} \quad \text{for} \quad j = 0, 1, 2. \tag{6}$$

Finally, the C^2 continuity conditions are

$$\omega_i p^i_{1,1-j,j+1} + \omega_{i+1} p^i_{2,1-j,j} + \omega_{i+2} p^i_{1,2-j,j} = \bar{\omega}_i p^{i+1}_{1-j,2,j} + \bar{\omega}_{i+1} p^{i+1}_{1-j,1,j+1} + \bar{\omega}_{i+2} p^{i+1}_{2-j,1,j} \quad \text{for} \quad j = 0, 1, \tag{7}$$

where $(\omega_i, \omega_{i+1}, \omega_{i+2})$ are the barycentric coordinates of V_i with respect to triangle (S, V_{i+1}, V_{i+2}), and $(\bar{\omega}_i, \bar{\omega}_{i+1}, \bar{\omega}_{i+2})$ are the barycentric coordinates of V_{i+1} with respect to triangle (V_i, S, V_{i+2}). Namely, with $\tau_i > 0$ for $i = 0, 1, 2$, i.e., the split point S is interior to $\mathcal{T}$, we have

$$\omega_i = 1/\tau_i, \quad \omega_{i+1} = -\tau_{i+1}/\tau_i, \quad \omega_{i+2} = -\tau_{i+2}/\tau_i, \tag{8}$$

and

$$\bar{\omega}_i = -\tau_i/\tau_{i+1}, \quad \bar{\omega}_{i+1} = 1/\tau_{i+1}, \quad \bar{\omega}_{i+2} = -\tau_{i+2}/\tau_{i+1}. \tag{9}$$

More information can be found e.g. in [2, 13, 23, 29].

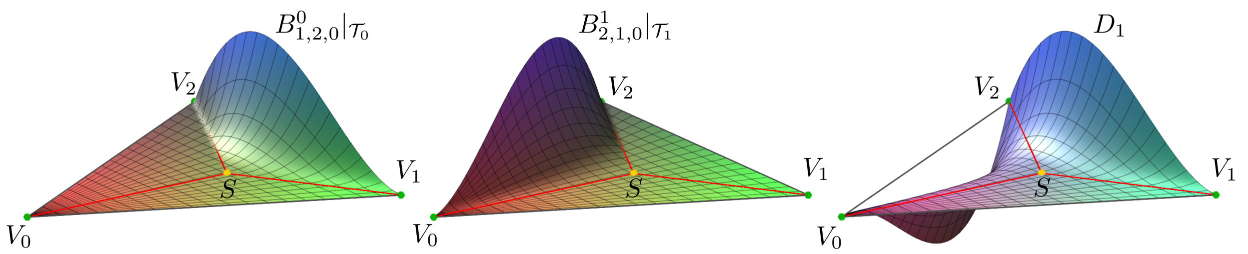

Figure 2: Bernstein-Bézier basis functions restricted to micro-triangles $\mathcal{T}_0$ (left) and $\mathcal{T}_1$ (middle), respectively, are only C^0 across micro-edge SV_2. Their proper blend D_1, defined in (15), however, forms a C^1-continuous function over $\mathcal{T}$ (outlined by grey edges) that vanishes over $\mathcal{T}_3$ and also along the micro-edge SV_2 (right).

2.3. Hammer-Stroud quadrature for cubic polynomials over simplices

We now recall the result of Hammer and Stroud [14] that derives quadrature rules for polynomials of total degree three over a simplex in $\mathbb{R}^n$.

Theorem 2.1. *[14, Theorem 1] Let $\mathcal{S}_n$ be a simplex in $\mathbb{R}^n$, $n \geq 1$, with vertices $V_0, V_1, \ldots, V_n$, with $C = \sum_{i=0}^n V_i/(n+1)$ its barycentre (centroid), and Δ_n its hyper-volume. Then the quadrature formula*

$$Q^{\mathrm{HS}}[f] = c_n f(C) + w_n \sum_{i=0}^{n} f(U_i) \tag{10}$$

with weights

$$c_n = \frac{-(n+1)^2}{4(n+2)}\Delta_n, \quad w_n = \frac{(n+3)^2}{4(n+1)(n+2)}\Delta_n \tag{11}$$

and quadrature points

$$U_i = \frac{2}{n+3}V_i + \frac{n+1}{n+3}C, \quad i = 0, \ldots, n \tag{12}$$

is exact for any cubic polynomial f over $\mathcal{S}_n$.

In the planar case ($n = 2$), the dimension of the polynomial space of total degree d is $\binom{d+2}{2}$, and specifically $\dim(\Pi_3) = 10$.

The quadrature rule of Hammer and Stroud (10) uses the minimal number of quadrature points in the sense that there is no exact quadrature with fewer nodes. However, the total number of degrees of freedom of the quadrature is 12 (4 nodes in $\mathbb{R}^2$ and 4 weights). Thus, a natural question arises: can the space Π_3 of cubic polynomials over a triangle be extended by two linearly independent functions $D_1, D_2 \notin \Pi_3$ such that the quadrature (10) is exact also for D_1 and D_2?

In the following section, we give an affirmative answer to this question and show that the quadrature (10) is also exact for the C^1 cubic Clough-Tocher spline space over a macro-triangle if and only if the split-point is the barycentre of the macro-triangle.

3. Gaussian quadrature for C^1 cubic Clough-Tocher macro-triangles

We aim to prove that under the condition that the split-point is the barycentre, the Hammer-Stroud quadrature (10) is also exact for the C^1 cubic Clough-Tocher macro-triangle. To this end, we show that there exist specific C^1 cubic Clough-Tocher splines that vanish along the three micro-edges of $\mathcal{T}$, and whose integrals over $\mathcal{T}$ vanish as well.

We first define two one-parameter families of piece-wise cubic functions $D_1^{\mu_1}$ and $D_2^{\mu_2}$, $\mu_1, \mu_2 \in \mathbb{R}$, that arise from blending Bernstein-Bézier basis functions acting on two different micro-triangles:

$$\begin{aligned} D_1^{\mu_1} &:= B^0_{1,2,0}|_{\mathcal{T}_0} - \mu_1 B^1_{2,1,0}|_{\mathcal{T}_1}, \\ D_2^{\mu_2} &:= B^2_{2,1,0}|_{\mathcal{T}_2} - \mu_2 B^1_{1,2,0}|_{\mathcal{T}_1}, \end{aligned} \tag{13}$$

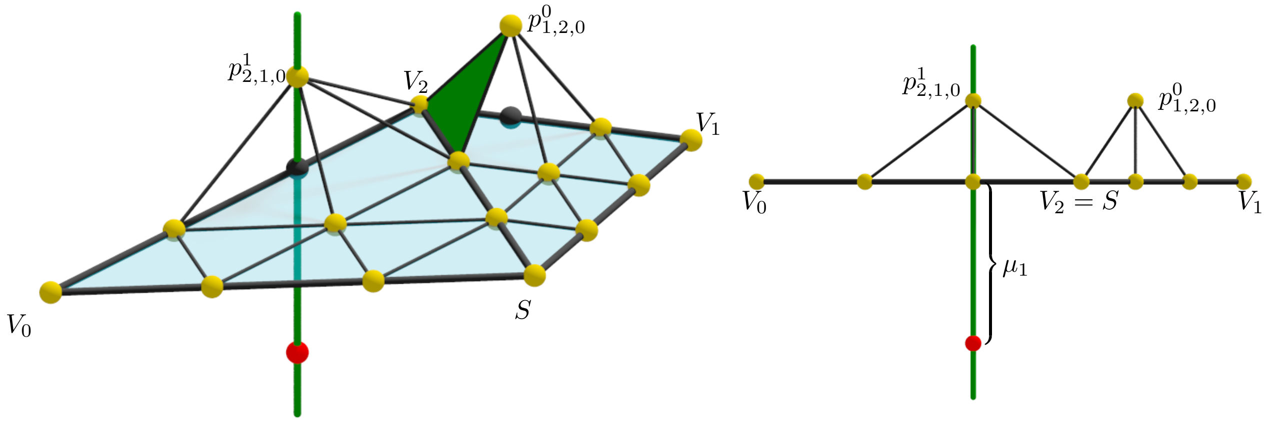

Figure 3: A geometric proof of Lemma 3.1. Left (3D view): Bernstein-Bézier coefficients (yellow) over their Greville abscissae (black) of two basis functions $B^0_{1,2,0}$ and $B^1_{2,1,0}$ are shown. All the Greville abscissae lie in the V_0SV_2 plane, and the only non-zero cofficents are $p^0_{1,2,0}$ and $p^1_{2,1,0}$. The C^1-continuity of the blend D_1 in (15) is achieved by a proper choice of μ_1, which geometrically corresponds to the intersection (red dot) of the plane of the green triangle with the ordinate line of $p^1_{2,1,0}$. Note that for illustration purposes, the split-point S is shown outside of $\mathcal{T}$. Right (2D view): An orthogonal projection of the situation onto a plane perpendicular to SV_2, which shows that μ_1, and thus $D_1 \in C^1$, always exists.

where the superscript of the Bernstein-Bézier basis functions corresponds to their original micro-triangle. These functions are restricted to their respective micro-triangle and are constant zero elsewhere; see Fig. 2.

By construction, $D_1^{\mu_1}$ and $D_2^{\mu_2}$ are C^0 (c.f. (5)) for any value of μ_i, $i = 1, 2$ and vanish along the micro-edges of $\mathcal{T}^\star$. Moreover, for a specific choice of these parameters they become C^1 over $\mathcal{T}$.

Lemma 3.1. *Let S be an interior point of $\mathcal{T}$. Then there exist μ_1, $\mu_2 \in \mathbb{R}$, dependent on S, such that $D_i^{\mu_i} \in S^1_3(\mathcal{T}^\star)$, $i = 1, 2$.*

Proof. In the case of $D_1^{\mu_1}$, the C^1 conditions (6) across micro-edge SV_2 simplify to just one equation, namely

$$0 = \tau_0(-\mu_1) + \tau_1, \tag{14}$$

from which we obtain $\mu_1 = \tau_1/\tau_0$. Similarly, for $D_2^{\mu_2}$ we get $\mu_2 = \tau_1/\tau_2$. $\square$

This result leads to two C^1 cubic splines (cf. (13))

$$\begin{aligned} D_1 &:= B^0_{1,2,0}|\mathcal{T}_0 - \tfrac{\tau_1}{\tau_0} B^1_{2,1,0}|\mathcal{T}_1 \quad \text{and} \\ D_2 &:= B^2_{2,1,0}|\mathcal{T}_2 - \tfrac{\tau_1}{\tau_2} B^1_{1,2,0}|\mathcal{T}_1 \end{aligned} \tag{15}$$

on $\mathcal{T}^\star$.

Remark 1. Although the proof of Lemma 3.1 is rigorous and short, it provides little geometric insight. We now present an alternative proof based on geometric arguments.

The condition of C^1-continuity between two Bernstein-Bézier basis functions (6) is geometrically expressed by the fact that the corresponding triangles consisting of the control points along the common edge (shaded in grey in Fig. 1) have to be coplanar; see Fig. 3. Consider $D_1^{\mu_1}$ in (13). The coefficients of the two Bernstein-Bézier basis functions to be blended all vanish except for $p^0_{1,2,0} = 1$ and $p^1_{2,1,0} = 1$. The coplanarity constraint can be geometrically interpreted as a plane-line intersection. The plane, α, is defined by the triangle corresponding to $p^0_{1,2,0}$, $p^0_{0,3,0}$, and $p^0_{0,2,1}$, and the line is the ordinate line of $p^1_{2,1,0}$ with parameter μ_1. Since the split point lies strictly inside $\mathcal{T}$, α has a finite slope with respect to the $z = 0$ plane and therefore it intersects the ordinate line. An analogous argument applies to $D_2^{\mu_2}$ as well.

Recall that Π_3 is a 10-dimensional linear space, $S_3^1(\mathcal{T}^\star)$ is 12-dimensional, and $\Pi_3 \subset S_3^1(\mathcal{T}^\star)$. The following lemma shows that D_1 and D_2 extend Π_3 to $S_3^1(\mathcal{T}^\star)$.

Lemma 3.2. *It holds that*

$$S_3^1(\mathcal{T}^\star) = \Pi_3 \oplus \operatorname{span}\{D_1, D_2\}. \tag{16}$$

Proof. As Π_3 is spanned by (global) polynomials and D_1 vanishes on $\mathcal{T}_2$, it follows that D_1 is linearly independent from Π_3 on $\mathcal{T}$. The same argument holds for D_2 as well. And since by construction $D_1, D_2 \in S_3^1(\mathcal{T}^\star)$, it remains to show that no non-trivial linear combination of D_1 and D_2 is in Π_3.

We show this by contradiction. Let us assume that there exist two non-zero coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 D_1 + \alpha_2 D_2 =: G \in \Pi_3$. We now employ the C^2 conditions (7) and note that all involved ordinates are equal to zero except

$$p^0_{1,2,0} = \alpha_1, \quad p^1_{2,1,0} = -\alpha_1 \frac{\tau_1}{\tau_0}, \quad p^1_{1,2,0} = -\alpha_2 \frac{\tau_1}{\tau_2}, \quad p^2_{2,1,0} = \alpha_2. \tag{17}$$

At micro-edge SV_2, using (8), (9), and (17), the C^2 condition simplifies to

$$\frac{\alpha_1}{\alpha_2} = -\frac{1}{2}\frac{\tau_0^2}{\tau_2^2}. \tag{18}$$

Similarly, at micro-edge SV_0 we get

$$\frac{\alpha_1}{\alpha_2} = -2\frac{\tau_0^2}{\tau_2^2}. \tag{19}$$

Because these two C^2 conditions cannot be satisfied simultaneously, G is not C^2 over $\mathcal{T}^\star$ for any non-zero parameters α_1 and α_2, and thus $G \notin \Pi_3$. This completes the proof. □

The next lemma establishes a connection between the integrals of D_1 and D_2 and the split-point S.

Lemma 3.3. *The integrals of D_1 and D_2 over $\mathcal{T}$ vanish, i.e.,*

$$\int_{\mathcal{T}} D_i \,\mathrm{d}\boldsymbol{\tau} = 0, \quad i = 1, 2, \tag{20}$$

if and only if S is the barycentre of $\mathcal{T}$.

Proof. The integrals of Bernstein-Bézier basis functions depend linearly on the triangle area [13]. In our setting with $d = 3$ and $\mathcal{T}$ being of unit area, we have

$$\int_{\mathcal{T}_j} B^j_{\mathbf{i}} \,\mathrm{d}\boldsymbol{\tau} = \frac{1}{10}\tau_j, \quad j = 1, 2, 3. \tag{21}$$

Consequently, we obtain

$$\int_{\mathcal{T}} D_1 \,\mathrm{d}\boldsymbol{\tau} = \int_{\mathcal{T}_0} B^0_{1,2,0} \,\mathrm{d}\boldsymbol{\tau} - \frac{\tau_1}{\tau_0}\int_{\mathcal{T}_1} B^1_{2,1,0} \,\mathrm{d}\boldsymbol{\tau} = \frac{1}{10}\frac{\tau_0^2 - \tau_1^2}{\tau_0} \tag{22}$$

and similarly

$$\int_{\mathcal{T}} D_2 \,\mathrm{d}\boldsymbol{\tau} = \frac{1}{10}\frac{\tau_2^2 - \tau_1^2}{\tau_2}. \tag{23}$$

It follows that both integrals can vanish if and only if $\tau_0 = \tau_1 = \tau_2$, i.e., exactly when the split-point coincides with the barycentre of $\mathcal{T}$. □

Before we prove our main result, we need the following definition regarding quadrature rules for polynomials and spline-spaces over (macro-)triangles.

Definition 3.1. We call a quadrature over a triangle a *micro-edge quadrature* if, for a selected internal split-point S, all its quadrature nodes lie on the union of the three line segments connecting S to the vertices of $\mathcal{T}$.

Note that the quadrature of Hammer and Stroud (10) in two dimensions is a micro-edge quadrature since the barycentre is one of the nodes and all other nodes lie on micro-edges, one on each.

We are now ready to formalise our main theorem.

Theorem 3.1. *A micro-edge quadrature exact on $S_3^1(\mathcal{T})$ exists if and only if the split-point is the barycentre of $\mathcal{T}$.*

Proof. By construction, both D_1 and D_2 vanish along all three micro-edges of $\mathcal{T}$.

($\Leftarrow$) For any micro-edge quadrature it holds

$$Q[D_i] = 0, \quad i = 1, 2. \tag{24}$$

If S is not the barycentre, then by Lemma 3.3 at least one of the functions D_1 or D_2 has a non-vanishing integral over $\mathcal{T}$. Therefore, $Q[D_i] \neq \int_{\mathcal{T}} D_i \mathrm{d}\boldsymbol{\tau}$ for at least one $i = 1, 2$ and thus no micro-edge quadrature can be exact on $S_3^1(\mathcal{T})$.

($\Rightarrow$) If the split-point S is the barycentre of $\mathcal{T}$, it follows from Lemma 3.3 that the integrals of D_1 and D_2 over $\mathcal{T}$ vanish. Additionally, the (micro-edge) quadrature (10) also vanishes when applied to D_1 and D_2. In other words,

$$Q^{HS}[D_i] = 0 = \int_{\mathcal{T}} D_i \mathrm{d}\boldsymbol{\tau}, \quad i = 1, 2. \tag{25}$$

Therefore, by Lemma 3.2, the micro-edge quadrature of Hammer and Stroud is exact not only on Π_3, but also on $S_3^1(\mathcal{T}^\star)$. $\square$

Our theoretical result implies that it is now possible to save a considerable amount of quadrature points when integrating over macro-triangles. Instead of using the traditional four quadrature points per micro-triangle and thus twelve altogether per macro-triangle, we can now use only four quadrature points in a macro-triangle to integrate over it exactly. This reduces the number of quadrature points needed by a factor of three when compared to the traditional approach. Additionally, our result applies to all Clough-Tocher variants of the C^1 cubic macro-triangle, including its original version, the reduced space, and other variants (see [20] for a summary) as long as the split-point is the barycentre.

Remark 2. When S is the barycentre, the rule (10) is exact for any function from $S_3^1(\mathcal{T}^\star)$. For functions $f \notin S_3^1(\mathcal{T}^\star)$, one can mimic the analysis conducted in [31, Section 5.3] to compute a-priori estimates of the error of the rule. The calculation of the actual error constants by integrating the kernel function ([31, Eq. (5.8-4)]) is rather technical and, as it relies on embedding the current domain (the macro-triangle) into a rectangle, the actual bounds are expected to be less practical.

One must not misinterpret our findings: we considered only micro-edge quadratures as defined in Definition 3.1. Our result says nothing about the existence of Gaussian quadrature with nodes not constrained to lie on the micro-edges for the C^1 cubic Clough-Tocher macro-triangle. We have also tried to attack the problem of finding a Gaussian quadrature rule when the split-point is not in the barycentre, fixing only the split point to be a quadrature point, and allowing the other three nodes to move freely on the three micro-triangles. Unfortunately, we did not discover any simplifications like those presented above and the arising system of polynomial equations with the barycentric coordinates of the split-point as parameters is beyond the current symbolic capabilities of the technical computing software Maple.

Nevertheless, one can still assemble the corresponding quadrature system and solve it numerically. Based on the count of degrees of freedom in Section 2.3, we can expect that even when the split-point is not at the barycentre of $\mathcal{T}$, four quadrature points (also called nodes) should suffice to integrate $S_3^1(\mathcal{T}^\star)$ exactly. However, to obtain a polynomial system of equations (and not a piece-wise polynomial one), we need to place the four quadrature points into the three micro-triangles of $\mathcal{T}$. Due to the fact that D_1 and D_2 vanish over one of the micro-triangles each, not all four quadrature points can lie in one of the micro-triangles. Up

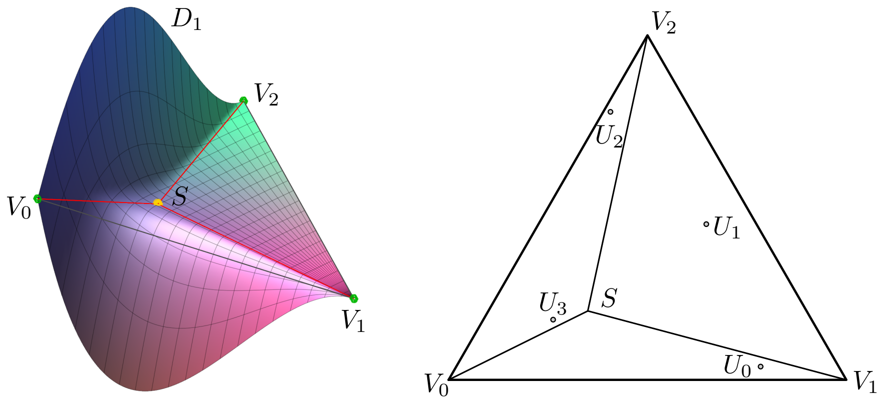

Figure 4: Left: The basis function D_1 over $\mathcal{T}$ with the split-point S away from the barycentre. Right: The positions of the quadrature points U_i, $i = 0, \ldots, 3$, corresponding the the nodal layout (1,2,1). Their barycentric coordinates can be found in Table 1.

to permutations, this leaves us with three so-called nodal layouts: $(3, 1, 0)$, $(2, 2, 0)$, and $(2, 1, 1)$, where each triple denotes how many quadrature points live in each of the three micro-triangles.

For our numerical example, we chose the split point S to be given by $(\tau_0, \tau_1, \tau_2) = (\frac{11}{20}, \frac{1}{4}, \frac{1}{5})$; see Figure 4. And we fixed the nodal layout to be $(1, 2, 1)$. This leads to a well-constrained system of 12 equations (one for each basis function; see (16)) in 12 unknowns. These unknowns are one weight ω_i and two barycentric coordinates u_i and v_i of each of the four nodes U_i, $i = 0, \ldots, 3$. While the corresponding system admits many solutions (the maximal total degree of the Gröbner basis polynomials with one specific ordering of unknowns is 66), only one ensures that all the U_i lie in their prescribed micro-triangles, i.e., $0 < u_i, v_i < 1$ and $u_i + v_i < 1$ for all i. This numerical solution is reported in Table 1.

4. 3D and beyond

Our main result shows that the Hammer-Stroud quadrature rule over triangles is exact not only on the space of cubic polynomials, but also on the larger C^1 cubic Clough-Tocher macro-triangle (when the split-point is its barycentre). As the Hammer-Stroud quadrature rule applies to cubics over simplices of arbitrary dimension, one can naturally ask whether our result also generalises to higher dimensions.

Consider a non-degenerate tetrahedron in $\mathbb{R}^3$ and the 20-dimensional space of cubic polynomials defined on it. The associated Hammer-Stroud quadrature rule requires five quadrature points, which corresponds to 20 degrees of freedom (5 nodes in $\mathbb{R}^3$ and 5 weights). Expressing the exactness of the rule by an algebraic system of equations, this scenario leads to a well-constrained (20×20) system which already indicates that one cannot expect the quadrature to be exact on a larger space than cubic polynomials.

Table 1: A numerical example with $(\tau_0, \tau_1, \tau_2) = (\frac{11}{20}, \frac{1}{4}, \frac{1}{5})$; see Figure 4. The computed quadrature points U_i, given in terms of barycentric coordinates $(u_i, v_i, 1 - u_i - v_i)$ in their respective micro-triangles $V_j V_{j+1} S$ (nodal layout is $(1, 2, 1)$) and the corresponding weights ω_i (last column), are listed here. As the area of $\mathcal{T}$ is assumed to be one, the weights ω_i sum to one.

	u_i	v_i	ω_i
U_0	0.09039700369290119251	0.71710167289537160945	0.16101639368924650306
U_1	0.36472134323958328063	0.40592878510910588031	0.34503323833126581026
U_2	0.16455363488938706592	0.76341032510061727023	0.14086634590842244384
U_3	0.26378417182428927673	0.03430857491526249826	0.35308402207106524282

To obtain a larger space over a tetrahedron, one could consider a generalisation of the Clough-Tocher-Hsieh split from the triangular case. One such split uses a single split-point (e.g. the barycentre) to split the original tetrahedron into four micro-tetrahedra. This is called the Alfeld split [22]. However, in order to obtain a non-trivial C^1 spline space over such macro-tetrahedra, degree 5 is needed [1]. Additionally, the obtained space has 65 degrees of freedom.

Another option is to employ the Worsey-Farin split [22]. This split, on top of an internal split-point, also uses four face split-points, one in each face of the tetrahedron. This results into a macro-tetrahedron composed from twelve micro-tetrahedra. This split supports C^1 cubic splines and has 28 degrees of freedom [32]. This could indicate that 7 quadrature nodes might suffice. However, the choice of the face split-points means that the whole construction is not, in general, affine invariant and thus a general quadrature rule has to depend on certain parameters corresponding to the choice of the face split-points.

These facts show that there is no direct generalisation of our result into three (and higher) dimensions. Finding quadrature rules for the spline spaces of Alfeld [1] and Worsey-Farin [32] thus remains an interesting avenue for future research.

For other spline spaces over planar triangulations, one may ask a similar question as we posed here, namely if a certain polynomial bivariate quadrature rule integrates a larger space, and if so, under what conditions. In the case of C^1 quadratic Powell-Sabin 6-split macro-triangles, the underlying spline space admits more degrees of freedom and consequently the 3-node Gaussian quadrature(s) for quadratics can be generalised to the C^1 quadratic Powell-Sabin 6-split spline space over a macro-triangle for a two-parameter family of inner split-points [4], not just the barycentre as it is in the case of the Clough-Tocher spline space considered here.

5. Conclusion

We have investigated the existence of Gaussian rules for C^1 cubic Clough-Tocher macro-triangles and have shown that the Hammer-Stroud quadrature rule for cubic polynomials generalises to such a rule if and only if the split point is the barycentre of the macro-triangle. This result brings the reduction of the number of quadrature points by a factor of three when compared to the traditional element-wise integration. In terms of future research, additionally to the already mentioned challenges of Alfeld and Worsey-Farin splits, one may also focus on semi-Gaussian rules for Clough-Tocher splines over unions of macro-triangles.

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