

ON THE REGULARITY OF SOLUTIONS TO THE k -GENERALIZED KORTEWEG-DE VRIES EQUATION

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ABSTRACT. This work is concerned with special regularity properties of solutions to the k -generalized Korteweg-de Vries equation. In [4] it was established that if the initial datum $u_0 \in H^l((b, \infty))$ for some $l \in \mathbb{Z}^+$ and $b \in \mathbb{R}$, then the corresponding solution $u(\cdot, t)$ belongs to $H^l((\beta, \infty))$ for any $\beta \in \mathbb{R}$ and any $t \in (0, T)$. Our goal here is to extend this result to the case where $l > 3/4$.

1. INTRODUCTION

In this note we study the regularity of solutions to the initial value problem (IVP) associated to the k -generalized Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, & x, t \in \mathbb{R}, k \in \mathbb{Z}^+, \\ u(x, 0) = u_0(x). \end{cases} \quad (1.1)$$

The starting point is a property found by Isaza, Linares and Ponce [4] concerning the propagation of smoothness in solutions of the IVP (1.1). To state it we first recall the following well-posedness (WP) result for the IVP (1.1):

Theorem A1. *If $u_0 \in H^{3/4^+}(\mathbb{R})$, then there exist $T = T(\|u_0\|_{\frac{3}{4^+}, 2}; k) > 0$ and a unique solution $u = u(x, t)$ of the IVP (1.1) such that*

$$\begin{aligned} \text{(i)} \quad & u \in C([-T, T] : H^{3/4^+}(\mathbb{R})), \\ \text{(ii)} \quad & \partial_x u \in L^4([-T, T] : L^\infty(\mathbb{R})), \quad (\text{Strichartz}), \\ \text{(iii)} \quad & \sup_x \int_{-T}^T |J^r \partial_x u(x, t)|^2 dt < \infty \quad \text{for } r \in [0, 3/4^+], \\ \text{(iv)} \quad & \int_{-\infty}^{\infty} \sup_{-T \leq t \leq T} |u(x, t)|^2 dx < \infty, \end{aligned} \quad (1.2)$$

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with $J = (1 - \partial_x^2)^{1/2}$. Moreover, the map data-solution, $u_0 \rightarrow u(x, t)$ is locally continuous (smooth) from $H^{3/4+}(\mathbb{R})$ into the class $X_T^{3/4+}$ defined in (1.2).

If $k \geq 2$, then the result holds in $H^{3/4}(\mathbb{R})$. If $k = 1, 2, 3$, then T can be taken arbitrarily large.

For the proof of Theorem A1 we refer to [6], [1] and [3]. The proof of our main result Theorem 1.1 is based on an energy estimate argument for which the estimate (ii) in (1.2) (i.e. the time integrability of $\|\partial_x u(\cdot, t)\|_\infty$) is essential. However, we remark that from the WP point of view is not optimal. For a detailed discussion on the WP of the IVP (1.1) we refer to [7], Chapters 7-8.

Now we enunciate the result obtained in [4] regarding propagation of regularities which motivates our study here:

Theorem A2 ([4]). *Let $u_0 \in H^{3/4+}(\mathbb{R})$. If for some $l \in \mathbb{Z}^+$ and for some $x_0 \in \mathbb{R}$*

$$\|\partial_x^l u_0\|_{L^2((x_0, \infty))}^2 = \int_{x_0}^{\infty} |\partial_x^l u_0(x)|^2 dx < \infty, \quad (1.3)$$

then the solution $u = u(x, t)$ of the IVP (1.1) provided by Theorem A1 satisfies that for any $v > 0$ and $\epsilon > 0$

$$\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^{\infty} (\partial_x^j u)^2(x, t) dx < c, \quad (1.4)$$

for $j = 0, 1, \dots, l$ with $c = c(l; \|u_0\|_{3/4+, 2}; \|\partial_x^l u_0\|_{L^2((x_0, \infty))}; v; \epsilon; T)$.

In particular, for all $t \in (0, T]$, the restriction of $u(\cdot, t)$ to any interval of the form (a, ∞) belongs to $H^l((a, \infty))$.

Moreover, for any $v \geq 0$, $\epsilon > 0$ and $R > 0$

$$\int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + R - vt} (\partial_x^{l+1} u)^2(x, t) dx dt < c, \quad (1.5)$$

with $c = c(l; \|u_0\|_{3/4+, 2}; \|\partial_x^l u_0\|_{L^2((x_0, \infty))}; v; \epsilon; R; T)$.

Theorem A2 tells us that the H^l -regularity ($l \in \mathbb{Z}^+$) on the right hand side of the data travels forward in time with infinite speed. Notice that since the equation is reversible in time a gain of regularity in $H^s(\mathbb{R})$ cannot occur so at $t > 0$, so $u(\cdot, t)$ fails to be in $H^l(\mathbb{R})$ due to its decay at $-\infty$. In this regard, it was also shown in [4] that for any $\delta > 0$ and $t \in (0, T)$ and $j = 1, \dots, l$

$$\int_{-\infty}^{\infty} \frac{1}{\langle x_- \rangle^{j+\delta}} (\partial_x^j u)^2(x, t) dx \leq \frac{c}{t},$$

with $c = c(\|u_0\|_{3/4^+, 2}; \|\partial_x^j u_0\|_{L^2((x_0, \infty))}; x_0; \delta)$, $x_- = \max\{0; -x\}$ and $\langle x \rangle = (1 + x^2)^{1/2}$.

The aim of this note is to extend Theorem A2 to the case where the local regularity of the datum u_0 in (1.3) is measure with a fractional exponent. Thus, our main result is :

Theorem 1.1. *Let $u_0 \in H^{3/4^+}(\mathbb{R})$. If for some $s \in \mathbb{R}$, $s > 3/4$, and for some $x_0 \in \mathbb{R}$*

$$\|J^s u_0\|_{L^2((x_0, \infty))}^2 = \int_{x_0}^{\infty} |J^s u_0(x)|^2 dx < \infty, \quad (1.6)$$

then the solution $u = u(x, t)$ of the IVP (1.1) provided by Theorem A1 satisfies that for any $v > 0$ and $\epsilon > 0$

$$\sup_{0 \leq t \leq T} \int_{x_0 + \epsilon - vt}^{\infty} (J^r u)^2(x, t) dx < c, \quad (1.7)$$

for $r \in (3/4, s]$ with $c = c(l; \|u_0\|_{3/4^+, 2}; \|J^r u_0\|_{L^2((x_0, \infty))}; v; \epsilon; T)$.

Moreover, for any $v \geq 0$, $\epsilon > 0$ and $R > 0$

$$\int_0^T \int_{x_0 + \epsilon - vt}^{x_0 + R - vt} (J^{s+1} u)^2(x, t) dx dt < c, \quad (1.8)$$

with $c = c(l; \|u_0\|_{3/4^+, 2}; \|J^s u_0\|_{L^2((x_0, \infty))}; v; \epsilon; R; T)$.

From the results in section 2 it will be clear that we need only consider the case $s \in (3/4, \infty) - \mathbb{Z}^+$.

The rest of this paper is organized as follows : section 2 contains some preliminary estimates required for Theorem 1.1, whose proof will be given in section 3.

2. PRELIMINARY ESTIMATES

Let T_a be a pseudo-differential operator whose symbol

$$\sigma(T_a) = a(x, \xi) \in S^r, \quad r \in \mathbb{R}, \quad (2.1)$$

so that

$$T_a f(x) = \int_{\mathbb{R}^n} a(x, \xi) \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi. \quad (2.2)$$

The following result is the singular integral realization of a pseudo-differential operator, whose proof can be found in [8] Chapter 4.

Theorem A3. *Using the above notation (2.1)-(2.2) one has that*

$$T_a f(x) = \int_{\mathbb{R}^n} k(x, x - y) f(y) dy, \quad \text{if } x \notin \text{supp}(f) \quad (2.3)$$

where $k \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n - \{0\})$ satisfies : $\forall \alpha, \beta \in (\mathbb{Z}^+)^n \forall N \geq 0$

$$|\partial_x^\alpha \partial_z^\beta k(x, z)| \leq A_{\alpha, \beta, N, \delta} |z|^{-(n+m+|\beta|+N)}, \quad |z| \geq \delta, \quad (2.4)$$

if $n + m + |\beta| + N > 0$ uniformly in $x \in \mathbb{R}^n$.

To simplify the exposition from now on we restrict ourselves to the one-dimensional case $x \in \mathbb{R}$ where in the next section these results will be applied.

As a direct consequence of Theorem A3 one has :

Corollary 2.1. *Let $m \in \mathbb{Z}^+$ and $l \in \mathbb{R}$. If $g \in L^2(\mathbb{R})$ and $f \in L^p(\mathbb{R})$, $p \in [2, \infty]$, with*

$$\text{distance}(\text{supp}(f); \text{supp}(g)) \geq \delta > 0,$$

then

$$\|f \partial_x^m J^l g\|_2 \leq c \|f\|_p \|g\|_2. \quad (2.5)$$

Next, let $\theta_j \in C^\infty(\mathbb{R}) - \{0\}$ with $\theta'_j \in C_0^\infty(\mathbb{R})$ for $j = 1, 2$ and

$$\text{distance}(\text{supp}(1 - \theta_1); \text{supp}(\theta_2)) \geq \delta > 0. \quad (2.6)$$

Lemma 2.2. *Let $f \in H^s(\mathbb{R})$, $s < 0$, and T_a be a pseudo-differential operator of order zero ($a \in S^0$). If $\theta_1 f \in L^2(\mathbb{R})$, then*

$$\theta_2 T_a f \in L^2(\mathbb{R}). \quad (2.7)$$

Proof of Lemma 2.2. Since

$$\theta_2 T_a f = \theta_2 T_a(\theta_1 f) + \theta_2 T_a((1 - \theta_1)f),$$

combining the hypothesis and the continuity of T_a in $L^2(\mathbb{R})$ it follows that $\theta_2 T_a(\theta_1 f) \in L^2(\mathbb{R})$. Also

$$\begin{aligned} & \theta_2(x) T_a((1 - \theta_1)f)(x) \\ &= \int_{-\infty}^{\infty} \theta_2(x) a(x, \xi) \widehat{((1 - \theta_1)f)}(\xi) e^{2\pi i x \xi} d\xi \\ &= \int \underbrace{\left(\int \theta_2(x) a(x, \xi_1 + \xi_2) \widehat{((1 - \theta_1)f)}(\xi_1) e^{2\pi i x \xi_1} d\xi_1 \right)}_{b(x, \xi)} \widehat{f}(\xi_2) e^{2\pi i x \xi_2} d\xi_2 \\ &= T_b f(x) = \int \theta_2(x) k(x, x - z) (1 - \theta_1(z)) f(z) dz \\ &= \int \theta_2(x) k(x, x - z) (1 - \theta_1(z)) J^{2m} J^{-2m} f(z) dz \end{aligned} \quad (2.8)$$

with $-2m < s$, $m \in \mathbb{Z}^+$ and $k(\cdot, \cdot)$ as in (2.4), so integration by parts and Theorem A3 yield the desired result.

Proposition 2.3. *Let $f \in L^2(\mathbb{R})$ and*

$$J^s f = (1 - \partial_x^2)^{s/2} f \in L^2(\{x > a\}) \quad s > 0,$$

then for any $\epsilon > 0$ and any $r \in (0, s]$

$$J^r f \in L^2(\{x > a + \epsilon\}). \quad (2.9)$$

Proof of Proposition 2.3. Define

$$g = J^s f \in L^2(\{x > a\}),$$

thus $J^s f \in H^{-s}(\mathbb{R})$. Let $\theta_j \in C^\infty(\mathbb{R})$, $j = 1, 2$, with $\theta_1(x) = 1$ for $x \geq a + \epsilon/4$, $\text{supp } \theta_1 \subset \{x > a\}$, and $\theta_2(x) = 1$ for $x \geq a + \epsilon$ and $\text{supp } \theta_2 \subset \{x > a + \epsilon/2\}$, therefore $\theta_1 g \in L^2(\mathbb{R})$. Let $T = J^{i\beta}$, $\beta \in \mathbb{R}$. By Lemma 2.2

$$\theta_2 T f g = \theta_2 J^{s+i\beta} f \in L^2(\mathbb{R}),$$

and since $f \in L^2(\mathbb{R})$

$$\theta_2 J^{i\beta} f \in L^2(\mathbb{R}).$$

Hence, by the Three Lines Theorem it follows that

$$\theta_2 J^z f \in L^2(\mathbb{R}), \quad z = r + i\beta, \quad r \in [0, s], \quad \beta \in \mathbb{R},$$

which completes the proof.

Remark 2.4. *In a similar manner one has: for $\epsilon > 0$ let $\varphi_\epsilon \in C^\infty(\mathbb{R})$ with $\varphi_\epsilon(x) = 1$, $x \geq \epsilon$, $\text{supp } \varphi_\epsilon \subset \{x > \epsilon/2\}$ and $\varphi'_\epsilon(x) \geq 0$. Then*

(I) *If $m \in \mathbb{Z}^+$ and $\varphi_\epsilon J^m f \in L^2(\mathbb{R})$, then $\forall \epsilon' > 2\epsilon$*

$$\varphi_{\epsilon'} \partial_x^j f \in L^2(\mathbb{R}), \quad j = 0, 1, \dots, m.$$

(II) *If $m \in \mathbb{Z}^+$ and $\varphi_\epsilon \partial_x^j f \in L^2(\mathbb{R})$, $j = 0, 1, \dots, m$, then $\forall \epsilon' > 2\epsilon$*

$$\varphi_{\epsilon'} J^m f \in L^2(\mathbb{R}).$$

(III) *If $s > 0$, and $J^s(\varphi_\epsilon f)$, $f \in L^2(\mathbb{R})$, then $\forall \epsilon' > 2\epsilon$*

$$\varphi_{\epsilon'} J^s f \in L^2(\mathbb{R}).$$

(IV) *If $s > 0$, and $\varphi_\epsilon J^s f$, $f \in L^2(\mathbb{R})$, then $\forall \epsilon' > 2\epsilon$*

$$J^s(\varphi_{\epsilon'} f) \in L^2(\mathbb{R}).$$

The same results hold with θ_1 , θ_2 , as in (2.6), instead of χ_ϵ , $\chi_{\epsilon'}$.

Next, we recall some inequalities obtained in [5] :

Theorem A4 ([5]). *If $s > 0$ and $p \in (1, \infty)$, then*

$$\|J^s(fg)\|_p \leq c(\|f\|_\infty \|J^s g\|_p + \|J^s f\|_p \|g\|_\infty), \quad (2.10)$$

and

$$\begin{aligned} \| [J^s; f]g \|_p &= \| J^s(fg) - fJ^s g \|_p \\ &\leq c(\|\partial f\|_\infty \|J^{s-1}g\|_p + \|J^s f\|_p \|g\|_\infty). \end{aligned} \quad (2.11)$$

Also we shall use the following elementary estimate whose proof is similar to that found in [2], Chapter 6.

Lemma 2. *Let $\phi \in C^\infty(\mathbb{R})$ with $\phi' \in C_0^\infty(\mathbb{R})$. If $s \in \mathbb{R}$, then for any $l > |s - 1| + 1/2$*

$$\| [J^s; \phi]f \|_2 + \| [J^{s-1}; \phi]\partial_x f \|_2 \leq c \|J^l \phi'\|_2 \|J^{s-1}f\|_2. \quad (2.12)$$

3. PROOF OF THEOREM 1.1

Without loss of generality $x_0 = 0$. For $\epsilon > 0$ and $b \geq 5\epsilon$ define the families of functions

$$\chi_{\epsilon,b}, \phi_{\epsilon,b}, \widetilde{\phi}_{\epsilon,b}, \psi_\epsilon \in C^\infty(\mathbb{R}),$$

with $\chi'_{\epsilon,b} \geq 0$, $\chi_{\epsilon,b}(x) = 0$, $x \leq \epsilon$, $\chi_{\epsilon,b}(x) = 1$, $x \geq b$,

$$\chi'_{\epsilon,b}(x) \geq \frac{1}{10(b-\epsilon)} 1_{[2\epsilon, b-2\epsilon]}(x),$$

$$\text{supp}(\psi_\epsilon), \text{supp}(\widetilde{\psi}_{\epsilon,b}) \subset [\epsilon/4, b],$$

$$\phi_{\epsilon,b}(x) = \widetilde{\phi}_{\epsilon,b}(x) = 1, \quad x \in [\epsilon/2, \epsilon], \quad (3.1)$$

$$\text{supp}(\psi_\epsilon) \subset (-\infty, \epsilon/2]$$

$$\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_\epsilon(x) = 1, \quad x \in \mathbb{R},$$

$$\chi_{\epsilon,b}^2(x) + \widetilde{\phi}_{\epsilon,b}^2(x) + \psi_\epsilon(x) = 1, \quad x \in \mathbb{R}.$$

Hence,

$$\text{distance}(\text{supp}(\chi_{\epsilon,b}); \text{supp}(\psi_\epsilon)) \geq \epsilon/2.$$

Formally, we apply the operator J^s to the equation in (1.1) and multiply by $J^s u \chi_\epsilon^2(x + vt)$ to obtain after integration by parts the

identity

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int (J^s u)^2(x, t) \chi^2(x + vt) dx \\
& - v \underbrace{\int (J^s u)^2(x, t) \chi \chi'(x + vt) dx}_{A_1} \\
& + \frac{3}{2} \int (\partial_x J^s u)^2(x, t) \chi \chi'(x + vt) dx \\
& - \frac{1}{2} \underbrace{\int (J^s u)^2(x, t) \partial_x^3 (\chi^2(x + vt)) dx}_{A_2} \\
& + \underbrace{\int J^s(u \partial_x u) J^s u(x, t) \chi^2(x + vt) dx}_{A_3} = 0
\end{aligned} \tag{3.2}$$

where in χ the index ϵ, b were omitted. We shall do that now on.

Case: $s \in (3/4, 1)$.

First, we observe that combining (1.2) and the results in section 2 it follows that for any $R > 0$

$$\int_0^T \int_{-R}^R |J^r u(x, t)|^2 dx dt < \infty \quad \forall r \in [0, 7/4^+]. \tag{3.3}$$

Thus, after integration in time the terms A_1 and A_2 in (3.2) are bounded. So it only remains to handle A_3 .

Thus,

$$\begin{aligned}
J^s(u \partial_x u) \chi &= J^s(u \partial_x u \chi) - [J^s; \chi](u \partial_x u) \\
&= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x u - [J^s; \chi](u \partial_x u) \\
&= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x (u(\chi + \phi + \psi)) - [J^s; \chi](u \partial_x u) \\
&= B_1 + B_2 + B_3 + B_4 + B_5.
\end{aligned} \tag{3.4}$$

Inserting B_1 in (3.2) one obtains a term which can be estimated by integration by parts, Gronwall's inequality and (1.2). Using (2.11) it follows that

$$\|B_2\|_2 \leq c \|\partial_x(u\chi)\|_\infty \|J^s(u\chi)\|_2, \tag{3.5}$$

and

$$\|B_3\|_2 \leq c(\|\partial_x(u\chi)\|_\infty \|J^s(u\phi)\|_2 + \|\partial_x(u\phi)\|_\infty \|J^s(u\chi)\|_2). \tag{3.6}$$

To bound B_4 and B_5 we apply Corollary 2.1 and (2.12), respectively to get

$$\|B_4\|_2 = \|u\chi J^s \partial_x(u\psi)\|_2 \leq c\|u\|_\infty \|u\|_2 \quad (3.7)$$

and

$$\|B_5\|_2 \leq c\|u\|_\infty \|u\|_2. \quad (3.8)$$

Collecting the above information (3.4)-(3.8) in (3.2) we obtain (1.7) for any $r \in (3/4, 1)$, $v > 0$ and $\epsilon > 0$, and that for any $v > 0$, $\epsilon > 0$,

$$\int_0^T \int_{\epsilon-vt}^{R-vt} (J^s \partial_x u)^2 dx dt < \infty,$$

from which using the results, Remark 2.4, one obtains (1.8).

Case: $s \in (m, m+1)$, $m \in \mathbb{Z}^+$.

We assume (1.7) and (1.8) with $s \leq m$. Hence, from the results in section it follows that for any $\epsilon > 0$, $R > 0$ and $r \in [0, m]$

$$\int_0^T \int_{\epsilon-vt}^{R-vt} (J^r \partial_x u)^2 dx dt < \infty. \quad (3.9)$$

Again the starting point is the energy estimate identity (3.2). After integrating in time the terms A_1 and A_2 can be easily bounded using (3.9). So it suffices to consider A_3 . Thus, using the notation introduced in (3.1) we have

$$\begin{aligned} \chi J^s(u \partial_x u) &= J^s(u \chi \partial_x u) - \frac{1}{2} [J^s; \chi] \partial_x(u^2) \\ &= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x u - \frac{1}{2} [J^s; \chi] \partial_x(u^2) \\ &= u \chi J^s \partial_x u + [J^s; u \chi] \partial_x(u(\chi + \phi + \psi)) \\ &\quad - \frac{1}{2} [J^s; \chi] \partial_x((u^2)(\chi^2 + (\tilde{\phi})^2 + \psi)) \\ &= E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7. \end{aligned} \quad (3.10)$$

Inserting E_1 in (3.2) one obtains a term which can be estimated by integration by parts, Gronwall's inequality and (1.2). From (2.11) we see that

$$\|E_2\|_2 \leq c \|\partial_x(u\chi)\|_\infty \|J^s(u\chi)\|_2 \quad (3.11)$$

and

$$\|E_3\|_2 \leq c(\|\partial_x(u\chi)\|_\infty \|J^s(u\phi)\|_2 + \|\partial_x(u\phi)\|_\infty \|J^s(u\chi)\|_2). \quad (3.12)$$

For E_4 it follows that from Corollary 2.1 that

$$\|E_4\|_2 = \|u\chi J^s \partial_x(u\psi)\|_2 \leq c\|u\|_\infty \|u\|_2. \quad (3.13)$$

For E_5 and E_6 a combination of (2.10) and (2.12) yields the estimates

$$\begin{aligned} \|E_5\|_2 &\leq \| [J^s; \chi] \partial_x ((u\chi)^2) \|_2 \\ &\leq c \| J^s ((u\chi)^2) \|_2 \leq c \| u \|_\infty \| J^s (u\chi) \|_2, \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} \|E_6\|_2 &\leq \| [J^s; \chi] \partial_x ((u\tilde{\phi})^2) \|_2 = \| J^s ((u\tilde{\phi})^2) \|_2 \\ &\leq c \| u \|_\infty \| J^s (u\tilde{\phi}) \|_2. \end{aligned} \quad (3.15)$$

Finally, using Corollary 2.1 we write

$$\|E_7\|_2 \leq \| [J^s; \chi] \partial_x (u^2\psi) \|_2 = \| \chi J^s \partial_x (u^2\psi) \|_2 \leq c \| u \|_\infty \| u \|_2. \quad (3.16)$$

To complete the estimates in (3.11), (3.12), (3.14) and (3.15) we observe that

$$J^s(u\chi) = J^s u \chi + [J^s; \chi](u(\chi + \phi + \psi)) = G_1 + G_2,$$

where G_1 is the term whose L^2 -norm we are estimating and G_2 is of lower order, (hence bounded by assumption), and $\|J^2(u\phi)\|_2$ is bounded by (1.8) (assumption).

Collecting the above information in (3.2) we obtain the desired result.

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