GENERAL TÊTE-À-TÊTE GRAPHS AND SEIFERT MANIFOLDS

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ABSTRACT. Tête-à-tête graphs and relative tête-à-tête graphs were introduced by N. A'Campo in 2010 to model monodromies of isolated plane curves. By recent work of Fdez de Bobadilla, Pe Pereira and the author, they provide a way of modeling the periodic mapping classes that leave some boundary component invariant. In this work we introduce the notion of general tête-à-tête graph and prove that they model all periodic mapping classes. We also describe algorithms that take a Seifert manifold and a horizontal surface and return a tête-à-tête graph and vice versa.

CONTENTS

1. INTRODUCTION

In [A'C10] N. A'Campo introduced the notion of pure tête-à-tête graph in order to model monodromies of plane curves. These are metric ribbon graphs Γ without univalent vertices that satisfy a special property called the tête-à-tête property. One usually sees the ribbon graph as a strong deformation retract of a surface Σ with non-empty boundary, which is called the *thickening*. The tête-à-tête property says that if you pick a point p, then walk distance of π in any direction from that point and you always turn right at vertices, you get to the same point no matter the initial direction. This property defines an element in the mapping class group $\mathrm{MCG}^+(\Sigma, \partial\Sigma)$ which is freely periodic.

In [Gra15], C. Graf proved that if one allows univalent vertices in tête-à-tête graphs, then the set of mapping classes produced by tête-à-tête graphs are all freely periodic mapping classes of $\mathrm{MCG}^+(\Sigma, \partial\Sigma)$ with positive fractional Dehn twist coefficients. In [FdBPPPC17] this result was improved by showing that one does not need to enlarge the original class of metric ribbon graphs used to prove the same theorem.

A bigger class of graphs was introduced in [A'C10], the relative tête-à-tête graphs. These are pairs (Γ, A) formed by a metric ribbon graph Γ and a subset $A \subset \Gamma$ which is a collection of

circles. Seen as a strong deformation retract of a surface, this pair is properly embedded, i.e. $(\Gamma, A) \hookrightarrow (\Sigma, \partial\Sigma)$ and $\partial\Sigma \setminus A \neq \emptyset$. They satisfy the *relative tête-à-tête property* which is similar to the tête-à-tête property and defines an element in $\mathrm{MCG}^+(\Sigma, \partial\Sigma \setminus A)$ which is freely periodic. In [FdBPPPC17] it was proved that the set of mapping classes modeled by relative tête-à-tête graphs are all freely periodic mapping classes of $\mathrm{MCG}^+(\Sigma, \partial\Sigma \setminus A)$ with positive fractional Dehn twist coefficients at the boundary components in $\partial\Sigma \setminus A$.

At this point there is a natural question which was already posed in [Gra14], *how can one complement the definition of tête-à-tête graph to be able to model all periodic mapping classes?* (even if they do not leave any boundary component invariant). To cover these cases, we introduce general tête-à-tête graphs (see Definition 2.4). These are metric ribbon graphs with some special univalent vertices $\mathcal{P} \subset \partial\Sigma$ and a permutation acting on these vertices. An analogous general tête-à-tête property is defined. A general tête-à-tête graph defines a periodic mapping class in $\mathrm{MCG}^+(\Sigma)$.

As our main result, we prove:

Theorem A. *The mapping class of any periodic automorphism $\phi : \Sigma \to \Sigma$ of a surface can be realized via general tête-à-tête graphs. Moreover, the general tête-à-tête graph can be extended to a pure or relative tête-à-tête graph, thus realizing the automorphism as a restriction to Σ of a periodic automorphism on a surface $\hat{\Sigma} \supset \Sigma$ that leaves each boundary component invariant.*

The mapping torus of a periodic surface automorphism is a Seifert manifold and a orientable horizontal surface of a fiber-oriented Seifert manifold has a periodic monodromy induced on it. Hence, it is natural to assign a tête-à-tête graph to a Seifert manifold and a horizontal surface on it and vice versa. The rest of the work is devoted to understanding this relation.

In Section 3 we briefly review the theory of Seifert manifolds and plumbing graphs. The theory of Seifert manifolds is classical and there is plenty of literature about it (see for example [Neu81], [NR78], [Neu97], [JN83], [HNK71], [Hat07] or [Ped09]). Because of this, we try to avoid repeating well-known results. However, there is not such thing as *standard conventions* in Seifert manifolds. Since the conventions that we choose are very important for Section 5, we take some time to fix them carefully.

In Section 4, we review the theory about horizontal surfaces in a Seifert manifold M. This has been studied from different points of view in the literature. For example, in [EN85] it is proved a classification in the more general case when M is an integral homology sphere. In [Pic01] Pichon provides existence of fibrations of any graph manifold M by producing algorithmically a complete list of the conjugacy and isotopy invariants of the automorphisms whose associated mapping torus is diffeomorphic to M. We review some of this results and write them in a language that best suits our notation and conventions. Among these results is a classification of horizontal surfaces of a Seifert manifold with boundary.

In Section 5 we detail two algorithms. One takes a Seifert manifold and a horizontal surface as input and returns as output a general, relative or pure tête-à-tête graph realizing the horizontal surface and its monodromy. This algorithms differs from similar results in the literature in that our method produces directly the monodromy (in this case the tête-à-tête graph) without computing the conjugacy and isotopy invariants of the corresponding periodic mapping class. The other algorithm works in the opposite direction by taking a general, relative or pure tête-à-tête graph and producing the corresponding Seifert manifold and horizontal surface.

Finally, Section 6 contains a couple of detailed examples in which we apply the two algorithms.

2. GENERAL TÊTE-À-TÊTE STRUCTURES

In this section we study any orientation preserving periodic homeomorphism. Let $\phi : \Sigma \to \Sigma$ be such homeomorphism. We realize its boundary-free isotopy type and its conjugacy class in $\mathrm{MCG}(\Sigma)$ by a generalization of tête-à-tête graphs, using a technique that reduces to the case of homeomorphisms of a larger surface that leave all boundary components invariant.

Contrarily to what was done in [FdBPPPC17], we allow ribbon graphs with some *special* univalent vertices.

Definition 2.1. *A ribbon graph with boundary is a pair $(\Gamma, \mathcal{P})$ where Γ is a ribbon graph, and $\mathcal{P}$ is the set of univalent vertices, with the following additional property: given any vertex v of valency greater than 1 in the cyclic ordering of adjacent edges $e(v)$ there are no two consecutive edges connecting v with vertices in $\mathcal{P}$.*

In order to define the thickening of a ribbon graph with boundary we need the following construction:

Let Γ' be a ribbon graph (without univalent vertices) and let Σ be its thickening. Let

$$g_{\Gamma'} : \Sigma_{\Gamma'} \to \Sigma$$

be the gluing map. The surface $\Sigma_{\Gamma'}$ splits as a disjoint union of cylinders $\coprod_i \widetilde{\Sigma}_i$. Let w be a vertex of Γ'. The cylinders $\widetilde{\Sigma}_i$ such that w belongs to $g_{\Gamma'}(\widetilde{\Sigma}_i)$ are in a natural bijection with the pairs of consecutive edges (e', e'') in the cyclic order of the set $e(w)$ of adjacent edges to w.

Let $(\Gamma, \mathcal{P})$ be a ribbon graph with boundary. The graph Γ' obtained by erasing from Γ the set E of all vertices in $\mathcal{P}$ and its adjacent edges is a ribbon graph. Consider the thickening surface Σ of Γ'. Let e be an edge connecting a vertex $v \in \mathcal{P}$ with another vertex w, let e' and e'' be the immediate predecessor and successor of e in the cyclic order of $e(w)$. By the defining property of ribbon graphs with boundary they are consecutive edges in $e(w) \setminus E$, and hence determine a unique associated cylinder which will be denoted by $\widetilde{\Sigma}_i(v)$.

Each cylinder $\widetilde{\Sigma}_i$ has two boundary components, one, denoted by $\widetilde{\Gamma}_i$ corresponds to the boundary component obtained by cutting the graph, and the other, called C_i, corresponds to a boundary component of Σ. Fix a cylinder $\widetilde{\Sigma}_i$. Let $\{v_1, ..., v_k\}$ be the vertices of $\mathcal{P}$ whose associated cylinder is $\widetilde{\Sigma}_i$. Let $\{e_1, ..., e_k\}$ be the corresponding edges, let $\{w_1, ..., w_k\}$ be the corresponding vertices at Γ', and let $\{w_1', ..., w_k'\}$ be the set of preimages by $g_{\Gamma'}$ contained in $\widetilde{\Sigma}_i$. The defining property of ribbon graphs with boundary imply that w_i' and w_j' are pairwise different if $i \neq j$. Furthermore, since $\{w_1', ..., w_k'\}$ is included in the circle $\widetilde{\Gamma}_i$, which has an orientation inherited from Σ, the set $\{w_1', ..., w_k'\}$, and hence also $\{e_1, ..., e_k\}$ and $\{v_1, ..., v_k\}$ has a cyclic order. We assume that our indexing respects it.

Fix a product structure $\mathbb{S}^1 \times I$ for each cylinder $\widetilde{\Sigma}_i$, where $\mathbb{S}^1 \times \{0\}$ corresponds to the boundary component $\widetilde{\Gamma}_i$, and $\mathbb{S}^1 \times \{1\}$ corresponds to the boundary component of C_i.

Using this product structure we can embed Γ in Σ: for each vertex $v \in \mathcal{P}$ consider the corresponding cylinder $\widetilde{\Sigma}_{i(v)}$, let w' be the point in $\widetilde{\Gamma}_{i(v)}$ determined above. We embed the segment $g_{\Gamma'}(w' \times I)$ in Σ.

Doing this for any vertex v we obtain an embedding of Γ in Σ such that all the vertices $\mathcal{P}$ belong to the boundary $\partial\Sigma$, and such that Σ admits Γ as a regular deformation retract.

Definition 2.2. *Let $(\Gamma, \mathcal{P})$ be a ribbon graph with boundary. We define the thickening surface Σ of $(\Gamma, \mathcal{P})$ to be the thickening surface of Γ' together with the embedding $(\Gamma, \mathcal{P}) \subset (\Sigma, \partial\Sigma)$ constructed above. We say that $(\Gamma, \mathcal{P})$ is a* general spine *of $(\Sigma, \partial\Sigma)$.*

Definition 2.3 (General safe walk)**.** *Let $(\Gamma, \mathcal{P})$ be a metric ribbon graph with boundary. Let σ be a permutation of $\mathcal{P}$.*

We define a general safe walk *in $(\Gamma, \mathcal{P}, \sigma)$ starting at a point $p \in \Gamma \setminus v(\Gamma)$ to be a map $\gamma_p : [0, \pi] \to \Gamma$ such that*

1) *$\gamma_p(0) = p$ and $|\gamma_p'| = 1$ at all times.*

2) *when γ_p gets to a vertex of valency ≥ 2 it continues along the next edge in the cyclic order.*

3) *when γ gets to a vertex in $\mathcal{P}$, it continues along the edge indicated by the permutation σ.*

Definition 2.4 (General tête-à-tête graph)**.** *Let $(\Gamma, \mathcal{P}, \sigma)$ be as in the previous definition. Let γ_p, ω_p be the two safe walks starting at a point p in $\Gamma \setminus v(\Gamma)$.*

  We say Γ has the general tête-à-tête property *if*

  • *for any $p \in \Gamma \setminus v(\Gamma)$ we have $\gamma_p(\pi) = \omega_p(\pi)$*

Moreover we say that $(\Gamma, \mathcal{P}, \sigma)$ gives a general tête-à-tête structure *for $(\Sigma, \partial\Sigma)$ if $(\Sigma, \partial\Sigma)$ is the thickening of $(\Gamma, \mathcal{P})$.*

In the following construction we associate to a general tête-à-tête graph $(\Gamma, \mathcal{P}, \sigma)$ a homeomorphism of $(\Gamma, \mathcal{P})$ which restricts to the permutation σ in $\mathcal{P}$; we call it the *general tête-à-tête homeomorphism of $(\Gamma, \mathcal{P}, \sigma)$*. We construct also a homeomorphism of the thickening surface which leaves Γ invariant and restricts on Γ to the general tête-à-tête homeomorphism of $(\Gamma, \mathcal{P}, \sigma)$. We construct the homeomorphism on the graph and on its thickening simultaneously.

Consider the homeomorphism of $\Gamma' \setminus v(\Gamma)$ defined by

$$p \mapsto \gamma_p(\pi).$$

The same proof of [FdBPPPC17, Lemma 3.6] shows that there is an extension of this homeomorphism to a homeomorphism

$$\sigma_\Gamma : \Gamma \to \Gamma.$$

The restriction of the general tête-à-tête homeomorphism that we are constructing to Γ' coincides with σ_Γ. The mapping σ_Γ leaves Γ' invariant for being a homeomorphism. Let $\widetilde{\Gamma}'$ be the union of the circles $\widetilde{\Gamma}_i$. The homeomorphism $\sigma_\Gamma|_{\Gamma'}$ lifts to a periodic homeomorphism

$$\tilde{\sigma} : g_{\Gamma'}^{-1}(\widetilde{\Gamma}') \to g_{\Gamma'}^{-1}(\widetilde{\Gamma}'),$$

which may exchange circles in the following way. For any $p \in \widetilde{\Gamma}'$, the points in $g_{\Gamma'}^{-1}(p)$ corresponds to the starting point of safe walks in $\widetilde{\Gamma}'$ starting at p. A safe walk starting at p is determined by the point p and an starting direction at an edge containing p.

As we have seen, if $(\Gamma, \mathcal{P}, \sigma)$ is a general tête-à-tête structure for $(\Sigma, \partial\Sigma)$ then the surface $\Sigma_{\Gamma'}$ is a disjoint union of cylinders. The lifting $\tilde{\sigma}$ extends to $\Sigma_{\Gamma'}$ similarly as with the definition of the homeomorphism corresponding to a tête-à-tête structure defined in [FdBPPPC17]. This extension interchanges some cylinders $\widetilde{\Sigma}_i$ and goes down to an homeomorphism of Σ. We denote it by $\phi_{(\Gamma, \mathcal{P}, \sigma)}$. If necessary, we change the embedding of the part of Γ not contained in Γ' in Σ such that it is invariant by $\phi_{(\Gamma, \mathcal{P}, \sigma)}$. This is done by an adequate choice of the trivializations of the cylinders.

Definition 2.5. *The homeomorphism $\phi_{(\Gamma, \mathcal{P}, \sigma)}$ is by definition the homeomorphism of the thickening, and its restriction to Γ the* general tête-à-tête homeomorphism of $(\Gamma, \mathcal{P}, \sigma)$.

With the notation and definitions introduced we are ready to state and proof the main result of the work.

Theorem 2.6. *Given a periodic homeomorphism ϕ of a surface with boundary $(\Sigma, \partial\Sigma)$ which is not a disk or a cylinder, the following assertions hold:*

  (i) *There is a general tête-à-tête graph $(\Gamma, \mathcal{P}, \sigma)$ such that the thickening of $(\Gamma, \mathcal{P})$ is $(\Sigma, \partial\Sigma)$, the homeomorphism ϕ leaves Γ invariant and we have the equality $\phi|_\Gamma = \phi_{(\Gamma, \mathcal{P}, \sigma)}|_\Gamma$.*

  (ii) *We have the equality of boundary-free isotopy classes $[\phi|_\Gamma] = [\phi_{(\Gamma, \mathcal{P}, \sigma)}]$.*

  (iii) *The homeomorphisms ϕ and $\phi_{(\Gamma, \mathcal{P}, \sigma)}$ are conjugate.*

Proof. In the first part of the proof we extend the homeomorphism ϕ to a homeomorphism $\hat{\phi}$ of a bigger surface $\hat{\Sigma}$ that leaves all the boundary components invariant. Then, we find a tête-à-tête graph $\hat{\Gamma}$ for $\hat{\phi}$ such that $\hat{\Gamma} \cap \Sigma$, with a small modification in the metric and a suitable permutation, is a general tête-à-tête graph for ϕ.

Let n be the order of the homeomorphism. Consider the permutation induced by ϕ in the set of boundary components. Let $\{C_1, ..., C_m\}$ be an orbit of cardinality strictly bigger than 1, numbered such that $\phi(C_i) = C_{i+1}$ and $\phi(C_m) = C_1$. Take an arc $\alpha \subset C_1$ small enough so that it is disjoint from all its iterations by ϕ. Define the arcs $\alpha^i := \phi^i(\alpha)$ for $i \in \{0, ..., n-1\}$, which are contained in $\cup_i C_i$. Obviously we have the equalities $\alpha^{i+1} = \phi(\alpha^i)$ and $\phi(\alpha^{n-1}) = \alpha^0 = \alpha$.
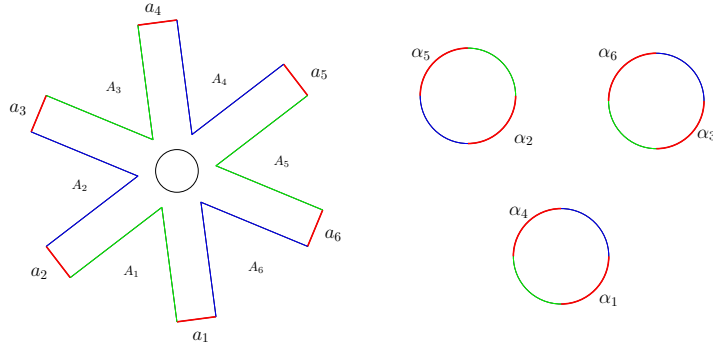

Figure 2.1. Example of a star-shaped piece S with 6 arms on the left and boundary components components on the right. The arcs along which the two pieces are glued, are marked in red. In blue and red are the boundaries of the two disks that we used to cap off the new boundaries.

We consider a star-shaped piece S of n arms as in Figure 2.1. We denote by D the central boundary component. Let $a^0, \ldots, a^{n-1}$ be the boundary of the arms of the star-shaped piece labeled in the picture, oriented counterclockwise. We consider the rotation r of order n acting on this piece such that $r(a^i) = a^{i+1}$. Note that this rotation leaves D invariant.

We consider the surface $\hat{\Sigma}$ obtained by gluing Σ and S identifying a^i with α^i reversing the orientation, and such that ϕ and the rotation r glue to a periodic homeomorphism $\hat{\phi}$ in the resulting surface.

The boundary components of the new surface are precisely the boundary components of Σ different from $\{C_1, ..., C_m\}$, the new boundary component D, and the boundary components $C_1', ..., C_k'$ that contain the part of the C_i's not included in the union $\cup_{i=0}^{n-1} \alpha^i$.

The homeomorphism $\hat{\phi}$ leaves D invariant and may interchange the new boundary components $C_1', ..., C_k'$. We cup each component C_i' with a disk D_i and extend the homeomorphism by the Alexander trick, obtaining a homeomorphism $\hat{\phi}$ of a bigger surface $\hat{\Sigma}$. The only new ramification points that the action of $\hat{\phi}$ may induce are the centers t_i of these disks. We claim that, in fact, each of the t_i's is a ramification point.

Denote the quotient map by

$$p : \hat{\Sigma} \to \hat{\Sigma}^{\hat{\phi}}.$$

In order to prove the claim notice that the difference $\hat{\Sigma} \setminus \Sigma$ is homeomorphic to a closed surface with $m+1$ disks removed. On the other hand the difference of quotient surfaces $\hat{\Sigma}^{\hat{\phi}} \setminus \Sigma^\phi$ is homeomorphic to a cylinder. Since m is strictly bigger than 1, Hurwitz formula for p forces the existence of ramification points. Since p is a Galois cover each t_i is a ramification point.

The new boundary component of $\hat{\Sigma}^{\hat{\phi}}$ corresponds to $p(D)$, where D is invariant by $\hat{\phi}$. The point $q_1 := p(t_i)$ is then a branch point of p.

We do this operation for every orbit of boundary components in Σ of cardinality greater than 1. Then we get a surface $\hat{\Sigma}$ and an extension $\hat{\phi}$ of ϕ that leaves all the boundary components invariant. The quotient surface $\hat{\Sigma}^{\hat{\phi}}$ is obtained from Σ^ϕ attaching some cylinders $\mathcal{C}_j$ to some boundary components. Let

$$p : \hat{\Sigma} \to \hat{\Sigma}^{\hat{\phi}}$$

denote the quotient map. Comparing $p|_\Sigma$ and $p|_{\hat{\Sigma}}$, we see that we have only one new branching point q_j in every cylinder $\mathcal{C}_j$.

Now we construct a tête-à-tête graph for $\hat{\phi}$ modifying slightly the construction of [FdBPPPC17, Theorem 5.12].

To fix ideas we consider the case in which the genus of the quotient $\hat{\Sigma}^{\hat{\phi}}$ is positive. The modification of the genus 0 case is exactly the same. As in [FdBPPPC17, Theorem 5.12] we use a planar representation of Σ^ϕ as a convex $4g$-gon in $\mathbb{R}^2$ with r disjoint open disks removed from its convex hull and whose edges are labeled clockwise like $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$, we number the boundary components $C_i \subset \partial\Sigma$, $1 \le i \le r$, we denote by d the arc $a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$, and we consider $l_1,...,l_{r-1}$ arcs as in Figure 2.2. We denote by $c_1,...,c_r$ the edges in which a_1^{-1} (and a_1) is subdivided according to the component $p(C_i)$ they enclose.

We impose the further condition that each of the regions in which the polygon is subdivided by the l_i's encloses not only a component $p(C_i)$, but also the branching point q_i that appears in the cylinder $\mathcal{C}_i$. We assume that the union of d, $a_1 b_1 a_1^{-1} b_1^{-1}$ and the l_i's contains all the branching points of p except the q_i's.

In order to be able to lift the retraction we need that the spine that we draw in the quotient contains all branching points. In order to achieve this we add an edge s_i joining q_i and some interior point q_i' of l_i for $i = 1, \ldots, r-1$ and joining q_r with some interior point q_r' of l_{r-1}. We may assume that q_i' is not a branching point. We consider the circle $p(C_i)$ and ask s_i to meet it transversely to it at only 1 point. See Figure 2.3. We consider the graph Γ' as the union of the previous segments and the s_i's. Clearly the quotient surface retracts to it. Since it contains all branching points, its preimage $\hat{\Gamma}$ is a spine for $\hat{\Sigma}$. It has no univalent vertices since the q_i's are branching points of a Galois cover.
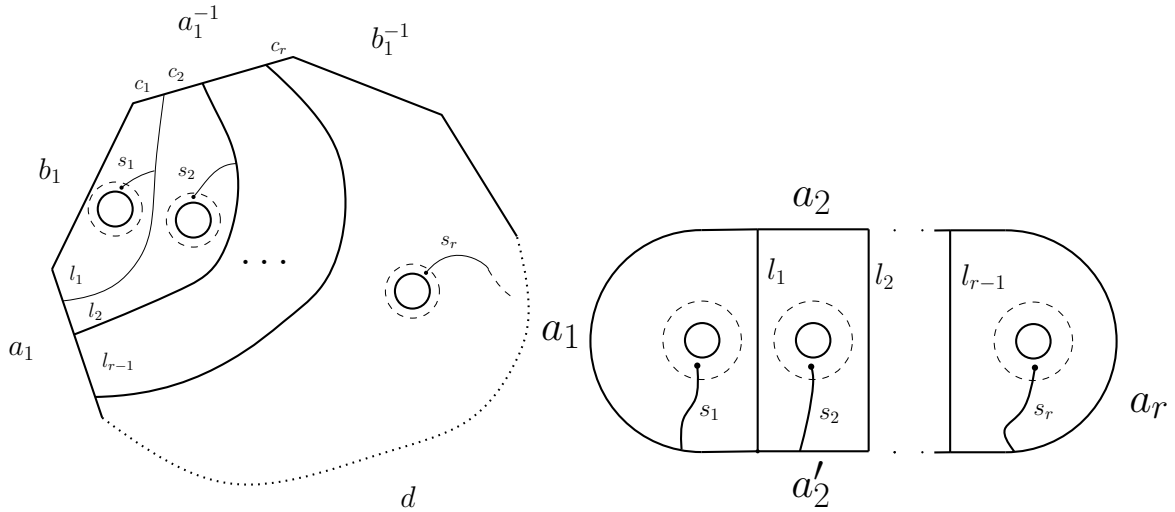

Figure 2.2. Drawing of Γ' for the case $genus(\hat{\Sigma}^{\hat{\phi}}) \ge 1$ in the first image and $genus(\hat{\Sigma}^{\hat{\phi}}) = 0$ in the second.

In order to give a metric in the graph we proceed as follows. We give the segments d and C_i's the same length they had in the proof of [FdBPPPC17, Theorem 5.12]. We impose every s_i to have length some small enough ϵ and the part of s_i inside the cylinder $\mathcal{C}_i$ to have length $\epsilon/2$ (see Figure 2.3). We give each segment l_i length $L - 2\epsilon$. It is easy to check that the preimage graph $\hat{\Gamma}$ with the pullback metric is tête-à-tête .

Now we consider the graph $\Gamma := \hat{\Gamma} \cap \Sigma$ with the restriction metric, except on the edges meeting $\partial\Sigma$ whose length is redefined to be ϵ. Along the lines of the proof of [FdBPPPC17, Theorem 5.12] we get that $\phi_{(\Gamma,\mathcal{P},\sigma)}$ and ϕ are isotopic and conjugate. If we denote by $\mathcal{P}$ the set of univalent vertices of Γ, it is an immediate consequence of the construction that $(\Gamma, \mathcal{P}, \sigma)$ with the obvious permutation σ of $\mathcal{P}$ is a general tête-à-tête graph with $\phi_{(\Gamma,\mathcal{P},\sigma)}|_\Gamma = \phi|_\Gamma$.   □
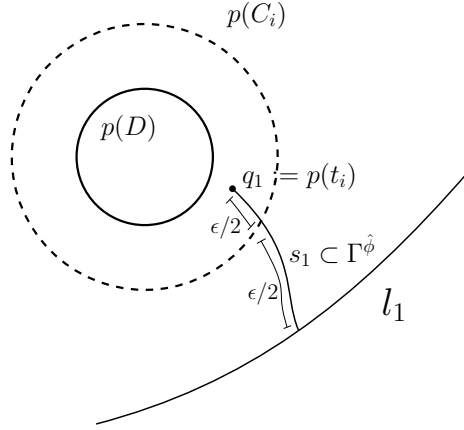
Figure 2.3. Neighborhood of $q_1 = p(t_i)$ in $\hat{\Sigma}^{\hat{\phi}}$ and edge s_1 joining q_1 and l_1.

Example 2.7. We show an example that illustrates these ideas. Let Σ be surface of genus 1 and 3 boundary components C_0, C_1, C_2 embedded in $\mathbb{R}^3$ as in the picture 2.4. Let $\phi : \Sigma \to \Sigma$ be the restriction of the space rotation of order 3 that exchanges the 3 boundary components. We observe that in particular $\phi^3|_{C_i} = id$ for $i = 0, 1, 2$.

We consider the star-shaped piece S with 3 arms together with the order 3 rotation r that exchanges the arms (see the picture Figure 2.4).
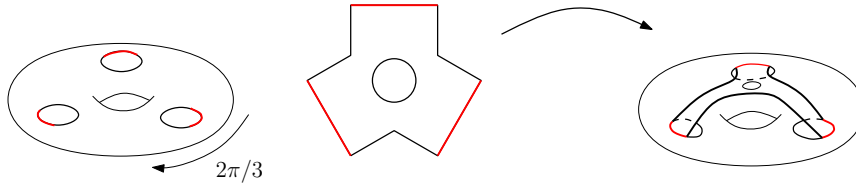

Figure 2.4. On the left, the torus Σ with 3 disks removed and the orbit of an arc marked, that is, 3 arcs in red. In the center, the star-shaped piece S with 3 arms to be glued to the torus along those arcs. On the right, the surface we get after gluing, with 2 boundary components, one of them invariant by the induced homeomorphism.

We glue S to Σ as the theorem indicates: we mark a small arc $\alpha^0 \subset C_2$ and all its iterated images by the rotation. Then we glue $\alpha^0, \alpha^1, \alpha^2$ to a^0, a^1, a^2 respectively by orientation reversing homeomorphisms. We get a new surface $\hat{\Sigma} := \Sigma \cup S$ with 2 boundary components. We cap the boundary component that intersects $C_0 \cup C_1 \cup C_2$ with a disk D^2 and extend the homeomorphism to the interior of the disk getting a new surface $\hat{\Sigma}$ and a homeomorphism $\hat{\phi}$.

Using Hurwitz formula $2 - 2g - 4 = 0 - 2$ we get that the surface we are gluing to Σ has genus 0 and hence it is a sphere with 4 boundary components. See picture Figure 2.5. Three of them are identified with C_0, C_1, C_2, and the 4-th is called C and is the only boundary component of $\hat{\Sigma}$.

We compute the orbit space $\hat{\Sigma}^{\hat{\phi}}$ by the extended homeomorphism $\hat{\phi}$ and get a torus with 1 boundary component. We consider the graph Γ' as in picture Figure 2.6. We put a metric in this graph. We set every edge of the hexagon to be $\pi/6 - \epsilon/3$ long and the path joining the hexagon with the branch point to be ϵ long. In this way, if we look at the result of cutting $\hat{\Sigma}^{\hat{\phi}}$ along the graph $\hat{\phi}$ we see that the only boundary component that maps to the graph by the gluing map has length $6(\pi/6 - \epsilon/3) + 2\epsilon = \pi$.

The preimage $\hat{\Gamma}$ of Γ' by the quotient map is a tête-à-tête graph whose thickening is $\hat{\Sigma}$. Its associated homeomorphism $\hat{\phi}$ leaves Σ invariant and its restriction to it coincides with the rotation ϕ. Moreover $(\hat{\Gamma} \cap \Sigma, \hat{\Gamma} \cap \partial\Sigma)$ is a general spine of $(\Sigma, \partial\Sigma)$. Modifying the induced metric
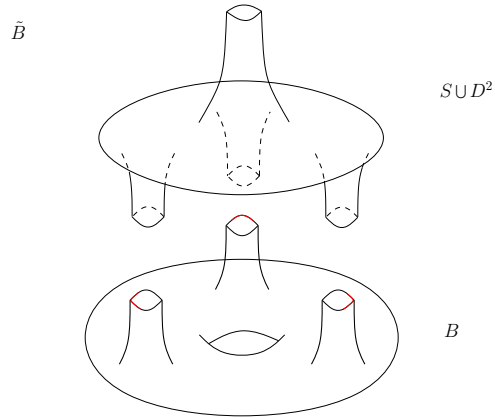
Figure 2.5. On the left, the torus with 3 disks removed and 3 the orbit of an arc marked. On the right, the star-shaped piece with 3 arms to be glued to the torus along those arcs.

Figure 2.6. On the lower part we have the original surface. On the upper part we have the surface that we attach, in this case a sphere with 4 holes removed.

in $\hat{\Gamma} \cap \Sigma$ as in the proof of the Theorem and adding the order 3 cyclic permutation to the valency 1 vertices we obtain a tête-à-tête graph whose associated homeomorphism equals ϕ.
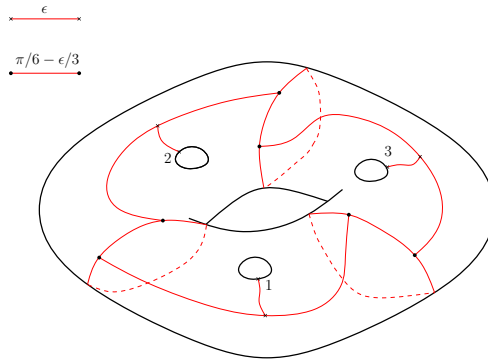
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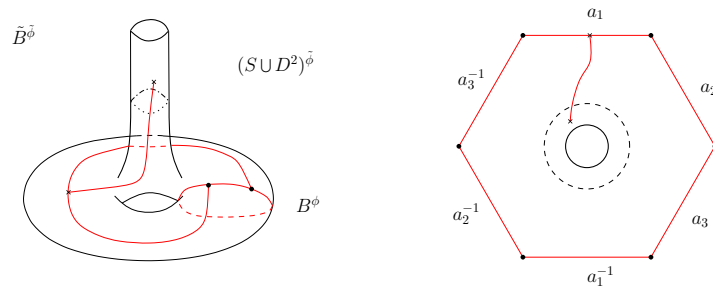
3. Seifert manifolds and plumbing graphs

In this subsection we recall some theory about Seifert manifolds and plumbing graphs and fix the conventions used in this work. For more on this topic, see [Neu81], [NR78], [Neu97], [JN83], [HNK71], [Hat07] or [Ped09]. In many aspects we follow [Ped09].

3.1. **Seifert manifolds.** Let $p, q \in \mathbb{Z}$ with $q > 0$ and $\gcd(p, q) = 1$. Let $D^2 \times [0, 1]$ be a solid cylinder. We consider the natural orientation on $D^2 \times [0, 1]$.

Let (t, θ) be polar coordinates for D^2. Let $r_{p/q} : D^2 \to D^2$ be the rotation $(t, \theta) \mapsto (t, \theta + 2\pi p/q)$. Let $T_{p,q}$ be the mapping torus of D^2 induced by the rotation $r_{p/q}$, that is, the quotient space

$$\frac{D^2 \times [0,1]}{(t, \theta, 1) \sim (t, r_{p/q}(\theta), 0)}.$$

If $p, p' \in \mathbb{Z}$ with $p \equiv p' \mod q$, then the rotations $r_{p/q}$ and $r_{p'/q}$ are exactly the same map so $T_{p',q} = T_{p,q}$. The resulting space is diffeomorphic to a solid torus naturally foliated by circles which we call *fibers*. We call this space a solid (p,q)-torus or a solid torus of type (p,q). It has an orientation induced from the orientation of $D^2 \times [0,1] \subset \mathbb{R}^3$. The torus $\partial T_{p,q}$ is oriented as boundary of $T_{p,q}$.

We call the image of $\{(0,0)\} \times [0,1] \subset D^2 \times [0,1]$ in $T_{p,q}$ *the central fiber*. We say that q is the *multiplicity* of the central fiber. If $q = 1$ we call the central fiber a *typical fiber* and if $q > 1$ we call the central fiber a *special fiber*. Also any other fiber than the central fiber is called a typical fiber.

If a and b are two closed curves in $\partial T_{p,q}$, let $[a] \cdot [b]$ denote the oriented intersection number of their classes in $H_1(\partial T_{p,q}; \mathbb{Z})$. We describe 4 classes of simple closed curves on $H_1(\partial T_{p,q}, \mathbb{Z})$:

(1) A *meridian* curve $m := \partial D^2 \times \{y\}$. We orient it as boundary of $D^2 \times \{y\}$.
(2) A *fiber* f on the boundary $\partial T_{p,q}$. We orient it so that the radial projection on the central fiber is orientation preserving. It satisfies that $[m] \cdot [f] = q$.
(3) A *longitude* l is a curve such that $[l]$ is a generator of $H_1(T_{p,q}; \mathbb{Z})$ and $[m] \cdot [l] = 1$.
(4) A *section* s. That is a closed curve that intersects each fiber exactly once. It is well defined up integral multiples of f. It is oriented so that $[s] \cdot [f] = -1$.
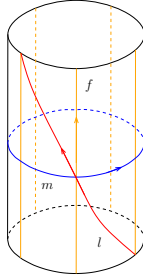

Figure 3.1. A torus $T_{2,5}$ with some closed curves marked on its boundary. In orange a fiber f, in blue a meridian m and in red a longitude l.

We have defined two basis of the homology of $\partial T_{p,q}$, so we have that there must exist unique $a, b \in \mathbb{Z}$ such that the equation

$$(3.1) \qquad ([s][f]) = ([m][l]) \begin{pmatrix} a & p \\ b & q \end{pmatrix}$$

holds in $H_1(\partial T_{p,q}; \mathbb{Z})$. The matrix is nothing but a change of basis.

The matrix of the equation has determinant -1 because $[s], [f]$ is a negative basis in the homology group $H_1(\partial T_{p,q}; \mathbb{Z})$. Therefore $bp \equiv 1 \mod q$. Changing the class $[s]$ by adding integer multiples of $[f]$ to it, changes b by integer multiples of q.

We now fix conventions on Seifert manifolds. Let B' be an oriented surface of genus g and $r + k$ boundary components, $M' := B' \times \mathbb{S}^1$ and $s' : M' \times \mathbb{S}^1 \to B'$ the projection onto B'. Let $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ be k pairs of integers with $\alpha_i > 0$ for all $i = 1, \ldots, k$. Let $N_1, \ldots N_k$ be k boundary tori on M'. On each of them consider the following two curves $s_i := B' \times \{0\} \cap N_i$ and any fiber $f_i \subset N_i$. Orient them so that $\{[s_i], [f_i]\}$ is a positive basis of N_i as boundary of M'. For each i, consider a solid torus $T_i = D^2 \times \mathbb{S}^1$ and the curves $m = \partial D^2 \times \{0\}$ and $l := \{pt\} \times \mathbb{S}^1$ oriented so that $\{[m], [l]\}$ is a positive basis of T_i. Attach T_i to N_i along its boundary by

$$\begin{pmatrix} -\alpha_i & c \\ -\beta_i & d \end{pmatrix} : \partial T_i \to N_i$$

with respect to the two given basis. The numbers c and d are integers such that the matrix has determinant -1. Note that, since the first column defines the attaching of the meridian, the gluing is well defined up to isotopy.

The foliation on N_i extends to all T_i and gives it a structure of a fibered solid torus. After gluing and extending the foliation to all k tori, we get a manifold M and a collapsing map $s : M \to B$ where B is the surface of genus g and r boundary components.

If a manifold M can be constructed like this, we say that it is a *Seifert manifold* and the map $s : M \to B$ is a *Seifert fibering* for M. We denote the resulting manifold after gluing k tori by

$$M(g, r, (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)).$$

Each pair (α_i, β_i) is called *Seifert pair* and we say that it is normalized when $0 \leq \beta_i < \alpha_i$.

We have, by definition and the discussion above, the following lemma and corollary.

Lemma 3.2. *Let $M \to B$ be a Seifert fibering. If a fiber f has a neighborhood diffeomorphic to a (p, q)-solid torus, then the there exists $b \in \mathbb{Z}$ such that the (possibly unnormalized) Seifert invariant corresponding to f is $(q, -b)$ with $bp \equiv 1 \mod q$. Conversely, the special fiber f corresponding to a Seifert pair (α, β) has a neighborhood diffeomorphic as a circle bundle to a $(-c, \alpha_i)$-solid torus with $c\beta \equiv 1 \mod \alpha$.*

Corollary 3.3. *Let $\phi : \Sigma \to \Sigma$ be an orientation preserving periodic automorphism of a surface Σ of order n and let Σ_ϕ be the corresponding mapping torus. Let $x \in \Sigma$ be a point whose isotropy group in the group $<\phi>$ has order k with $n = k \cdot s$. Then ϕ^s acts as a rotation in a disk around x with rotation number p/k for some $p \in \mathbb{Z}$ and the (possibly unnormalized) Seifert pair of M_ϕ corresponding to the fiber passing through x is $(k, -b)$ with $bp \equiv 1 \mod k$.*

Proof. That ϕ^s acts as a rotation in a disk $D \subset \Sigma$ around x with rotation number p/k for some $p \in \mathbb{Z}_{>0}$ follows from the fact that x is a fixed point for $\phi^{n/k}$. By construction of the mapping torus of Σ we observe that the two mapping tori $M_{\phi|_D} \simeq D_{\phi^{n/k}}$ are diffeomorphic where D is a small disk around x. By definition of fibered torus we have that $D_{\phi^{n/k}} \simeq T_{p,k}$. The rest follows from Lemma 3.2 above. $\qquad\square$

3.2. **Plumbing graphs.** A plumbing graph is a decorated graph that encodes the information to recover the topology of a certain 3-manifold. As with Seifert manifolds, we fix notation and conventions.

Definition 3.4. *A plumbing graph is a decorated graph Λ that encodes the information to recover the topology of a certain 3-manifold.*

It has vertex set $\mathcal{V} \sqcup \mathcal{A}$ with $\mathcal{A}$ possibly empty. The valency r_v of a vertex v is the number of edges stemming out of v where loops are counted twice. Vertices in $\mathcal{A}$ are represented by arrowheads and always have valency 1.

Now we describe the decoration of Λ and its meaning.

(1) *Each vertex $v \in \mathcal{V}$ is decorated with 2 integers e_v (placed on top) and g_v placed on bottom. Let r_v be its valency. It represents the circle bundle Y_v over the surface of genus g_i and r_i boundary components such that (after picking a section on each boundary torus) the global Euler number is well defined and equal to e_i. When g is omitted it is assumed to be 0. We consider the sections as part of the data that comes with a plumbing graph. One can always change the section at a given boundary torus provided that the section of another boundary torus in the same circle bundle is also changed accordingly to preserve the Euler number. A collection of sections defines an identification (up to isotopy) of each boundary torus with $\mathbb{S}^1 \times \mathbb{S}^1$ that sends the section to the first factor and the fiber to the second. This identification is called* frame.*

(2) *Each edge is weighted by a sign $+$ or $-$ (when omitted, $+$ is assumed). It tells us that the circle bundles corresponding to the ends of the edge are glued along a boundary torus by the gluing map $J(x, y) = (y, x)$ if the sign is $+$, and $-J$ if the sign is $-$. This map is defined with respect to the chosen frame on each boundary torus. The section has*

been fixed in (1) and the fiber does not change the diffeomorphism type of the resulting manifold after the gluing.

(3) An arrowhead (or vertex in $\mathcal{A}$) represents that an open fibered solid torus is removed from the corresponding circle bundle from where the edge comes out.

The construction of the 3-manifold associated to a plumbing graph is clear from the description of its decoration above. Also, in this work we only consider plumbing graphs with non-empty arrowhead-set $\mathcal{A}$. Furthermore, the plumbing graphs that we use are a special subset of the plumbing graphs:those that represent Seifert manifolds.

We point out a minor correction to an argument in [Neu81] and reprove a known lemma which is crucial in Section 5 (see discussion afterwards in Remark 3.8).

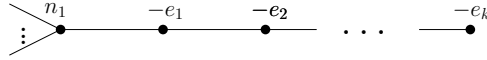**Lemma 3.5.** *Let Λ be a plumbing graph.*

1) *If a portion of Λ has the following form:*
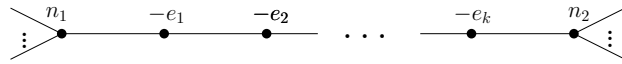
Then the piece corresponding to the node n_1 is glued to the piece corresponding to the node n_2 along a torus by the matrix $G = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\det(G) = -1$ and where $-b/a = [e_1, \ldots, e_k]$ with the numbers in brackets being the continued fraction

$$\frac{-b}{a} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{e_3 - \frac{1}{\cdots}}}.$$

2) *If a portion of Λ has the following form:*

Then the piece corresponding to the node n_i is glued along a torus to the boundary of a solid torus $D^2 \times \mathbb{S}^1$ by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $-d/c = [e_1, e_2, \ldots, e_k]$.

Proof. Let $T := D^2 \times \mathbb{S}^1$ be a solid torus naturally foliated by circles by its product structure. Let s be the closed curve $\partial D^2 \times \{0\}$ and let f be any fiber on the boundary of the solid torus. Orient them so that $\{[s], [f]\}$ is a positive basis of $H_1(\partial T; \mathbb{Z})$. If T, T' are two copies of the solid torus. Then M_i is the S^1-bundle $T \sqcup_{E_i} T'$ where $E_i : \partial T \to \partial T'$ is the matrix

$$\begin{pmatrix} -1 & 0 \\ e_i & 1 \end{pmatrix}$$

used in the gluing along the boundaries. In particular $[f] = [f']$ in $H_1(M_i; \mathbb{Z})$. The -1 in the upper left part reflects the fact that s inherits different orientations from the two tori.

We treat the case 1) first. If M_i is the piece corresponding to the node n_i with $i = 1, 2$ we have that the gluing from M_1 to M_2 is

$$M_1 \sqcup_J (A \times \mathbb{S}^1 \sqcup_{E_1} A \times \mathbb{S}^1) \sqcup_J \cdots \sqcup_J (A \times \mathbb{S}^1 \sqcup_{E_k} A \times \mathbb{S}^1) \sqcup_J M_2$$

Where $A \times \mathbb{S}^1$ is the trivial circle bundle over the annulus $A := [1/2, 1] \times \mathbb{S}^1$. Let (r, θ) be polar coordinates for A. The two tori forming the boundary of $A \times \mathbb{S}^1$ are oriented as boundaries of $A \times \mathbb{S}^1$. Observe that the map $r((1/2, \theta), \eta) = ((1, \theta), \eta)$ is orientation reversing.

We define $s = \{S^1_{1/2}\} \times \{0\}$ and $f = \{(1/2, 0) \times \mathbb{S}^1$ and orient them so that the ordered basis $\{[s], [f]\}$ is a positive basis for $H_1(S^1_{1/2} \times \mathbb{S}^1; \mathbb{Z})$. We define similarly $s' = \{S^1_1\} \times \{0\}$, $f' = \{(1, 0) \times \mathbb{S}^1$ and orient them so that $\{[s'].[f']\}$ is a positive basis for $H_1(S^1_1 \times \mathbb{S}^1; \mathbb{Z})$. Then the homology classes $[r(s)]$ and $[r(f)]$ form a negative basis. In fact $[s'] = -[r(s)]$ and $[f] = [r(f)]$. This is the reason of the matrices $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in the Equation (3.6) below.

So the gluing matrix G from a torus in the boundary of M_1 to a torus in the boundary of M_2 is given by the following composition of matrices:

$$G = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ e_k & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ e_1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

(3.6)
$$= \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ -e_k & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ -e_1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

$$= \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & -e_k \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} 0 & -1 \\ 1 & -e_1 \end{smallmatrix}\right)$$

Observe that each matrix in the definition of G has determinant -1 so $\det(G) = -1$ because there is an odd number. Hence G inverts orientation on the boundary tori, preserving the orientation on the global 3 manifold. The result about the continued fraction follows easily by induction on k.

Now we treat similarly the case 2). The gluing from a boundary torus from M_1 to $\partial D^2 \times \mathbb{S}^1$ is

$$M_1 \sqcup_J (A \times \mathbb{S}^1 \sqcup_{E_1} A \times \mathbb{S}^1) \sqcup_J \cdots \sqcup_J (A \times \mathbb{S}^1 \sqcup_{E_k} D^2 \times \mathbb{S}^1).$$

Hence, by a similar argument to the previous case, the matrix that defines the gluing is

$$G = \left(\begin{smallmatrix} -1 & 0 \\ e_k & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ e_1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$$

(3.7)
$$= \left(\begin{smallmatrix} -1 & 0 \\ e_k & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & -e_{k-1} \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} 0 & -1 \\ 1 & -e_1 \end{smallmatrix}\right)$$

$$= \left(\begin{smallmatrix} 0 & 1 \\ 1 & -e_k \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & -e_{k-1} \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} 0 & -1 \\ 1 & -e_1 \end{smallmatrix}\right)$$

By the expression in the last line we see that all matrices involved but the one on the left, have determinant 1 so we get $det(G) = -1$. Again, by induction on k the result on the continued fraction follows straight from the las line. $\qquad\square$

Remark 3.8. Note the differences of the Lemma above with Lemma 5.2 and the discussion before it in [Neu81]: there the author does not observe that in each piece $A \times \mathbb{S}^1$, the natural projection from one boundary torus to the other is orientation reversing. So the matrices $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are not taken into account there.

In a more extended manner. The problem is with the claim that the matrix C (in equation $(*)$ pg.319 of [Neu81]) is the gluing matrix. The equation above equation $(*)$ in that page, describes the gluing between the two boundary tori as a concatenated gluing of several pieces. In particular you glue a piece of the form $A \times S^1$ with another piece of the same form using the matrix H_k and then you glue these pieces a long J- matrices. Then it is claimed that "since $A \times S^1$ is a collar" then the gluing matrix (up to a sign) is $JH_kJ\cdots JH_1J$. However, notice that each piece $A \times S^1$ has two boundary tori, and they inherit "opposite" orientations. More concretely, the natural radial projection from one boundary torus to the other is orientation reversing. So even, if they are a collar (which they are), they interfere somehow in the gluing. That is why we add the matrices $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ between each J and each H_k matrix.

4. Horizontal surfaces in Seifert manifolds

In this section, we study and classify horizontal surfaces of Seifert fiberings up to isotopy. The results contained here are known. The exposition that we choose to do here is useful for Section 5.

We consider only Seifert manifolds that are orientable with orientable base space and with non-empty boundary in this section. Let $s : Y \to B$ be a Seifert fibering of a manifold Y with non-empty boundary and Seifert invariants $(g, r, (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$.

Definition 4.1. *Let H be a surface with non-empty boundary which is properly embedded in Y i.e. $H \cap \partial Y = \partial H$. We say that H is a horizontal surface of Y if it is transverse to all the fibers of Y.*

Horizontal surfaces in a orientable Seifert manifold with orientable base space are always orientable since the map s restricted to the horizontal surface is a branched cover over the base space. Therefore, by our assumptions, only orientable horizontal surfaces appear in this work.

Definition 4.2. *Let $H(Y)$ be the set of all horizontal surfaces of Y, we define*

$$\mathcal{H}(Y) := H(Y)/\sim$$

where two elements $H_1, H_2 \in H(Y)$ are related $H_1 \sim H_2$ if their inclusion maps are isotopic.

Let $n := \mathrm{lcm}(\alpha_1, \ldots, \alpha_k)$. We consider the action of the subgroup of the unitary complex numbers given by the n-th roots of unity $c_n := \{e^{2\pi i m/n}\}$ with $m = 0, \ldots, n-1$ on the fibers of Y. The element $e^{2\pi i m/n}$ acts on a typical fiber by a rotation of $2\pi m/n$ radians and acts on a special fiber with multiplicity α_i by a rotation of $2\pi m \alpha_i/n$ radians.

We quotient Y by the action of this group and denote $\hat{Y} = Y/c_n$ the resulting quotient space. By definition, the action of c_n preserves the fibers and is effective. The manifold $\hat{Y}$ is then a Seifert manifold where we have *killed* the multiplicity of all the special fibers of Y. Hence it is a locally trivial S^1-fibration over B and since $\partial B \neq \emptyset$, it is actually a trivial fibration so $\hat{Y}$ is diffeomorphic to $B \times \mathbb{S}^1$.

Let $\pi : Y \to \hat{Y}$ be the quotient map induced by the action of c_n. Observe that $\hat{Y}$, seen as a Seifert fibering with no special fibers, has the same base space as Y because the action given by c_n preserves fibers. In particular we find the following commutative diagram

(4.3)
$$
\begin{array}{ccc}
Y & \xrightarrow{\ \pi\ } & \hat{Y} \\
{\scriptstyle s}\big\downarrow & \swarrow {\scriptstyle \hat{s}} & \\
B & &
\end{array}
$$

Where s (resp. $\hat{s}$) is the projection map from the Seifert fibering Y (resp. $\hat{Y}$) onto its base space B.

Definition 4.4. *Let H be a horizontal surface in Y. We say that H is well embedded if it is invariant by the action of c_n.*

A horizontal surface H defines a linear map $H_1(Y;\mathbb{Z}) \to \mathbb{Z}$ by considering its Poincaré dual. If H intersects a generic fiber m times, then it intersects a special fiber with multiplicity α, $m/\alpha \in \mathbb{Z}$ times (in particular m is a multiple of n). This is because a generic fiber covers that special fiber α times. Hence, by isotoping any horizontal surface, we can always find well-embedded representatives $\hat{H} \in [H]$.

Remark 4.5. Observe that if H and and H' are two well-embedded surfaces with $[H] = [H']$, then we can always find a fiber-preserving isotopy h that takes the inclusion $i : H \hookrightarrow Y$ to an homeomorphism $h(\cdot, 1) : H \to H'$ such that $h(H, t)$ is a well-embedded surface for all t. This helps us prove the following:

Lemma 4.6. *There is a bijection $\pi_\sharp : \mathcal{H}(Y) \to \mathcal{H}(\hat{Y})$ induced by π.*

Proof. Let $[H] \in \mathcal{H}(Y)$ and suppose that $H \in [H]$ is a well-embedded representative. Then clearly $\pi(H) \in H(\hat{Y})$. If H' is another well-embedded representative of the same class, then by Remark 4.5 we have that $[\pi(H)] = [\pi(H')]$ in $\mathcal{H}(\hat{Y})$. Hence the map $\pi_\sharp([H]) := [\pi(H)]$ is well defined.

The map $\pi_\sharp$ is clearly surjective because $\pi^{-1}(\hat{H})$ is a well-embedded surface for any horizontal surface $\hat{H} \in H(\hat{Y})$ and hence $\pi_\sharp([\pi^{-1}(\hat{H})]) = [\hat{H}]$.

Now we prove that the natural candidate for inverse $\pi_\sharp^{-1}([\hat{H}]) := [\pi^{-1}(\hat{H})]$ is well-defined. Let $[\hat{H}] \in \mathcal{H}(\hat{Y})$ with $\hat{H} \in [\hat{H}]$ a representative of the class. Let $H := \pi^{-1}(\hat{H})$. If $[\hat{H}] = [\hat{H}']$ for some $\hat{H}'$ in $H(\hat{Y})$ then $[\pi^{-1}(\hat{H}')] = [\pi^{-1}(\hat{H})]$ by just pulling back the isotopy between $\hat{H}$ and $\hat{H}'$ to Y by the map π. Hence the map is well defined. By construction, it is clear that for any $H \in \mathcal{H}(Y)$ we have that $\pi_\sharp^{-1}(\pi_\sharp([H]) = [H]$ so we are done. $\qquad\square$

The objective of this section is to study $\mathcal{H}(Y)$ but because of Lemma 4.6 above, it suffices to study $\mathcal{H}(\hat{Y})$.

Fix a trivialization $\hat{Y} \simeq B \times \mathbb{S}^1$ once and for all. We observe that since $\partial B \neq \emptyset$, the surface B is homotopically equivalent to a wedge of $\mu = 2g + r - 1$ circles, denote this wedge by $\tilde{B}$. Observe that $\mathcal{H}(Y)$ is in bijection with multisections of $\tilde{B} \times \mathbb{S}^1 \to \tilde{B}$ up to isotopy. Multisections are multivalued continuous maps from $\tilde{B}$ to $\tilde{B} \times \mathbb{S}^1$.

Lemma 4.7. *The elements in $\mathcal{H}(\tilde{B} \times \mathbb{S}^1)$ are in bijection with elements of*

$$H^1(\tilde{B} \times \mathbb{S}^1; \mathbb{Z}) = H^1(\tilde{B}; \mathbb{Z}) \oplus \mathbb{Z}$$

that are not in $H^1(\tilde{B}; \mathbb{Z}) \oplus \{0\}$. Oriented horizontal surfaces that intersect positively any fiber of $\tilde{B} \times \mathbb{S}^1$ are in bijection with elements of $H^1(\tilde{B}; \mathbb{Z}) \oplus \mathbb{Z}_{>0}$.

Proof. We have that $H^1(B; \mathbb{Z}) \oplus \mathbb{Z} = \mathbb{Z}^\mu \oplus \mathbb{Z}$. Take an element

$$(p_1, \ldots, p_\mu, q) = k((p_1', \ldots, p_\mu', q')) \in \mathbb{Z}^\mu \times \mathbb{Z}$$

with $q \neq 0$ and $(p_1', \ldots, p_\mu', q')$ irreducible (seeing $\mathbb{Z}^\mu \times \mathbb{Z}$ as a $\mathbb{Z}$-module). Let $\frac{p_j'}{q'} = \frac{k_j p_j''}{k_j q''}$ with p_j''/q'' an irreducible fraction. Consider in each $S_k^1 \times \mathbb{S}^1$, k_j disjoint copies of the closed curve of slope p_j''/q''. We denote the union of these k_j copies by $\tilde{H}_j$. We observe that $\tilde{H}_j$ intersects C in $k_j q'' = q'$ points for each j. We can, therefore, isotope the connected components of each $\tilde{H}_j$ so that $\bigcup_j \tilde{H}_j$ intersects C in just q' points. We do so and consider the set $\bigcup_j \tilde{H}_j$. The horizontal surface $\tilde{H}$ of $\tilde{B} \times \mathbb{S}^1$ associated to $k((p_1', \ldots, p_\mu', q')$ is k disjoint parallel copies of $\bigcup_j \tilde{H}_j$.

On the other direction, given an element $[H] \in \mathcal{H}(\tilde{B} \times \mathbb{S}^1)$. Let $[H]$ denote also the class of any horizontal surface in $H_1(\tilde{B} \times \mathbb{S}^1; \mathbb{Z})$. Then $q = [H] \cdot C$ and $p_i = [H] \cdot [S_i^1 \times \{0\}]$. That is, the corresponding element in $H^1(B; \mathbb{Z}) \oplus \mathbb{Z}$ is the Poincaré dual of the class of H in the homology of $\tilde{B} \times \mathbb{S}^1$. $\square$

Lemma 4.8. *$\tilde{H}$ is connected if and only if the element $(p_1, \ldots, p_\mu, q)$ is irreducible in $H^1(\hat{Y}; \mathbb{Z}) \simeq H^1(\tilde{B}; \mathbb{Z}) \oplus \mathbb{Z}$.*

Proof. We know that by construction $\tilde{H} \cap C$ are q points. It is enough to show that these q points lie in the same connected component since any other part of $\tilde{H}$ intersects some of these points. We label the points cyclically according to the orientation of C. So we have $c_1, \ldots, c_q \in C$. We recall that $S_j^1 \times \mathbb{S}^1 \cap \tilde{H}$ is formed by k_j parallel copies of the closed curve of slope p_j'/q' with $\frac{k_j p_j'}{k_j q'} = \frac{p_j}{q}$. Hence the point x_i is connected by these curves with the points $c_{i+tk_j \mod q}$. Since $(p_1, \ldots, p_\mu, q)$ is irreducible then $\gcd(p_1, \ldots, p_\mu, q) = 1$ and hence $\gcd(k_1, \ldots, k_\mu) = 1$. Therefore the equation

$$i + t_1 k_1 + \cdots + t_\mu k_\mu = j \mod q$$

admits an integer solution on the variables $t_1, \ldots, t_\mu$ for any two $i, j \in \{1, \ldots, q\}$. This proves that the points c_i and c_j are in the same connected component in $\tilde{H}$.

Conversely if the element is not irreducible, then $(p_1, \ldots, p_\mu, q) = k(p_1', \ldots, p_\mu', q')$ for $(p_1', \ldots, p_\mu', q')$ irreducible and $k > 1$. Then, by construction, $\tilde{H}$ is formed by k disjoint copies of the connected horizontal surface associated to $(p_1', \ldots, p_\mu', q')$ $\square$

Handy model of a Seifert fibering. We describe a particularly handy model of the Seifert fibering that we use in Section 5. The idea is taken from a construction in [Hat07]. For each $i = 1, \ldots, k$ let $x_i \in B$ be the image by $s : Y \to B$ of the special fiber F_i. We pick one boundary component of the base space and denote it by L. For each $i = 1, \ldots, k$ pick an arc l_i properly embedded in B and with the end points in L (i.e. with $l_i \cap L = \partial l_i$) in such a way that cutting

along l_i cuts off a disk D_i that contains x_i and no other point from $\{x_1, \ldots, x_k\}$. We pick a collection of such arcs $l_1, \ldots, l_k$ pairwise disjoint. We define

$$B' := B \setminus \bigsqcup_i int(D_i)$$

where $int(\cdot)$ denotes the interior. See Figure 4.1 below and observe that B and B' are diffeomorphic.
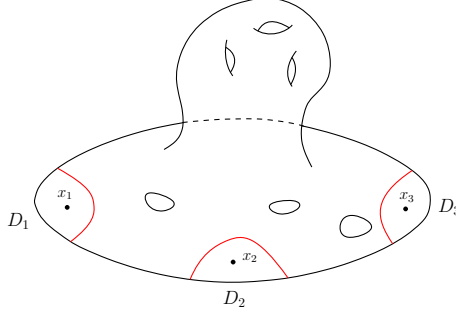

Figure 4.1. We see the base space B of a Seifert manifold. It has genus 3 and 4 boundary components. The 3 points are the image of the special fibers by the projection s and if we cut along the three red arcs, we get the surface B'.

Let $Y' := s^{-1}(B')$. Since Y' contains no special fibers and $\partial B' \neq \emptyset$ then Y' is diffeomorphic as a circle bundle to $B' \times \mathbb{S}^1$. Recall that $s^{-1}(D_i)$ is a solid torus of type (p_i, α_i) with $p_i \beta_i \equiv 1$ mod α_i (see Lemma 3.2).

Summarizing, the *handy model* consists of:

   i) A system of arcs $l_1, \ldots, l_k$ as explained.
   ii) A trivialization of Y', that is an identification of Y' with $B' \times \mathbb{S}^1$.
   iii) Identifications of $s^{-1}(D_i)$ with the corresponding model T_{p_i, α_i} for each $i = 1, \ldots, k$.

Remark 4.9. Let A_i be the vertical annulus $s^{-1}(l_i)$. A properly embedded horizontal disk $D \subset s^{-1}(D_i)$ intersects A_i in α_i disjoint arcs by definition of the number α_i. Since a horizontal surface H intersects each typical fiber the same number of times we get that H must meet each fiber $t \cdot \mathrm{lcm}(\alpha_1, \ldots, \alpha_k) = t \cdot n$ times for some $t \in \mathbb{Z}_{>0}$. If a horizontal surface meets $t \cdot n$ times a typical fiber, then it meets $t \cdot n/\alpha_i$ times the special fiber F_i.

Lemma 4.10. *There is a bijection between $\mathcal{H}(Y)$ and $HS(Y) := \{\gamma \in H^1(Y; \mathbb{Z}) : \gamma([C]) \neq 0\}$ where C is a generic fiber of Y.*

Proof. Clearly, an element $[H] \in \mathcal{H}(Y)$ can be seen as the dual of a 1-form γ with $\gamma_H(C) \neq 0$.

To see that there is a bijection, take a handy model for Y (we use notation described there). Then we observe that given a $\gamma \in HS(Y)$, it restricts to a 1-form in $H^1(Y'; \mathbb{Z})$. The manifold Y' is diffeomorphic to a product, so by Lemma 4.7, there is a horizontal surface in $\mathcal{H}(Y')$ representing the restriction of γ to Y'. It also restricts as a 1-form in $H^1(s^{-1}(B); \mathbb{Z})$ where we recall that $s^{-1}(B)$ is a disjoint union of tori, each containing a special fiber of Y. If $\gamma([C]) = n$ then, $\gamma([F_i]) = n/\alpha_i \in \mathbb{Z}$ so we can see the dual of $\gamma|_{s^{-1}(B)}$ as an union of n/α_i disks in each of the tori $s^{-1}(D_i)$ for all i. Each of these disks intersects α_i times the annulus $s^{-1}(l_i)$. So we can glue the horizontal surface represented by $\gamma|_{Y'}$ with these disks to produce a horizontal surface in all Y. By construction, this horizontal surface represents the given $\gamma \in H^1(Y; \mathbb{Z})$. $\square$

Lemma 4.11. *Let $\hat{H}$ be a horizontal surface in $\hat{Y}$, that is, $\hat{H} \in H(\hat{Y})$ and let $H := \pi^{-1}(\hat{H})$. Then H is connected if and only if $\hat{H}$ is connected.*

Proof. If H is connected, then so is $\hat{H}$ because π is a continuous map.

Suppose now that $\hat{H}$ is connected. If $\pi^{-1}(\hat{H})$ is not connected, then it is formed by parallel copies of diffeomorphic horizontal surfaces. Each of them is sent by π onto $\hat{H}$ and each of them represents the same element in $HS(Y)$. But, by Lemma 4.10 $HS(Y)$ is in bijection with $\mathcal{H}(Y)$ which, by Lemma 4.6, is in bijection with $\mathcal{H}(\hat{Y})$. So we get to a contradiction. $\square$

By construction, we have established the $1:1$ correspondences

$$(4.12) \qquad HS(Y) \longleftrightarrow \mathcal{H}(Y) \longleftrightarrow \mathcal{H}(\hat{Y}) \longleftrightarrow H^1(B;\mathbb{Z}) \oplus \mathbb{Z} \setminus H^1(B;\mathbb{Z}) \oplus \{0\}$$

Where the first correspondence is Lemma 4.10, the second is Lemma 4.6 and the last one is Lemma 4.7.

Actually if we fix an orientation on the manifold and the fibers and we restrict ourselves to oriented horizontal surfaces that intersect positively the fibers of Y, these are parametrized by elements in $H^1(B;\mathbb{Z}) \oplus \mathbb{Z}_{>0}$. From now on we restrict ourselves to oriented horizontal surfaces H with $H \cdot C > 0$, that is, those whose oriented intersection product with any typical fiber is positive. Also the fibers are assumed to be oriented. This orientation induces a monodromy on each horizontal surface.

Remark 4.13. Let Σ be a surface with boundary and $\phi : \Sigma \to \Sigma$ a periodic automorphism and let Σ_ϕ be the corresponding mapping torus which is a Seifert manifold. The manifold Σ_ϕ fibers over S^1 and we can see Σ as a horizontal surface of Σ_ϕ by considering any of the fibers of $f : \Sigma_\phi \to \mathbb{S}^1$. Now let Σ^ϕ be the orbit space of Σ which is also the base space of Σ_ϕ. Let m be the lcm of the multiplicities of the special fibers of the Seifert fibering and let Σ_ϕ/c_m be the quotient space resulting from the action of c_m on Σ_ϕ. We observe, as before, that Σ_ϕ/c_m is diffeomorphic to $\Sigma^\phi \times \mathbb{S}^1$ but there is not a preferred diffeomorphism between them. A trivialization is given by a choice of a section of $\Sigma_\phi/c_m \to \Sigma^\phi$.

Let $[S_1], \ldots, [S_\mu]$ be a basis of the homology group $H_1(\Sigma^\phi; \mathbb{Z})$ where each S_i is a simple closed curve in Σ^ϕ. Let C be any fiber of of Σ_ϕ/c_m. Let $w, \hat{w} : \Sigma^\phi \to \Sigma_\phi/c_m$ be two sections, then we have two different basis of the homology of $H_1(\Sigma_\phi/c_m; \mathbb{Z})$ induced by these two sections. For instance

$$\{[w(S_1)], \ldots, [w(S_\mu)], [C]\} \text{ and } \{[\hat{w}(S_1)], \ldots, [\hat{w}(S_\mu)], [C]\}.$$

Let $\Sigma = f^{-1}(0)$ be the horizontal surface that we are studying and let $\hat{\Sigma} := \pi(\Sigma)$ where π is the quotient map $\Sigma_\phi \to \Sigma_\phi/c_m$. Then $\hat{\Sigma}$ is represented with respect to the (duals of the) two basis by integers $(p_1, \ldots, p_\mu, q)$ and $(\hat{p}_1, \ldots, \hat{p}_\mu, q)$ respectively and $p_i \equiv \hat{p}_i \mod q$ for all $i = 1, \ldots, \mu$ because a section differs from another section in a integer sum of fibers at the level of homology.

So the numbers $p_1, \ldots, p_\mu$ are well defined modulo $\mathbb{Z}_q$ regardless of the trivialization chosen for Σ_ϕ/c_m. Also by the discussion above, we see that if we fix a basis of $H_1(B; \mathbb{Z})$, then all the elements of the form $(p_1 + n_1 q, \ldots, p_\mu + n_\mu q, q)$ represent diffeomorphic horizontal surfaces with the same monodromy. That there exists an diffeomorphism of Y preserving the fibers that sends H_1 to H_2 comes from the fact that on a torus $S^1 \times \mathbb{S}^1$, there exist a diffeomorphism preserving the vertical fibers $\{t\} \times \mathbb{S}^1$ that sends the curve of type (p, q) to the curve of type $(p + kq, q)$ for any $k \in \mathbb{Z}$: the k-th power of a left handed Dehn twist along some fiber $\{pt\} \times \mathbb{S}^1$ that is different from C.

5. Translation Algorithms

Every mapping torus arising from a tête-à-tête graph is a Seifert manifold so it admits a (star-shaped) plumbing graph. The monodromies induced on horizontal surfaces of Seifert manifolds are periodic.

In this section we describe an algorithm that, given a general tête-à-tête graph, produces a star-shaped plumbing graph together with the element in cohomology modulo $\mathbb{Z}_q$ corresponding to the horizontal surface given by the tête-à-tête graph. We also describe the algorithm that goes in the opposite direction.

5.1. **From general tête-à-tête graph to star-shaped plumbing graph.** We first state a known proposition that used in the algorithm. It can be found in several references in the literature. See for example [NR78] or [Ped09] .

Proposition 5.1. *Let $M(g, r; (\hat{\alpha}_1, \hat{\beta}_1), \ldots, (\hat{\alpha}_k, \hat{\beta}_k))$ be a Seifert fibering. Then it is diffeomorphic as a circle bundle to a Seifert fibering of the form*

$$M(g, r; (1, b), (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$$

where $0 \leq \beta_i < \alpha_i$ for all $i = 1, \ldots, k$. If the surface admits a horizontal surface and $r = 0$, then $b = -\sum_i \frac{\beta_i}{\alpha_i}$. If $r \neq 0$ but a framing is given in each of the r boundary components, then b is well-defined and satisfies the same equation. If $r \neq 0$ then the statement is still true but the number b is not well-defined: any b produces diffeomorphic Seifert manifolds with boundary.

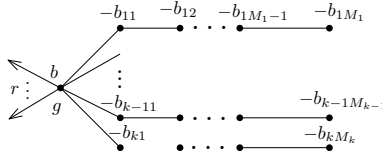The corresponding plumbing graph associated to the Seifert manifold is

Figure 5.1. Plumbing graph for a Seifert manifold

where the numbers b_{ij} are the continuous fraction expansion for α_i / β_i, that is

$$\frac{\alpha_i}{\beta_i} = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{b_{i3} - \frac{1}{\cdots}}}$$

Let $(\Gamma, \mathcal{P}, \sigma)$ be a general tête-à-tête structure. This includes as particular cases pure tête-à-tête graphs and relative tête-à-tête graphs. Let ϕ_Γ be a truly periodic representative of the tête-à-tête automorphism and let Σ_{ϕ_Γ} be the mapping torus of the the diffeomorphism $\phi_\Gamma : \Sigma \to \Sigma$. The mapping torus given by a periodic diffeomorphism of a surface is a Seifert manifold. We describe an algorithm that takes $(\Gamma, \mathcal{P}, \sigma)$ as input and returns as output:

(1) The invariants of a Seifert manifold:

$$M(g, r, (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$$

diffeomorphic to the mapping torus Σ_{ϕ_Γ}. It is represented by a star-shaped plumbing graph Λ corresponding to the Seifert manifold and

(2) a tuple $(p_1, \ldots, p_\mu, q)$ with $p_i \in \mathbb{Z}/q\mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ representing the horizontal surface given by Σ with respect to some basis of the homology $H_1(B; \mathbb{Z}) \simeq \mathbb{Z}^\mu$ of the base space B of M. In particular $\mu = 2g + r - 1$.

Step 1. We consider Γ^{ϕ_Γ}, that is the quotient space $\Gamma / \sim$ where $\sim$ is the equivalence relation induced by the action of the safe walks on the graph. This graph is nothing but the image of Γ by the projection of the branched cover $p : \Sigma \to \Sigma^{\phi_\Gamma}$ onto the orbit space.

The map $p|_\Gamma : \Gamma \to \Gamma^{\phi_\Gamma}$ induces a ribbon graph structure on Γ^{ϕ_Γ}. We can easily get the genus g and number of boundary components r of the thickening of Γ^{ϕ_Γ} from the combinatorics of the graph. This gives us the first two invariants of the Seifert manifold.

Step 2. Let $sv(\Gamma)$ be the set of points with non trivial isotropy subgroup in $< \phi_\Gamma >$. This is the set of branch points of $p : \Sigma \to \Sigma^{\phi_\Gamma}$ by definition.

Let $v \in sv(\Gamma)$. Then there exists $m < n$ with $n = m \cdot s$ such that v is a fixed point of ϕ_Γ^m (take m the smallest natural number satisfying that property). We can therefore use Corollary 3.3. We get that ϕ_Γ^m acts as rotation with rotation number p/s in a small disk centered at v. So around the fiber corresponding to the vertex v, the Seifert manifold is diffeomorphic fiberwise to a p, s-torus. and the corresponding Seifert pair (α_v, β_v) is given by $(\alpha_v, \beta_v) = (s, -b)$ with $bp \equiv 1 \mod q$.

We do this for every vertex in $sv(\Gamma)$ and we get all the Seifert pairs.

Step 3. Since we have already found a complete set of Seifert invariants, we have that

$$M(g, r; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$$

is diffeomorphic to the mapping torus of the pair (Σ, ϕ_Γ).

Step 4. For this step, recall Section 4 and notation introduced there. Fix a basis $[S_1^1], \ldots, [S_\mu^1]$ of $H_1(B; \mathbb{Z})$ where S_i^1 is a simple closed curve contained in Γ^{ϕ_Γ}.

Let $m := \mathrm{lcm}(\alpha_1, \ldots, \alpha_k)$. Observe that necessarily $m | n$ so $n = m \cdot q$ (with n the order of ϕ_Γ).

This number q that we have found is the last term of the cohomology element we are looking for. Of course, it is also the oriented intersection number of $\hat{\Gamma} = \pi(\Gamma)$ with C. (recall π was the projection $M \to M/c_m =: \hat{M}$).

Now if we intersect the graph $\Gamma/c_m \cap S_i^1 \times C$ with the torus over one of the representatives of the basis, we get a collection of k_i closed curves, each one isotopic to the curve of slope p_i'/q_i' where $q_i' \cdot k_i = q$ and $p_i = p_i' \cdot k_i = p_i$. This number, p_i, is the $i-th$ coordinate of the cohomology element with respect to the fixed basis.

We can compute $p_1, \ldots, p_k$ directly. Let S_i^1 be one of the generators of $H_1(\Gamma^{\phi_\Gamma}; \mathbb{Z})$. Let $\hat{S}_i^1 := \Gamma/c_m \cap S_i^1 \times \mathbb{S}^1$ where $S_i^1 \times \mathbb{S}^1 \subset \Gamma^{\phi_\Gamma} \times \mathbb{S}^1$; and let $\tilde{S}_i^1 := p|_\Gamma^{-1}(\hat{S}_i^1)$. Observe that $\hat{S}_i^1$ consists of k_i disjoint circles and that k_i divides q. Let $q_i' = q/k_i$.

Pick a point $z \in S_i^1$ which is not in the image by π of a special fiber. Then $\hat{\pi}^{-1}(z) \cap \Gamma/c_m$ consists of q points lying in the k_i connected components of $\hat{S}_i^1$. Pick one of these connected components and enumerate the corresponding q_i' points in it using the orientation induced on that connected component by the given orientation of S_i^1. Then we have the points $z_1, \ldots, z_{q_i'}$. We observe that by construction, these points lie on the same fiber in $\hat{M}$ and this fiber is oriented. Follow the fiber from z_1 in the direction indicated by the orientation, the next point is z_{t_i}, with $t_i \in \{1, \ldots, q_i'\}$. We therefore find that this connected component of $\hat{S}_i^1$ lies in $S_i^1 \times \mathbb{S}^1$ as the curve with slope $(t_i - 1)/q_i'$ and so $p_i = (t_i - 1) \cdot k_i$.

5.2. **From star-shaped plumbing graph to tête-à-tête graphs.** The input that we have is:

  i) A Seifert fibering of a manifold M.
  ii) A horizontal surface given by an element in $H^1(B \times \mathbb{S}^1; \mathbb{Z})$ that does not vanish on a typical Seifert fiber.

The output is:

  (1) A general, relative or pure tête-à-tête graph such the induced mapping torus is diffeomorphic to the given plumbing manifold in the input. And such that the thickening of the graph, represents the horizontal surface given.

Step 1. We start with a Seifert fibering $M(g, r; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$.

We fix a model of the Seifert fibering as in Handy model of a Seifert fibering. We recall that the model consists of the following data:

  i) The Seifert fibering $s : M \to B$ where B is a surface of genus g and r boundary components.
  ii) A collection of arcs $\{l_i\}$ with $i = 1, \ldots, k$ properly embedded in B where the boundary of these arcs lie in one chosen boundary component of B. These satisfy that when we cut along one of them, say l_i we cut off a disk denoted by D_i from B that contains the image of exactly one special fiber, we denote the image of this fiber by x_i.
  iii) We have an identification of each solid torus $s^{-1}(D_i)$ with the corresponding fibered solid torus T_{p_i, q_i} with $q_i = \alpha_i$ and $-p_i \beta_i \equiv 1 \mod \alpha_i$ and $0 < p_i < q_i$.

Step 2. Observe that B is homotopic to a wedge of $\mu = 2g - r + 1$ circles that does not contain any x_i for $i = 1, \ldots, i$. We can see this wedge as a spine embedded in B. Denote by c the common point of all the circles. Now we embed disjoint segments e_i with $i = 1, \ldots, k$ where

each one satisfies that one of its ends lies in the spine and the other end lies in x_i. Also, they do not intersect the wedge of circles at any other point and they also e_i does not intersect any D_j for $j \neq i$. We denote the union of the wedge and these segments by $\tilde{\Lambda}$ and observe that $\tilde{\Lambda}$ is a spine of B.

Step 3. We suppose that the element in $H^1(\hat{M}; \mathbb{Z})$ given is irreducible, otherwise if it is of the form $k(p_1', \ldots, p_\mu', q')$ with $(p_1', \ldots, p_\mu', q')$ irreducible, we take the irreducible part, carry out the following construction of the corresponding horizontal surface and then take k parallel copies of this surface.

Recall Equation (4.3) for the definition of the maps $s, \hat{s}$ and π.

Once and for all, fix a trivialization $\hat{M} \simeq B \times \mathbb{S}^1$. We assume that the element $(p_1, \ldots, p_\mu, q) \in H^1(\hat{M}; \mathbb{Z})$ is expressed with respect to the dual basis $[S_1], \ldots, [S_\mu], [C]$ where the first μ are circles of the wedge embedded in B and $[C]$ is the homology class of $C := \hat{s}^{-1}(c)$.

For each $i = 1, \ldots, \mu$, consider the torus $\hat{s}^{-1}(S_i)$ which is naturally trivialized by the trivialization of $\hat{M}$. We pick in it k_i copies of the curve of slope p_i'/q' where $p_i/q = k_i p_i'/k_i q'$. For each i, the curves constructed this way in $\hat{s}^{-1}(S_i)$ intersect q times the curve C. Hence we can isotope them so that all of them intersect C in the same q points. We denote the union of these curves by $\hat{\Lambda}'$. We assume that $\hat{\Lambda}'$ projects to $\tilde{\Lambda} \setminus \bigcup e_i$ by $B \times \mathbb{S}^1 \to B$.

By construction, $\hat{\Lambda}'$ is a ribbon graph for the surface horizontal surface $\hat{H} \subset \hat{M}$. Observe that $s(\hat{\Lambda}') \neq \tilde{\Lambda}$. However $s(\hat{\Lambda}')$ is also a spine of B (it coincides with the wedge of circles in B).

Define $\Lambda' := \pi^{-1}(\hat{\Lambda}')$. By the definition of π, this graph can also be constructed by taking in each of the tori $\pi^{-1}(\hat{s}^{-1}(S_i)) = s^{-1}(S_i)$, k_i copies of the curve of slope p_i'/n. Which by construction all intersect in n points in $s^{-1}(c)$.

Step 4. Now we describe $\pi^{-1}(\hat{s}|_{\hat{H}}^{-1}(e_i))$ for each $i = 1, \ldots, k$. First we observe that it is equal to $s|_H^{-1}(e_i)$ which is a collection of $q \cdot n/\alpha_i$ disjoint start shaped graphs. Each star-shaped piece has α_i. To find out the gluings of these arms with Λ' one looks as the structure of $s^{-1}(D_i)$ as a (c, α_i)- solid torus. To visualize it, place the $q \cdot n/\alpha_i$ star-shaped pieces in a solid cylinder $D^2 \times [0,1]$ and identify top with bottom by a $c/\alpha_i)$ rotation. The fibers of the fibered torus give the monodromy on the end of the arms and the attaching to Λ'.

We define Λ as the union of Λ' with these star-shaped pieces

Step 5. The embedding of H in the Seifert manifold defines a diffeomorphism $\phi : H \to H$ in the following way. Let $x \in H$ and follow the only fiber of the Seifert manifold that passes through x in the direction indicated by its orientation, we define $\phi(x)$ as the next point of intersection of that fiber with H.

To describe ϕ up to isotopy it is enough to give the rotation numbers of ϕ around each boundary component of H plus some spine invariant by ϕ. By construction Λ is an invariant graph. The fibers of the Seifert fibering give us an automorphism on the graph Λ. To get the rotation numbers, we cut the thickening H along Λ and we get a collection of cylinders $\Lambda_j \times [0,1]$ with $j = 1, \ldots, r'$.

Now we invoke [FdBPPPC17, Theorem 5.12] if the monodromy leaves at least 1 boundary component invariant and we invoke Theorem 2.6 if the monodromy does not leave any boundary component invariant. This gives us a constructive method to find a graph (which in general will be different from $\bigcup_{i=1}^\mu S_i \bigcup_{j=1}^k e_j$) containing all branch points in B such that it is a retract of B and such that it admits a metric that makes its preimage a tête-à-tête graph.

6. EXAMPLES

We apply the algorithms developed in the previous sections to two examples.

Example 6.1. Suppose we are given the bipartite complete graph Γ of type $(4, 11)$ with the cyclic ordering induced by placing 4 and 11 vertices in two horizontal parallel lines in the plane and taking the joint of the two sets in that plane. Give each edge length $\pi/2$. This metric

makes it into a tête-à-tête graph as we already know. Let ϕ_Γ be a periodic representative of the mapping class induced by the tête-à-tête structure.

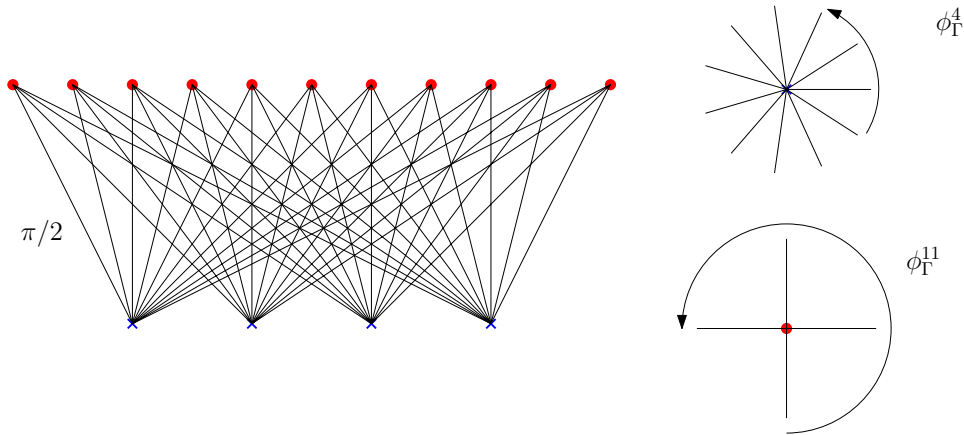**Figure 6.1.** On the left we see the tête-à-tête graph $K_{4,11}$. On the right we see a small neighborhood of a vertex of valency 11 where ϕ_Γ^4 acts as the rotation $r_{4/11}$ radians. Equivalently, for a vertex of valency 4, we see that ϕ_Γ^{11} acts as the rotation $r_{3/4}$.

Let's find the associated invariants. One can easily check that the orbit graph is just a segment joining the only two branch points so the orbit surface is a disk and hence $g = 0$ and $r = 1$.

The map $p : \Sigma \to \Sigma^{\phi_\Gamma}$ has two branch points that correspond to two Seifert pairs. Let r_1 be the branching point in which preimage lie the 4 points of valency 11. We choose any of those 4 points and denote it p_1, now ϕ^4 acts as a rotation with rotation number $4/11$ in a small disk around p_1 . Hence, the associated normalized Seifert pair is $(11, 8)$. Note that $8 \cdot 4 \equiv -1$ mod 11 and that $0 < 8 < 11$. Equivalently for the other point we find that ϕ_Γ^{11} is a rotation with rotation number $3/4$ when restricted to a disk around any of the 11 vertices of valency 4. Hence, the corresponding normalized Seifert pair is $(4, 1)$.

Computing the continued fraction we have that $\frac{11}{8} = [2, 2, 3, 2]$ and $\frac{4}{1} = [4]$. For computing the number b we think of the surface resulting from extending the periodic automorphism to a disk capping off the only boundary component of Σ. By a similar argument, since the rotation number induced on the boundary is $-1/44$, this would lead to a new Seifert pair $(44, 1)$.

So the plumbing diagram corresponding to the mapping torus of Σ by ϕ_Γ is the following
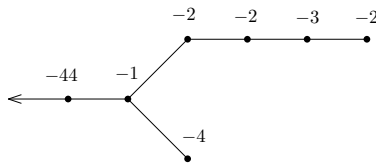
Note that contracting the bamboo that ends in the arrowhead does not change the diffeomorphism type of the manifold. The weight -1 on the central node only makes sense if we consider the manifold resulting from removing the arrow (which corresponds with the mapping torus of the periodic extension of ϕ_Γ a disk capping off the only boundary component).

Finally, we are going to compute the element that the surface Σ represents in the homology group $H_1(\Sigma^{\phi_\Gamma}) \oplus \mathbb{Z}$. First observe that since Σ^{ϕ_Γ} is a disk, the group is isomorphic to $0 \oplus \mathbb{Z}$. This tells us that the only possible choices of multisections in the bundle $\Sigma^{\phi_\Gamma} \times \mathbb{S}^1$ are classified (up to isotopy) by the elements $(0, k)$ with $k \neq 0$. The element $(0, k)$ corresponds to k parallel copies of the disk Σ^{ϕ_Γ}. In our case, there is only one such disk so the element is $(0, 1)$.

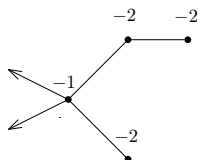**Example 6.2.** Suppose we are given the following plumbing graph:

Figure 6.2

We are indicated two of the invariants of the Seifert manifold: the genus of the base space $g = 0$ and its number of boundary components $r = 2$. The base space B is therefore an annulus.

We compute the Seifert invariants by interpreting the weights on the two bamboos of the plumbing graph as numbers describing continued fractions. We get $[2,2] = 3/2$ and $[2] = 2$ so the Seifert pairs are $(3,2)$ and $(2,1)$. So the corresponding Seifert fibering $s : M \to B$ has two special fibers F_1 (for the pair $(2,1)$ and F_2 (for the pair $(3,2)$). Using Lemma 3.2 we have that the Seifert fiber corresponding to the pair $(3,2)$ has a tubular neighborhood diffeomorphic to the fibered solid torus $T_{1,3}$; this is because $-2 \cdot 1 \equiv 1 \mod 3$. Analogously, the fiber corresponding to the Seifert pair $(2,1)$ has a tubular neighborhood diffeomorphic to the fibered solid torus $T_{1,2}$.

Now we fix a model for our Seifert manifold. Take an annulus as in Figure 6.3. Now we use the kind of model explained in Figure 4.1 (the handy model). We choose a boundary component and we pick properly embedded arcs (with their boundaries lying on the chosen boundary component) in such a way that cutting along one of them cuts off a disk containing only one of the two images by s of the special fibers; over those disks in M lie the two corresponding fibered solid tori. Let d be the point lying under $\pi(F_1)$ and let a be the point lying under the fiber $\pi(F_2)$. We pick an embedded graph which is a spine of B as in Figure 6.3 below, that is, the graph is the union of: a circle whose class generates the homology of the base space. We denote it by S; a segment joining a point $c \in S$ with the vertex d. We denote this segment by D and a segment joining a point $b \in S$ with the vertex a. We denote this segment by A. See Figure 6.3.
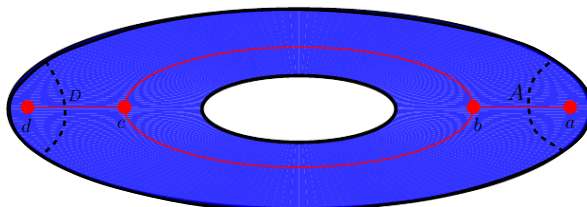
We denote this graph by $\tilde{\Lambda}$.

Figure 6.3. This is the base space B of the Seifert fibering. In red we see $\tilde{\Gamma}$ which is formed by a circle and two segments attached to it that end at the image by s of the special fibers. The dashed lines represents the properly embedded arcs

Now we consider $\hat{M}$ which is diffeomorphic to $B \times \mathbb{S}^1$ which is homotopically equivalent to $\tilde{\Gamma} \times \mathbb{S}^1$. We denote the projection on B by $\hat{s} : \hat{M} \to B$. The map $\pi : M \to \hat{M}$ satisfies that $\hat{s} \circ \pi = s$.

The piece of information missing from the input is the horizontal surface. Suppose we are given the element $(1,2) \in H^1(B;\mathbb{Z}) \oplus \mathbb{Z}$ with respect to the basis formed by the class of S. Then, the intersection of the horizontal surface $\hat{H} \subset \hat{M}$ with the torus $\hat{S} := \hat{s}^{-1}(S)$ is a curve of slope $1/2$. We also have that $\hat{s}^{-1}(A)$ consists of two segments, as well as $\hat{s}^{-1}(D)$. See figure Figure 6.4.

The horizontal surface that we are looking for is $H := \pi^{-1}(\hat{H})$ that is the thickening of $\pi^{-1}(\hat{\Gamma})$. To know the topology of H and the action on it of the monodromy, we construct the ribbon graph $\pi^{-1}(\hat{\Gamma})$. We observe that $\text{lcm}(2,3) = 6$ so $\pi^{-1}(\hat{S})$ is the curve of slope $1/12$ on the torus $s^{-1}(S)$. We also have that $s^{-1}(a) = (\pi \circ \hat{s})^{-1}(a)$ consists of 4 and $s^{-1}(A)$ consists of 12 segments separated in groups giving valency 3 to each of the points in points. Equivalently
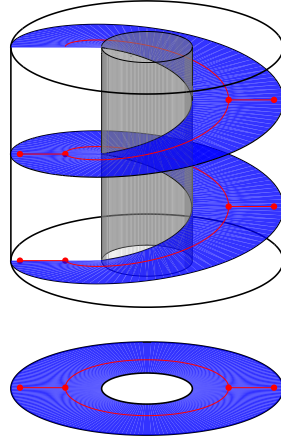
Figure 6.4. This is $\hat{M}$ together with the base space under it. Lying over the graph $\tilde{\Gamma}$ we can see the graph $\hat{G}$ whose thickening is the horizontal surface $\hat{H}$ (the blue helicoidal ramp on the figure). Also we see that lying over the circle of $\tilde{\Gamma}$ lies the closed curve in $\hat{\Gamma}$ that is a curve of slope $1/2$ in the torus $\hat{s}^{-1}(S_1)$.

$s^{-1}(d)$ consists of 6 vertices and $s^{-1}(D)$ of 12 segments naturally separated by pairs. The fact that $s^{-1}(S)$ is a curve of slope $1/12$, give us the combinatorics of the graph. Using notation of 6.5, and the rotation numbers associated to each of the two Seifert pairs, we have that the graph is that of Figure 6.5.
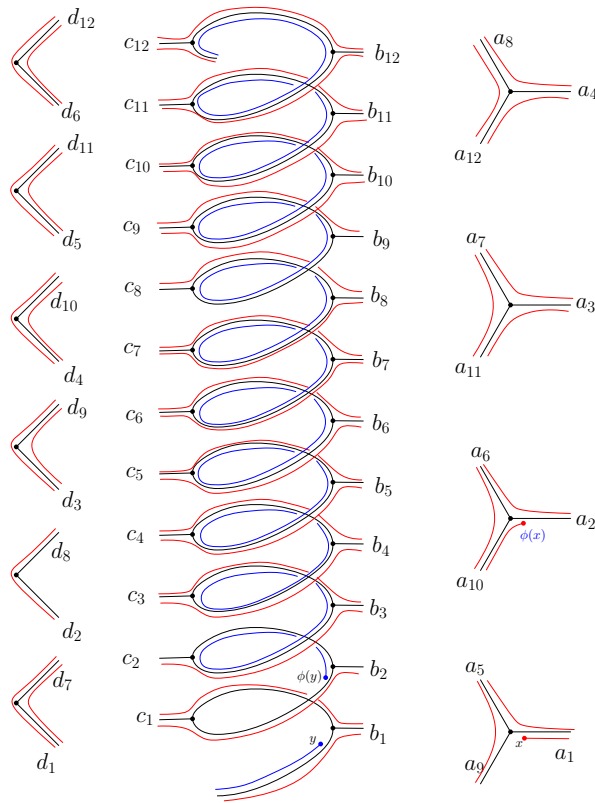

Figure 6.5. The graph $\pi^{-1}(\hat{\Lambda})$ in black. The letters with subindices are interpreted like this: the vertex a_i is glued to the vertex b_i and the vertex d_i is glued to the vertex d_i. In red we see a path from x to $\phi(x)$ used to compute the rotation number of ϕ with respect to the outer boundary component; we observe that the outer boundary component retracts to 72 edges (each edge is counted twice if the boundary component retracts to both sides of the edge), and the red path covers 66 of these edges.

You can easily compute from the ribbon graph that the surface has 2 boundary components and genus 7. Since it has only two boundary components, each of them is invariant by the action of the monodromy induced by the orientation on the fibers. We compute their rotation numbers as explained on Step 5 of the algorithms. We observe that the "outer" boundary component retracts to 72 edges where an edge is counted twice if the boundary component retracts to both sides of it. We pick a point x and observe that the monodromy indicated by the orientation of the fibers takes it to the point immediately above it $\phi(x)$. Now we consider a path "turning right" starting at x and observe that it goes along 66 edges before reaching $\phi(x)$. Hence, the rotation number of ϕ with respect to this boundary component is $11/12$. Similarly, we observe that the other boundary component retracts to 24 edges and by a similar procedure we can check that ϕ also has a rotation number $11/12$ with respect to this other boundary component. See figure Figure 6.5.

Following the construction in [FdBPPPC17, Theorem 5.12], we should put a metric on $\hat{G}$ so that the part where the outer boundary component retracts has a length of $\pi/11$ and the same for the other boundary component. But this is impossible given the combinatorics of the graph. That means that this graph does not accept a tête-à-tête metric. However theorem [FdBPPPC17, Theorem 5.12] gives us a procedure to find a graph admitting a tête-à-tête metric producing the given monodromy. In this case, it is enough to consider the following graph.
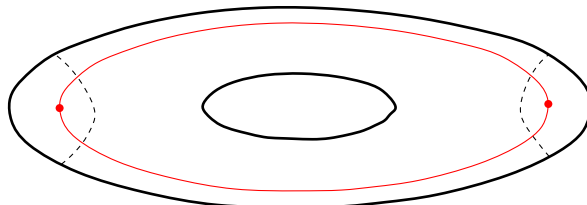

Figure 6.6. Graph $\tilde{\Gamma}$ that admits a tête-à-tête metric.

If we call this graph $\tilde{\Gamma}$ we see that that $\Gamma := p^{-1}(\tilde{\Gamma})$ is the following graph:
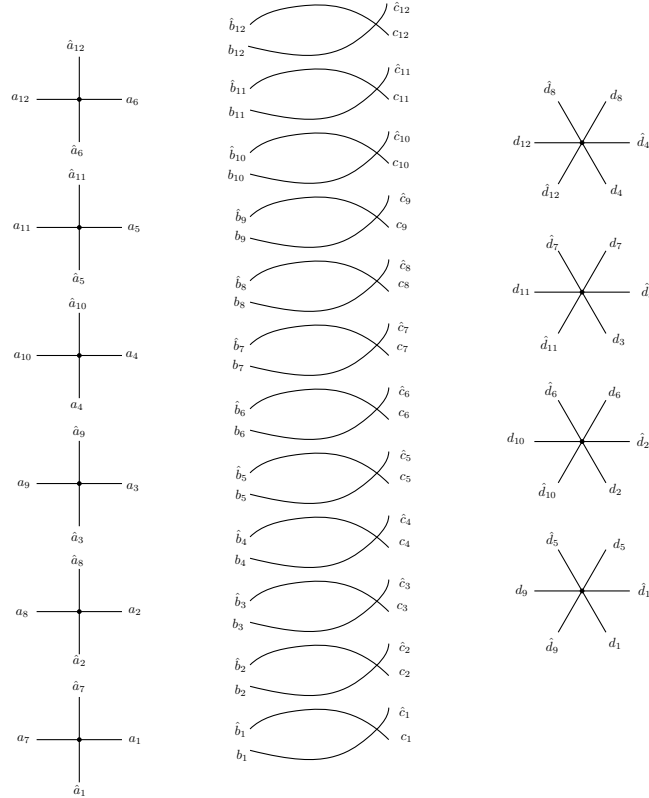
Figure 6.7. The tête-à-tête graph Γ. The notation means that a_i is glued to b_i, $\hat{a}_i$ to $\hat{b}_i$, c_i to d_i and $\hat{c}_i$ to $\hat{d}_i$ for all $i = 1, \ldots, 12$.

By setting each of the two edges of the circle $\tilde{\Gamma}$ has length $\pi/22$, then Γ is a pure tête-à-tête graph modeling the action of the monodromy on the horizontal surface H.

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