

# A PROOF OF THE DIFFERENTIABLE INVARIANCE OF THE MULTIPLICITY USING SPHERICAL BLOWING-UP

Dedicated to Professor Felipe Cano on the occasion of his 60th birthday

J. EDSON SAMPAIO

ABSTRACT. In this paper we use some properties of spherical blowing-up to give an alternative and more geometric proof of Gau-Lipman Theorem about the differentiable invariance of the multiplicity of complex analytic sets. Moreover, we also provide a generalization of the Ephraim-Trotman Theorem.

## 1. INTRODUCTION

Let  $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be the germ of a reduced analytic function at the origin with  $f \not\equiv 0$ . Let  $(V(f), 0)$  be the germ of the zero set of  $f$  at the origin. **The multiplicity of  $V(f)$  at the origin**, denoted by  $m(V(f), 0)$ , is defined as follows: we write

$$f = f_m + f_{m+1} + \cdots + f_k + \cdots$$

where each  $f_k$  is a homogeneous polynomial of degree  $k$  and  $f_m \neq 0$ . Then,  $m(V(f), 0) := m$ .

In 1971, Zariski in [17] asked if the multiplicity of complex analytic hypersurface was an invariant of the embedded topology, more precisely, he asked the following

**Question A:** Let  $f, g: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be germs of reduced holomorphic functions at the origin. If there is a homeomorphism  $\varphi: (\mathbb{C}^n, V(f), 0) \rightarrow (\mathbb{C}^n, V(g), 0)$ , is it true that  $m(V(f), 0) = m(V(g), 0)$ ?

This problem is known as Zariski's problem and as Zariski's Multiplicity Conjecture in its stated version. It is still opened, but there are some partial answers. For example, in 1932, Zariski in [18] already had proved that his problem had a positive answer when  $n = 2$ . For any  $n$ , Ephraim in [5] and independently Trotman in [14] showed that the Zariski's problem has a positive answer if the homeomorphism  $\varphi$  and its inverse are  $C^1$ .

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Since the notion of multiplicity is defined for any complex analytic set with pure dimension (see, for example, [3] for a definition of multiplicity in higher codimension), we can get the same Zariski's problem in any codimension. However, it is easy to produce examples of complex analytic sets  $X, Y \subset \mathbb{C}^n$  with codimension greater than 1, being embedded homeomorphic and having different multiplicities. For instance, there is a homeomorphism  $\varphi : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$  sending the cusp  $X = \{(x, y, z) \in \mathbb{C}^3; y^2 = x^3 \text{ and } z = 0\}$  over the complex line  $Y = \{(x, y, z) \in \mathbb{C}^3; y = x = 0\}$ , so that in this case,  $m(X, 0) = 2$  and  $m(Y, 0) = 1$ . Therefore Zariski's problem in codimension larger than 1 has a negative answer. However, in 1983, Gau and Lipman in [9], showed that if  $X, Y \subset \mathbb{C}^n$  are complex analytic sets and there exists a homeomorphism  $\varphi : (\mathbb{C}^n, V(f), 0) \rightarrow (\mathbb{C}^n, V(g), 0)$  such that  $\varphi$  and  $\varphi^{-1}$  are differentiable at the origin, then  $m(X, 0) = m(Y, 0)$ . This result will be called here Gau-Lipman Theorem. In particular, Gau-Lipman Theorem generalizes the quoted above result proved by Ephraim and Trotman, called here Ephraim-Trotman Theorem. In order to know more about Zariski's problem see, for example, [7].

The aim of this paper is to give a short and geometric proof of Gau-Lipman Theorem. To this end, we present some definitions and results in Section 2, and in Section 3 we prove some more results and we present a proof of Gau-Lipman Theorem. Finally, in Section 4, we present a generalization of Ephraim-Trotman Theorem.

## 2. PRELIMINARIES

This Section is closely related with the paper [13].

**Definition 2.1.** *Let  $A \subset \mathbb{R}^n$  be a subanalytic set such that  $x_0 \in \bar{A}$  is a non-isolated point. A vector  $v \in \mathbb{R}^n$  is tangent to  $A$  at  $x_0$  if there is a sequence of points  $\{x_i\} \subset A \setminus \{x_0\}$  tending to  $x_0 \in \mathbb{R}^n$  and there is a sequence of positive numbers  $\{t_i\} \subset \mathbb{R}^+$  such that*

$$\lim_{i \rightarrow \infty} \frac{1}{t_i} (x_i - x_0) = v.$$

*Let  $C(A, x_0)$  denote the set of all tangent vectors of  $A$  at  $x_0 \in \mathbb{R}^n$ . We call  $C(A, x_0)$  the **tangent cone of  $A$  at  $x_0$** .*

Notice that  $C(A, x_0)$  is the cone  $C_3(A, x_0)$  as defined by Whitney (see [16]).

**Remark 2.2.** *If  $A$  is a complex analytic set of  $\mathbb{C}^n$  such that  $x_0 \in A$  then  $C(A, x_0)$  is the zero locus of finitely many homogeneous polynomials (See [16], Chapter 7, Theorem 4D). In particular,  $C(A, x_0)$  is a union of complex lines passing through 0.*

Another way to present the tangent cone of a subset  $X \subset \mathbb{R}^n$  at the origin  $0 \in \mathbb{R}^n$  is via the spherical blow-up of  $\mathbb{R}^n$  at the point 0. Let us consider the **spherical blowing-up** (at the origin) of  $\mathbb{R}^n$

$$\begin{aligned} \rho_n : \mathbb{S}^{n-1} \times [0, +\infty) &\longrightarrow \mathbb{R}^n \\ (x, r) &\longmapsto rx \end{aligned}$$

Note that  $\rho_n : \mathbb{S}^{n-1} \times (0, +\infty) \rightarrow \mathbb{R}^n \setminus \{0\}$  is a homeomorphism with inverse mapping  $\rho_n^{-1} : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1} \times (0, +\infty)$  given by  $\rho_n^{-1}(x) = (\frac{x}{\|x\|}, \|x\|)$ . The **strict transform** of the subset  $X$  under the spherical blowing-up  $\rho_n$  is  $X' := \overline{\rho_n^{-1}(X \setminus \{0\})}$ . The subset  $X' \cap (\mathbb{S}^{n-1} \times \{0\})$  is called the **boundary** of  $X'$  and it is denoted by  $\partial X'$ .

**Remark 2.3.** If  $X$  is a subanalytic set of  $\mathbb{R}^n$ , then  $\partial X' = \mathbb{S}_0 X \times \{0\}$ , where  $\mathbb{S}_0 X = C(X, 0) \cap \mathbb{S}^{n-1}$ .

**Definition 2.4.** Let  $(X, 0)$  and  $(Y, 0)$  be subsets germs, respectively at the origin of  $\mathbb{R}^n$  and  $\mathbb{R}^p$ .

- A continuous mapping  $\varphi : (X, 0) \rightarrow (Y, 0)$ , with  $0 \notin \varphi(X \setminus \{0\})$ , is a **blow-spherical morphism** (shortened as **blow-morphism**), if the mapping

$$\rho_p^{-1} \circ \varphi \circ \rho_n : X' \setminus \partial X' \rightarrow Y' \setminus \partial Y'$$

extends as a continuous mapping  $\varphi' : X' \rightarrow Y'$ .

- A **blow-spherical homeomorphism** (shortened as **blow-isomorphism**) is a blow-morphism  $\varphi : (X, 0) \rightarrow (Y, 0)$  such that the extension  $\varphi'$  is a homeomorphism. In this case, we say that the germs  $(X, 0)$  and  $(Y, 0)$  are **blow-spherical equivalent** or **blow-spherical homeomorphic** (or **blow-isomorphic**).

The authors Birbrair, Fernandes and Grandjean in [2] defined blow-spherical morphisms and homeomorphisms with the additional hypotheses that they are required to also be subanalytic. Here, we work with the same definition already presented in [13].

**Remark 2.5.** Let  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^p$  be two subanalytic subsets. If  $\varphi : (X, 0) \rightarrow (Y, 0)$  is a blow-spherical homeomorphism, then we have a homeomorphism  $\nu_\varphi : \mathbb{S}_0 X \rightarrow \mathbb{S}_0 Y$  such that  $\varphi'(x, 0) = (\nu_\varphi(x), 0)$  for all  $(x, 0) \in \partial X'$ . Moreover, the mapping  $\nu_\varphi$  induces a homeomorphism  $d_0 \varphi : C(X, 0) \rightarrow C(Y, 0)$  given by

$$d_0 \varphi(x) = \begin{cases} \|x\| \nu_\varphi\left(\frac{x}{\|x\|}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Let  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^m$  be two subsets. Let us recall the following definition: a mapping  $f : X \rightarrow Y$  is a  $C^k$  mapping, if for each  $x \in X$ , there exist an open  $U \subset \mathbb{R}^n$  and a mapping  $F : U \rightarrow \mathbb{R}^m$  such that  $x \in U$ ,  $F|_{X \cap U} = f|_{X \cap U}$  and  $F$  is a  $C^k$  mapping.

The next result was proved in [13], but we sketch its proof here.

**Proposition 2.6.** *Let  $X, Y \subset \mathbb{R}^m$  be two subanalytic subsets. If  $\varphi : (\mathbb{R}^m, X, 0) \rightarrow (\mathbb{R}^m, Y, 0)$  is a homeomorphism such that  $\varphi$  and  $\varphi^{-1}$  are differentiable at the origin, then  $\varphi : (X, 0) \rightarrow (Y, 0)$  is a blow-spherical homeomorphism.*

*Proof.* Observe that  $\nu : \mathbb{S}_0 X \rightarrow \mathbb{S}_0 Y$  given by

$$\nu(x) = \frac{D\varphi_0(x)}{\|D\varphi_0(x)\|}$$

is a homeomorphism with inverse

$$\nu^{-1}(x) = \frac{D\varphi_0^{-1}(x)}{\|D\varphi_0^{-1}(x)\|}.$$

Using that  $\varphi(tx) = tD\varphi_0(x) + o(t)$ , we obtain

$$\lim_{t \rightarrow 0^+} \frac{\varphi(tx)}{\|\varphi(tx)\|} = \frac{D\varphi_0(x)}{\|D\varphi_0(x)\|} = \nu(x)$$

Then the mapping  $\varphi' : X' \rightarrow Y'$  given by

$$\varphi'(x, t) = \begin{cases} \left( \frac{\varphi(tx)}{\|\varphi(tx)\|}, \|\varphi(tx)\| \right), & t \neq 0 \\ (\nu(x), 0), & t = 0, \end{cases}$$

is a homeomorphism. Therefore,  $\varphi$  is a blow-spherical homeomorphism.  $\square$

**Definition 2.7.** *Let  $X \subset \mathbb{R}^n$  be a subanalytic set such that  $0 \in X$ . We say that  $x \in \partial X'$  is a **simple point of  $\partial X'$** , if there is an open  $U \subset \mathbb{R}^{n+1}$  with  $x \in U$  such that:*

- a) *the germs at  $x$  of the connected components of  $(X' \cap U) \setminus \partial X'$ , say  $X_1, \dots, X_r$ , are topological manifolds with  $\dim X_i = \dim X$ , for all  $i = 1, \dots, r$ ;*
- b)  *$(X_i \cup \partial X') \cap U$  is a topological manifold with boundary, for all  $i = 1, \dots, r$ ;*

*Let  $Smp(\partial X')$  be the set of all simple points of  $\partial X'$ .*

**Remark 2.8.** By Theorem 2.2 proved in [12], we get that  $Smp(\partial X')$  is dense in  $\partial X'$  if  $\dim \partial X' = \dim X - 1$  and  $X$  has pure dimension (see also [1]).

**Definition 2.9.** *Let  $X \subset \mathbb{R}^n$  be a subanalytic set such that  $0 \in X$ . We define  $k_X : Smp(\partial X') \rightarrow \mathbb{N}$ , with  $k_X(x)$  is the number of components of  $\rho^{-1}(X \setminus \{0\}) \cap U$ , for  $U$  an open sufficiently small containing  $x$ .*

**Remark 2.10.** *It is clear that the function  $k_X$  is locally constant. In fact,  $k_X$  is constant in each connected component  $C_j$  of  $\text{Smp}(\partial X')$ . Then, we define  $k_X(C_j) := k_X(x)$  with  $x \in C_j \cap \text{Smp}(\partial X')$ .*

**Remark 2.11.** *When  $X$  is a complex analytic set, there is a complex analytic set  $\Sigma$  with  $\dim \Sigma < \dim X$ , such that  $X_j \setminus \Sigma$  intersect only one connected component  $C_i$  (see [3], pp. 132-133), for each irreducible component  $X_j$  of tangent cone  $C(X, 0)$ , then we define  $k_X(X_j) := k_X(C_i)$ .*

**Remark 2.12.** *The number  $k_X(C_j)$  is the integer number  $n_j$  defined by Kurdyka and Raby in [10, pp. 762], and is also equal to the integer number  $k_j$  defined by Chirka in [3, pp. 132-133], when  $X$  is a complex analytic set.*

**Remark 2.13** ([3, p. 133, Proposition]). Let  $X$  be a complex analytic set of  $\mathbb{C}^n$  and let  $X_1, \dots, X_r$  be the irreducible components of  $C(X, 0)$ . Then

$$(1) \quad m(X, 0) = \sum_{j=1}^r k_X(X_j) \cdot m(X_j, 0).$$

Since the multiplicity is equal to the density (see [4, Theorem 7.3]), Equation (1) was also proved by Kurdyka and Raby in [10].

**Theorem 2.14.** *Let  $X$  and  $Y \subset \mathbb{C}^m$  be complex analytic subsets of  $\mathbb{C}^n$  of pure dimension  $p = \dim X = \dim Y$ , and let  $X_1, \dots, X_r$  and  $Y_1, \dots, Y_s$  be the irreducible components of the tangent cones  $C(X, 0)$  and  $C(Y, 0)$ , respectively. If there is a blow-spherical homeomorphism  $\varphi : (X, 0) \rightarrow (Y, 0)$  such that  $d_0\varphi(X_j) = Y_j$ , for  $j = 1, \dots, r$ , then  $k_X(X_j) = k_Y(Y_j)$ , for  $j = 1, \dots, r$ .*

*Proof.* Fix  $j \in \{1, \dots, r\}$ , let  $p \in \mathbb{S}_0 X_j \times \{0\}$  generic and  $U \subset X'$  a small neighborhood of  $p$ . As  $\varphi' : X' \rightarrow Y'$  is a homeomorphism, the image  $V := \varphi'(U)$  is a small neighborhood of  $\varphi'(p) \in \mathbb{S}_0 Y_j \times \{0\}$ . Moreover,  $\varphi'(U \setminus \partial X') = V \setminus \partial Y'$ , since (by definition)  $\varphi'|_{\partial X'} : \partial X' \rightarrow \partial Y'$  is a homeomorphism. Using once more that  $\varphi'$  is a homeomorphism, we obtain that the number of connected components of  $U \setminus \partial X'$  is equal to  $V \setminus \partial Y'$ , showing that  $k_X(X_j) = k_Y(Y_j)$ .  $\square$

**Proposition 2.15.** *Let  $\varphi : A \rightarrow B$  be a  $C^1$  homeomorphism between two complex analytic sets with pure dimension. If  $X$  is an irreducible component of  $A$ , then  $\varphi(X)$  is an irreducible component of  $B$ .*

To prove this result, we recall a well known result by Milnor [11].

**Proposition 2.16** ([11], page 13). *Let  $X$  be a complex analytic set of  $\mathbb{C}^n$ . If  $X$  is  $C^1$ -smooth at  $x \in X$ , then  $X$  is analytically smooth at  $x$ .*

*Proof of Proposition 2.15.* By Proposition 2.16,  $\varphi(\text{Sing}(A)) = \text{Sing}(B)$  and then  $\varphi|_{A \setminus \text{Sing}(A)} : A \setminus \text{Sing}(A) \rightarrow B \setminus \text{Sing}(B)$  is, in particular, a homeomorphism. Moreover, we know that if  $Y$  is a complex analytic set of pure dimension, then each connected component of  $Y \setminus \text{Sing}(Y)$  is open and dense in exactly one irreducible component of  $Y$ . Let  $X_*$  be the connected component of  $A \setminus \text{Sing}(A)$  such that  $X_* \subset X$  and  $\overline{X_*} = X$ . Thus,  $\varphi(X_*)$  is a connected component  $B \setminus \text{Sing}(B)$ . Therefore,  $\overline{\varphi(X_*)}$  is an irreducible component of  $B$ . As  $\varphi$  is a homeomorphism, we get that  $\varphi(X)$  is an irreducible component of  $B$ .  $\square$

### 3. DIFFERENTIABLE INVARIANCE OF THE MULTIPLICITY

In this section, we present an alternative proof of Gau-Lipman Theorem ([9]), about the differentiable invariance of the multiplicity.

**Remark 3.1.** Let  $X$  be a complex analytic set of  $\mathbb{C}^n$  and let  $c_n : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be the conjugation map given by  $c_n(z_1, \dots, z_n) = (\bar{z}_1, \dots, \bar{z}_n)$ . Then  $c_n(X)$  is a complex analytic set and  $m(c_n(X), 0) = m(X, 0)$ . In particular,  $m(X \times c_n(X), 0) = m(X, 0)^2$ .

Let  $X$  be a real analytic subset of  $\mathbb{R}^n$  with  $0 \in X$ . We denote by  $X_{\mathbb{C}}$  the complexification of the germ  $(X, 0)$  in  $\mathbb{C}^n$ ; (For more about complexification, see [6] and [15]).

**Lemma 3.2** ([6], Proposition 2.9). *Let  $X$  be an irreducible complex analytic set of  $\mathbb{C}^n$ . Then,  $X_{\mathbb{C}}$  is complex analytic isomorphic to  $X \times c_n(X)$ .*

**Proposition 3.3.** *Let  $X$  and  $Y$  be complex analytic sets of  $\mathbb{C}^n$ . If  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a  $\mathbb{R}$ -linear isomorphism such that  $\varphi(X) = Y$ , then  $m(X, 0) = m(Y, 0)$ .*

*Proof.* By additivity of the multiplicity and by Proposition 2.15, we can suppose that  $X$  and  $Y$  are irreducible. Since  $\varphi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is a  $\mathbb{R}$ -linear isomorphism (with the usual identification  $\mathbb{C}^n = \mathbb{R}^{2n}$ ), it is easy to see that its complexification  $\varphi_{\mathbb{C}} : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$  is a  $\mathbb{C}$ -linear isomorphism and  $\varphi_{\mathbb{C}}(X_{\mathbb{C}}) = Y_{\mathbb{C}}$ . Then, by Lemma 3.2,  $X_{\mathbb{C}}$  is complex analytic isomorphic to  $X \times c_n(X)$  and  $Y_{\mathbb{C}}$  is analytic isomorphic to  $Y \times c_n(Y)$ . Thus,  $m(X \times c_n(X), 0) = m(Y \times c_n(Y), 0)$ , since the multiplicity is an analytic invariant. Therefore, by Remark 3.1,  $m(X, 0) = m(Y, 0)$ .  $\square$

**Theorem 3.4.** *Let  $X$  and  $Y$  be complex analytic sets of  $\mathbb{C}^n$ . If there is a homeomorphism  $\varphi : (\mathbb{C}^n, X, 0) \rightarrow (\mathbb{C}^n, Y, 0)$  such that  $\varphi$  and  $\varphi^{-1}$  are differentiable at the origin, then  $m(X, 0) = m(Y, 0)$ .*

*Proof.* Observe that  $D\varphi_0 : (\mathbb{C}^n, C(X, 0), 0) \rightarrow (\mathbb{C}^n, C(Y, 0), 0)$  is a  $\mathbb{R}$ -linear isomorphism. Then, by Proposition 2.15,  $D\varphi_0$  maps bijectively the irreducible components of  $C(X, 0)$  over the irreducible components of  $C(Y, 0)$ . Thus, let  $X_1, \dots, X_r$  and  $Y_1, \dots, Y_r$  be the irreducible components of  $C(X, 0)$  and  $C(Y, 0)$ , respectively, such that  $Y_j = D\varphi_0(X_j)$ ,  $j = 1, \dots, r$ . As  $D\varphi_0$  is a  $\mathbb{R}$ -linear isomorphism, by Proposition 3.3, we have that  $m(X_j, 0) = m(Y_j, 0)$ ,  $j = 1, \dots, r$ .

Furthermore, by Proposition 2.6,  $\varphi$  is a blow-spherical homeomorphism, then by Theorem 2.14,  $k_X(X_j) = k_Y(Y_j)$ , for all  $j = 1, \dots, r$ . By remark 2.13,

$$m(X, 0) = \sum_{j=1}^r k_X(X_j) \cdot m(X_j, 0)$$

and

$$m(Y, 0) = \sum_{j=1}^r k_Y(Y_j) \cdot m(Y_j, 0).$$

Therefore,  $m(X, 0) = m(Y, 0)$ . □

#### 4. A GENERALIZATION OF EPHRAIM-TROTMAN THEOREM

It is clear that Theorem 3.4 generalizes Ephraim-Trotman Theorem. In this Section, we prove Theorem 4.2 which is also slightly more general than Ephraim-Trotman Theorem.

**Lemma 4.1** ([5], Theorem 2.6 and Theorem 2.7). *If  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  is an irreducible homogeneous complex polynomial, then  $H_1(\mathbb{C}^n \setminus V(f); \mathbb{Z}) \cong \mathbb{Z}$ . Moreover,  $f_* : H_1(\mathbb{C}^n \setminus V(f); \mathbb{Z}) \rightarrow H_1(\mathbb{C} \setminus \{0\}; \mathbb{Z})$  is an isomorphism.*

**Theorem 4.2.** *Let  $f, g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be two complex analytic functions. Suppose that there are a complex line  $L \subset \mathbb{C}^n$  such that  $L \cap C(V(f), 0) = \{0\}$  and a blow-spherical homeomorphism  $\varphi : (\mathbb{C}^n, V(f), 0) \rightarrow (\mathbb{C}^n, V(g), 0)$  such that  $d_0\varphi(L)$  is a real plane. Then,  $m(V(f), 0) \leq m(V(g), 0)$ .*

*Proof.* By Theorem 2.14, we can suppose that  $f$  and  $g$  are irreducible homogeneous complex polynomials and  $\varphi$  send  $L$  over a real plane  $\tilde{L} = \varphi(L)$ .

Let  $\gamma$  be a generator of  $H_1(L \setminus \{0\}; \mathbb{Z})$ . Then,  $(f|_{L \setminus \{0\}})_*(\gamma) = \pm m(V(f), 0)$  and  $(g|_{L \setminus \{0\}})_*(\gamma) = \pm m(V(g), 0)$ . In particular,  $i_*(\gamma) = \pm m(V(f), 0)$ , where  $i : L \setminus \{0\} \rightarrow \mathbb{C}^n \setminus V(f)$  is the inclusion map, since

$$f_* : H_1(\mathbb{C}^n \setminus V(f); \mathbb{Z}) \rightarrow H_1(\mathbb{C} \setminus \{0\}; \mathbb{Z})$$

is an isomorphism. However,  $\varphi_* : H_1(\mathbb{C}^n \setminus V(f); \mathbb{Z}) \rightarrow H_1(\mathbb{C}^n \setminus V(g); \mathbb{Z})$  is also an isomorphism, then  $\varphi_*(i_*(\gamma)) = \pm m(V(f), 0)$ . Therefore,

$$(2) \quad ((g \circ \varphi)|_{L \setminus \{0\}})_*(\gamma) = \pm m(V(f), 0),$$

since  $g_* : H_1(\mathbb{C}^n \setminus V(g); \mathbb{Z}) \rightarrow H_1(\mathbb{C} \setminus \{0\}; \mathbb{Z})$  is an isomorphism, as well.

Since  $\tilde{L} = \varphi(L)$  is a real plane, there exists a complex plane  $H \subset \mathbb{C}^n$  such that  $\tilde{L} \subset H$ . Then  $h = g|_H : H \cong \mathbb{C}^2 \rightarrow \mathbb{C}$  is a homogeneous polynomial with degree  $k = m(V(g), 0)$ . Therefore,  $h$  factorizes as  $h = g_1 \cdots g_k$ , where each  $g_r$  is complex linear. However, by hypothesis,  $L \cap V(f) = \{0\}$  and  $\varphi(V(f)) = V(g)$ , then for each  $r \in \{1, \dots, k\}$ , we have

$$\tilde{L} \cap V(g_r) = \varphi(L) \cap \varphi(V(f)) = \varphi(L \cap V(f)) = \{0\},$$

which means that  $\tilde{L}$  and  $V(g_r)$  are two transversal real planes in  $H \cong \mathbb{R}^4$  and, in particular,  $H = \tilde{L} \oplus V(g_r)$ . Thus, if  $P : H \rightarrow \tilde{L}$  is the linear projection over  $\tilde{L}$  such that  $\text{Ker}(P) = V(g_r)$ , we can see that the inclusion map  $j : \tilde{L} \setminus \{0\} \rightarrow H \setminus V(g_r)$  is a homotopy equivalence and  $P|_{H \setminus V(g_r)} : H \setminus V(g_r) \rightarrow \tilde{L} \setminus \{0\}$  is a homotopy inverse of  $j$ . Moreover,  $\eta = \varphi(\gamma)$  is a generator of  $H_1(\tilde{L} \setminus \{0\}; \mathbb{Z})$  and of  $H_1(H \setminus V(g_r); \mathbb{Z})$ , since  $\varphi : L \setminus \{0\} \rightarrow \tilde{L} \setminus \{0\}$  is a homeomorphism. Furthermore, by Lemma 4.1,  $g_{r*} : H_1(H \setminus V(g_r); \mathbb{Z}) \rightarrow H_1(\mathbb{C} \setminus \{0\}; \mathbb{Z})$  is an isomorphism, for all  $r = 1, \dots, k$ . Therefore,  $|g_{r*}(\eta)| = 1$  and, then,

$$|g_*(\eta)| \leq k = m(V(g), 0),$$

since  $g_*(\eta) = \sum_{i=1}^k g_{i*}(\eta)$ . However,

$$|g_*(\eta)| = |((g \circ \varphi)|_{L \setminus \{0\}})_*(\gamma)| \stackrel{(2)}{=} m(V(f), 0)$$

and therefore  $m(V(f), 0) \leq m(V(g), 0)$ .  $\square$

As a first consequence we get

**Corollary 4.3.** *Let  $f, g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be complex analytic functions. Suppose that there are two complex lines  $L, L' \subset \mathbb{C}^n$  such that*

$$L \cap C(V(f), 0) = \{0\} \text{ and } L' \cap C(V(g), 0) = \{0\}$$

*and a blow-spherical homeomorphism  $\varphi : (\mathbb{C}^n, V(f), 0) \rightarrow (\mathbb{C}^n, V(g), 0)$  such that  $d_0\varphi(L)$  and  $d_0\varphi^{-1}(L')$  are real planes. Then,  $m(V(f), 0) = m(V(g), 0)$ .*

As another consequence we obtain the following

**Corollary 4.4.** *Let  $f, g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be complex analytic functions. If there is a homeomorphism  $\varphi : (\mathbb{C}^n, V(f), 0) \rightarrow (\mathbb{C}^n, V(g), 0)$  such that  $\varphi$  and  $\varphi^{-1}$  are differentiable at the origin, then  $m(V(f), 0) = m(V(g), 0)$ .*

*Proof.* In this case,  $D\varphi_0$  and  $D\varphi_0^{-1}$  are  $\mathbb{R}$ -linear isomorphisms from  $\mathbb{C}^n$  to  $\mathbb{C}^n$ . Then,  $D\varphi_0(L)$  and  $D\varphi_0^{-1}(L')$  are real planes whenever  $L$  is a real plane. Moreover,



if  $L$  is a real plane, we get that  $D\varphi_0(L) = d_0\varphi(L)$  and  $D\varphi_0^{-1}(L) = d_0\varphi^{-1}(L)$ . By Theorem 4.2, the result follows.  $\square$

It is easy to produce an example of a blow-spherical homeomorphism such that it sends real planes over real planes, but it is not differentiable at the origin. Here, we finish this paper presenting a simple example of a such blow-spherical homeomorphism.

**Example 4.5.** Let  $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a linear isomorphism. We verify that the mapping  $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  given by

$$\varphi(x) = \begin{cases} \|x\|^{\frac{1}{2}} \cdot \frac{\psi(x)}{\|\psi(x)\|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a blow-spherical homeomorphism that sends real planes over real planes but it is not differentiable at the origin.

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(J. Edson Sampaio) BCAM - BASQUE CENTER FOR APPLIED MATHEMATICS, MAZARREDO, 14 E48009 BILBAO, BASQUE COUNTRY - SPAIN. E-MAIL: [esampaio@bcamath.org](mailto:esampaio@bcamath.org)

AND

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO CEARÁ, RUA CAMPUS DO PICI, S/N, BLOCO 914, PICI, 60440-900, FORTALEZA-CE, BRAZIL.

E-MAIL: [edsonsampaio@mat.ufc.br](mailto:edsonsampaio@mat.ufc.br)