Goal-Oriented p-Adaptivity using Unconventional Error Representations for a 1D Steady State Convection-Diffusion Problem*

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Abstract

This work proposes the use of an alternative error representation for Goal-Oriented Adaptivity (GOA) in context of steady state convection dominated diffusion problems. It introduces an arbitrary operator for the computation of the error of an alternative dual problem. From the new representation, we derive element-wise estimators to drive the adaptive algorithm. The method is applied to a one dimensional (1D) steady state convection dominated diffusion problem with homogeneous Dirichlet boundary conditions. This problem exhibits a boundary layer that produces a loss of numerical stability. The new error representation delivers sharper error bounds. When applied to a p-GOA Finite Element Method (FEM), the alternative error representation captures earlier the boundary layer, despite the existing spurious numerical oscillations.

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1 Introduction

The need for accurate computations with limited computational resources is becoming increasingly urging in many engineering applications. Over the last decades, algorithms designed to compute optimal meshes arose for various problems (e.g., [11, 12, 5]).

Classical adaptive algorithms are driven by global norms such as the energy-norm. They are intended to provide optimal discrete spaces in the sense that they minimize the number of unknowns needed to achieve a prescribed tolerance error in the energy-norm. Unfortunately, these adaptive strategies may be unuseful when the quantity of interest (QoI) is only loosely related to the energy of the problem, as it occurs in a large variety of engineering applications (see, e.g., [14]). For those cases, GOA algorithms are employed (see [1, 15, 13, 16, 4, 10, 8]).

GOA algorithms iterate along the following steps. Given an initial coarse mesh, they estimate the error either using an *a posteriori* error technique or by simply approximating it over a finer mesh. Then, the error in the quantity of interest is represented with the help of the global error functions of the original and adjoint problems. Such global representation provides the so-called *influence function*, which expresses how much the error in a particular point is affecting to the error in the QoI. This influence function alone is insufficient to drive refinements, since an enrichment (refinement) on the discrete space does not guarantee a decrease of the influence function at any particular point. Thus, one builds an element-wise based upper bound of the influence function in terms of local norms whose energy decreases as refinements occur. This upper bound is the one driving refinements: one simply enriches those elements with largest contributions to that upper bound. In this way, the next coarse grid is built, and the entire process can be iterated until a given tolerance error is reached.

In our previous work [6], we developed and analyzed an alternative error representation that provides sharper upper bounds of the error in the QoI for a 1D Helmholtz equation. The key of such representation was to introduce a new operator to compute the error of an alternative dual problem. As a result, the adaptive algorithm arising from the new error representation was more efficient when compared to a classical one.

This work extends our previous results to the case of convection dominated diffusion problems, where numerical instabilities occur due to the presence of a boundary layer [9, 12]. Thus, the main contribution here is to study the effect of the alternative error representation developed in [6] for the case of convection dominated diffusion problems. It is well-known that the rapid capture of the boundary layer is essential to re-gain stability. Numerical results show that our proposed strategy captures such layer faster than classical GOA methods (see, e.g., [15, 13]).

In this work, we employ a classical FEM. Other methods such as Streamline upwind/Petrov-Galerkin (SUPG) [3] or the Discontinuous Petrov–Galerkin DPG [7] formulations have been designed and studied for the case of convection dominated diffusion problems. Our aim here is to present an unsophisticated technique that works is often used by for practitioners.

The rest of this article is organized as follows. We introduce the problem in Section 2 and, in Section 3, we recall the method described in [6]. Section 4 describes the algorithm. Section 5 is the core of this article and shows the numerical results of the application of the alternative estimator for the convection-diffusion case. Finally, Section 6 draws the main conclusions of this investigation.

2 Model problem and formulation

We consider the following 1D steady state convection dominated diffusion problem with homogeneous Dirichlet conditions:

> For a given source $F \in L^2(0,1)$ and $0 < \alpha \ll 1$, find u such that,
> $$\begin{cases} -\alpha \partial_{xx} u - \partial_x u = F(x) & \text{on } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

An variational problem associated is:

> Find $u \in H_0^1(0,1)$ such that,
> $$b(u, \phi) = f(\phi), \quad \forall \phi \in H_0^1(0,1) \tag{2.1}$$

where b is defined as

$$b(u, v) = \alpha \langle \partial_x u \,, \partial_x v \rangle_{L^2(0,1)} - \langle \partial_x u \,, v \rangle_{L^2(0,1)} \,, \quad \forall u, v \in H_0^1(0,1).$$

and $\langle \cdot \,, \cdot \rangle_{L^2}$ is the standard L^2 scalar product. In above, we denote by $L^2(0,1)$ the space of square integrable functions on $(0,1)$, $H^1(0,1)$ the Sobolev space of functions with $L^2(0,1)$ first derivatives on $(0,1)$ and by $H_0^1(0,1)$ the subspace of functions from $H^1(0,1)$ that vanish on the boundaries.

3 Goal Oriented Adaptivity

Let l, a linear continuous form, denote the operator associated to the QoI. The goal is to construct a mesh that accurately approximates $l(u)$ using a minimum number of degrees of freedom (DoF). We use a FEM to approximate the exact solution u by a discrete solution u_h that belongs to a finite dimensional Galerkin approximation space $V_h \subset H_0^1(0,1)$. We split the interval $(0,1)$ into a partition of open elements K such that $(0,1) = \bigcup K$.

To reach the optimum grid, we use adaptivity, which employs upper bounds of $l(e) = l(u - u_h) = l(u) - l(u_h)$. As mentioned in the introduction, this work focuses on constructing sharp upper bounds.

3.1 Classical error representation

The classical method introduces the adjoint state problem:

> Find $v \in H_0^1(0,1)$ such that
> $$b(\phi, v) = l(\phi), \quad \forall \phi \in H_0^1(0,1) \tag{3.1}$$

The approximation of v in V_h is denoted by v_h and its error as $\varepsilon = v - v_h$.
By plugging the error e of the direct problem (2.1) into the adjoint problem (3.1) and using the Galerkin orthogonality, we obtain the following element-wise upper bound:

$$|l(e)| = |b(e, \varepsilon)| \leqslant \sum_K |b_K(e, \varepsilon)| \,. \tag{3.2}$$

3.2 Alternative error representation

We introduce a symmetric positive definite bilinear (localizable) form $\widetilde{b}$ in order to compute a Riesz representation of the adjoint problem residual as follows:

Find $\widetilde{\varepsilon} \in H_0^1(0,1)$ such that

$$\widetilde{b}(\phi, \widetilde{\varepsilon}) = l(\phi) - b(\phi, v_h), \quad \forall \phi \in H_0^1(0,1)$$

In our case, we select the operator associated to the Laplace equation, i.e.:

$$\widetilde{b}(u, v) = \langle \partial_x u \,, \partial_x v \rangle_{L^2(0,1)} \,, \quad \forall u, v \in H_0^1(0,1). \tag{3.3}$$

The corresponding error representation and upper bound then becomes::

$$|l(e)| = \left| \widetilde{b}(e, \widetilde{\varepsilon}) \right| \leqslant \sum_K \left| \widetilde{b}_K(e, \widetilde{\varepsilon}) \right| . \tag{3.4}$$

Remark: Mathematically, upper error bounds (3.2) and (3.4) can be controlled by an element-wise sum of norm product by applying Cauchy-Schwarz inequality when the bilinear forms are symmetric and positive definite (for non-positive definite forms, a slightly more sophisticated approach is needed). Those bounds exhibit a strictly monotonous behavior as we refine the grid (i.e., enrich the space), while (3.2) and (3.4) can present some oscillations. However, in practice, those norm-based upper bounds involve an additional loss of sharpness. Thus, most practitioners prefer to use bounds (3.2) and (3.4) for computations.

4 p-Adaptive goal oriented algorithm.

We consider the following p-adaptive algorithm with the new error representations given by Equations (3.2) and (3.4). Given an initial mesh, we first solve the direct and adjoint problem. We then create a finer mesh by increasing uniformly the polynomial order of approximation p by two with respect to the orders set for the initial coarse mesh.

Then, we compute errors e, ε and $\widetilde{\varepsilon}$, which are the errors of the direct, adjoint, and alternative dual problems, respectively. They are solutions of the following variational formulations:

Find $e \in H_0^1(0,1)$ such that

$$b(e, \phi) = f(\phi) - b(u_h, \phi), \quad \forall \phi \in H_0^1(0,1)$$

Find $\varepsilon \in H_0^1(0,1)$ such that

$$b(\phi, \varepsilon) = l(\phi) - b(\phi, v_h), \quad \forall \phi \in H_0^1(0,1)$$

Find $\widetilde{\varepsilon} \in H_0^1(0,1)$ such that

$$\widetilde{b}(\phi, \widetilde{\varepsilon}) = l(\phi) - b(\phi, v_h), \quad \forall \phi \in H_0^1(0,1)$$

Subsequently, we compute the element-wise indicators $|b_K(e,\varepsilon)|$ (resp. $\left|\widetilde{b}_K(e,\widetilde{\varepsilon})\right|$). Then, we mark those elements that exhibit an error estimator larger than a given ratio $1-\beta$ of the largest error estimator. Finally, we increase by one the order of the marked elements. Figure 1 sketches the main steps of the algorithm. This strategy has been chosen for its simplicity of implementation; more sophisticated marking processes and refinements strategies can be designed, including those shown in [16, 2].

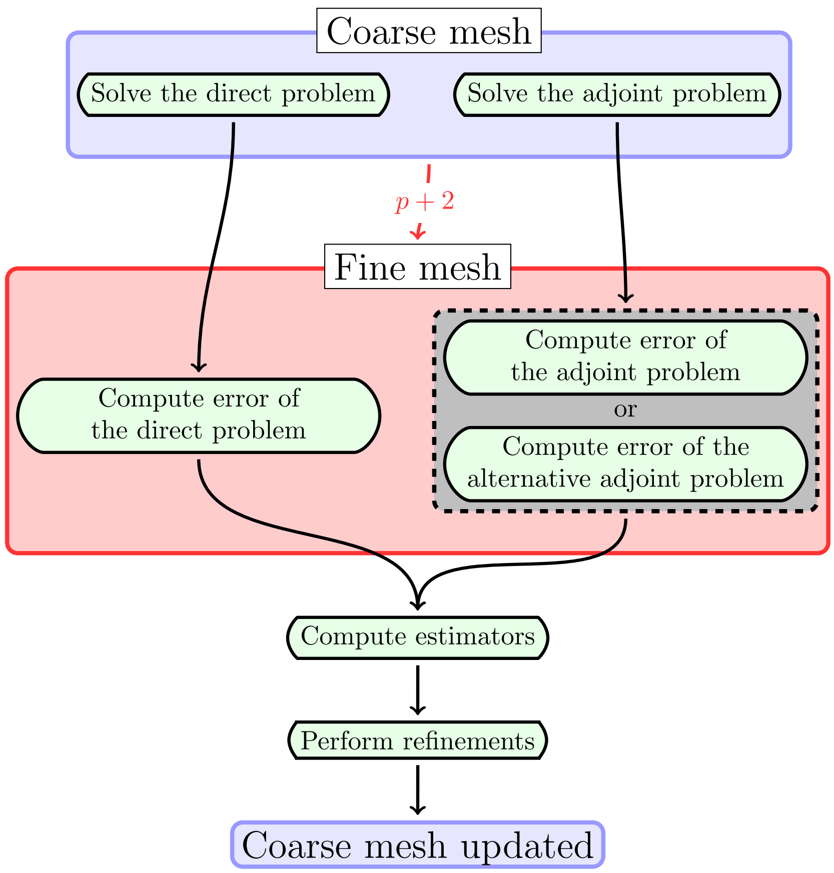


Figure 1 – Goal oriented adaptive algorithm.

5 Numerical results

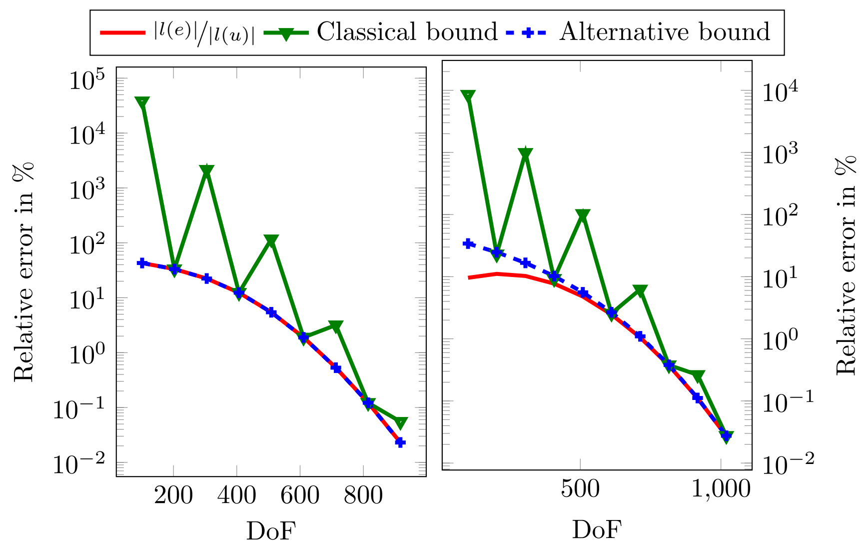
The initial coarse mesh contains 100 elements uniformly distributed in the log scale between 10^{-4} and 1 with a uniform polynomial order $p_{init}=1$. We define the source $F\equiv 1$ over the domain and the QoI as the integration of the i^{th} derivative, ($i=0$ or $i=1$) on a subset that includes the boundary layer, specifically $(0, 0.05) \subset (0, 1)$. That is, $\forall \phi \in H_0^1(0,1)$,

$$f(\phi) = \int_0^1 F \cdot \phi = \int_0^1 \phi,$$
$$l_i(\phi) = \frac{1}{0.05} \int_0^{0.05} \phi^{(i)}.$$

We set the diffusion coefficient to $\alpha = 10^{-6}$. The bilinear form we use to compute the alternative bounds is the one derived from the Laplace equation with homogeneous Dirichlet boundary conditions given by Equation (3.3).

5.1 Uniform p-refinement

Figure 2 illustrates the behavior of the bounds when performing uniform p-refinements. Figure 2a shows that the alternative bound coincides exactly with the error in the QoI when considering l_0. The sign of every estimator is the same, and thus, the triangular inequality that provides the element wise upper bound generates no loss. Figure 2b illustrates the behavior of the upper bounds when considering l_1. In both cases, the alternative bound is sharper than the classical one. Consequently, we expect the adaptive process driven by the alternative error estimators to be more efficient. In the following, we only consider l_0 as QoI.



(a) The QoI is l_0

(b) The QoI is l_1

Figure 2 – Bounds obtained using uniform p-refinements.

5.2 Adaptive process

Since the exact solution is almost linear everywhere except on the proximity of the boundary layer, we expect that the optimal mesh will select a very large p on the element(s) containing the boundary layer, and it will be $p = 1$ elsewhere. Furthermore, since the alternative adjoint error $\widetilde{\varepsilon}$ is zero at the nodes due to the choice of Laplace equation as the alternative dual operator (see [6] for details), we would expect to solve the linear part exactly when considering the exact solution as the reference one. However, the selected fine mesh is not adapted to this kind of problem, since the initial mesh is coarse and does not match the boundary layer, as it occurs in practical problems. Thus, the algorithm starts in the pre-asymptotic regime.

We ran the adaptive processes described in Section 4 with a stopping tolerance in the QoI of 0.01% of the direct fine solution in the QoI and a maximum approximation order of 27. Figure 3 tracks the error evolution on the QoI throughout the adaptive process with the percentage used for the marking process set to $\beta = 60\%$ (left panel) and $\beta = 1\%$ (right panel). The alternative estimators employs less DoF than both the classical adaptive process and uniform p-refinements. We have plotted the final meshes (see Figure 4) for the cases where $\beta = 60\%$ (left panel) and $\beta = 1\%$ (right panel). The respective direct solution computed on those meshes are displayed in Figure 5.

We observe that the classical algorithm fails at eliminating the spurious oscillations in both cases, especially, when $\beta = 1\%$. It is completely miss-driven by the spurious oscillations. Also the alternative method identifies the boundary layer more efficiently than the classical one. Indeed, for the case $\beta = 60\%$, the algorithm with the classical estimators executes more refinements in the linear part of the solution, while the alternative estimator refines more intensively around the boundary layer. For $\beta = 1\%$, the classical method fails to catch both the boundary layer and the QoI, producing thus an erroneous solution, whereas the alternative method almost succeeds at eliminating the oscillations by computing the optimal mesh. Note that the classical method concentrates refinements around the point 0.05, where the QoI ends.

For the figures, we have employed an "exact solution" computed using an overkilling mesh (containing around 800 elements).

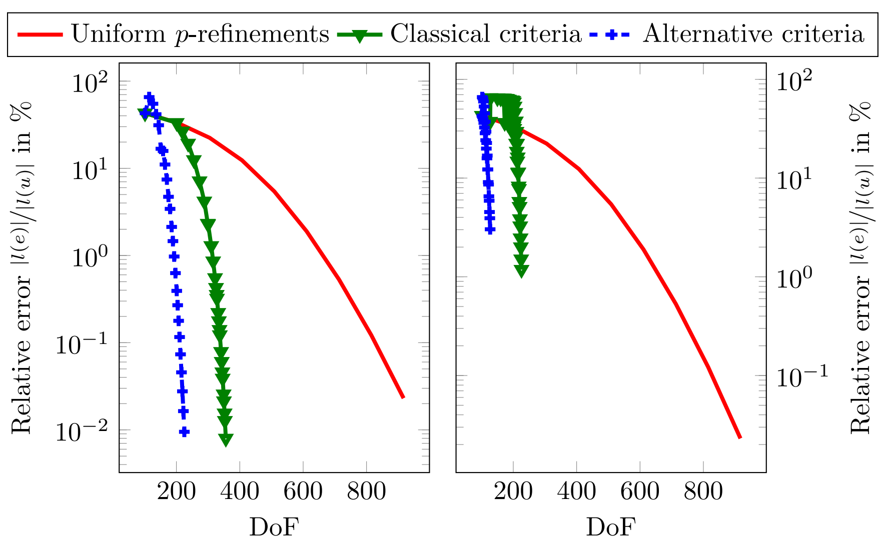


Figure 3 – Error in the QoI obtained using adaptive p-refinements for different refinements ratios β. Left panel: $\beta = 60\%$, right panel: $\beta = 1\%$.

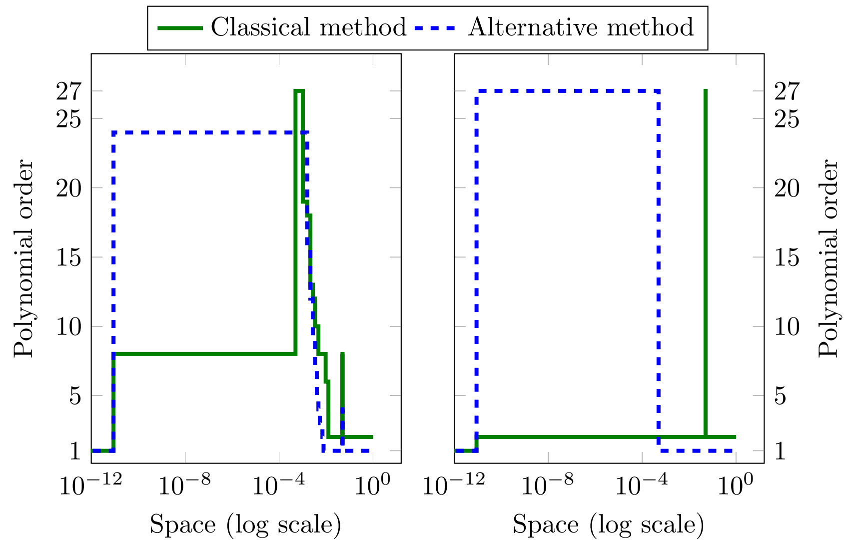


Figure 4 – Final adapted coarse meshes after the adaptive process for different ratios of refinement β. Left panel: $\beta = 60\%$, right panel: $\beta = 1\%$.

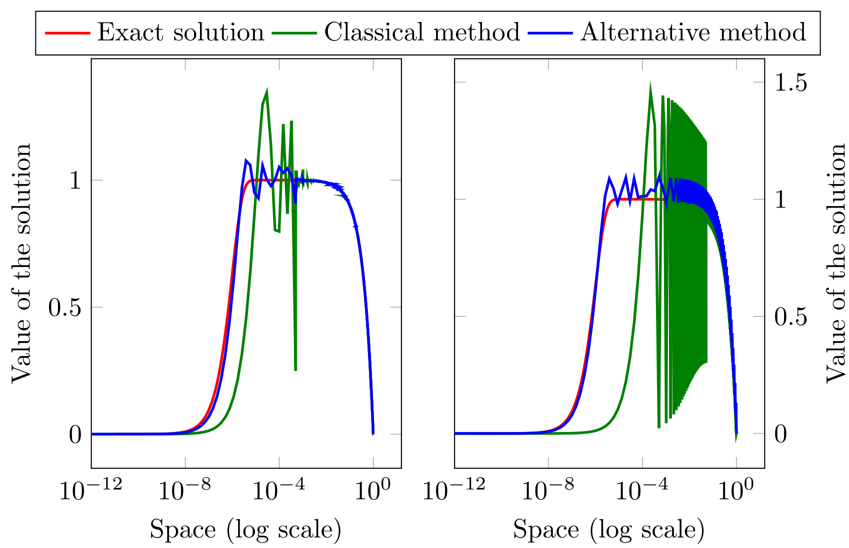


Figure 5 – Solution of the direct problem on the adapted coarse mesh for different ratios of refinement β. Left panel: $\beta = 60\%$, right panel: $\beta = 1\%$.

6 Conclusions

In this work, we have studied the behavior of a simple p-adaptive process when using both a classical error representation in the QoI (as described in [13, 15]) and an alternative error representation, for the case of a 1D convection dominated diffusion problem. The well-known particularity of this problem is that a boundary layer generates spurious oscillations on the numerical solution for coarse meshes [9]. For a coarse initial mesh with few DoF, our alternative GOA algorithm still produces a final mesh that catches the boundary layer while reducing significantly the spurious oscillations. As future work, we plan to incorporate unrefinements within our adaptive strategy in order to correct possibly misplaced refinements at a later stage, and thus, obtain optimal meshes for any ratio of refinement β. We are also working in 2D and 3D problems.

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