Darrieus-Landau instabilities in the framework of the G-equation

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Abstract
We consider a model formulation of the flame front propagation in turbulent premixed combustion based on stochastic fluctuations imposed to the mean flame position. In particular, the mean flame motion is described by a G-equation, while the fluctuations are described according to a probability density function which characterizes the underlying stochastic motion of the front. The proposed approach reproduces as special cases the G-equation along the motion of the mean flame position, when the stochastic fluctuations are removed, and the Zimont & Lipatnikov model, when a Gaussian density for fluctuations is used together with the assumption of a plane front. The potentiality of the approach is here investigated further focusing on the Darrieus-Landau (hydrodynamic) instabilities. In particular, this model formulation is set to lead to the Michelson-Sivashinsky equation. Furthermore, a formula that connects the consumption speed and the front curvature is established.

Introduction

It is known that Darrieus-Landau (hydrodynamic) instabilities in turbulent premixed combustion are described by the so-called Michelson-Sivashinsky equation [1,2,3]. A preliminary study to derive this equation from the framework of the G-equation was presented at the XXXIX Meeting of the Italian Section of the Combustion Institute [4]. Here we continue that study.

This derivation is performed within a model formulation introduced in References [5,6,7]. In particular, this proposed approach is based on the G-equation to describe the motion of the mean flame front and stochastic fluctuations are imposed on its position according to a probability density function (PDF) which characterizes the underlying random motion [5,6,7]. Hence the formulation is based on the prior knowledge of the PDF of the fluctuations of the front position.

After the introduction of stochastic fluctuations for the flame position, the evolution equation of the resulting effective burned fraction emerges to be a reaction-diffusion equation. This shows that the two classical approaches to model turbulent premixed combustion, considered so far alternatives to each other, i.e., the G-equation and the reaction-diffusion equation, are indeed complementary one to each other. Moreover, if the stochastic fluctuations are removed, such formulation reduces to the standard G-equation for the motion of the mean flame position and, when a Gaussian density for fluctuations is used together with the assumption of a plane front, the model of Zimont & Lipatnikov [8,9] is recovered. Considered these special cases, other efforts are pursued to further investigate this formulation.

Modeling Approach

Let the scalar function $G(\boldsymbol{x},t), \boldsymbol{x}\in\mathbb{R}^n$ be a level surface that represents the front $\Gamma(t), t\geq 0$, which neatly separates burned and unburned domains. Let $\boldsymbol{x}_c$ be a point on the level surface $G=c$ at the instant t_0, such that the corresponding front is $\Gamma_0=\Gamma(t_0)=\{\boldsymbol{x}=\boldsymbol{x}_\mathbf{0}\in S|G(\boldsymbol{x}_\mathbf{0},t_0)=c\}$, where $S\subseteq\mathbb{R}^n$. From now on we consider the zero-isocontour only, i.e., $c=0$. The level surface propagates with a consumption speed given by the laminar burning velocity s_L in the normal direction relative to the mixture element and its evolution is described by the following Hamilton-Jacobi equation, known in combustion literature as G-equation,

$$\frac{\partial G}{\partial t}+u\cdot\nabla G=s_L\|\nabla G\|, \qquad (1)$$

where **u** is the velocity field of the flow.
In equation (1) the propagation in the normal direction is stated by

$$s_L\mathbf{n}=-s_L\frac{\nabla G}{\|\nabla G\|}, \qquad (2)$$

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where $\boldsymbol{n}$ denotes the normal vector.

Let the front motion be described by the random process $\boldsymbol{X}_c^{\omega}(\hat{\boldsymbol{x}},t)$ where ω labels any independent realization, such that the random front contour is

$$\Gamma^{\omega}(t)=\left\{x=X_c^{\omega}(t)\in S|G^{\omega}(X_c^{\omega},t)=c\right\}. \quad (3)$$

Let the mean value of $\boldsymbol{X}_c^{\omega}$ be denoted by $\langle \boldsymbol{X}^{\omega}(\hat{\boldsymbol{x}},t)\rangle=\hat{\boldsymbol{x}}(t)$, then if $P_c\left(\boldsymbol{x}_c;t|\hat{\boldsymbol{x}}\right)$ is the corresponding PDF, with initial condition $P_c\left(\boldsymbol{x}_c;t_0|\hat{\boldsymbol{x}}\right)=\delta\left(\boldsymbol{x}-\boldsymbol{x}_0\right)$, the mean flame position is obtained by applying the integral formula

$$\langle x_c\rangle=\int_{\mathbb{R}^n} x_c P_c(x_c;t|\hat{x})dx_c=\hat{x}(t). \quad (4)$$

Introducing $\check{G}(\hat{x},t)$, with initial condition $\check{G}(\hat{x},t_0)=\check{G}(x_0,t_0)=c$, as the implicit formulation of the mean flame position $\hat{\boldsymbol{x}}$, the ensemble averaging of (1) gives

$$\frac{\partial\check{G}}{\partial t}+\hat{\boldsymbol{u}}\cdot\nabla\check{G}=-\widehat{s_L\boldsymbol{n}}\cdot\nabla\check{G}. \quad (5)$$

This procedure was previously proposed by Oberlack *et al.* [10].

Since the G-equation can be derived on the basis of considerations about symmetries, there is a unique model for the RHS term of equation (5) providing a relation between the laminar burning velocity s_L and the turbulent burning velocity s_T [10], i.e.,

$$\widehat{s_L\boldsymbol{n}}=s_T\check{\boldsymbol{n}},\ \check{\boldsymbol{n}}=-\frac{\nabla\check{G}}{\|\nabla\check{G}\|}. \quad (6)$$

Finally, combining equation (5) and model (6), the *G*-equation that describes the surface motion along the mean flame position is

$$\frac{\partial\check{G}}{\partial t}+\hat{\boldsymbol{u}}\cdot\nabla\check{G}=s_T\|\nabla\check{G}\|. \quad (7)$$

It is here stressed the difference between the mean of the normal vectors to the random flame front, i.e., $\hat{\boldsymbol{n}}$, and the normal vector of the mean flame front, i.e., $\check{\boldsymbol{n}}$, which is an important feature of the process. In particular, the mean of the random level surface $\langle G^{\omega}\rangle$ is generally different from the level surface $\check{G}$ depicted by the mean position of the flame [2].

By using the sifting property of the Dirac δ-function, it holds

$$G^{\omega}(X_c^{\omega},t)=\int_{\mathbb{R}^n} G(x,t)\delta(x-X_c^{\omega}(\hat{x},t))dx, \quad (8)$$

and also the following formula including the stochastic fluctuations around the front depicted by the mean flame position, i.e.,

$$\phi^{\omega}(x,t)=\int_{\mathbb{R}^n} G(\hat{x},t)\delta(x-X_c^{\omega}(\hat{x},t))d\hat{x}. \quad (9)$$

We expect that further investigation of the relationships between formula (8) and formula (9) leads to an explicit formula relating $\langle G^{\omega}\rangle$ and $\check{G}$. In the following, the evolution equation associated to (9) is derived.

Given the level surface $\check{G}$, then the inner domain $\check{\Omega}(t)$ enclosed by the front contour $\check{\Gamma}(t)=\{\boldsymbol{x}\in\partial\check{\Omega}(t)\}$ can be understood as the volume occupied by the reacted fraction and that outside as occupied by the still unreacted fraction. Then the following indicator function is introduced

$$I_{\Omega}(t)=\begin{cases}1, x\in\Omega(t)\\ 0, x\notin\Omega(t)\end{cases} \quad (10)$$

In analogy with (9), the random indicator associated to the surfaces which enclose the volume of the burned fraction is given by the following formula

$$I_{\check{\Omega}}^{\omega}(x,t)=\int_{\mathbb{R}^n} I_{\check{\Omega}}(\hat{x},t)\delta(x-X_c^{\omega}(\hat{x},t))d\hat{x} = \\ =\int_{\check{\Omega}}\delta(\boldsymbol{x}-\boldsymbol{X}_c^{\omega}(\hat{\boldsymbol{x}},t))d\hat{\boldsymbol{x}} \quad (11)$$

Finally, ensemble averaging of (11) gives

$$\langle I_{\check{\Omega}}^{\omega}(\boldsymbol{x},t)\rangle=\int_{\check{\Omega}}\langle\delta(\boldsymbol{x}-\boldsymbol{X}_c^{\omega}(\hat{\boldsymbol{x}},t))\rangle d\hat{\boldsymbol{x}} \\ =\int_{\check{\Omega}(t)} P_c(\boldsymbol{x};t|\hat{\boldsymbol{x}})d\hat{\boldsymbol{x}}=V_e(\boldsymbol{x},t) \quad (12)$$

The observable $V_e(x,t)$ can be understood as the effective fraction of burned mass. $V_e(x,t)$ ranges in the compact interval $[0,1]$ and the front contourn is given by selecting a threshold value.

When applying the Reynolds transport theorem to formula (12) we obtain the following evolution equation of reaction-diffusion type

$$\frac{\partial V_e}{\partial t} = \int_{\check{\Omega}(t)} \frac{\partial P_c}{\partial t} d\hat{\boldsymbol{x}} + \int_{\check{\Omega}(t)} \nabla_{\hat{x}} \cdot \left[s_T \check{\boldsymbol{n}} P_c(\boldsymbol{x};t|\hat{\boldsymbol{x}}) \right] d\hat{\boldsymbol{x}} \ . \quad (13)$$

Equation (13) had been derived with a different argument in Reference [7].

Considering the general kinetic equation for P_c

$$\frac{\partial P_c}{\partial t} = -\nabla J, \quad (14)$$

where J is the flux, equation (13) becomes

$$\frac{\partial V_e}{\partial t} = -\nabla \int_{\check{\Omega}(t)} J(\boldsymbol{x};t|\hat{\boldsymbol{x}}) d\hat{\boldsymbol{x}} + \int_{\check{\Omega}(t)} \nabla_{\hat{x}} \cdot \left[s_T \check{\boldsymbol{n}} P_c(\boldsymbol{x};t|\hat{\boldsymbol{x}}) \right] d\hat{\boldsymbol{x}} \quad (15)$$

If a flux-gradient relation is assumed, i.e.,

$$J(x,t) = -D \nabla P_c, \quad (16)$$

where D is the diffusion coefficient, equation (15) can be re-written as

$$\frac{\partial V_e}{\partial t} = D \nabla^2 V_e + \int_{\check{\Omega}(t)} \nabla_{\hat{x}} \cdot \left[s_T \check{\boldsymbol{n}} P_c(\boldsymbol{x}-\hat{\boldsymbol{x}};t) \right] d\hat{\boldsymbol{x}} \quad (17)$$

which reduces to the G-equation when no diffusion is assumed [7].

Moreover, when the mean front curvature $\kappa = \nabla \cdot \boldsymbol{n}/2$ is taken into account, and turbulent burning velocity reads $s_T(\boldsymbol{x},t) = s_T(\kappa,t)$, equation (17) becomes

$$\frac{\partial V_e}{\partial t} = D \nabla^2 V_e - \nabla \cdot \int_{\Omega(t)} s_T \check{n} \, \mathrm{P}_c(x-\hat{x};t) d\hat{x} + \int_{\check{\Omega}(t)} P_c \left\{ \frac{\partial s_T}{\partial \kappa} (\nabla_{\hat{x}} \kappa) \cdot \check{\boldsymbol{n}} + 2 s_T \kappa \right\} d\hat{\boldsymbol{x}}. \quad (18)$$

Hence, if we consider a plane front, such that $\kappa = 0$, we have that $s_T = s_T(t)$ and $\boldsymbol{n} = \boldsymbol{n}(t)$, equation (18) reduces to

$$\frac{\partial V_e}{\partial t} = D \nabla^2 V_e + s_T(t) \|\nabla V_e\|, \quad (19)$$

that is the same equation derived by Zimont & Lipatnikov [8] and studied in [9]. This shows that equation (18) can be considered as the natural extension of Zimont & Lipatnikov model to the multidimensional case with non null mean curvature.

Specific Objectives: Derivation of the Michelson-Sivashinsky equation

The Michelson-Sivashinsky equation in the one-dimensional case reads, see, e.g., [1,2,3],

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} - \left(\frac{\partial g}{\partial x} \right)^2 - D_x^1 g, \quad (20)$$

where D_x^1 is the fractional derivative of order 1 in the Riesz-Feller sense with Fourier symbol is $-|\xi|$. It differs from the classical first derivative, and it is related to the Hilbert transform by the formula

$$D_x^1 g = \frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{g(x')}{(x'-x)} dx' \ . \quad (21)$$

We also recall here the fractional Laplacian $-(-\Delta)^s, s \in (0,1)$, defined by its Fourier symbol $-|\xi|^{2s}$. The classical Laplacian is recovered when $s = 1$.

It is well-known that the dispersion relation of equation (20) is

$$e^{-\xi^2 t + |\xi| t}. \quad (22)$$

This suggests the following fractional differential equation for the PDF of the fluctuations of the front position:

$$\begin{cases} \dfrac{\partial P_c}{\partial t} = \Delta P_c - (-\Delta)^{1/2} P_c, \\ P_c(\boldsymbol{x},0) = \delta(\boldsymbol{x}). \end{cases} \quad (23)$$

Actually, the dispersion relation (22) is the Fourier transform of the Green function of (23).

It is here highlighted that the PDF of fluctuations which solves (23) emerges to be a quasiprobability distribution showing negative values that requires high care. However, these negative values can be interpreted as due to local extinction phenomena when local reversibility of the value of the progress variable occurs following from the entering of fresh mixture into a volume just now fully burned. This effect can be ascribed to the so-called counter-gradient that is generated by the density difference between reactants and products.

We define the field $g(\boldsymbol{x},t)$ in analogy with (12), i.e.,

$$g(x,t)=\int_{\check{\Omega}(t)} \check{G}(\hat{x},t)P_c(x-\hat{x},t)\,d\hat{x}. \tag{24}$$

To simplify the following computations, we will consider the evolution of the deterministic front only on the hyper surface of unitary gradient embedding the burning front as its zero-isocontour:

$$\frac{\partial \check{G}}{\partial t}=s_T(\hat{x},t)\|\nabla\check{G}\|;\|\nabla\check{G}\|=1. \tag{25}$$

Formulae (24) and (25) will give the evolution equation of the function $g(\boldsymbol{x},t)$ in the following form:

$$\frac{\partial g}{\partial t}=\Delta g-\left(-\Delta^{1/2}\right)g+\int_{\check{\Omega}(t)} s_T(\hat{x},t)P(x-\hat{x}t)\,d\hat{x} \tag{26}$$

It is worth noting that in one dimension, it reduces to

$$\frac{\partial g}{\partial t}=\frac{\partial^2 g}{\partial x^2}-D_x^1 g+\int s_T(\hat{x},t)P(x-\hat{x},t)\,d\hat{x}. \tag{27}$$

Equation (27) recasts the Michelson-Sivashinsky (20) except for the last term on the RHS. Then we impose to such term of model equation (27) to be equal to the negative square of the space derivative of $g(\boldsymbol{x},t)$ in order to stick to the Michleson-Sivashinsky equation (20), i.e.,

$$-\left(\frac{\partial g}{\partial x}\right)^2=\int s_T(\hat{x}t)P(x-\hat{x}t)\,d\hat{x}. \tag{28}$$

By using the Fourier integral transform marked with $\sim$ we have

$$\frac{\partial g}{\partial x}=\frac{i}{2\pi}\int e^{-i\xi x}\xi\,\tilde{P}(\xi;t)\tilde{G}(\xi,t)\,d\xi, \tag{29}$$

And finally

$$-\left(\frac{\partial g}{\partial x}\right)^2=\left(\frac{1}{2\pi}\int e^{-ikx}k\,\tilde{P}(k;t)\tilde{G}(k,t)\,dk\right)^2 \tag{30}$$

Results and Discussion

The formulas derived so far allow us to make at least three attempts in order to couple the proposed formulation with the at Darrieus-Landau instability and get during this process useful information on the speed of propagation of the flame. We will address from now on the RHS of equation (30) as $\omega(x,t)$. Using simple properties of the Fourier transform, we can rewrite (28) as

$$\tilde{s}_T=\frac{\tilde{\omega}}{\tilde{P}}=\exp\left(\xi^2 t-|\xi|t\right)\tilde{\omega} \tag{31}$$

That is, taking the anti-transform of the latter formula,

$$s_T(x,t)=\frac{1}{2\pi}\int e^{-i\xi x}e^{\xi^2 t-|\xi|t}\tilde{\omega}(\xi,t)\,d\xi \tag{32}$$

We obtain the desired formula for the speed to be plugged into in (26). With this first approach a possible algorithm for the computation of a front evolution would be a splitting method where at a first time the $\check{G}(\boldsymbol{x},t)$ field is advanced in (7), and then $\omega(x,t)$ is computed and used in (32) to get another time step for the $s_T(x,t)$ field.

Let $u(\boldsymbol{x},t)$ be defined by its Fourier transform $\tilde{u}=\frac{1}{\tilde{P}}=e^{\left(\xi^2 t-|\xi|t\right)}$. Another approach to get $s_T(x,t)$ would be the solution of the evolution equation for $u(\mathbf{x},t)$:

$$\frac{\partial u}{\partial t}=\frac{-\partial^2 u}{\partial x^2}+D_x^1 u;u(\boldsymbol{x},0)=\delta(\boldsymbol{x}) \tag{33}$$

The consumption speed $s_T(x,t)$ is then computed by performing the following convolution integral

$$s_T(x,t)=\int u(x-\xi,t)\left(\frac{\partial g}{\partial \xi}\right)^2 d\xi \tag{34}$$

In this case the splitting algorithm would start at first with $\check{G}(x,t)$, in order to obtain $g(x,t)$. The latter has to be used in (34) along with $u(x,t)$ (who comes from the integration of (33)) to compute $s_T(x,t)$. Once the speed is known it would be possible the advancing of $\check{G}(x,t)$ for another time step.

A third use of the theoretical asset developed so far concerns a multidimensional analysis.

We recall that we still stick to a hypothesis of unitary norm for the gradient of $\breve{G}(x,t)$. Under such assumption the curvature reads

$$\kappa=\nabla\cdot n=\nabla^2 G \qquad (35)$$

so that in the Fourier domain we have the relation

$$\widetilde{\kappa}=-\xi^2\widetilde{G} \qquad (36)$$

Recalling for the G-equation with the unitary-gradient assumption, in the Fourier domain, it holds

$$\frac{\partial\widetilde{G}}{\partial t}=\widetilde{s_T} \qquad (37)$$

Then after multiplying both sides by the square of ξ we have

$$\frac{\partial(\xi^2\widetilde{G})}{\partial t}=\xi^2\widetilde{s_T} \qquad (38)$$

And finally

$$\widetilde{s_T}=-\left(\frac{\partial}{\partial t}\frac{\widetilde{\kappa}}{\xi^2}\right) \qquad (39)$$

Anti-transorming the last equation it holds

$$s_T(\boldsymbol{x},t)=\frac{-1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp(-i\xi x)\frac{\partial}{\partial t}\left(\frac{\widetilde{\kappa}}{\xi^2}\right)d\xi \qquad (40)$$

Equation (40) bears some explicit result: in order to avoid a null consumption speed the curvature field κ cannot be constant in time; if the curvature depends on time with a power law then the consumption speed with a power law less of 1 (which means that for a curvature filed with linear time dependence the consumption speed is constant in time); if the curvature field is proportional to ξ^2 then the consumption speed is spatially reduced to a delta-function.

Conclusions

In the present proceedings a promising approach has been further explored to derive the Michelson-Sivashinsky equation in the framework of the G-equation. This has been possible by making use of a Fourier analysis of a G-field averaged by a PDF, with the latter that is assumed from the physics of the process. At least three possibilities of further improvement of the model have been derived, two of them that stick to a mono-dimensional analysis while the third one with results concerning the relationship between speed and curvature.

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