Front Curvature Evolution and Hydrodynamics Instabilities

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Abstract
It is known that hydrodynamic instabilities in turbulent premixed combustion are described by the Michelson-Sivashinsky (MS) equation. A model of the flame front propagation based on the G-equation and on stochastic fluctuations imposed to the mean flame position is considered. By comparing the governing equation of this model and the MS equation, an equation is derived for the front curvature computed in the mean flame position. The evolution in time of the curvature emerges to be driven by the inverse of the dispersion relation and by the nonlinear term of the MS equation.

Introduction

In this Extended Abstract we proceed further with the research presented in [1,2,3,4]. A model to study turbulent premixed combustion is developed on the basis of the G-equation and the introduction of stochastic fluctuations for the flame position [1]. The evolution equation for the resulting observable can be reduced to equations for the progress variable by choosing the proper probability density function (PDF) of the fluctuations. This approach reduces to the G-equation when there are no fluctuations [1] and to the Zimont equation when there are Gaussian fluctuations [1,2]. The proper PDF of fluctuations for reduction to the Michelson-Sivashinsky (MS) equation was derived in [3,4]. Within this framework, the evolution equation of the front curvature computed in the mean flame position is derived.

G-Equation analysis

Let the scalar function $G(\boldsymbol{x},t), \boldsymbol{x} \in \mathbb{R}^n$, be a level surface that represents the front which splits the considered domain into burned and unburned sub-domains. Let $\boldsymbol{x}_c$ be a point on the level surface $G=c$ at the instant t_0 . The level surface propagates with a consumption speed given by the laminar burning velocity s_L in the normal direction ***n*** relative to the mixture element and its evolution is described by the following Hamilton-Jacobi equation where the flow velocity field is ***u***

$$\frac{\partial G}{\partial t}+\boldsymbol{u}\cdot\nabla G=s_L\|\nabla G\| \quad , \quad \boldsymbol{n}=\frac{\nabla G}{\|\nabla G\|} \quad . \tag{1}$$

Let the front motion be described by the random process $\boldsymbol{X}_{\boldsymbol{c}}^{\boldsymbol{\omega}}(\hat{\boldsymbol{x}},t)$ where ω labels any independent realization and the mean value of $\boldsymbol{X}_{\boldsymbol{c}}^{\boldsymbol{\omega}}$ be denoted by $\langle \boldsymbol{X}^{\boldsymbol{\omega}}(\hat{\boldsymbol{x}},t)\rangle=\hat{\boldsymbol{x}}(t)$, then if $P_c\left(\boldsymbol{x}_{\boldsymbol{c}};t|\hat{\boldsymbol{x}}\right)$ is the corresponding PDF, with initial condition $P_c\left(\boldsymbol{x}_{\boldsymbol{c}};t_0|\hat{\boldsymbol{x}}\right)=\delta\left(\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{0}}\right)$, the mean flame position is given by the integral

$$\langle \boldsymbol{x}_{\boldsymbol{c}}\rangle=\int_{\mathbb{R}^n}\boldsymbol{x}_{\boldsymbol{c}}P_c(\boldsymbol{x}_{\boldsymbol{c}};t|\hat{\boldsymbol{x}})\,d\boldsymbol{x}_{\boldsymbol{c}}=\hat{\boldsymbol{x}}(t) \ . \tag{2}$$

Introducing $\check{G}(\hat{\boldsymbol{x}},t)$, with $\check{G}(\hat{\boldsymbol{x}},t_0)=\check{G}(\boldsymbol{x}_{\boldsymbol{0}},t_0)=c$, as the implicit formulation of the mean flame position $\hat{\boldsymbol{x}}$, the ensemble averaging of (1) gives [5]

$$\frac{\partial \check{G}}{\partial t}+\hat{\boldsymbol{u}}\cdot\nabla\check{G}=\widehat{s_L\boldsymbol{n}}\cdot\nabla\check{G} \, . \tag{3}$$

Since the G-equation can be derived on the basis of considerations about symmetries, there is a unique model for the RHS term of equation (3) providing a relation between the laminar burning velocity s_L and the turbulent burning velocity s_T [5], i.e.

$$\widehat{s_L\boldsymbol{n}}=s_T\check{\boldsymbol{n}} \quad , \quad \check{\boldsymbol{n}}=\frac{\nabla\check{G}}{\|\nabla\check{G}\|} \quad . \tag{4}$$

Finally, combining equation (3) and (4), the *G*-equation that describes the surface motion along the mean flame position results to be

$$\frac{\partial \check{G}}{\partial t}+\hat{\boldsymbol{u}}\cdot\nabla\check{G}=s_T\|\nabla\check{G}\| \ . \tag{5}$$

Let the curvature of the front be defined as $\check{\kappa}=-\nabla\cdot\frac{\nabla\check{G}}{\|\nabla\check{G}\|}$. Under the assumptions of null-mean velocity field and unitary-gradient G field, i.e., $\hat{u}=\boldsymbol{0}$, $\|\nabla\check{G}\|=1$, the evolution equation of the front curvature is

$$\frac{\partial\check{\kappa}}{\partial t}=-\Delta s_T \ , \tag{6}$$

and in the Fourier domain

$$\tilde{\kappa}=\xi^2\tilde{G} \ , \tag{7}$$

where the Fourier integral transform and anti-transform are defined as

$$\tilde{f}(\xi)=\int_{\mathbb{R}^n}\exp(+i\xi x)f(x)dx \ , \ f(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp(-i\xi x)\tilde{f}(\xi)d\xi \ , \tag{8}$$

and the turbulent burning velocity reads

$$s_T(\boldsymbol{x},t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp(-i\xi x)\frac{\partial\tilde{\kappa}}{\partial t}\frac{d\xi}{\xi^2} \ . \tag{9}$$

Michelson-Sivashinsky Equation analysis

The 1D M-S equation reads [6]

$$\frac{\partial g}{\partial t}=\frac{\partial^2 g}{\partial x^2}-\left(\frac{\partial g}{\partial x}\right)^2-D_x^1 g \ , \tag{10}$$

where D_x^1 is the fractional derivative of order 1 in the Riesz-Feller sense with Fourier symbol $-|\xi|$, which differs from the classical first derivative, being related to the Hilbert transform by the formula

$$D_x^1 g=\frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{+\infty}\frac{g(x')}{(x'-x)}dx' \ . \tag{11}$$

Let us introduce the field $g(\boldsymbol{x},t)$ in analogy with [1], i.e.,

$$g(x,t)=\int_{-\infty}^{+\infty}\check{G}(\hat{x},t)P_c(x-\hat{x},t)d\hat{x} \ . \tag{12}$$

It is well-known that the dispersion relation of equation (10) is, see Fig. 1(a),

$$e^{-\xi^2 t+|\xi|t} . \tag{13}$$

This suggests the following fractional differential equation for the PDF of the fluctuations of the front position:

$$\frac{\partial P_C}{\partial t} = \Delta P_C - (-\Delta)^{1/2} P_C, \quad P_C(x,0) = \delta(x). \tag{14}$$

Actually, the dispersion relation (13) is the Fourier transform of the Green function of (14).
It is here highlighted that the PDF of fluctuations which solves (14) emerges to be a quasi-probability distribution showing negative values that requires high care, see Fig. 1(b). However, these negative values can be interpreted as due to local extinction phenomena when local reversibility of the value of the progress variable occurs following from the entering of fresh mixture into a volume just now fully burned. This effect can be ascribed to the so-called counter-gradient that is generated by the density difference between reactants and products.

(a) (b)

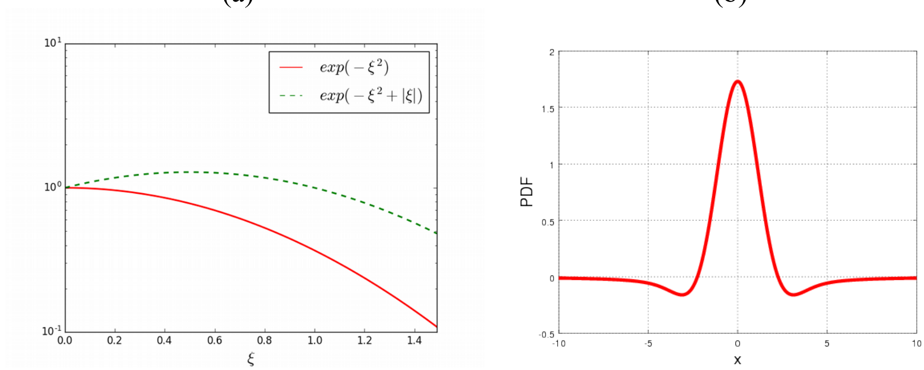

Figure 1: (a) Influence of non-locality on the dispersion relation (13). (b) Quasi-probability distribution solution of (14).

The evolution equation of the function $g(x,t)$ is:

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} - D_x^1 g + \int_{-\infty}^{+\infty} s_T(\hat{x},t) P_C(x-\hat{x},t)\, d\hat{x}. \tag{15}$$

Comparing (10) and (15) we have

$$-\left(\frac{\partial g}{\partial x}\right)^2 = \int_{-\infty}^{+\infty} s_T(\hat{x}t) P_C(x-\hat{x}t)\, d\hat{x} \equiv \omega, \tag{16}$$

that in Fourier domain reads

$$\tilde{\omega}=\tilde{s}_T\tilde{P}_c=\tilde{s}_T\exp(-\xi^2 t+|\xi|t) \ , \tag{17}$$

and the turbulent burning velocity turns out to be

$$s_T(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-i\,\xi\,x)\exp(\xi^2 t-|\xi|t)\,\tilde{\omega}(\xi,t)\,d\xi \ . \tag{18}$$

Front Curvature analysis

Equalling the RHS of (9) and (18) we obtain the following relationship between the nonlinear term and the dispersion relation of M-S equation (10), and the front curvature:

$$\frac{\partial\tilde{\kappa}}{\partial t}=\exp(\xi^2 t-|\xi|t)\,\xi^2\tilde{\omega} \quad , \tag{19}$$

that is,

$$\tilde{\kappa}=\tilde{\kappa}_0+\xi^2\int_0^t\exp(\xi^2\tau-|\xi|\tau)\,\tilde{\omega}\,d\tau \quad , \tag{20}$$

and after anti-transforming we get

$$\check{k}(x,t)=\check{k}_0(x)-\int_0^t\left\{\int_{-\infty}^{+\infty}\omega(x-\mu,\tau)\frac{\partial^2 h}{\partial\mu^2}\,d\mu\right\}d\tau \ , \tag{21}$$

where $h(x,t)$ is the solution of

$$\frac{\partial h}{\partial t}=-\frac{\partial^2 h}{\partial x^2}+D_x^1 h; \quad h(x,0)=\delta(x) \ . \tag{22}$$

To conclude, the evolution equation of the flame curvature in the mean flame position is

$$\frac{\partial\check{k}(x,t)}{\partial t}=-\int_{-\infty}^{+\infty}\omega(x-\mu,t)\frac{\partial^2 h}{\partial\mu^2}\,d\mu \ . \tag{23}$$

Fixed $\xi=\xi_0$, $t=1$ and defining $\delta\tilde{\kappa}=\tilde{\kappa}-\tilde{\kappa}_0$ we note that if $|\xi_0|<1$

$$\xi^2\int_0^t\exp(\xi^2 t-|\xi|t)\,\tilde{\omega}\,d\tau<\int_0^t\tilde{\omega}\,d\tau \ , \tag{24}$$

and inequality (24) is inverted in the case $|\xi_0|>1$.
It is worth noting that, when $|\xi_0|=1$, the reference length scale is considered and the evolution is driven only by the non-linear term ω of the M-S equation.
We finally get to the chain of inequalities

$$\delta\tilde{\kappa}_{|\xi|>1} > \delta\tilde{\kappa}_{|\xi|=1} > \delta\tilde{\kappa}_{|\xi|<1} \ . \qquad (25)$$

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Corrigendum
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