SOME REMARKS ON THE L^p REGULARITY OF SECOND DERIVATIVES OF SOLUTIONS TO NON-DIVERGENCE ELLIPTIC EQUATIONS AND THE DINI CONDITION

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Abstract. In this note we prove an end-point regularity result on the L^p integrability of the second derivatives of solutions to non-divergence form uniformly elliptic equations whose second derivatives are a priori only known to be integrable. The main assumption on the elliptic operator is the Dini continuity of the coefficients. We provide a counterexample showing that the Dini condition is somehow optimal. We also give a counterexample related to the BMO regularity of second derivatives of solutions to elliptic equations. These results are analogous to corresponding results for divergence form elliptic equations in [3, 15].

1. Introduction

In this note we investigate some regularity issues for solutions to non-divergence form elliptic equations whose second derivatives are locally integrable. Given an open bounded domain $\Omega \subset \mathbb{R}^n$, we will assume that $A(x) = (a_{ij}(x))$ is a real symmetric matrix such that there is a $\lambda > 0$ verifying

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda^{-1}\,|\xi|^2\,, \quad \text{for any } \xi \in \mathbb{R}^n,\ x \in \Omega.$$

Here we deal with solutions of operators of the form

$$\mathcal{L}u = tr\left(AD^2u\right) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u, \tag{1.1}$$

where the entries of the matrix A are continuous functions in $\overline{\Omega}$.

We recall the reader the following regularity fact [13, Lemma 9.16]:

Lemma 1. *Let p, q be such that $1 < p < q < \infty$ and f be in $L^q(\Omega)$. If u in $W^{2,p}_{loc}(\Omega)$ verifies $\mathcal{L}u = f$ in Ω, then $u \in W^{2,q}_{loc}(\Omega)$.*

The previous result does not cover the case $p = 1$ and, as far as we know, this case has not been considered in the literature. It is the purpose of this note to deal with it. We remark that Lemma 1 is true under the mere assumption of the continuity of the coefficients. However, as we shall see, this mild assumption is

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not enough in order to improve the integrability of the second derivatives of $W^{2,1}_{loc}$ solutions. On the contrary, a Dini-type condition on the coefficients is sufficient for this purpose and it is optimal. Here we will consider the following Dini condition:

Definition 1. A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *Dini continuous* in $\overline{\Omega}$ if there is continuous a non-decreasing function $\theta : [0, +\infty) \to [0, +\infty)$ verifying

$$|f(x) - f(y)| \leq \theta(|x-y|), \text{ for any } x, y \in \Omega$$

and such that

$$\int_0^1 \frac{\theta(t)}{t}\, dt < +\infty \tag{1.2}$$

and

$$\theta(2t) \leq 2\theta(t), \text{ for } t \in (0, \tfrac{1}{2}). \tag{1.3}$$

We will say that θ is the *Dini modulus of continuity* of f.

Condition (1.3) is not restrictive. In fact, as we learnt from [1, Remark 1], any modulus of continuity satisfying (1.2) can be dominated by

$$\tilde{\theta}(t) = t \sup_{\tau \in [t,1]} \frac{\theta(\tau)}{\tau},$$

which is again a Dini modulus of continuity such that $\tilde{\theta}(t)/t$ is non-increasing. The later implies (1.3) for $\tilde{\theta}$.

Before stating our results we first briefly review the case of elliptic equations in divergence form. In this situation, motivated by a question raised in [21] and the results in [14], H. Brezis proved the following [3, Theorems 1 and 2].

Theorem 1. *Let A be a uniformly elliptic matrix such that A is Dini continuous in $\overline{\Omega}$. Let u in $W^{1,1}(\Omega)$ solve*

$$\int_\Omega A\nabla u \cdot \nabla \varphi \, dx = 0, \quad \textit{for any } \varphi \textit{ in } C_0^\infty(\Omega).$$

Then, for any $1 < p < \infty$, u is in $W^{1,p}_{loc}(\Omega)$ and

$$\|u\|_{W^{1,p}(K)} \leq C\|u\|_{W^{1,1}(\Omega)} \tag{1.4}$$

for any compact subset $K \subset \Omega$, where C depends on n, p, K, the ellipticity constant, Ω and the uniform modulus of continuity of the coefficients, but not on the Dini modulus of continuity.

The independence of the constant in (1.4) with respect to the Dini modulus of continuity by no means implies that this result is true when the coefficients are merely continuous in $\overline{\Omega}$: a counterexample to such assertion is given in [15].

In the context of non-divergence form elliptic equations, the main result proved in this note is the following.

Theorem 2. *Assume that the coefficients of $\mathcal{L}$ are Dini continuous in $\overline{\Omega}$ and let u in $W^{2,1}(\Omega)$ satisfy $\mathcal{L}u = f$, a.e. in Ω with f in $L^p(\Omega)$, for some $1 < p < \infty$. Then u is in $W^{2,p}_{loc}(\Omega)$ and*

$$\|u\|_{W^{2,p}(K)} \leq C\left[\|u\|_{W^{2,1}(\Omega)} + \|f\|_{L^p(\Omega)}\right],$$

for any compact subset $K \subset \Omega$, where C depends on n, p, K, λ, Ω and the uniform modulus of continuity of the coefficients, but not on the Dini modulus of continuity.

Similarly to the case of divergence form elliptic equations, the Dini condition on A is the optimal to derive such a result. Here we give a counterexample inspired by [8, Section 3], showing that Theorem 2 is false when the coefficients of $\mathcal{L}$ are not Dini continuous.

Theorem 3. *There is an operator $\mathcal{L}$ with continuous coefficients in $\overline{B}_1$, which are not Dini continuous at $x = 0$, and a solution u in $W^{2,1}(B_1) \cap W_0^{1,1}(B_1)$ of $\mathcal{L}u = 0$ such that u is not in $W^{2,p}(B_{\frac{1}{2}})$, for any $p > 1$.*

Concerning the other end-point in the scale of L^p spaces, we recall that the singular integrals theory [23, Chapter IV] allows to prove that weak solutions [13, Chapter 8] to $\Delta u = f$ in B_2 have generalized second order derivatives in $BMO(B_1)$ when $f \in L^\infty(B_2)$. Moreover, the Laplace operator can be perturbed in order to obtain similar results for elliptic operators (1.1) with Dini continuous coefficients [7] or with A verifying

$$|A(x) - A(y)| \leq C/[1 + |\log|x - y||], \tag{1.5}$$

for some $C > 0$ sufficiently small [5, Theorem A, ii and Corollary 4.1].

As far as we know, there are no counterexamples in the literature showing that mere continuity of the coefficients is not enough to prove that the second derivatives of solutions of elliptic equations do not belong to BMO in general. The next counterexample, which is a modification of [15, Proposition 1.6], fills this gap.

Theorem 4. *There exists an operator $\mathcal{L}$ with continuous coefficients in $\overline{B}_1$, which are not Dini continuous at $x = 0$, and a solution u in $W^{2,p}(B_1) \cap W_0^{1,p}(B_1)$ of $\mathcal{L}u = 0$, $1 < p < \infty$, such that D^2u is not in $BMO(B_{\frac{1}{2}})$.*

The counterexample in Theorem 4 is sharp because its coefficient matrix A verifies (1.5) for x, y in B_1, for some fixed $C > 0$.

The main ingredients in the proof of Theorem 2 are the Sobolev inequality and the boundedness of solutions to equations involving the formal adjoint operator $\mathcal{L}^*$ given by

$$\mathcal{L}^* v = \sum_{i,j=1}^{n} \partial_{ij}(a^{ij} v).$$

In order to make sense of the solutions associated to the operator $\mathcal{L}^*$ when the coefficients of $\mathcal{L}$ are only continuous we must consider distributional or weak solutions to the *adjoint* equation. For our purposes, we need to deal with boundary value problems of the form

$$\begin{cases} \mathcal{L}^* w = div^2\Phi + \eta, & \text{in } \Omega, \\ w = \psi + \frac{\Phi\nu\cdot\nu}{A\nu\cdot\nu}, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Phi = (\varphi^{kl})_{k,l=1}^n$, $div^2\Phi = \sum_{k,l=1} \partial_{kl}\varphi^{kl}$, with

$$\Phi \text{ in } L^p(\Omega),\ \eta \text{ in } L^p(\Omega),\ \psi \text{ in } L^p(\partial\Omega, d\sigma),\ 1 < p < \infty. \tag{1.7}$$

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain with unit exterior normal vector $\nu = (\nu_1, \ldots, \nu_n)$, Φ, ψ and η verify (1.7), let $\mathcal{L}$ be as in (1.1), $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We say that w in $L^p(\Omega)$ is an *adjoint solution* of (1.6) if w satisfies

$$\int_\Omega w\,\mathcal{L}u\,dy = \int_\Omega tr(\Phi D^2u)\,dy + \int_\Omega \eta u\,dy + \int_{\partial\Omega} \psi A\nabla u\cdot\nu\,d\sigma(y), \tag{1.8}$$

for any u in $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$.

Later we shall explain why this definition makes sense. At first, the boundary conditions in (1.6) may look strange. However, if we formally multiply (1.6) by a test function u in $C^\infty(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$, assume that w is in $C^\infty(\overline{\Omega})$ and integrate by parts, taking into account that $\nabla u = (\nabla u \cdot \nu)\nu$ on $\partial\Omega$, we arrive at (1.8).

We will also consider *local* adjoint solutions of

$$\mathcal{L}^* w = div^2 \Phi + \eta \quad \text{in } \Omega,$$

i.e., solutions which do not satisfy any specified boundary condition. Such local solutions are those in $L^p_{loc}(\Omega)$ that verify (1.8), when u is in $W_0^{2,p'}(\Omega)$; thus, the boundary integrals in (1.8) are omitted.

This kind of adjoint solutions have been already studied in the literature. For instance, in [22, 2, 12, 11, 9, 19] solutions of (1.6) with $\Phi = 0$ are studied under low regularity assumptions on either the coefficients of $\mathcal{L}$ or the boundary of the domain. Moreover, when the data and the boundary of the domain involved in (1.6) are smooth, the weak formulation (1.8) can be recasted in such a way that the regularity theory in [18] or [20] can be used to prove that w is smooth and solves (1.6) in a classical sense.

For our purposes we need to prove the existence and uniqueness of such adjoint solutions.

Lemma 2. *Let $1 < p < \infty$ and assume that (1.7) holds. Then, there exists a unique adjoint solution w in $L^p(\Omega)$ of (1.6). Moreover, the following estimate holds*

$$\|w\|_{L^p(\Omega)} \le C \left[\|\Phi\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|\psi\|_{L^p(\partial\Omega)} \right], \tag{1.9}$$

where C depends on Ω, p, n, λ and the continuity of A.

This result follows from the so-called *transposition* or duality method [18, 20], which relies on the existence and uniqueness of $W^{2,p'} \cap W_0^{1,p'}(\Omega)$ solutions to $\mathcal{L}u = f$.

Finally, the proof of Theorem 2 requires the boundedness of certain adjoint solutions to problems of the form (1.6) with $\Phi = 0$. It is at this point where the Dini continuity of the coefficients plays a role. However, and similarly to what it was done in [3], we only employ the boundedness of these adjoint solutions in a qualitative form, that is, we do not need an specific estimate of the boundedness of those adjoint solutions.

In order to prove the boundedness of the specific adjoint solutions, we employ a perturbative technique based on ideas first stablished in [4, 6] and used in [17] to prove the continuity of the gradient of solutions to divergence-form second order elliptic systems with Dini continuous coefficients. Accordingly, we do not only prove that those adjoint solutions are bounded but also their continuity.

Lemma 3. *Let $\zeta \in C_0^\infty(B_3)$, $1 < p < \infty$ and assume that the elliptic operator $\mathcal{L}$ has Dini continuous coefficients in B_4. Then, if v in $L^p(B_4)$ satisfies*

$$\int_{B_4} v\, \mathcal{L}u \, dx = \int_{B_4} \zeta u \, dx, \ \textit{for any} \ \ u \in W^{2,p'}(B_4) \cap W_0^{1,p'}(B_4),$$

v is continuous in $\overline{B}_3$.

The paper is organized as follows: in Section 2 we give the counterexamples stated in Theorems 3 and 4; in Section 3 we prove Lemma 2 using the duality method; in Section 4 we prove that certain adjoint solutions are continuous and in Section 5 we prove Theorem 2.

2. Counterexamples

In this section we give two counterexamples. Both of them arise as solutions of uniformly elliptic operators of the form

$$\mathcal{L}_\alpha u = tr\left[\left(I + \alpha(r)\frac{x}{r} \otimes \frac{x}{r}\right) D^2 u\right], \tag{2.1}$$

where $(x \otimes x)_{ij} = x_i x_j$, $r = |x|$, with α is a continuous radial function in $\overline{B_1}$, $\alpha(0) = 0$.

Proof of Theorem 3. If we look for a radial solution u of (2.1), we find that u must satisfy

$$\mathcal{L}_\alpha u = (\alpha(r) + 1)u'' + \frac{n-1}{r}u' = 0. \tag{2.2}$$

We choose

$$u(r) = \int_r^1 t^{1-n}\left(\log\frac{R}{t}\right)^{-\gamma} dt, \quad \gamma > 1,$$

with $R > 1$ to be chosen. Then

$$u'(r) = -r^{1-n}\left(\log\frac{R}{r}\right)^{-\gamma}$$

$$u''(r) = r^{-n}\left(\log\frac{R}{r}\right)^{-\gamma}\left[n - 1 - \gamma\left(\log\frac{R}{r}\right)^{-1}\right].$$

Hence, $u \in W^{2,1}(B_1) \cap W_0^{1,1}(B_1)$ but $D^2 u \notin L^p(B_1)$ for any $p > 1$, when $\gamma > 1$ and $R > 1$. Solving (2.2) for α we obtain

$$\alpha(r) = \frac{\gamma}{(n-1)\log\frac{R}{r} - \gamma},$$

which ensures the uniform ellipticity and the continuity of the coefficients of $\mathcal{L}_\alpha$ over $\overline{B}_1$, when R is sufficiently large. However, α is not Dini continuous at $x = 0$. $\square$

Proof of Theorem 4. Let $\varphi \in C^2((0,1))$, $\alpha \in C([0,1])$ and define

$$u(x) = x_1 x_2 \varphi(r).$$

A computation shows that

$$\mathcal{L}_\alpha u = \frac{x_1 x_2}{r^2}\left[(n+3)r\varphi' + r^2\varphi'' + \alpha(2\varphi + 4r\varphi' + r^2\varphi'')\right].$$

Choosing $\varphi(r) = \left(\log\frac{R}{r}\right)^2$ for some $R > 1$ yields

$$\mathcal{L}_\alpha u = \frac{x_1 x_2}{r^2}\left[1 + \alpha - (2 + n + 3\alpha)\log\frac{R}{r} + \alpha\left(\log\frac{R}{r}\right)^2\right],$$

which is identically zero in $B_1(0)$ provided that

$$\alpha(r) = \frac{(2+n)\log\frac{R}{r} - 1}{\left(\log\frac{R}{r}\right)^2 - 3\log\frac{R}{r} + 1},$$

and $R > 1$ is taken large enough in order to ensure the uniform ellipticity and the continuity of the coefficients of $\mathcal{L}_\alpha$ in $\overline{B}_1$. A computation shows that

$$\partial_{12}u \geq \frac{1}{2}\left(\log\frac{R}{r}\right)^2 \text{ on } \overline{B}_1,$$

when $R > 1$ is large enough. Moreover, for any $c \in \mathbb{R}$ there is $\varepsilon = \varepsilon(c)$ such that $\left(\log\frac{R}{r}\right)^2 \geq 4|c|$ in B_ε. Thus

$$\int_{B_{\frac{1}{2}}} e^{N|\partial_{12}u - c|}dx \geq \int_{B_\varepsilon} e^{\frac{N}{4}(\log\frac{R}{r})^2}dx = +\infty, \text{ for any } N > 0,\ c \in \mathbb{R}.$$

By the John-Nirenberg inequality [16], $\partial_{12}u$ cannot belong to $BMO(B_1)$. □

3. Existence of adjoint solutions

We recall the following well known existence result for the Dirichlet problem for non-divergence form elliptic equations [13, Theorem 9.15, Lemma 9.17].

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{1,1}$ domain, f be in $L^p(\Omega)$ and $1 < p < \infty$. Then, there exists a unique $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\mathcal{L}u = f$ a.e. in Ω. Moreover, there is a constant $C > 0$ depending on Ω, p, n, λ and the modulus of continuity of A such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{3.1}$$

An easy consequence of Lemma 4 is the existence and uniqueness of adjoint solutions to (1.6) stated in Lemma 2.

Proof of Lemma 2. We construct the solution by means of *tranposition.* If p' is the conjugate exponent of p, we define the functional $T : L^{p'}(\Omega) \to \mathbb{R}$ by

$$T(f) = \int_\Omega tr\left(\Phi D^2 u\right)\, dx + \int_\Omega \eta u\, dx + \int_{\partial\Omega} \psi A\nabla u \cdot \nu\, d\sigma, \tag{3.2}$$

where u in $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ verifies $\mathcal{L}u = f$, a.e. in Ω. Combining (3.1), the trace inequality [10, §5.5, Theorem 1], (3.2) and Hölder's inequality, it is straightforward to check that

$$|T(f)| \leq C\|f\|_{L^{p'}(\Omega)}\left[\|\Phi\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|\psi\|_{L^p(\partial\Omega)}\right],$$

where $C = C(A, \Omega, p, n)$. Hence T is a bounded functional on $L^{p'}(\Omega)$ and by the Riesz representation Theorem, there is a unique w in $L^p(\Omega)$ such that

$$T(f) = \int_\Omega wf\, dx, \text{ for any } f \in L^{p'}(\Omega). \tag{3.3}$$

Moreover,

$$\|w\|_{L^p(\Omega)} \leq C\left[\|\Phi\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|\psi\|_{L^p(\partial\Omega)}\right].$$

Now, combining (3.2) and (3.3), it is clear that w is the unique adjoint solution to (1.6). □

4. Proof of Lemma 3

For the proof of Lemma 3 we need first the following Lemma.

Lemma 5. *Let* $\Phi \in L^p(B_1)$, $\eta \in L^\infty(B_1)$, $w \in L^p(B_1)$, $1 < p < \infty$ *and* $\mathcal{L}$ *be an operator like* (1.1) *with continuous coefficients and* $A(0) = I$, *the identity matrix. Then, if*

$$\mathcal{L}^* w = div^2 \Phi + \eta, \ \ in \ B_1,$$

there exists a harmonic function h *in* $B_{\frac{3}{4}}$ *such that*

$$\begin{aligned} &\|h\|_{L^p(B_{\frac{3}{4}})} \le M \|w\|_{L^p(B_1)}, \\ &\|w - h\|_{L^p(B_{\frac{3}{4}})} \le M \left[\|\Phi\|_{L^p(B_1)} + \|A - I\|_{L^\infty(B_1)} \|w\|_{L^p(B_1)} + \|\eta\|_{L^\infty(B_1)} \right], \end{aligned} \tag{4.1}$$

where M *depends on* p, n, λ *and the modulus of continuity of* A.

Proof of Lemma 5. We first prove Lemma 5 assuming that the coefficients of $\mathcal{L}$ and data are smooth in $\overline{B}_1$. However, the constants in the estimate will only depend on p, λ, n and the modulus of continuity of A. Under these assumptions, the regularity theory [20, 18], implies that w is smooth in B_1. By Fubini's theorem, there is $\frac{3}{4} \le t \le 1$ such that

$$\|w\|_{L^p(\partial B_t)} \le 4^{-\frac{1}{p}} \|w\|_{L^p(B_1)}. \tag{4.2}$$

Using Lemma 2 we can find a function h such that

$$\begin{cases} \Delta^* h = 0, & \text{in } B_t, \\ h = w, & \text{on } \partial B_t, \end{cases}$$

in the sense of (1.6). Of course, h is harmonic in the interior of B_t. Moreover, the estimate provided by Lemma 2 together with (4.2) imply

$$\|h\|_{L^p(B_t)} \le M \|w\|_{L^p(\partial B_t)} \le M 4^{-\frac{1}{p}} \|w\|_{L^p(B_1)}, \tag{4.3}$$

with $M = M(p, n)$. Then $w - h$ satisfies

$$\begin{aligned} \int_{B_t} (w - h)\, \mathcal{L} u dx &= \int_{B_t} tr \left[h(I - A) D^2 u \right] dx + \int_{B_t} tr \left[\Phi D^2 u \right] dx \\ &+ \int_{B_t} \eta u dx + \int_{\partial B_t} w \,(A - I)\, \nabla u \cdot \nu d\sigma \\ &= \int_{B_t} tr \left[h(I - A) D^2 u \right] dx + \int_{B_t} tr \left[\Phi D^2 u \right] dx \\ &+ \int_{B_t} \eta u dx + \int_{\partial B_t} w \frac{(A - I)\, \nu \cdot \nu}{A \nu \cdot \nu} A \nabla u \cdot \nu d\sigma \end{aligned} \tag{4.4}$$

for any $u \in W^{2,p'}(B_t) \cap W_0^{1,p'}(B_t)$. Therefore, $w - h$ is an adjoint solution to a problem which falls into the conditions of Lemma 2 and we can apply (1.9) to the equation (4.4) to get that

$$\begin{aligned} \|w - h\|_{L^p(B_t)} \le M \big[&\|A - I\|_{L^\infty(B_t)} \|h\|_{L^p(B_t)} \\ &+ \|\Phi\|_{L^p(B_t)} + \|A - I\|_{L^\infty(B_t)} \|w\|_{L^p(\partial B_t)} + \|\eta\|_{L^p(B_t)} \big], \end{aligned}$$

which together with (4.3) imply the desired estimate. Finally, an approximation argument allows us to derive the same estimate under the more general conditions mentioned above. ◻

The perturbative technique used in the proof of Lemma 3 is based on the local smallness of certain quantities. We may assume that $A(0) = I$ and that θ is a Dini modulus of continuity for A on B_4. For this reason, if v and ζ verify the conditions in Lemma 3, it is handy to define for $0 < t,\ \delta \le 1$,

$$\omega(t) = t^2 + \theta(t), \quad \overline{\delta} = M^{-1}\delta^{\frac{n}{p}} \frac{\omega(\delta)}{1 + \|v\|_{L^p(B_1)} + \|\zeta\|_{L^\infty(B_1)}},$$

where M is the constant in (4.1), and to consider the rescaled functions

$$v_\delta(x) = \overline{\delta} v(\delta x), \quad \zeta_\delta(x) = \overline{\delta}\delta^2\zeta(\delta x). \tag{4.5}$$

From (1.3)

$$\omega(4t) \le 16\,\omega(t), \text{ for } t \le 1/4 \tag{4.6}$$

and the dilation and rescaling yield

$$\|v_\delta\|_{L^p(B_1)} \le M^{-1}\omega(\delta), \quad \|\zeta_\delta\|_{L^\infty(B_1)} \le M^{-1}\delta^2\omega(\delta). \tag{4.7}$$

Also,

$$\mathcal{L}_\delta^* v_\delta = \zeta_\delta, \text{ in } B_1, \quad \text{with } \mathcal{L}_\delta u = tr\left(A(\delta x) D^2 u\right). \tag{4.8}$$

Next, we show by induction that there are $C > 0$, $0 < \delta \le 1$ and harmonic functions h_k in $4^{-k}B_{\frac{3}{4}}$, $k \ge 0$, such that

$$\begin{aligned} &C^{-1}\|h_k\|_{L^p(4^{-k}B_{\frac{3}{4}})} + \|v - \sum_{j=0}^{k} h_j\|_{L^p(4^{-k}B_{\frac{1}{4}})} \le 4^{-k\frac{n}{p}}\omega(4^{-k}\delta), \\ &\|h_k\|_{L^\infty(4^{-k}B_{\frac{1}{2}})} + 4^{-k}\|\nabla h_k\|_{L^\infty(4^{-k}B_{\frac{1}{2}})} \le C\omega(4^{-k}\delta), \end{aligned} \tag{4.9}$$

where C depends on n, p, λ and the modulus of continuity of A.

When $k = 0$, (4.7), (4.8) and Lemma 5 applied to v_δ show that there is a harmonic function h_0 in $B_{\frac{3}{4}}$ such that

$$\|h_0\|_{L^p(B_{\frac{3}{4}})} \le M\|v_\delta\|_{L^p(B_1)} \le \omega(\delta),$$
$$\|v_\delta - h_0\|_{L^p(B_{\frac{3}{4}})} \le M\left[\theta(\delta)\|v_\delta\|_{L^p(B_1)} + \|\zeta_\delta\|_{L^\infty(B_1)}\right] \le \omega(\delta)^2.$$

By regularity of harmonic functions [10, §2.2.3c]

$$\|h_0\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla h_0\|_{L^\infty(B_{\frac{1}{2}})} \le C(n,p)\|h_0\|_{L^p(B_{\frac{3}{4}})} \le C(n,p)\omega(\delta).$$

Thus, (4.9) holds for $k = 0$, when C and δ satisfy

$$C^{-1} + \omega(\delta) \le 1 \text{ and } C \ge C(n,p). \tag{4.10}$$

Now, assume that (4.9) holds up to some $k \ge 0$ and define

$$A_{k+1}(x) = A(4^{-k-1}\delta x), \quad \mathcal{L}_{k+1}u = tr(A_{k+1}(x)D^2u)$$
$$G_{k+1}(x) = (I - A_{k+1}(x))\sum_{j=0}^{k} h_j(4^{-k-1}x).$$

Then, $W_{k+1}(x) = v_\delta(4^{-k-1}x) - \sum_{j=0}^{k} h_j(4^{-k-1}x)$ solves

$$\mathcal{L}_{k+1}^* W_{k+1}(x) = div^2 G_{k+1} + 4^{-2k-2}\zeta_\delta(4^{-k-1}x), \text{ in } B_1. \tag{4.11}$$

Using the induction hypothesis (4.9) and (4.6) , one finds that G_{k+1} satisfies

$$
\begin{aligned}
\|G_{k+1}\|_{L^p(B_1)} &\le |B_1|^{\frac{1}{p}}\theta(4^{-k-1}\delta)\sum_{j=0}^{k}\|h_j(4^{-k-1}\cdot)\|_{L^\infty(B_1)} \\
&\le \left[32\,C|B_1|^{\frac{1}{p}}\int_0^\delta \frac{\omega(t)}{t}\,dt\right]\theta(4^{-k-1}\delta).
\end{aligned} \tag{4.12}
$$

Besides, the inequality in the first line of (4.9) gives

$$
\|W_{k+1}\|_{L^p(B_1)} \le 4^{\frac{n}{p}}\omega(4^{-k}\delta). \tag{4.13}
$$

From (4.6), (4.11), (4.12) and (4.13), apply Lemma 5 to W_{k+1} to find that with the same M, there is a harmonic function $\tilde h_{k+1}$ in $B_{\frac34}$ such that

$$
\|\tilde h_{k+1}\|_{L^p(B_{\frac34})} \le 4^{2+\frac{n}{p}}M\omega(4^{-k-1}\delta). \tag{4.14}
$$

and

$$
\|W_{k+1}-\tilde h_{k+1}\|_{L^p(B_{\frac34})} \le M\left[32\,|B_1|^{\frac{1}{p}}C\int_0^\delta \frac{\omega(t)}{t}\,dt+\omega(\delta)\right]\omega(4^{-k-1}\delta).
$$

From standard interior estimates for harmonic functions and (4.14)

$$
\|\tilde h_{k+1}\|_{L^\infty(B_{\frac12})}+\|\nabla\tilde h_{k+1}\|_{L^\infty(B_{\frac12})} \le C(n,p)4^{2+\frac{n}{p}}M\omega(4^{-k-1}\delta).
$$

Setting, $h_{k+1}(x)=\tilde h_{k+1}(4^{k+1}x)$, the last three formulae and (4.10) show that the induction hypothesis holds when $C=2\,C(n,p)\left[4^{2+\frac{n}{p}}M+1\right]$ and δ is determined by the condition

$$
2M\left[32\,|B_1|^{\frac{1}{p}}C\int_0^\delta \frac{\omega(t)}{t}\,dt+\omega(\delta)\right] \le 1.
$$

On the other hand, for $|x|\le 4^{-k-1}$

$$
\begin{aligned}
|\sum_{j=0}^{k}h_j(x)-\sum_{j=0}^{\infty}h_j(0)| &\le \sum_{j=k+1}^{\infty}|h_j(0)|+4^{-k-1}\sum_{j=0}^{k}\|\nabla h_j\|_{L^\infty(4^{-k}B_{\frac14})} \\
&\le 16\,C\left(\int_0^{4^{-k}\delta}\frac{\omega(t)}{t}\,dt+4^{-k-1}\delta\int_{4^{-k-1}\delta}^{\delta}\frac{\omega(t)}{t^2}\,dt\right)
\end{aligned} \tag{4.15}
$$

Therefore, (4.9) together with (4.15) and (4.6) imply

$$
\begin{aligned}
&\fint_{4^{-k-1}B_1}|v_\delta(x)-\sum_{j=0}^{\infty}h_j(0)|\,dx \le \\
&\le 4^4C\left[\int_0^{4^{-k-1}\delta}\frac{\omega(t)}{t}dt+4^{-k-1}\delta\int_{4^{-k-1}\delta}^{\delta}\frac{\omega(t)}{t^2}dt+\omega(4^{-k-1}\delta)\right],
\end{aligned} \tag{4.16}
$$

when $k\ge 0$. Using Fubini's theorem it is easy to check that $t\int_t^1\frac{\omega(s)}{s^2}\,ds$ is a Dini modulus of continuity, one can verify that

$$
\sigma(t)=\int_0^t\frac{\omega(s)}{s}\,ds+t\int_t^1\frac{\omega(s)}{s^2}\,ds+\omega(t)
$$

is non-decreasing and derive that $\lim_{t\to 0^+}\sigma(t)\to 0$. Hence, from (4.16) and (4.5), we have proved that there are $C>0$, depending on λ, n and the modulus of Dini continuity of A, and a number $a(0)$ such that

$$\fint_{B_r} |v(x)-a(0)|dx \le C\sigma(r)\left[\|v\|_{L^p(B_1)}+\|\zeta\|_{L^\infty(B_1)}\right], \text{ when } 0<r\le 1. \tag{4.17}$$

Since $v\in L^p(B_4)$ is an adjoint solution in B_4, we can repeat the proof of (4.17) in balls of radius 1 centered at any point $\overline{x}$ in B_3. We note that the constant C and the modulus of continuity σ in (4.16) do not depend on the center of the ball, hence, for each $\overline{x}$ in B_3, we find a number $a(\overline{x})$ such that

$$\fint_{B_r(\overline{x})} |v(x)-a(\overline{x})|dx \le C\sigma(r)\left[\|v\|_{L^p(B_4)}+\|\zeta\|_{L^\infty(B_4)}\right], \text{ when } \ 0<r\le 1.$$

By Lebesgue's differentiation theorem, u and a are equal a.e. in B_3. Now, if $\overline{x}$ and $\overline{y}$ are in B_3 and $\frac{r}{2}\le|\overline{x}-\overline{y}|\le r$, we have

$$\begin{aligned}
|u(\overline{x})-u(\overline{y})| &\le \fint_{B_r(\overline{x})}|u(\overline{x})-u(x)|dx+\fint_{B_r(\overline{x})}|u(x)-u(\overline{y})|dx\\
&\lesssim \fint_{B_r(\overline{x})}|u(\overline{x})-u(x)|dx+\fint_{B_r(\overline{y})}|u(x)-u(\overline{y})|dx\\
&\lesssim \sigma(2r)\left[\|v\|_{L^p(B_4)}+\|\zeta\|_{L^\infty(B_4)}\right], \text{ when } \ 0<r\le 1/2.
\end{aligned}$$

which proves Lemma 3.

5. Proof of Theorem 2

It suffices to show that if u in $W^{2,1}(B_4)$ verifies $\mathcal{L}u=f$, with f in $L^p(B_4)$, $1<p<\infty$, then $u\in W^{2,q}(B_1)$, for some $q>1$. Let then η be a function in $C_0^\infty(B_2)$ with $\eta=1$ in B_1 and $0\le\eta\le 1$. Set $q=\min\left\{\frac{n}{n-1},p\right\}$ and let φ be in $C_0^\infty(B_3)$ with $\|\varphi\|_{L^{q'}(B_3)}\le 1$. We shall show that

$$\left|\int_{B_4}\partial_{kl}(u\eta)\varphi\,dx\right|\le C\left[\|f\|_{L^p(B_4)}+\|u\|_{W^{2,1}(B_4)}\right], \tag{5.1}$$

where C only depends on q, p, λ, n and the uniform modulus of continuity of the coefficients A, but not on the Dini modulus of continuity of A.

Let u_ε in $C^\infty(B_4)$ be a sequence of functions converging to u in $W^{2,1}_{loc}(B_4)$ as $\varepsilon\to 0$, then for any φ in $C_0^\infty(B_3)$ we have

$$\int_{B_4}\partial_{kl}(u\eta)\varphi\,dx=\lim_{\varepsilon\to 0}\int_{B_4}\partial_{kl}(u_\varepsilon\eta)\varphi\,dx.$$

By Lemma 2 with $\Omega=B_4$ and $p=q'$, for $k,l\in\{1,\dots,n\}$, there is a unique weak adjoint solution v in $L^{q'}(B_4)$ to

$$\begin{cases}\mathcal{L}^*v=\partial_{kl}\varphi, & \text{on } B_4,\\ v=0, & \text{on } \partial B_4.\end{cases}$$

That is, a function v in $L^{q'}(B_4)$ such that

$$\int_{B_4} v\,\mathcal{L}w\,dy=\int_{B_4}\varphi\,\partial_{kl}w\,dy,$$

for any w in $W^{2,q}(B_4) \cap W_0^{1,q}(B_4)$ and

$$\|v\|_{L^{q'}(B_4)} \leq C\|\varphi\|_{L^{q'}(B_3)} \leq C. \tag{5.2}$$

Observe that $u_\varepsilon \eta$ is in $W^{2,q}(B_4) \cap W_0^{1,q}(B_4)$, for any $\varepsilon > 0$. Thus,

$$\int_{B_4} \partial_{kl}(u_\varepsilon \eta)\varphi\, dx = \int_{B_4} v\, \mathcal{L}(u_\varepsilon \eta)\, dx. \tag{5.3}$$

Now, we want to take limits in (5.3) as $\varepsilon \to 0$. A priori, we only know that $\partial_{kl} u$ is in $L^1(B_4)$, so we can just assert that $\mathcal{L}(u_\varepsilon \eta) \to \mathcal{L}(u\eta)$ in $L^1(B_4)$ as $\varepsilon \to 0$. However, in order to take the limit as $\varepsilon \to 0$ inside of the integral in the right-hand side of (5.3) and because of the support properties of the functions involved, we only need to know that v is bounded in $\overline{B}_3$, which indeed is the case because of Lemma 3, with $\zeta = \partial_{kl}\varphi$. Hence, we obtain

$$\begin{aligned} \int_{B_4} \partial_{kl}(u\eta)\varphi\, dx = \int_{B_4} v\, \mathcal{L}(u\eta)\, dx = \int_{B_4} v\eta\, \mathcal{L}u\, dx + \int_{B_4} vu\, \mathcal{L}\eta\, dx \\ + 2\int_{B_4} vA\nabla u \cdot \nabla \eta\, dx \triangleq J_1 + J_2 + J_3. \end{aligned}$$

Now, Hölder's inequality, Sobolev's inequality and (5.2) yield

$$\begin{aligned} |J_1| &\leq \|v\|_{L^{q'}(B_4)} \|\mathcal{L}u\|_{L^q(B_4)} \leq C\|f\|_{L^p(B_4)}, \\ |J_2| &\leq M\|v\|_{L^{q'}(B_4)} \|u\|_{L^q(B_4)} \leq C\|u\|_{W^{1,1}(B_4)}, \\ |J_3| &\leq M\|v\|_{L^{q'}(B_4)} \|\nabla u\|_{L^q(B_4)} \leq C\|u\|_{W^{2,1}(B_4)}, \end{aligned}$$

which implies (5.1), and by density and duality

$$\|\partial_{kl}(u\eta)\|_{L^q(B_3)} \leq C\left[\|f\|_{L^p(B_4)} + \|u\|_{W^{2,1}(B_4)}\right].$$

Therefore, u is in $W^{2,q}(B_1)$ and

$$\|u\|_{W^{2,q}(B_1)} \leq C\left[\|f\|_{L^p(B_4)} + \|u\|_{W^{2,1}(B_4)}\right],$$

which is the desired estimate.

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