

EULER REFLEXION FORMULAS FOR MOTIVIC MULTIPLE ZETA FUNCTIONS

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ABSTRACT. We introduce a new notion of \boxtimes -product of two integrable series with coefficients in distinct Grothendieck rings of algebraic varieties, preserving the integrability and commuting with the limit of rational series. In the same context, we define a motivic multiple zeta function with respect to an ordered family of regular functions, which is integrable and connects closely to Denef-Loeser's motivic zeta functions. We also show that the \boxtimes -product is associative in the class of motivic multiple zeta functions.

Furthermore, a version of the Euler reflexion formula for motivic zeta functions is nicely formulated to deal with the \boxtimes -product and motivic multiple zeta functions, and it is proved for both univariate and multivariate cases by using the theory of arc spaces. As an application, taking the limit for the motivic Euler reflexion formula we recover the well known motivic Thom-Sebastiani theorem.

1. INTRODUCTION

We study extensions of Denef-Loeser's motivic zeta functions under motivations from a nice simple formula concerning multiple zeta values ζ and from a problem on poles of the Igusa local zeta function of a Thom-Sebastiani type function. The latter may involve the monodromy conjecture, the highest interest of ours so that the present work is just a start. The relation between real numbers $s_1, s_2 \geq 2$ presented through the single and double zeta values as

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

is widely known as the Euler reflexion formula, whose further important generalizations can be found in Zagier's works, such as [16]. This beauty partially inspires us to consider an analogous phenomenon in the framework of motivic zeta functions, which probably provides more profound relations than the motivic Thom-Sebastiani theorem does.

In [2] and [8], Denef and Denef-Veys discuss poles of the Igusa local zeta function $Z_\Phi(s, \chi, f)$ of a polynomial f with respect to a Schwartz-Bruhat function Φ and

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to a character χ . It is proved that there exists a function $A(s, \chi)$ depending on a character such that, for polynomials f and g and Schwartz-Bruhat functions Φ and Ψ , the poles of $A(s, \chi)Z_{\Phi\Psi}(s, \chi, f(x) + g(y))$ are of the form $s_1 + s_2$, where s_1 and s_2 are poles of $A(s, \chi_1)Z_{\Phi}(s, \chi_1, f)$ and $A(s, \chi_2)Z_{\Psi}(s, \chi_2, g)$, respectively, for some $\chi_1\chi_2 = \chi$. Naturally, we can ask whether a similar result still holds for motivic zeta functions, and, hopefully, a motivic Euler reflexion formula may be the first step to answer it.

The motivic zeta function of a regular function was developed in the background of Denef-Loeser's motivic integration [3, 4, 5]. Afterwards, a version for a family of regular functions was also discussed in [9] and [11]. Such a motivic zeta function for r regular functions f_i on a smooth algebraic variety X over a field k of characteristic zero is a formal series $Z_{f_1, \dots, f_r}(T_1, \dots, T_r)$ with coefficients in a certain monodromic Grothendieck ring $\mathcal{M}_{X_0}^{\mu}$, where X_0 is the common zero set of the family of f_i . Originally, it is defined as follows

$$Z_{f_1, \dots, f_r}(T_1, \dots, T_r) = \sum [\mathcal{X}_{n_1, \dots, n_r}] \mathbb{L}^{-d \sum n_i} T_1^{n_1} \dots T_r^{n_r},$$

where the sum is taken over $\mathbb{N}_{>0}^r$ and $\mathcal{X}_{n_1, \dots, n_r}$ is the set of arcs $\varphi \in \mathcal{L}_{\sum n_i}(X)$ such that $f_i(\varphi) = t^{n_i}$ modulo t^{n_i+1} . When looking for a motivic analogue of the Euler reflexion formula, we recognize that Z_{f_1, \dots, f_r} is still rather far to be an appropriate one, even letting the sum run over the "optimal" subset Δ of $\mathbb{N}_{>0}^r$ defined by $1 \leq n_1 < \dots < n_r$. This requires a solid improvement in many aspects, including motivic zeta functions and products of them. In our approach, we replace the conditions $f_i(\varphi) = t^{n_i}$ modulo t^{n_i+1} by $\text{ord} f_i > n_i$ for every $2 \leq i \leq r$, and take the sum over Δ , where the resulting motivic zeta function will be denoted by $\zeta_{f_1, \dots, f_r}(T_1, \dots, T_r)$. This new notation still covers classical motivic zeta functions $Z_{f_1}(T_1)$, thus from now on we shall write $\zeta_{f_1}(T_1)$ in stead of $Z_{f_1}(T_1)$ for the coherence in literature. The integrability of $\zeta_{f_1, \dots, f_r}(T_1, \dots, T_r)$ will be proved in Corollary 5.9.

We introduce a new product of two integrable series (e.g., motivic zeta functions) in different rings of formal series. More precisely, if $a(\mathbf{T}) \in \overline{\mathcal{M}}_X^{\mu}[[\mathbf{T}]]$ and $b(\mathbf{U}) \in \overline{\mathcal{M}}_Y^{\mu}[[\mathbf{U}]]$ are integrable series in several variables, we define a reasonable element $a(\mathbf{T}) \boxtimes b(\mathbf{U})$ in $\overline{\mathcal{M}}_{X \times Y}^{\mu}[[\mathbf{T}, \mathbf{U}]]$ which is also an integrable series (Definitions 3.1 and 5.10, Corollary 5.9). Here, for a technical reason, we work in an appropriate localization $\overline{\mathcal{M}}_X^{\mu}$ of \mathcal{M}_X^{μ} for any base X . Roughly speaking, the \boxtimes -product is an object lying in the middle of the external product and the convolution. When \mathbf{T} and \mathbf{U} reduce to univariates T and U , the commuting of \boxtimes with $\lim_{T=U \rightarrow \infty}$ will be stated in Theorem 3.2 and given a complete proof. This product allows us to describe the motivic zeta function of a Thom-Sebastiani type regular function in terms of motivic multiple zeta functions.

The following is the statement of the most important results of the present article, the motivic Euler reflexion formulas. Let X and Y be smooth algebraic k -varieties, on which it admits regular functions f and g with the zero loci X_0 and Y_0 , respectively. Let $f \oplus g$ be the function on $X \times Y$ defined by the sum $f(x) + g(y)$. Denote by ι the inclusion of $X_0 \times Y_0$ in $X \times Y$. The motivic Euler reflexion formula in this case states that the identity

$$\zeta_f(T) \boxtimes \zeta_g(U) = \zeta_{f,g}(T, U) + \zeta_{g,f}(U, T) + \iota^* \zeta_{f \oplus g}(TU),$$

holds in $\overline{\mathcal{M}}_{X_0 \times Y_0}^{\hat{\mu}}[[T, U]]$. This formula is given in Theorem 4.1. As an application, taking $T = U$ and using the fact that \boxtimes and $\lim_{T \rightarrow \infty}$ commute, we can deduce from the motivic Euler reflexion formula the motivic Thom-Sebastiani theorem, which was proved previously in [5], [15] and [13].

More generally, we consider ordered families of regular functions $\mathbf{f} = (f_1, \dots, f_r)$ and $\mathbf{g} = (g_1, \dots, g_s)$ on smooth algebraic k -varieties X_1, \dots, X_r and Y_1, \dots, Y_s , with common zero loci X_0 and Y_0 , respectively, and formulate the general motivic Euler reflexion formula as follows

$$\zeta_{\mathbf{f}}(\mathbf{T}) \boxtimes \zeta_{\mathbf{g}}(\mathbf{U}) = \sum \iota^* \zeta_{p_1, \dots, p_\eta}(T_{\alpha_1}^{a_1} U_{\beta_1}^{b_1}, \dots, T_{\alpha_\eta}^{a_\eta} U_{\beta_\eta}^{b_\eta}),$$

where the context of the identity is $\overline{\mathcal{M}}_{X_0 \times Y_0}^{\hat{\mu}}[[\mathbf{T}, \mathbf{U}]]$, and the sum is taken over all the ordered families of regular functions (p_1, \dots, p_η) satisfying

$$p_i = a_i f_{\alpha_i} \oplus b_i g_{\beta_i}, \quad 1 \leq i \leq \eta,$$

with $(a_i, b_i) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, $\sum (a_i + b_i) = r + s$, and $\{\alpha_i\}_{a_i=1}$ and $\{\beta_i\}_{b_i=1}$ being strictly monotonic increasing sequences, and ι is the inclusion of $X_0 \times Y_0$ in $\prod_{i=1}^r X_i \times \prod_{j=1}^s Y_j$ (see Theorem 5.12). An direct corollary of this formula is the associativity of the \boxtimes -product in the class of motivic multiple zeta functions (see Corollaries 5.13 and 5.14).

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2. PRELIMINARIES

2.1. Grothendieck rings and rings of formal series. Let k be a field of characteristic zero, X an algebraic k -variety and Var_X the category of X -varieties. The *Grothendieck group* $K_0(\text{Var}_X)$ of X -varieties is an abelian group generated by symbols $[Y \rightarrow X]$ for objects $Y \rightarrow X$ in Var_X modulo the following relations

$$[Y \rightarrow X] = [V \rightarrow X]$$

if $Y \rightarrow X$ and $V \rightarrow X$ are isomorphic in Var_X , and

$$[Y \rightarrow X] = [V \rightarrow X] + [Y \setminus V \rightarrow X]$$

if V is Zariski closed in Y . Furthermore, $K_0(\text{Var}_X)$ has structure of a ring with unit with product induced by fiber product of X -varieties and the unit being the class of the identity morphism $X \rightarrow X$. Let \mathcal{M}_X be the localization of $K_0(\text{Var}_X)$ with respect to the multiplicative system of \mathbb{L}^i with $i \in \mathbb{N}$, where $\mathbb{L} := [\mathbb{A}_X^1] = [\mathbb{A}_k^1 \times X \rightarrow X]$. In this situation and from now on, whenever writing $X \times X'$ for k -schemes X and X' we means the fiber product $X \times_k X'$.

Let $\mu_n = \mu_n(k)$ be the group scheme of n th roots of unity in k , $\text{Spec}(k[t]/(t^n - 1))$. The family of all μ_n , $n \in \mathbb{N}_{>0}$, forms a projective system with respect to morphisms $\mu_{nm} \rightarrow \mu_n$ given by $\xi \mapsto \xi^m$, we denote its projective limit by $\hat{\mu}$. By definition, a *good* μ_n -action on an X -variety Y is a group action $\mu_n \times Y \rightarrow Y$, which is a morphism of X -varieties, such that each orbit is contained in an affine k -subvariety of Y ; a *good* $\hat{\mu}$ -action on Y is an action of $\hat{\mu}$ on Y factoring through a good μ_n -action.

The *monodromic Grothendieck group* $K_0^{\hat{\mu}}(\text{Var}_X)$ of X -varieties endowed with good $\hat{\mu}$ -action is an abelian group generated by the $\hat{\mu}$ -equivariant isomorphism

classes $[Y \rightarrow X, \sigma]$, σ being a good $\hat{\mu}$ -action on X -variety Y , modulo the following conditions

$$[Y \rightarrow X, \sigma] = [V \rightarrow X, \sigma|_V] + [Y \setminus V \rightarrow X, \sigma|_{Y \setminus V}]$$

if V is Zariski closed in Y and

$$[Y \times \mathbb{A}_k^n \rightarrow X, \sigma] = [Y \times \mathbb{A}_k^n \rightarrow X, \sigma']$$

if σ, σ' lift the same $\hat{\mu}$ -action on $Y \rightarrow X$ to an affine action on $Y \times \mathbb{A}_k^n \rightarrow X$. When no confusion may happen, we write $[Y, \sigma]$ for $[Y \rightarrow X, \sigma]$ for simplicity. Thanks to fiber product of X -varieties, $K_0^{\hat{\mu}}(\text{Var}_X)$ has the natural structure of a ring. Define

$$\mathcal{M}_X^{\hat{\mu}} := K_0^{\hat{\mu}}(\text{Var}_X)[\mathbb{L}^{-1}],$$

the $\hat{\mu}$ -equivariant version of the ring \mathcal{M}_X . We also consider the ring $\mathcal{M}_X^{\hat{\mu}^r}$ when working with good $\hat{\mu}^r$ -actions. Let $\overline{\mathcal{M}}_X^{\hat{\mu}}$ be the localization of $\mathcal{M}_X^{\hat{\mu}}$ with respect to the multiplicative family generated by the elements $1 - \mathbb{L}^n$ with $n \in \mathbb{N}_{>0}$. There is a natural morphism $\text{loc} : \mathcal{M}_X^{\hat{\mu}} \rightarrow \overline{\mathcal{M}}_X^{\hat{\mu}}$, which has not been proved or disproved to be injective; however, for simplicity of notation, if necessary, we shall identify a with $\text{loc}(a)$, that is, consider $a \in \mathcal{M}_X^{\hat{\mu}}$ as an element of $\overline{\mathcal{M}}_X^{\hat{\mu}}$.

For a morphism of k -varieties $f : X \rightarrow X'$, one defines group morphisms $f_! : \mathcal{M}_X^{\hat{\mu}} \rightarrow \mathcal{M}_{X'}^{\hat{\mu}}$ and $f_! : \overline{\mathcal{M}}_X^{\hat{\mu}} \rightarrow \overline{\mathcal{M}}_{X'}^{\hat{\mu}}$ by composition, also defines ring morphisms $f^* : \mathcal{M}_{X'}^{\hat{\mu}} \rightarrow \mathcal{M}_X^{\hat{\mu}}$ and $f^* : \overline{\mathcal{M}}_{X'}^{\hat{\mu}} \rightarrow \overline{\mathcal{M}}_X^{\hat{\mu}}$ by fiber product. If $X' = \text{Spec}k$, $f_!$ is usually denoted by \int_X .

Let \mathcal{M} be a $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$ -module, and let $\mathbf{T} = (T_1, \dots, T_r)$ be a multivariate. We shall consider $\mathcal{M}[[\mathbf{T}]]$ and the following sub- $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$ -modules

$$\begin{aligned} \mathcal{M}[[\mathbf{T}]]_{\text{sr}} &:= \mathcal{M}[\mathbf{T}] \left[(1 - \mathbb{L}^m \mathbf{T}^{\mathbf{n}})^{-1} \right]_{(m, \mathbf{n}) \in \mathbb{Z} \times (\mathbb{N}^r \setminus \{(0, \dots, 0)\})}, \\ \mathcal{M}[[\mathbf{T}]]_{\text{ssr}} &:= \mathcal{M}[\mathbf{T}] \left[(1 - \mathbb{L}^m \mathbf{T}^{\mathbf{n}})^{-1} \right]_{(m, \mathbf{n}) \in \mathbb{Z}_{\leq 0} \times (\mathbb{N}^r \setminus \{(0, \dots, 0)\})}, \text{ and} \\ \mathcal{M}[[\mathbf{T}]]_{\text{int}} &:= \mathcal{M}[\mathbf{T}] \left[(1 - \mathbb{L}^m \mathbf{T}^{\mathbf{n}})^{-1} \right]_{(m, \mathbf{n}) \in \mathbb{Z}_{< 0} \times (\mathbb{N}^r \setminus \{(0, \dots, 0)\})}. \end{aligned}$$

The identity

$$\frac{1}{1 - \mathbb{L}^m \mathbf{T}^{\mathbf{n}}} = \sum_{l \geq 0} (\mathbb{L}^m \mathbf{T}^{\mathbf{n}})^l$$

induces canonical embeddings of the previous modules in $\mathcal{M}[[\mathbf{T}]]$. Elements of $\mathcal{M}[[\mathbf{T}]]_{\text{sr}}$ are called *rational series*, elements of $\mathcal{M}[[\mathbf{T}]]_{\text{ssr}}$ are called *strongly rational series*, and elements of $\mathcal{M}[[\mathbf{T}]]_{\text{int}}$ are called *integrable series*, over \mathcal{M} . It is immediate that an integrable series is also a strongly rational series and a strongly rational series is also a rational series. The terminology “integrable” is inspired from the discussions of Cluckers and Loeser on integrable constructible functions in Section 4, especially Theorem 4.5.4, of their article [1].

In particular, if we fix a k -variety X and let \mathcal{M} be one of two rings $\mathcal{M}_X^{\hat{\mu}}$ and $\overline{\mathcal{M}}_X^{\hat{\mu}}$, then the previous rings can be obviously viewed as \mathcal{M} -modules. If this is the case, and if \mathbf{T} reduces to a univariate T , we get that every integrable series is also of *finite mass* in the sense of Looijenga [15]. Moreover, as shown in [3], there exists a unique $\mathcal{M}_X^{\hat{\mu}}$ -linear morphism

$$\lim_{T \rightarrow \infty} : \mathcal{M}[[T]]_{\text{sr}} \rightarrow \mathcal{M}$$

such that $\lim_{T \rightarrow \infty} \frac{\mathbb{L}^m T^n}{1 - \mathbb{L}^m T^n} = -1$ for any $(m, n) \in \mathbb{Z} \times \mathbb{N}_{>0}$.

2.2. Arc spaces and motivic zeta functions. Let X be an algebraic k -variety. For any $n \in \mathbb{N}_{>0}$, let $\mathcal{L}_n(X)$ be the space of n -jet schemes of X , which is a k -scheme representing the functor sending a k -algebra A to the set of morphisms of k -schemes $\mathrm{Spec}(A[t]/(t^{n+1})) \rightarrow X$. For any pair $n \leq m$, the truncation defines a morphism of k -schemes

$$\pi_n^m : \mathcal{L}_m(X) \rightarrow \mathcal{L}_n(X)$$

and this is an affine morphism. If X is smooth of dimension d , the morphism π_n^m is a locally trivial fibration with fiber $\mathbb{A}_k^{(m-n)d}$. The n -jet schemes and the morphisms π_n^m form in a natural way a projective system of k -schemes, we denote its limit by $\mathcal{L}(X)$ and call this space *the arc space of X* . For any field extension $k \subset K$, the K -points of $\mathcal{L}(X)$ correspond one-to-one to the $K[[t]]$ -points of X .

Furthermore, the schemes $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ are endowed with a natural action of μ_n given by $\xi\varphi(t) := \varphi(\xi t)$. The profinite group scheme $\hat{\mu}$ acts on these schemes via μ_n 's.

Assume in the rest of this section that X is a smooth algebraic k -variety of pure dimension d . Let $f : X \rightarrow \mathbb{A}_k^1$ be a regular function with the zero locus X_0 . For $n \in \mathbb{N}_{>0}$, let $\mathcal{X}_n(f)$ be the set of arcs $\varphi \in \mathcal{L}_n(X)$ such that $f(\varphi) = t^n \pmod{t^{n+1}}$. Since the image of $\mathcal{X}_n(f)$ under the canonical morphism $\mathcal{L}_n(X) \rightarrow X$ is contained in X_0 , it is also an X_0 -variety. Furthermore, $\mathcal{X}_n(f)$ is stable for the action of μ_n on $\mathcal{L}_n(X)$, thus it defines a motivic class $[\mathcal{X}_n(f)] := [\mathcal{X}_n(f) \rightarrow X_0]$ in $\mathcal{M}_{X_0}^{\hat{\mu}}$. The *motivic zeta function of f* is defined as follows

$$Z_f(T) := \sum_{n \geq 1} [\mathcal{X}_n(f)] \mathbb{L}^{-nd} T^n,$$

which lives in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$. If x is a closed point in X_0 , we define the *local motivic zeta function* $Z_{f,x}(T)$ to be $x^*Z_f(T)$, where x^* stands for the pullback of the inclusion of x in X_0 . Clearly, the series $Z_{f,x}(T)$ is an element of $\mathcal{M}_k^{\hat{\mu}}[[T]]$.

Theorem 2.1 (Denef-Loeser). *The motivic zeta function $Z_f(T)$ is an integrable series.*

The proof of Theorem 2.1 by Denef and Loeser in [3] uses in a crucial way invariants of a log-resolution of X_0 . Let us now recall briefly their work with such a resolution $h : Y \rightarrow (X, X_0)$. The exceptional divisors and irreducible components of the strict transform for h will be denoted by E_i , where i is in a finite set A . For $\emptyset \neq I \subset A$, one puts

$$E_I = \bigcap_{i \in I} E_i \quad \text{and} \quad E_I^\circ = E_I \setminus \bigcup_{j \notin I} E_j.$$

Consider an affine covering $\{U\}$ of Y such that on each piece $U \cap E_I^\circ \neq \emptyset$ the pullback of f has the form $u \prod_{i \in I} y_i^{N_i}$ with u a unit and y_i a local coordinate defining E_i . Let m_I denote $\mathrm{gcd}(N_i)_{i \in I}$. Denef and Loeser study the unramified Galois covering $\pi_I : \tilde{E}_I^\circ \rightarrow E_I^\circ$ with Galois group μ_{m_I} defined locally with respect to $\{U\}$ as follows

$$\{(z, y) \in \mathbb{A}_k^1 \times (U \cap E_I^\circ) \mid z^{m_I} = u(y)^{-1}\}.$$

The local pieces are glued over $\{U\}$ as in the proof of [3, Lemma 3.2.2] to give \tilde{E}_I° and π_I as mentioned, and the definition of the covering π_I is independent of

the choice of the affine covering $\{U\}$. Furthermore, \widetilde{E}_I° is endowed with a μ_{m_I} -action by multiplication of the z -coordinate with elements of μ_{m_I} , defining a class $[\widetilde{E}_I^\circ] = [\widetilde{E}_I^\circ \rightarrow E_I^\circ \rightarrow X_0]$ in $\mathcal{M}_{X_0}^{\hat{\mu}}$ (cf. [7]). For each $i \in A$, we denote by $\nu_i - 1$ the multiplicity of E_i in the divisor of $h^*\omega_X$, where ω_X is a local generator of the sheaf of differential forms on X of maximal degree. Then Denef-Loeser's formula of motivic zeta function in terms of h is the following

$$(2.1) \quad Z_f(T) = \sum_{\emptyset \neq I \subset A} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^\circ] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}},$$

which holds in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$. This proves that $Z_f(T)$ is an integrable series.

The quantity

$$\mathcal{S}_f := - \lim_{T \rightarrow \infty} Z_f(T) = \sum_{\emptyset \neq I \subset A} (1 - \mathbb{L})^{|I|-1} [\widetilde{E}_I^\circ]$$

in $\mathcal{M}_{X_0}^{\hat{\mu}}$ is called *the motivic nearby cycles of f* . Also, the element $\mathcal{S}_{f,x} := x^* \mathcal{S}_f$ of $\mathcal{M}_k^{\hat{\mu}}$ is called *the motivic Milnor fiber of f at x* . Recently, \mathcal{S}_f and $\mathcal{S}_{f,x}$ have been getting more important in singularity theory because of their relations with various classical invariants, such as Euler characteristic, Hodge spectrum, monodromy zeta functions (cf. [6, 7], [9], [10]).

More generally, we are going to consider a modification of the motivic zeta function in several variables concerning a family of functions mentioned in Guibert [9]. The version with a rational polyhedral convex cone in $\mathbb{N}_{>0}^r$ was studied by Guibert-Loeser-Merle [11] for one variable with respect to an appropriate linear form on the cone. Let \mathbf{f} be an ordered family of r regular functions $f_i : X \rightarrow \mathbb{A}_k^1$. For simplicity of notation, we also write X_0 for $X_0(\mathbf{f})$, the common zeros of the family \mathbf{f} . For any $\mathbf{n} \in \mathbb{N}_{>0}^r$, let $|\mathbf{n}| = \sum_{i=1}^r n_i$, and we define

$$\mathcal{X}_{\mathbf{n}}(\mathbf{f}) := \{ \varphi \in \mathcal{L}_{|\mathbf{n}|}(X) \mid f_i(\varphi) = t^{n_i} \pmod{t^{n_i+1}}, 1 \leq i \leq r \}.$$

In the particular case where $X = X_1 \times \cdots \times X_r$ with X_i smooth algebraic k -varieties and, for every $1 \leq i \leq r$, f_i is a regular function on X_i , we define

$$\mathcal{D}_{\mathbf{n}}(\mathbf{f}) := \left\{ \varphi = (\varphi_1, \dots, \varphi_r) \in \mathcal{L}_{|\mathbf{n}|}(X) \mid \begin{array}{l} f_1(\varphi_1) = t^{n_1} \pmod{t^{n_1+1}} \\ \text{ord} f_i(\varphi_i) > n_i, 2 \leq i \leq r \end{array} \right\}.$$

It is clear that, for every $\mathbf{n} \in \mathbb{N}_{>0}^r$, $\mathcal{X}_{\mathbf{n}}(\mathbf{f})$ (resp. $\mathcal{D}_{\mathbf{n}}(\mathbf{f})$) is stable under the good $\mu_{\text{gcd}(\mathbf{n})}$ -action (resp. μ_{n_1} -action) on the space $\mathcal{L}_{|\mathbf{n}|}(X)$ given by $\xi\varphi(t) := \varphi(\xi t)$ (resp. $\xi\varphi(t) := (\varphi_1(\xi t), \varphi_2(t), \dots, \varphi_r(t))$), and that $\mathcal{X}_{\mathbf{n}}(\mathbf{f})$ (resp. $\mathcal{D}_{\mathbf{n}}(\mathbf{f})$) admits a morphism to X_0 . This fact thus gives rise to an element $[\mathcal{X}_{\mathbf{n}}(\mathbf{f})] := [\mathcal{X}_{\mathbf{n}}(\mathbf{f}) \rightarrow X_0]$ (resp. $[\mathcal{D}_{\mathbf{n}}(\mathbf{f})] := [\mathcal{D}_{\mathbf{n}}(\mathbf{f}) \rightarrow X_0]$) in $\mathcal{M}_{X_0}^{\hat{\mu}}$.

Let C be a rational polyhedral convex cone in $\mathbb{N}_{>0}^r$, let Δ be the special one among C 's which consists of $\mathbf{n} = (n_1, \dots, n_r)$ such that $1 \leq n_1 < \cdots < n_r$. Let \mathbf{T} denote the r -tuple (T_1, \dots, T_r) of variables, and let $\mathbf{T}^{\mathbf{n}}$ stand for $T_1^{n_1} \cdots T_r^{n_r}$.

Definition 2.2. The *motivic zeta function* $Z_{\mathbf{f}}^C(\mathbf{T})$ of a family \mathbf{f} of regular functions on X is the following series in $\mathcal{M}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]$:

$$Z_{\mathbf{f}}^C(\mathbf{T}) := \sum_{\mathbf{n} \in C} [\mathcal{X}_{\mathbf{n}}(\mathbf{f})] \mathbb{L}^{-|\mathbf{n}|} d\mathbf{T}^{\mathbf{n}}.$$

If $\mathbf{f} = (f_1, \dots, f_r)$ is an ordered family \mathbf{f} of regular functions $f_i : X_i \rightarrow \mathbb{A}_k^1$, $1 \leq i \leq r$, the *motivic multiple zeta function* $\zeta_{\mathbf{f}}(\mathbf{T})$ of \mathbf{f} is the series

$$\zeta_{\mathbf{f}}(\mathbf{T}) := \sum_{n \in \Delta} [\mathcal{D}_{\mathbf{n}}(\mathbf{f})] \mathbb{L}^{-|\mathbf{n}|d} \mathbf{T}^{\mathbf{n}}$$

in $\mathcal{M}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]$. For a closed point $x \in X_0$, we define the *local motivic* and the *local motivic multiple zeta functions* as $Z_{\mathbf{f},x}^C(\mathbf{T}) := x^* Z_{\mathbf{f}}^C(\mathbf{T})$ and $\zeta_{\mathbf{f},x}(\mathbf{T}) := x^* \zeta_{\mathbf{f}}(\mathbf{T})$, elements of $\mathcal{M}_k^{\hat{\mu}}[[\mathbf{T}]]$.

We refer to [11, Section 2.9] and [4, Lemma 3.4] to see that $Z_{\mathbf{f}}^C(\mathbf{T})$ is a rational series. Indeed, we can obtain the motivic zeta function $Z_{\mathbf{f}}^{C,\ell}(T)$ in [11], which depends on a linear form ℓ positive on the closure \bar{C} of C in $\mathbb{R}_{\geq 0}^r$ except at the origin, in terms of replacing $\mathbf{T}^{\mathbf{n}}$ in $Z_{\mathbf{f}}^C(\mathbf{T})$ by $T^{\ell(\mathbf{n})}$. There, Guibert, Loeser and Merle deduce the rationality of $Z_{\mathbf{f}}^{C,\ell}(T)$ thanks to [4, Lemma 3.4], and, fortunately, their arguments are definitely applicable to the rationality of $Z_{\mathbf{f}}^C(\mathbf{T})$. To be more precise, let us consider a log-resolution $h : Y \rightarrow (X, X_0(F))$, with $F = f_1 \cdots f_r$. Assume that

$$h^{-1}(X_0(F)) = \sum_{i \in A} N_i(F) E_i \quad \text{and} \quad K_Y = h^* K_X + \sum_{i \in A} (\nu_i - 1) E_i,$$

where E_i 's are irreducible components of $h^{-1}(X)$. As previous, we shall work with $[\tilde{E}_I^{\circ}]$ in $\mathcal{M}_{X_0(F)}^{\hat{\mu}}$ for any nonempty $I \subset A$. Denote by $N_i(f_j)$ the multiplicity of E_i in the divisor of $f_j \circ h$, and by N_i the vector $(N_i(f_1), \dots, N_i(f_r)) \in \mathbb{N}^r$. Denote by \mathcal{A} the set of all nonempty subsets I of A such that $h(E_I^{\circ}) \subset X_0 = X_0(\mathbf{f})$. For any $I \in \mathcal{A}$, we consider the linear morphisms $N_I : \mathbb{R}^I \rightarrow \mathbb{R}^r$ and $\nu_I : \mathbb{R}^I \rightarrow \mathbb{R}$ defined as follows: for any $\alpha = (\alpha_i)_{i \in I} \in \mathbb{R}^I$, $N_I(\alpha) := \sum_{i \in I} \alpha_i N_i$ and $\nu_I(\alpha) := \sum_{i \in I} \alpha_i \nu_i$. Using the same method as doing with (2.1) we obtain a formula for $Z_{\mathbf{f}}^C(\mathbf{T})$, which lives in $\mathcal{M}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]$, as follows

$$(2.2) \quad Z_{\mathbf{f}}^C(\mathbf{T}) = \sum_{I \in \mathcal{A}} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{\circ}] \sum_{\mathbf{k} \in N_I^{-1}(C)} \mathbb{L}^{-\nu_I(\mathbf{k})} \mathbf{T}^{N_I(\mathbf{k})}.$$

Thus $Z_{\mathbf{f}}^C(\mathbf{T})$ is integrable, since $\sum_{\mathbf{k} \in N_I^{-1}(C)} \mathbb{L}^{-\nu_I(\mathbf{k})} \mathbf{T}^{N_I(\mathbf{k})}$ is integrable, for any $I \in \mathcal{A}$, i.e., $Z_{\mathbf{f}}^C(\mathbf{T}) \in \mathcal{M}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$.

Furthermore, with ℓ being a linear form on \mathbb{R}^r positive on $\bar{C} \setminus \{0\}$, where \bar{C} is the closure of C in $\mathbb{R}_{\geq 0}^r$, it follows from (2.2) that

$$Z_{\mathbf{f}}^{C,\ell}(T) = \sum_{I \in \mathcal{A}} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{\circ}] \sum_{\mathbf{k} \in N_I^{-1}(C)} \mathbb{L}^{-\nu_I(\mathbf{k})} T^{\ell(N_I(\mathbf{k}))},$$

which means that $Z_{\mathbf{f}}^{C,\ell}(T)$ is integrable, thus rational, and we can take its limit

$$\lim_{T \rightarrow \infty} Z_{\mathbf{f}}^{C,\ell}(T) = \sum_{I \in \mathcal{A}} \chi(N_I^{-1}(C)) (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{\circ}].$$

Observe that the element $\lim_{T \rightarrow \infty} Z_{\mathbf{f}}^{C,\ell}(T)$ of $\mathcal{M}_{X_0}^{\hat{\mu}}$ is independent of the choice of such an ℓ , hence one usually writes $\mathcal{S}_{\mathbf{f}}^C$ for it. For a closed point $x \in X_0$, we define $\mathcal{S}_{\mathbf{f},x} := x^* \mathcal{S}_{\mathbf{f}}$, which evidently equals the limit $\lim_{T \rightarrow \infty} x^* Z_{\mathbf{f}}^{C,\ell}(T)$.

Similarly, we have

Proposition 2.3. *As an element of $\overline{\mathcal{M}}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]$ the motivic multiple zeta function $\zeta_{\mathbf{f}}(\mathbf{T})$ is integrable, i.e., $\zeta_{\mathbf{f}}(\mathbf{T}) \in \overline{\mathcal{M}}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$.*

To prove this proposition we may compute directly the series $\zeta_{\mathbf{f}}(\mathbf{T})$ in terms of a resolution of singularity as done for $Z_{\mathbf{f}}^{\mathcal{C}}(\mathbf{T})$ in (2.2), with a slight modification. More precisely, using the previous notation \tilde{E}_I° but changing the definition so that $m_I = \gcd(N_i)_{i \in I}$ is replaced by $m'_I = \gcd(N_i)_{i \in I_1}$, where I_1 is the subset of $i \in I$ coming from resolution of $\{f_1 = 0\}$, and by convenience, $m'_I = 1$ if $I_1 = \emptyset$, we get

$$\zeta_{\mathbf{f}}(\mathbf{T}) = \sum_{I \in \mathcal{A}} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{\circ}] \sum_{\beta \in N_I^{-1}(\{0\} \times \mathbb{N}_{>0}^{r-1})} \sum_{\substack{\mathbf{k}' \in N_I^{-1}(\Delta) \\ k'_i > -\beta_i, i \in I}} \mathbb{L}^{-\nu_I(\mathbf{k}' + \beta)} \mathbf{T}^{N_I(\mathbf{k}')}.$$

Remark that, in this setting, although \mathbf{k}' and β are rational vectors, $\mathbf{k}' + \beta$ are positively integral vectors. A generalization of Lemma 8.5.2 of [12] (see also [10, Section 2.9]) shows that

$$(2.3) \quad S(I; \mathbf{T}) := \sum_{\beta \in N_I^{-1}(\{0\} \times \mathbb{N}_{>0}^{r-1})} \sum_{\substack{\mathbf{k}' \in N_I^{-1}(\Delta) \\ k'_i > -\beta_i, i \in I}} \mathbb{L}^{-\nu_I(\mathbf{k}' + \beta)} \mathbf{T}^{N_I(\mathbf{k}')}.$$

viewed as a series in $\overline{\mathcal{M}}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]$, is integrable. This proves $\zeta_{\mathbf{f}}(\mathbf{T}) \in \overline{\mathcal{M}}_{X_0}^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$. We also notice that we shall provide another proof for Proposition 2.3 in Corollary 5.9.

Furthermore, in terms of (2.3), we get the following formula

$$\lim_{T \rightarrow \infty} S(I; T, \dots, T) = \chi(N_I^{-1}(\{0\} \times \mathbb{N}_{>0}^{r-1})) \chi(N_I^{-1}(\Delta)),$$

which allows to compute the limit $\lim_{T \rightarrow \infty} \zeta_{\mathbf{f}}(T, \dots, T)$ of the motivic multiple zeta function of \mathbf{f} (see Definition 2.4).

Definition 2.4. The *motivic multiple nearby cycles of the family \mathbf{f}* in Proposition 2.3, denoted by $\mathcal{S}_{\mathbf{f}}$, is defined to be the element

$$-\lim_{T \rightarrow \infty} \zeta_{\mathbf{f}}(T, \dots, T) = - \sum_{I \in \mathcal{A}} \chi(N_I^{-1}(\{0\} \times \mathbb{N}_{>0}^{r-1})) \chi(N_I^{-1}(\Delta)) (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{\circ}]$$

of the ring $\overline{\mathcal{M}}_{X_0}^{\hat{\mu}}$. For a closed point $x \in X_0$, we set

$$\mathcal{S}_{\mathbf{f},x} := (\{x\} \hookrightarrow X_0)^* \mathcal{S}_{\mathbf{f}}$$

and call it the *local motivic multiple nearby cycles of \mathbf{f} at x* .

3. HADAMARD PRODUCTS AND \boxtimes -PRODUCT

3.1. Convolution and Hadamard products. The standard concept of convolution product on the monodromic Grothendieck rings of algebraic varieties was given earlier in [5], [15] and [10]. To recall it explicitly, let us consider the Fermat varieties F_0^n and F_1^n in $\mathbb{G}_{m,k}^2$ defined by the equations $u^n + v^n = 0$ and $u^n + v^n = 1$, respectively. Note that the varieties F_0^n and F_1^n admit the obvious action of $\mu_n \times \mu_n$.

Let X , Y and Z be algebraic k -varieties endowed with good μ_n -action. Assume that there are μ_n -equivariant morphisms $Y \rightarrow X$ and $Z \rightarrow X$. Define operations

in $\mathcal{M}_X^{\mu_n}$ as follows

$$\begin{aligned} [Y \rightarrow X] *_0 [Z \rightarrow X] &:= [F_0^n \times^{\mu_n \times \mu_n} (Y \times_X Z)], \\ [Y \rightarrow X] *_1 [Z \rightarrow X] &:= [F_1^n \times^{\mu_n \times \mu_n} (Y \times_X Z)], \\ [Y \rightarrow X] * [Z \rightarrow X] &:= [Y \rightarrow X] *_0 [Z \rightarrow X] - [Y \rightarrow X] *_1 [Z \rightarrow X], \end{aligned}$$

where, for $i \in \{0, 1\}$, $F_i^n \times^{\mu_n \times \mu_n} (Y \times_X Z)$ is the quotient of $F_i^n \times (Y \times_X Z)$ with respect to the equivalence relation by which any two elements $(\xi u, \eta v, x, y)$ and $(u, v, \xi x, \eta y)$ are equivalent, for $\xi, \eta \in \mu_n$. The group scheme μ_n acts diagonally on $F_i^n \times^{\mu_n \times \mu_n} (Y \times_X Z)$. Then passing through the projective limit with respect to $n \in \mathbb{N}_{>0}$ we get the (*standard*) *convolution product* $*$ on $\mathcal{M}_X^{\hat{\mu}}$. We can also extend the $*$ -product to $\overline{\mathcal{M}}_X^{\hat{\mu}}$ in a natural way. By [10, Proposition 5.2], the convolution product $*$ is commutative and associative.

Let X, Y, Z and W be algebraic k -varieties which are endowed with good $\hat{\mu}$ -action and admit $\hat{\mu}$ -equivariant morphisms $Z \rightarrow X$ and $W \rightarrow Y$ (we may choose the trivial action of $\hat{\mu}$ on the bases X and Y). The cartesian product induces a morphism of rings $\mathcal{M}_X^{\hat{\mu}} \times \mathcal{M}_Y^{\hat{\mu}} \rightarrow \mathcal{M}_{X \times Y}^{\hat{\mu}^2}$, by which the diagonal action induces naturally a canonical morphism $\mathcal{M}_{X \times Y}^{\hat{\mu}^2} \rightarrow \mathcal{M}_{X \times Y}^{\hat{\mu}}$. Then the composition of these morphisms yields an *external product*

$$(3.1) \quad \mathcal{M}_X^{\hat{\mu}} \times \mathcal{M}_Y^{\hat{\mu}} \rightarrow \mathcal{M}_{X \times Y}^{\hat{\mu}},$$

where, by abuse of notation, we also denote it by \times . As previous, we let \mathbf{T} be an r -tuple of variables. The (*external*) *Hadamard $\times_{\mathcal{H}}$ -product of two series* $a(\mathbf{T}) = \sum_{\mathbf{n} \in \mathbb{N}^r} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$ in $\mathcal{M}_X^{\hat{\mu}}[[\mathbf{T}]]$ and $b(\mathbf{T}) = \sum_{\mathbf{n} \in \mathbb{N}^r} b_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$ in $\mathcal{M}_Y^{\hat{\mu}}[[\mathbf{T}]]$ is the series

$$(3.2) \quad a(\mathbf{T}) \times_{\mathcal{H}} b(\mathbf{T}) := \sum_{\mathbf{n} \in \mathbb{N}^r} a_{\mathbf{n}} \times b_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$$

in $\mathcal{M}_{X \times Y}^{\hat{\mu}}[[\mathbf{T}]]$. This product is commutative, and it is also associative in the following sense, where the verification is straightforward. If $a(\mathbf{T})$ is in $\mathcal{M}_X^{\hat{\mu}}[[\mathbf{T}]]$, $b(\mathbf{T})$ is in $\mathcal{M}_Y^{\hat{\mu}}[[\mathbf{T}]]$ and $c(\mathbf{T})$ is in $\mathcal{M}_Z^{\hat{\mu}}[[\mathbf{T}]]$, then the identity

$$(3.3) \quad (a(\mathbf{T}) \times_{\mathcal{H}} b(\mathbf{T})) \times_{\mathcal{H}} c(\mathbf{T}) = a(\mathbf{T}) \times_{\mathcal{H}} (b(\mathbf{T}) \times_{\mathcal{H}} c(\mathbf{T}))$$

holds in $\mathcal{M}_{X \times Y \times Z}^{\hat{\mu}}[[\mathbf{T}]]$. It is stated similarly as in Lemma 7.6 of [15] that, in the univariate case (i.e., $r = 1$), the \times -product is “anti-compatible” with the Hadamard $\times_{\mathcal{H}}$ -product via the morphism $\lim_{T \rightarrow \infty}$. Namely, if $a(T)$ is in $\mathcal{M}_X^{\hat{\mu}}[[T]]_{\text{sr}}$ and $b(T)$ is in $\mathcal{M}_Y^{\hat{\mu}}[[T]]_{\text{sr}}$, then $a(T) \times_{\mathcal{H}} b(T)$ is in $\mathcal{M}_{X \times Y}^{\hat{\mu}}[[T]]_{\text{sr}}$ and the identity

$$(3.4) \quad \lim_{T \rightarrow \infty} (a(T) \times_{\mathcal{H}} b(T)) = - \left(\lim_{T \rightarrow \infty} a(T) \right) \times \left(\lim_{T \rightarrow \infty} b(T) \right)$$

holds in $\mathcal{M}_{X \times Y}^{\hat{\mu}}$. An analogous assertion for an arbitrary r is also true when we replace the morphism $\lim_{T \rightarrow \infty}$ by the morphism $\lim_{T_1 = \dots = T_r \rightarrow \infty}$, the composition of $\lim_{T \rightarrow \infty}$ and the assignment $T = T_1 = \dots = T_r$.

The previous external product also deduces naturally the following external product, which we again denote by \times ,

$$\overline{\mathcal{M}}_X^{\hat{\mu}} \times \overline{\mathcal{M}}_Y^{\hat{\mu}} \rightarrow \overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}.$$

This product has the same properties as the previous ones that we have mentioned.

Let us now introduce a generalized (external) convolution product of the previous standard one. Using the external product, the *generalized (external) convolution product*

$$* : \mathcal{M}_X^{\hat{\mu}} \times \mathcal{M}_Y^{\hat{\mu}} \rightarrow \mathcal{M}_{X \times Y}^{\hat{\mu}}$$

(again by abuse of notation) is defined as follows

$$\begin{aligned} [Z \rightarrow X] *_0 [W \rightarrow Y] &:= ([Z \rightarrow X] \times [Y \rightarrow Y]) *_0 ([X \rightarrow X] \times [W \rightarrow Y]), \\ [Z \rightarrow X] *_1 [W \rightarrow Y] &:= ([Z \rightarrow X] \times [Y \rightarrow Y]) *_1 ([X \rightarrow X] \times [W \rightarrow Y]), \\ [Z \rightarrow X] * [W \rightarrow Y] &:= [Z \rightarrow X] *_0 [W \rightarrow Y] - [Z \rightarrow X] *_1 [W \rightarrow Y]. \end{aligned}$$

The *Hadamard $*_{\mathcal{H}}$ -product of two formal series* $a(\mathbf{T}) = \sum_{\mathbf{n} \in \mathbb{N}^r} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}} \in \mathcal{M}_X^{\hat{\mu}}[[\mathbf{T}]]$ and $b(\mathbf{T}) = \sum_{\mathbf{n} \in \mathbb{N}^r} b_{\mathbf{n}} \mathbf{T}^{\mathbf{n}} \in \mathcal{M}_Y^{\hat{\mu}}[[\mathbf{T}]]$ is the formal series

$$(3.5) \quad a(\mathbf{T}) *_{\mathcal{H}} b(\mathbf{T}) := \sum_{\mathbf{n} \in \mathbb{N}^r} a_{\mathbf{n}} * b_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$$

in $\mathcal{M}_{X \times Y}^{\hat{\mu}}[[\mathbf{T}]]$. The associativity of the Hadamard product $*_{\mathcal{H}}$ is obtained from that of the convolution product $*$. Similarly to [15, Lemma 7.6], the $*$ -product is anti-compatible with the Hadamard product $*_{\mathcal{H}}$ -product via the morphism $\lim_{T_1=\dots=T_r \rightarrow \infty}$. Namely, for $r = 1$ for instance, if $a(T)$ is in $\mathcal{M}_X^{\hat{\mu}}[[T]]_{\text{sr}}$ and $b(T)$ is in $\mathcal{M}_Y^{\hat{\mu}}[[T]]_{\text{sr}}$, then $a(T) *_{\mathcal{H}} b(T)$ is in $\mathcal{M}_{X \times Y}^{\hat{\mu}}[[T]]_{\text{sr}}$, and moreover,

$$(3.6) \quad \lim_{T \rightarrow \infty} (a(T) *_{\mathcal{H}} b(T)) = - \left(\lim_{T \rightarrow \infty} a(T) \right) * \left(\lim_{T \rightarrow \infty} b(T) \right).$$

The external convolution product can be extended to the following

$$* : \overline{\mathcal{M}}_X^{\hat{\mu}} \times \overline{\mathcal{M}}_Y^{\hat{\mu}} \rightarrow \overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}},$$

which remains the properties mentioned previously.

3.2. The \boxtimes -product of integrable univariate series. Let X and Y be two algebraic k -varieties, and let T and U be univariates. In this paragraph, we introduce a new product of two integrable series $a(T) \in \overline{\mathcal{M}}_X^{\hat{\mu}}[[T]]_{\text{int}}$ and $b(U) \in \overline{\mathcal{M}}_Y^{\hat{\mu}}[[U]]_{\text{int}}$, which is an element of $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}[[T, U]]_{\text{int}}$ and commutes with the morphism $\lim_{T=U \rightarrow \infty}$.

We use the augmentation map $\mathcal{M}_X^{\hat{\mu}} \rightarrow \mathcal{M}_X$ defined in [15, Section 5], with remark that in the context of $\hat{\mu}$ the characteristic zero ground field is not necessarily algebraically closed. There is a more effective way to obtain a generalized augmentation map $\mathcal{M}_X^{\hat{\mu}^r} \rightarrow \mathcal{M}_X^{\hat{\mu}^{r-1}}$ using [10, Proposition 2.6, Section 3.10]. The image of an element $z \in \mathcal{M}_X^{\hat{\mu}}$ under the augmentation map will be denoted by z' .

Definition 3.1. The \boxtimes -product of the series $a(T) = \sum_{n \geq 1} a_n T^n$ and $b(U) = \sum_{m \geq 1} b_m U^m$ is defined as follows

$$a(T) \boxtimes b(U) := \sum_{n, m \geq 1} c_{n, m} T^n U^m \in \overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}[[T, U]],$$

where $c_{n, m}$ equals

$$\begin{cases} (\mathbb{L} - 1) \sum_{l > m} a_n \times b'_l & \text{if } n < m, \\ (\mathbb{L} - 1) \sum_{l > n} a'_l \times b_m & \text{if } n > m, \\ -a_n * b_n + \sum_{l \leq n} \mathbb{L}^{l-n} a_l *_0 b_l + (\mathbb{L} - 1) \sum_{l > n} (a_n \times b'_l + a'_l \times b_n) & \text{if } n = m. \end{cases}$$

Remark that the integrability of $a(T)$ and $b(U)$ implies that $a(T) \boxtimes b(U)$ is well defined. Indeed, since $a(T)$ and $b(U)$ are integrable, they are of finite mass, a condition guarantees that $\sum_{l>n} a_l$ and $\sum_{l>m} b_l$ make sense and belong to $\overline{\mathcal{M}}_X^\mu$ and $\overline{\mathcal{M}}_Y^\mu$, respectively.

Theorem 3.2. *The \boxtimes -product preserves the integrability and commutes with the limit of integrable series. More precisely, if $a(T)$ is in $\overline{\mathcal{M}}_X^\mu[[T]]_{\text{int}}$ and $b(U)$ is in $\overline{\mathcal{M}}_Y^\mu[[U]]_{\text{int}}$, then $a(T) \boxtimes b(U)$ is in $\overline{\mathcal{M}}_{X \times Y}^\mu[[T, U]]_{\text{int}}$, and*

$$\lim_{T=U \rightarrow \infty} (a(T) \boxtimes b(U)) = \lim_{T \rightarrow \infty} a(T) * \lim_{U \rightarrow \infty} b(U).$$

Proof. The first statement that $a(T) \boxtimes b(U)$ is in $\overline{\mathcal{M}}_{X \times Y}^\mu[[T, U]]_{\text{int}}$ if $a(T)$ is in $\overline{\mathcal{M}}_X^\mu[[T]]_{\text{int}}$ and $b(U)$ is in $\overline{\mathcal{M}}_Y^\mu[[U]]_{\text{int}}$ will be proved in the general case for several variables in Theorem 5.11.

Let us prove the second one. Write the series $a(T)$, $b(U)$ and $a(T) \boxtimes b(U)$ as $\sum_{n \geq 1} a_n T^n$, $\sum_{m \geq 1} b_m U^m$ and $\sum_{n,m} c_{n,m} T^n U^m$, respectively. Take $T = U$ in $a(T) \boxtimes b(U)$ so that the resulting series can be written as

$$\sum_{n,m} c_{n,m} T^{n+m} = A_1 + A_2 + (\mathbb{L} - 1)(B_1 + B_2),$$

where, by definition,

$$\begin{aligned} A_1 &= - \sum_{n \geq 1} a_n * b_n T^{2n}, & A_2 &= \sum_{n \geq 1} \left(\sum_{l \leq n} \mathbb{L}^{l-n} a_l *_0 b_l \right) T^{2n}, \\ B_1 &= \sum_{1 \leq n \leq m} \left(a_n \times \sum_{l > m} b'_l \right) T^{n+m}, & B_2 &= \sum_{1 \leq m \leq n} \left(\sum_{l > n} a'_l \times b_m \right) T^{n+m}. \end{aligned}$$

Here the integrability of $a(T)$ and $b(U)$ implies that $\sum_{l>n} a'_l$ converges in $\overline{\mathcal{M}}_X$ and $\sum_{l>m} b'_l$ converges in $\overline{\mathcal{M}}_Y$. The first limit is computed to be

$$\lim_{T \rightarrow \infty} A_1 = \lim_{T \rightarrow \infty} a(T^2) * \lim_{T \rightarrow \infty} b(T^2) = \lim_{T \rightarrow \infty} a(T) * \lim_{T \rightarrow \infty} b(T)$$

by means of (3.6). It is quite easy to obtain that

$$\begin{aligned} \lim_{T \rightarrow \infty} A_2 &= \lim_{T \rightarrow \infty} \sum_{n \geq 1} \left(\sum_{l=1}^n (\mathbb{L}^{-1})^{n-l} a_l *_0 b_l \right) T^{2n} \\ &= \left(\lim_{T \rightarrow \infty} \sum_{n \geq 0} \mathbb{L}^{-n} T^{2n} \right) \cdot \left(\lim_{T \rightarrow \infty} \sum_{n \geq 1} a_n *_0 b_n T^{2n} \right) \end{aligned}$$

which vanishes in $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}$, since $\lim_{T \rightarrow \infty} \sum_{n \geq 0} \mathbb{L}^{-n} T^{2n}$ vanishes in $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}$. The limits of B_1 and B_2 require more computations. It is verified in $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}[[T]]_{\text{int}}$ that

$$\begin{aligned} B_1 &= \sum_{1 \leq n \leq m} (a_n \times b'(1)) T^{n+m} - \sum_{1 \leq n \leq m} \left(a_n \times \sum_{1 \leq l \leq m} b'_l \right) T^{n+m} \\ &= \sum_{n \geq 1} a_n \times b'(1) \sum_{1 \leq m \leq n} T^{n+m} - \sum_{1 \leq n \leq m} \left(a_n \times \sum_{1 \leq l \leq m} b'_l \right) T^{n+m} \\ &= \frac{a(T^2)}{1-T} \times b'(1) - \sum_{1 \leq n \leq m} \left(a_n \times \sum_{1 \leq l \leq m} b'_l \right) T^{n+m}, \end{aligned}$$

and, similarly, that

$$B_2 = \left(\frac{a'(1)}{1-T} \sum_{n \geq 1} T^{2n} \right) \times_{\mathcal{H}} \frac{b(T^2)}{1-T} - \sum_{1 \leq m \leq n} \left(\sum_{l \leq n} a'_l \times b_m \right) T^{n+m},$$

where $a'(T) := \sum_{n \geq 1} a'_n T^n$ and $b'(U) := \sum_{m \geq 1} b'_m U^m$. Note that we can extend naturally the augmentation map to a map $\mathcal{M}_{X \times Y}^{\hat{\mu}^2} \rightarrow \mathcal{M}_{X \times Y}^{\hat{\mu}}$, from which, for every m and n , the elements $(\mathbb{L} - 1)a_n \times b'_m$ and $(\mathbb{L} - 1)a'_n \times b_m$ in $\mathcal{M}_{X \times Y}^{\hat{\mu}}$ coincide, since both are the image of $(\mathbb{L} - 1)a_n \times b_m$ in $\mathcal{M}_{X \times Y}^{\hat{\mu}^2}$. We may also refer to [10, Proposition 2.6, Section 3.10] to see this fact. In $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}$, because $\mathbb{L} - 1$ is invertible, the identity $a_n \times b'_m = a'_n \times b_m$ holds. Thus, for each $\kappa \geq 1$, by combinatoric computation, we obtain the following identity in $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}$:

$$\begin{aligned} & \sum_{n+m=\kappa} \left(\sum_{1 \leq n, j \leq m} a_n \times b'_j + \sum_{1 \leq m, i \leq n} a'_i \times b_m \right) \\ &= \sum_{n+m=\kappa} \left(\sum_{1 \leq n, j \leq m} a_n \times b'_j + \sum_{1 \leq m, i \leq n} a_i \times b'_m \right) \\ &= \sum_{n+m=\kappa} \sum_{j \leq m} a_n \times b'_j + \sum_{i \leq \lfloor \frac{\kappa}{2} \rfloor} a_i \times \sum_{j \leq \lfloor \frac{\kappa}{2} \rfloor} b'_j, \end{aligned}$$

where $\lfloor \frac{\kappa}{2} \rfloor$ is the integer part of $\frac{\kappa}{2}$. It implies that, in $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}[[T]]_{\text{int}}$,

$$\begin{aligned} B_1 + B_2 &= \frac{a(T^2)}{1-T} \times b'(1) + a'(1) \times \frac{b(T^2)}{1-T} - \sum_{l \geq 1} a_l T^l \times \sum_{l \geq 1} \left(\sum_{m \leq l} b'_m \right) T^l \\ &\quad - (1+T) \sum_{l \geq 1} \left(\sum_{n \leq l} a_n \right) T^{2l} \times_{\mathcal{H}} \sum_{l \geq 1} \left(\sum_{m \leq l} b'_m \right) T^{2l} \\ &= \frac{a(T^2)}{1-T} \times b'(1) + a'(1) \times \frac{b(T^2)}{1-T} - a(T) \times \frac{b'(T)}{1-T} \\ &\quad - (1+T) \left(\frac{a(T^2)}{1-T^2} \times_{\mathcal{H}} \frac{b'(T^2)}{1-T^2} \right), \end{aligned}$$

since

$$\sum_{\kappa \geq 1} \left(\sum_{i \leq \lfloor \frac{\kappa}{2} \rfloor} a_i \times \sum_{j \leq \lfloor \frac{\kappa}{2} \rfloor} b'_j \right) T^\kappa = (1+T) \sum_{l \geq 1} \left(\sum_{n \leq l} a_n \times \sum_{m \leq l} b'_m \right) T^{2l}.$$

Here, for any two series $\alpha(T) \in \overline{\mathcal{M}}_X^\mu[[T]]$ and $\beta(T) \in \overline{\mathcal{M}}_Y^\mu[[T]]$, by $\alpha(T) \times \beta(T)$ we mean the usual product of formal series in which the multiplication for the coefficients uses the external product \times . Now it is easy to obtain the vanishing of $\lim_{T \rightarrow \infty} (B_1 + B_2)$ in $\overline{\mathcal{M}}_{X \times Y}^\mu$, and the theorem is proved. \square

4. A MOTIVIC ANALOGUE OF THE EULER REFLEXION FORMULA

4.1. Main theorem. In this paragraph we state and prove an analogue of the Euler reflexion formula for motivic zeta functions, the most important result of the present article.

Theorem 4.1. *Let X and Y be smooth algebraic k -varieties, let f and g be regular functions on X and Y with the zero loci X_0 and Y_0 , respectively. Define a function $f \oplus g$ on $X \times Y$ by $f \oplus g(x, y) = f(x) + g(y)$. Let ι be the inclusion of $X_0 \times Y_0$ in $X \times Y$. Then the following identity*

$$\zeta_f(T) \boxtimes \zeta_g(U) = \zeta_{f,g}(T, U) + \zeta_{g,f}(U, T) + \iota^* \zeta_{f \oplus g}(TU)$$

holds in $\overline{\mathcal{M}}_{X_0 \times Y_0}^\mu[[T, U]]$. It is called the motivic Euler reflexion formula for (f, g) .

Proof. Let d_1 and d_2 be the pure k -dimensions of X and Y , respectively, and let $d := d_1 + d_2$. For brevity of notation, we write a_n for $[\mathcal{X}_n(f)] \mathbb{L}^{-nd_1}$ in $\mathcal{M}_{X_0}^\mu$ and b_n for $[\mathcal{X}_n(g)] \mathbb{L}^{-nd_2}$ in $\mathcal{M}_{Y_0}^\mu$, we also ignore writing arrows to base for relative objects when they are clearly understood, e.g., let $[\mathcal{X}_n(f)]$ simply stand for $[\mathcal{X}_n(f) \rightarrow X_0]$. The motivic zeta functions of f and g can be rewritten as follows

$$\zeta_f(T) = \sum_{n \geq 1} a_n T^n \in \mathcal{M}_{X_0}^\mu[[T]] \quad \text{and} \quad \zeta_g(U) = \sum_{n \geq 1} b_n U^n \in \mathcal{M}_{Y_0}^\mu[[U]].$$

Let us consider the coefficients of the series $\iota^* \zeta_{f \oplus g}(TU)$. For $n \in \mathbb{N}_{>0}$, we have

$$[\iota^* \mathcal{X}_n(f \oplus g)] = [\{(\varphi, \psi) \in \mathcal{L}_n(X \times Y) \mid f(\varphi) + g(\psi) = t^n \pmod{t^{n+1}}\}]$$

that equals the sum $A_1^{(n)} + A_2^{(n)} + A_3^{(n)}$, where $A_1^{(n)}$, $A_2^{(n)}$ and $A_3^{(n)}$ are given by the expressions

$$\begin{aligned} A_1^{(n)} &= \left[\left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \mid \begin{array}{l} f(\varphi) + g(\psi) = t^n \pmod{t^{n+1}} \\ \text{ord} f(\varphi) = \text{ord} g(\psi) = n \end{array} \right\} \right], \\ A_2^{(n)} &= \left[\left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \mid \begin{array}{l} f(\varphi) + g(\psi) = t^n \pmod{t^{n+1}} \\ \text{ord} f(\varphi) \neq \text{ord} g(\psi) \end{array} \right\} \right], \\ A_3^{(n)} &= \sum_{1 \leq l < n} \left[\left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \mid \begin{array}{l} f(\varphi) + g(\psi) = t^n \pmod{t^{n+1}} \\ \text{ord} f(\varphi) = \text{ord} g(\psi) = l \end{array} \right\} \right]. \end{aligned}$$

It is useful for the rest of the proof to introduce another notation, B_n , so that

$$B_n = (\mathbb{L} - 1) \left[\left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \mid \begin{array}{l} f(\varphi) = t^n \pmod{t^{n+1}} \\ g(\psi) = -t^n \pmod{t^{n+1}} \end{array} \right\} \right].$$

Lemma 4.2. *The identities $a_n *_{1} b_n = A_1^{(n)} \mathbb{L}^{-nd}$ and $a_n *_{0} b_n = B_n \mathbb{L}^{-nd}$ hold in $\mathcal{M}_{X_0 \times Y_0}^\mu$.*

Proof of Lemma 4.2. We shall prove the first identity, that $a_n *_1 b_n = A_1^{(n)} \mathbb{L}^{-nd}$, proving the second one can be done in the same way. The mapping from the k -variety $\mathcal{X}_n(f) \times \mathcal{X}_n(g) \times F_1^n$ toward the k -variety

$$E := \left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \left| \begin{array}{l} \text{ord}f(\varphi) = \text{ord}g(\psi) = n \\ f(\varphi) + g(\psi) = t^n \pmod{t^{n+1}} \end{array} \right. \right\}$$

that sends $(\varphi(t), \psi(t); \xi, \eta)$ to $(\varphi(\xi t), \psi(\eta t))$ gives rise to a morphism θ of $(X_0 \times Y_0)$ -varieties

$$\mathcal{X}_n(f) \times \mathcal{X}_n(g) \times^{\mu_n \times \mu_n} F_1^n \rightarrow E.$$

It is clear that the source and the target of θ are endowed with the natural action of μ_n , and that θ is a μ_n -equivariant isomorphism. The desired identity $a_n *_1 b_n = A_1^{(n)} \mathbb{L}^{-nd}$ is now proved. The reader may also find in the proof of Lemma 5.2 in [13] to obtain more detailed arguments. \square

Lemma 4.3. *The identity $(\mathbb{L} - 1) \sum_{l > n} (a_n \times b'_l + a'_l \times b_n) = A_2^{(n)} \mathbb{L}^{-nd}$ holds in $\overline{\mathcal{M}}_{X_0 \times Y_0}^{\mu}$.*

Proof of Lemma 4.3. Note that the condition $\text{ord}f(\varphi) \neq \text{ord}g(\psi)$ in the definition of $A_2^{(n)}$ may be presented as

$$(\text{ord}f(\varphi) < \text{ord}g(\psi)) \vee (\text{ord}f(\varphi) > \text{ord}g(\psi)),$$

so we can write $A_2^{(n)}$ as follows

$$A_2^{(n)} = [\{\varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n \pmod{t^{n+1}}\}] \times [\{\psi \in \mathcal{L}_n(Y) \mid \text{ord}g(\psi) > n\}] \\ + [\{\varphi \in \mathcal{L}_n(X) \mid \text{ord}f(\varphi) > n\}] \times [\{\psi \in \mathcal{L}_n(Y) \mid g(\psi) = t^n \pmod{t^{n+1}}\}].$$

Let us denote by D the constructible subset $\{\varphi \in \mathcal{L}_n(X) \mid \text{ord}f(\varphi) > n\}$ of $\mathcal{L}_n(X)$. Then $\mu(\pi_n^{-1}(D)) = [D] \mathbb{L}^{-d_1}$, with μ being the motivic measure. Putting

$$D_l := \{\varphi \in \mathcal{L}(X) \mid \text{ord}f(\varphi) = l\},$$

for any $l > n$, we get $\pi_n^{-1}(D) = \bigcup_{l > n} D_l$, and, by σ -additivity of μ , we have

$$[D] = \mathbb{L}^{d_1} \mu(\pi_n^{-1}(D)) = \mathbb{L}^{d_1} \sum_{l > n} \mu(D_l) = \sum_{l > n} [\pi_l(D_l)] \mathbb{L}^{-ld_1}.$$

Since $\{\varphi \in \mathcal{L}_l(X) \mid \text{ord}f(\varphi) = l\}$ is isomorphic as an algebraic X_0 -variety to the quotient of $\mathcal{X}_l(f) \times \mathbb{G}_{m,k}$ by the μ_l -action given by $\xi \cdot (\varphi, \lambda) := (\xi\varphi, \xi^{-1}\lambda)$, we have $[\pi_l(D_l)] \mathbb{L}^{-ld_1} = (\mathbb{L} - 1) a'_l$, thus we get

$$[D] = (\mathbb{L} - 1) \sum_{l > n} a'_l,$$

and in the same way, $[\{\psi \in \mathcal{L}_n(Y) \mid \text{ord}g(\psi) > n\}]$ is equal to $(\mathbb{L} - 1) \sum_{l > n} b'_l$. The lemma is then proved. \square

Lemma 4.4. *The equality $\sum_{l < n} a_l *_0 b_l \mathbb{L}^{l-n} = A_3^{(n)} \mathbb{L}^{-nd}$ holds in $\overline{\mathcal{M}}_{X_0 \times Y_0}^{\mu}$.*

Proof of Lemma 4.4. For any $l < n$, let us consider the k -varieties

$$U_l = \left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \left| \begin{array}{l} f(\varphi) + g(\psi) = t^n \pmod{t^{n+1}} \\ \text{ord}f(\varphi) = \text{ord}g(\psi) = l \end{array} \right. \right\}, \\ \tilde{U}_l = \left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \left| \begin{array}{l} f(\varphi) + g(\psi) = t^n \pmod{t^{n+1}} \\ f(\varphi) = t^l \pmod{t^{l+1}} \\ g(\psi) = -t^l \pmod{t^{l+1}} \end{array} \right. \right\},$$

and

$$W_l = \left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \left| \begin{array}{l} \text{ord}(f(\varphi) + g(\psi)) = n \\ f(\varphi) = t^l \pmod{t^{l+1}} \\ g(\psi) = -t^l \pmod{t^{l+1}} \end{array} \right. \right\},$$

which admit evidently the natural action of μ_l . Here, the class of U_l is nothing else than the l -th term of the sum $A_3^{(n)}$. Again as in the proof of Lemma 4.3, since U_l is isomorphic as a $(X_0 \times Y_0)$ -variety to the quotient of $\tilde{U}_l \times \mathbb{G}_{m,k}$ by the μ_l -action given by $\xi \cdot (\varphi, \psi, \lambda) := (\xi\varphi, \xi\psi, \xi^{-1}\lambda)$, we get $[U_l] = (\mathbb{L} - 1)[\tilde{U}_l]'$ in $\mathcal{M}_{X_0 \times Y_0}^{\mu}$. Similarly, $[W_l] = (\mathbb{L} - 1)[\tilde{W}_l]'$ in $\mathcal{M}_{X_0 \times Y_0}^{\mu}$. Hence $[U_l] = [W_l]$ in $\mathcal{M}_{X_0 \times Y_0}^{\mu}$.

Now we write $[W_l] = [W_l^{\geq n}] - [W_l^{\geq n+1}]\mathbb{L}^{-d}$, where

$$W_l^{\geq n} = \left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \left| \begin{array}{l} \text{ord}(f(\varphi) + g(\psi)) \geq n \\ f(\varphi) = t^l \pmod{t^{l+1}} \\ g(\psi) = -t^l \pmod{t^{l+1}} \end{array} \right. \right\}.$$

Put

$$E_{n,l} = \left\{ (\varphi, \psi) \in \mathcal{L}_n(X \times Y) \left| \begin{array}{l} f(\varphi) = t^l \pmod{t^{l+1}} \\ g(\psi) = -t^l \pmod{t^{l+1}} \end{array} \right. \right\}$$

$$A = \{ \tau \in \mathcal{L}_{n-l}(\mathbb{A}_k^1) \mid \tau = t + \alpha_2 t^2 + \dots + \alpha_{n-l} t^{n-l} \}.$$

The $(X_0 \times Y_0)$ -morphism $W_l^{\geq n} \times A \rightarrow E_{n,l}$ sending (φ, ψ, τ) to $(\varphi \circ \tau, \psi)$ is an isomorphism, from which $[W_l^{\geq n}] = [E_{n,l}]\mathbb{L}^{l+1-n}$. Since $[E_{n,l}] = B_l(\mathbb{L} - 1)^{-1}\mathbb{L}^{(n-l)d}$, it follows from Lemma 4.2 that $[E_{n,l}] = a_l *_0 b_l(\mathbb{L} - 1)^{-1}\mathbb{L}^{nd}$, therefore

$$[W_l^{\geq n}] = a_l *_0 b_l(\mathbb{L} - 1)^{-1}\mathbb{L}^{nd+l+1-n}.$$

Consequently,

$$[W_l] = [W_l^{\geq n}] - [W_l^{\geq n+1}]\mathbb{L}^{-d} = a_l *_0 b_l \mathbb{L}^{nd+l-n}.$$

Then we get $A_3^{(n)}\mathbb{L}^{-nd} = \sum_{l < n} [W_l]\mathbb{L}^{-nd} = \sum_{l < n} a_l *_0 b_l \mathbb{L}^{l-n}$ as desired. \square

Let us continue the proof of Theorem 4.1. Using Lemmas 4.2, 4.3 and 4.4 gives the coefficient of $T^n U^n$ in $\iota^* \zeta_{f \oplus g}(TU)$, also the coefficient of $T^n U^n$ in the right hand side of the Euler reflexion formula, as follows

$$\begin{aligned} [\iota^* \mathcal{X}_n(f \oplus g)]\mathbb{L}^{-nd} &= a_n *_1 b_n + \sum_{l < n} \mathbb{L}^{l-n} a_l *_0 b_l + (\mathbb{L} - 1) \sum_{l > n} (a_n \times b'_l + a'_l \times b_n) \\ &= -a_n * b_n + \sum_{l \leq n} \mathbb{L}^{l-n} a_l *_0 b_l + (\mathbb{L} - 1) \sum_{l > n} (a_n \times b'_l + a'_l \times b_n). \end{aligned}$$

This quantity agrees with the coefficient of $T^n U^n$ in the left hand side, according to the \boxtimes -product of the motivic zeta functions $\zeta_f(T)$ and $\zeta_g(U)$ (see Section 3.2).

On the other hand, for $n < m$, the coefficient of $T^n U^m$ in the right hand side of the Euler reflexion formula is nothing else than $[\mathcal{D}_{n,m}(f, g)]\mathbb{L}^{-(n+m)d}$, which equals

$$[\mathcal{X}_n(f)]\mathbb{L}^{-nd_1} \times \sum_{l > m} \{ (\psi \in \mathcal{L}_l(Y) \mid \text{ord}g(\psi) = l) \} \mathbb{L}^{-ld_2} = (\mathbb{L} - 1) \sum_{l > m} a_n \times b'_l,$$

definitely coinciding the coefficient of $T^n U^m$ in the left hand side of the Euler reflexion formula. For the detail in proving these identities, see the proof of Lemma 4.3. The previous arguments obviously run for the case $n > m$, and Theorem 4.1 is now proved. \square

4.2. Motivic multiple nearby cycles and the motivic Thom-Sebastiani theorem. Let X, Y, f and g be as in Theorem 4.1. Let us now compute the motivic multiple zeta functions $\mathcal{S}_{f,g}$ and $\mathcal{S}_{g,f}$, which are the limit of the series $-\zeta_{f,g}(T, T)$ and $-\zeta_{g,f}(T, T)$, respectively. Afterward, together with the commuting of \boxtimes -product and $\lim_{T \rightarrow \infty}$, and the motivic Euler reflexion formula, we deduce the motivic Thom-Sebastiani theorem.

Proposition 4.5. *The identities $\mathcal{S}_{f,g} = -\mathcal{S}_f \times [Y_0]$ and $\mathcal{S}_{g,f} = -[X_0] \times \mathcal{S}_g$ hold in $\mathcal{M}_{X_0 \times Y_0}^{\dot{\mu}}$.*

Proof. It suffices to check for the first identity. As in the proof of Theorem 4.1, for brevity of notation, let a_n and b_n stand for $[\mathcal{X}_n(f)]\mathbb{L}^{-nd_1}$ and $[\mathcal{X}_n(g)]\mathbb{L}^{-nd_2}$, respectively. By definition,

$$\mathcal{S}_{f,g} = - \lim_{T \rightarrow \infty} \zeta_{f,g}(T, T),$$

we get the following

$$\begin{aligned} -\mathcal{S}_{f,g} &= (\mathbb{L} - 1) \lim_{T \rightarrow \infty} \sum_{1 \leq n < m} a_n \times \sum_{l > m} b'_l T^{n+m} \\ &= (\mathbb{L} - 1) \lim_{T \rightarrow \infty} \sum_{1 \leq n < m} a_n \times \sum_{l \geq 1} b'_l T^{n+m} - (\mathbb{L} - 1) \lim_{T \rightarrow \infty} \sum_{1 \leq n < m} a_n \times \sum_{l \leq m} b'_l T^{n+m} \\ &= \lim_{T \rightarrow \infty} \sum_{1 \leq n < m} a_n T^{n+m} \times [Y_0] - (\mathbb{L} - 1) \lim_{T \rightarrow \infty} \sum_{1 \leq n < m} a_n \times \sum_{l \leq m} b'_l T^{n+m} \\ &= \lim_{T \rightarrow \infty} \sum_{n \geq 1} a_n \frac{T^{2n+1}}{1-T} \times [Y_0] - (\mathbb{L} - 1) \lim_{T \rightarrow \infty} \sum_{1 \leq n < m} \sum_{l \leq m} a_n \times b'_l T^{n+m} \\ &= \lim_{T \rightarrow \infty} \frac{T \zeta_f(T^2)}{1-T} \times [Y_0] + (\mathbb{L} - 1) \lim_{T \rightarrow \infty} Z_{f,\text{id},g}^{C,\ell} \\ &= \mathcal{S}_f \times [Y_0] + (\mathbb{L} - 1) \mathcal{S}_{f,\text{id},g}^{C,\ell}, \end{aligned}$$

where C is the rational polyhedral convex cone

$$\{(n, l, m) \in \mathbb{N}^3 \mid 1 \leq n < m, 1 \leq l \leq m\},$$

$\ell(n, m, l) = n + m$, for $(n, m, l) \in \mathbb{R}^3$, and id is the identity morphism on \mathbb{A}_k^1 . According to [11, Section 2.9], in fact, $\mathcal{S}_{f,\text{id},g}^{C,\ell}$ is independent of the choice of ℓ provided ℓ is linear on \mathbb{R}^3 and positive on the closure of C in \mathbb{R}^3 outside the origin. By this, we may replace ℓ by ℓ' defined by $\ell'(n, m, l) = m$ to get $Z_{f,\text{id},g}^{C,\ell'}(T)$ so that $Z_{f,\text{id},g}^{C,\ell'}(T)$ has the same limit $\lim_{T \rightarrow \infty}$ as $Z_{f,\text{id},g}^{C,\ell}(T)$. More precisely,

$$-\mathcal{S}_{f,\text{id},g}^{C,\ell} = -\mathcal{S}_{f,\text{id},g}^{C,\ell'} = \lim_{T \rightarrow \infty} Z_{f,\text{id},g}^{C,\ell'} = \lim_{T \rightarrow \infty} \sum_{1 \leq n < m} \sum_{1 \leq l \leq m} a_n \times b_l T^m.$$

By applying Lemma 7.6 of [15] to the external product \times , which was already recalled in Section 3.1, together with the previous identity, we obtain a formula for $-\mathcal{S}_{f,\text{id},g}^{C,\ell}$

as follows

$$\begin{aligned}
-\mathcal{S}_{f,\text{id},g}^{C,\ell} &= \lim_{T \rightarrow \infty} \sum_{m \geq 1} \left(\sum_{n < m} a_n \times \sum_{l \leq m} b_l \right) T^m \\
&= - \lim_{T \rightarrow \infty} \sum_{m \geq 1} \left(\sum_{n < m} a_n \right) T^m \times \lim_{T \rightarrow \infty} \sum_{m \geq 1} \left(\sum_{l \leq m} b_l \right) T^m \\
&= - \lim_{T \rightarrow \infty} \sum_{m \geq 1} a_m T^{m+1} \times \lim_{T \rightarrow \infty} \sum_{m \geq 1} b_m T^m \\
&= \lim_{T \rightarrow \infty} \sum_{m \geq 1} a_m \frac{T^{m+1}}{1-T} \times \lim_{T \rightarrow \infty} \sum_{m \geq 1} b_m \frac{T^m}{1-T},
\end{aligned}$$

which vanishes because of the vanishing of the second factor of the last expression, completing the proof of Proposition 4.5. \square

Theorem 4.6 (Motivic Thom-Sebastiani theorem). *Using the assumption as in Theorem 4.1, the following identity*

$$\iota^* \mathcal{S}_{f \oplus g} = -\mathcal{S}_f * \mathcal{S}_g + \mathcal{S}_f \times [Y_0] + [X_0] \times \mathcal{S}_g$$

holds in $\overline{\mathcal{M}}_{X_0 \times Y_0}^{\mu}$.

Proof. This is a direct consequence of Theorems 4.1, 3.2 and Proposition 4.5. \square

5. GENERALIZATION OF \boxtimes -PRODUCT AND MOTIVIC EULER REFLEXION FORMULA

5.1. Integrable series. First of all, let us recall some basic results on integrability of formal series. We define

$$\mathbb{Z}[\mathbb{L}]_{\text{loc}} := \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, (1 - \mathbb{L}^n)^{-1}, n \geq 1].$$

Let \mathcal{M} and \mathcal{N} be $\mathbb{Z}[\mathbb{L}]_{\text{loc}}$ -modules, and let $\mathcal{M} \otimes \mathcal{N}$ denote $\mathcal{M} \otimes_{\mathbb{Z}[\mathbb{L}]_{\text{loc}}} \mathcal{N}$ for short.

Lemma 5.1. *If $a(\mathbf{T}) \in \mathcal{M}[[\mathbf{T}]]_{\text{int}}$ and $b(\mathbf{T}) \in \mathcal{N}[[\mathbf{T}]]_{\text{ssr}}$, then $a(\mathbf{T}) \otimes_{\mathcal{H}} b(\mathbf{T}) \in \mathcal{M} \otimes \mathcal{N}[[\mathbf{T}]]_{\text{int}}$.*

Proof. Looijenga gave a similar statement for the univariate case in [15, Lemma 7.6], which claims that the Hadamard product corresponding to tensor product on coefficients of two rational series is again rational. His arguments in fact still work in our situation. Moreover, there are methods more direct to prove this lemma, such as combinatorics or Cluckers-Loeser's computations for the constructible motivic functions in [1, Section 4] together with the version with action in [14]. \square

Lemma 5.2. *Let \mathcal{M} be $\mathbb{Z}[\mathbb{L}]_{\text{loc}}$ -modules, and \mathbf{T} and \mathbf{U} separated multivariates. Then*

$$\mathcal{M}[[\mathbf{T}]]_{\text{int}}[[\mathbf{U}]]_{\text{int}} \subset \mathcal{M}[[\mathbf{T}, \mathbf{U}]]_{\text{int}} \subset \mathcal{M}[[\mathbf{T}]]_{\text{int}}[[\mathbf{U}]],$$

where $\mathcal{M}[[\mathbf{T}]]_{\text{int}}[[\mathbf{U}]]$ is the set of formal series in \mathbf{U} over $\mathcal{M}[[\mathbf{T}]]_{\text{int}}$.

Proof of Lemma 5.2 is elementary and left to the readers.

For any pair of formal series with some variables mixed, namely, $a(\mathbf{T}, \mathbf{V}) = \sum_{\mathbf{n}, \mathbf{l}} a_{\mathbf{n}, \mathbf{l}} \mathbf{T}^{\mathbf{n}} \mathbf{V}^{\mathbf{l}}$ in $\mathcal{M}[[\mathbf{T}, \mathbf{V}]]$ and $b(\mathbf{U}, \mathbf{V}) = \sum_{\mathbf{m}, \mathbf{l}} b_{\mathbf{m}, \mathbf{l}} \mathbf{U}^{\mathbf{m}} \mathbf{V}^{\mathbf{l}}$ in $\mathcal{N}[[\mathbf{U}, \mathbf{V}]]$, their \mathbf{V} -Hadamard product is an element of $\mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{U}, \mathbf{V}]]$ given by

$$(5.1) \quad a(\mathbf{T}, \mathbf{V}) \otimes_{\mathcal{H}} b(\mathbf{U}, \mathbf{V}) := \sum_{\mathbf{n}, \mathbf{m}, \mathbf{l}} \mathbf{a}_{\mathbf{n}, \mathbf{l}} \otimes \mathbf{b}_{\mathbf{m}, \mathbf{l}} \mathbf{T}^{\mathbf{n}} \mathbf{U}^{\mathbf{m}} \mathbf{V}^{\mathbf{l}}.$$

Lemma 5.3. *If $a(\mathbf{T}, \mathbf{V})$ is in $\mathcal{M}[[\mathbf{T}, \mathbf{V}]]_{\text{int}}$ and $b(\mathbf{U}, \mathbf{V})$ is in $\mathcal{N}[[\mathbf{U}]]_{\text{int}}[[\mathbf{V}]]_{\text{ssr}}$, then the \mathbf{V} -Hadamard product $a(\mathbf{T}, \mathbf{V}) \otimes_{\mathcal{H}} b(\mathbf{U}, \mathbf{V})$ is in $\mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{U}, \mathbf{V}]]_{\text{int}}$.*

Proof. It is easy to see that the series $c(\mathbf{T}, \mathbf{U}, \mathbf{V}) := a(\mathbf{T}, \mathbf{V}) \otimes_{\mathcal{H}} b(\mathbf{U}, \mathbf{V})$ can be presented as the Hadamard product of two elements of $\mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{U}, \mathbf{V}]]$ as follows

$$(5.2) \quad c(\mathbf{T}, \mathbf{U}, \mathbf{V}) = \frac{a(\mathbf{T}, \mathbf{V})}{\prod(1 - U_j)} \otimes_{\mathcal{H}} \frac{b(\mathbf{U}, \mathbf{V})}{\prod(1 - T_i)},$$

where $\prod(1 - T_i) := (1 - T_1) \cdots (1 - T_r)$ and $\prod(1 - U_j) := (1 - U_1) \cdots (1 - U_s)$. By setting $b(\mathbf{U}, \mathbf{V}) = \sum_{\mathbf{m}} b_{\mathbf{m}}(\mathbf{V}) \mathbf{U}^{\mathbf{m}}$, we may write the factors in (5.2) as follows

$$\begin{aligned} \frac{a(\mathbf{T}, \mathbf{V})}{\prod(1 - U_j)} &= \sum_{\mathbf{m}} a(\mathbf{T}, \mathbf{V}) \mathbf{U}^{\mathbf{m}} \in \mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{V}]]_{\text{int}}[[\mathbf{U}]]_{\text{ssr}}, \text{ and} \\ \frac{b(\mathbf{U}, \mathbf{V})}{\prod(1 - T_i)} &= \sum_{\mathbf{m}} \tilde{b}_{\mathbf{m}}(\mathbf{T}, \mathbf{V}) \mathbf{U}^{\mathbf{m}} \in \mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{V}]]_{\text{ssr}}[[\mathbf{U}]]_{\text{int}}, \end{aligned}$$

where

$$\tilde{b}_{\mathbf{m}}(\mathbf{T}, \mathbf{V}) := \frac{b_{\mathbf{m}}(\mathbf{V})}{\prod(1 - T_i)}.$$

This together with Lemma 5.1 implies that $c(\mathbf{T}, \mathbf{U}, \mathbf{V}) \in \mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{V}]]_{\text{ssr}}[[\mathbf{U}]]_{\text{int}}$. Moreover, we have

$$c(\mathbf{T}, \mathbf{U}, \mathbf{V}) = \sum_{\mathbf{m}} a(\mathbf{T}, \mathbf{V}) \otimes_{\mathcal{H}} \tilde{b}_{\mathbf{m}}(\mathbf{T}, \mathbf{V}) \mathbf{U}^{\mathbf{m}},$$

which belongs to $\mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{V}]]_{\text{int}}[[\mathbf{U}]]$, by Lemma 5.1. It follows that $c(\mathbf{T}, \mathbf{U}, \mathbf{V})$ is an element of $\mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{V}]]_{\text{int}}[[\mathbf{U}]]_{\text{int}}$, hence an element of $\mathcal{M} \otimes \mathcal{N}[[\mathbf{T}, \mathbf{U}, \mathbf{V}]]_{\text{int}}$. \square

Let X_i , $1 \leq i \leq r$, be smooth algebraic k -varieties, and let $X := X_1 \times \cdots \times X_r$. As usual we use the multivariate $\mathbf{T} = (T_1, \dots, T_r)$. To each $1 \leq i \leq r$ and formal series $a(\mathbf{T}) = \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$ in $\mathcal{M}_X^{\hat{\mu}}[[\mathbf{T}]]$ associate a unique formal series $a_i(\mathbf{T}) := \sum_{\mathbf{n}} a_{\mathbf{n}}^{(i)} \mathbf{T}^{\mathbf{n}}$ in $\mathcal{M}_{X_i}^{\hat{\mu}}[[\mathbf{T}]]$ in such a way that $a_{\mathbf{n}}^{(i)} = (\text{pr}_i)_! a_{\mathbf{n}} \in \mathcal{M}_{X_i}$, where pr_i is the natural projection of X onto X_i .

Lemma 5.4. *If the series $a(\mathbf{T})$ is integrable, so are the series $a_i(\mathbf{T})$ for $1 \leq i \leq r$.*

Proof of this lemma is straightforward.

Definition 5.5. For any $r \in \mathbb{N}_{>0}$ and $1 \leq i \leq r$, a r -tuple $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ is said to have the $\Delta_{i, <}$ -property (resp. the $\Delta_{<}$ -property), written as $\mathbf{n} \in \Delta_{i, <}$ -property (resp. $\mathbf{n} \in \Delta_{<}$ -property) or simply as $\mathbf{n} \in \Delta_{i, <}$ (resp. $\mathbf{n} \in \Delta_{<}$), if

$$n_1 < \cdots < n_i = n_{i+1} = \cdots = n_r \quad (\text{resp. } n_1 < \cdots < n_r).$$

We denote by $\mathcal{M}_X^{i, <}[[\mathbf{T}]]$ (resp. $\mathcal{M}_X^{<}[[\mathbf{T}]]$) the subset of $\mathcal{M}_X^{\hat{\mu}}[[\mathbf{T}]]$ consisting of formal series of the form $\sum_{\mathbf{n} \in \Delta_{i, <}} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$ (resp. $\sum_{\mathbf{n} \in \Delta_{<}} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$). We also have an analogous definition for $\overline{\mathcal{M}}_X^{i, <}[[\mathbf{T}]]$ and $\overline{\mathcal{M}}_X^{<}[[\mathbf{T}]]$ as subset of $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]$. By definition, for any $a(\mathbf{T})$ in $\mathcal{M}_X^{i, <}[[\mathbf{T}]]$ (resp. in $\overline{\mathcal{M}}_X^{i, <}[[\mathbf{T}]]$), there exists a series $\tilde{a}(T_1, \dots, T_i)$ in $\mathcal{M}_X^{<}[[T_1, \dots, T_i]]$ (resp. in $\overline{\mathcal{M}}_X^{<}[[T_1, \dots, T_i]]$) such that

$$a(\mathbf{T}) = \tilde{a}(T_1, \dots, T_{i-1}, T_i \cdots T_r).$$

Let us now introduce a new notion of ordered cells. For an increasing sequence of positive integers $0 = r_0 < r_1 < \dots < r_i = r$ we define the *basic ordered cell* $\Delta_{(r_0, \dots, r_i)}$ to be the set

$$\{(n_1, \dots, n_r) \in \mathbb{N}^r \mid n_{r_{j-1}+1} = \dots = n_{r_j} \text{ and } n_{r_{j-1}} < n_{r_j}, 2 \leq j \leq i\}.$$

A subset Δ of \mathbb{N}^r is called an *ordered cell* if it is the image of a basic ordered cell $\Delta_{(r_0, \dots, r_i)}$ under a permutation map $\rho: \mathbb{N}^r \rightarrow \mathbb{N}^r$ that sends (n_1, \dots, n_r) to $(n_{\rho(1)}, \dots, n_{\rho(r)})$. It is easy to see that \mathbb{N}^r can be partitioned into all the ordered cells Δ . This implies that any formal series $a(\mathbf{T}) \in \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]$ can be uniquely decomposed as a finite sum of formal series

$$(5.3) \quad a(\mathbf{T}) = \sum_{\Delta} a_{\Delta}(\mathbf{T}) = \sum_{\Delta} a_{\Delta}^{\leq} \left(\prod_{l=1}^{r_1} T_{\rho(l)}, \dots, \prod_{l=r_{i-1}+1}^{r_i} T_{\rho(l)} \right),$$

where $a_{\Delta}(\mathbf{T}) := \sum_{\mathbf{n} \in \Delta} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$ and $a_{\Delta}^{\leq} \in \overline{\mathcal{M}}_X^{\leq}[[T_1, \dots, T_i]]$ in viewing X as

$$\prod_{l=1}^{r_1} X_{\rho(l)} \times \dots \times \prod_{l=r_{i-1}+1}^{r_i} X_{\rho(l)}.$$

Lemma 5.6. *If the series $a(\mathbf{T})$ is integrable, so are the series $a_{\Delta}(\mathbf{T})$ for all ordered cells Δ .*

Remark 5.7. Actually, in view of Cluckers-Loeser's theory on constructible motivic functions one can show that the lemma also works for any definable subset of \mathbb{N}^r , cf. [1, Lemma 4.5.8].

Proof of Lemma 5.6. It suffices to prove that $a_{\Delta}(\mathbf{T})$ is integrable for $\Delta = \Delta_{(r_0, \dots, r_i)}$ being a basic ordered cell. We can check easily that

$$a_{\Delta}(\mathbf{T}) = \varepsilon(\mathbf{T}) \cdot \mathcal{H} a(\mathbf{T}),$$

where, by definition,

$$\varepsilon(\mathbf{T}) := \frac{\prod_{j=2}^i \left(\prod_{l=r_{j-1}+1}^r T_l \right)}{\prod_{j=1}^i \left(1 - \prod_{l=r_{j-1}+1}^r T_l \right)},$$

which is strongly rational. Then the present lemma follows from Lemma 5.1. \square

We consider the morphism of $\overline{\mathcal{M}}_X^{\hat{\mu}}$ -modules

$$\Phi: \overline{\mathcal{M}}_X^{\leq}[[\mathbf{T}]] \rightarrow \overline{\mathcal{M}}_X^{\leq}[[\mathbf{T}]]$$

given by

$$\Phi \left(\sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}} \right) = (\mathbb{L} - 1)^{1-r} \sum_{\mathbf{n}} a_{\mathbf{n}}^{(1)} \times \prod_{i=2}^r (a_{\mathbf{n}-\mathbf{e}_i}^{(i)} - a_{\mathbf{n}}^{(i)}) \mathbf{T}^{\mathbf{n}},$$

where $a_{\mathbf{n}}^{(i)} := (\text{pr}_i)_! a_{\mathbf{n}} \in \overline{\mathcal{M}}_{X_i}$, pr_i is the natural projection of X onto X_i , and \mathbf{e}_i is the i -th standard vector in \mathbb{Z}^r , $1 \leq i \leq r$. Here by $\prod_{i=2}^r$ we mean temporarily the external products. It is clear that Φ can be extended to an endomorphism of $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]$,

$$(5.4) \quad \Phi: \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]] \rightarrow \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]],$$

by linearity, namely,

$$\Phi(a(\mathbf{T})) := \sum_{\Delta} \Phi(a_{\Delta}(\prod_{l=1}^{r_1} T_{\rho(l)}, \dots, \prod_{l=r_{i-1}+1}^{r_i} T_{\rho(l)})),$$

in terms of the decomposition of $a(\mathbf{T}) \in \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]$ into finitely many terms of the form (5.3).

Now we work with the restriction of Φ to the sub- $\overline{\mathcal{M}}_X^{\hat{\mu}}$ -module $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$ of $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]$.

Lemma 5.8. *The restriction of Φ to $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$ is an automorphism.*

Proof. Define the morphism $\Phi^{-1} : \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}} \rightarrow \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$ as follows

$$a(\mathbf{T}) = \sum_{\mathbf{n} \in \Delta_{<}} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}} \mapsto (\mathbb{L} - 1)^{r-1} \sum_{\mathbf{n} \in \Delta_{<}} a_{\mathbf{n}}^{(1)} \times \prod_{i=2}^r \left(\sum_{l>1} a_{\mathbf{n}+l\mathbf{e}_i}^{(i)} \right) \mathbf{T}^{\mathbf{n}},$$

with $\prod_{i=2}^r$ being the external products at the moment. Let us show that $\Phi^{-1}(a(\mathbf{T}))$ is an integrable series. We first prove that, for any $2 \leq i \leq r$,

$$(\mathbb{L} - 1) \sum_{\mathbf{n} \in \Delta_{<}} \sum_{l>1} a_{\mathbf{n}+l\mathbf{e}_i}^{(i)} \mathbf{T}^{\mathbf{n}} = \frac{\gamma_i(\mathbf{T})}{1 - T_i},$$

for some $\gamma_i(\mathbf{T}) \in \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$. Indeed, by setting $\hat{\mathbf{n}}_i := \mathbf{n} - n_i \mathbf{e}_i$ and $\hat{\mathbf{T}}_i := \mathbf{T} - (T_i - 1)\mathbf{e}_i$, $2 \leq i \leq r$, we have

$$\begin{aligned} (\mathbb{L} - 1) \sum_{\mathbf{n} \in \Delta_{<}} \sum_{l>1} a_{\mathbf{n}+l\mathbf{e}_i}^{(i)} \mathbf{T}^{\mathbf{n}} &= (\mathbb{L} - 1) \sum_{l>1} \sum_{\hat{\mathbf{n}}_i \in \Delta_{<}} a_{\hat{\mathbf{n}}_i+l\mathbf{e}_i}^{(i)} \hat{\mathbf{T}}_i^{\hat{\mathbf{n}}_i} \sum_{i \leq n_i < l} T_i^{n_i} \\ &= (\mathbb{L} - 1) \sum_{l>1} \sum_{\hat{\mathbf{n}}_i \in \Delta_{<}} a_{\hat{\mathbf{n}}_i+l\mathbf{e}_i}^{(i)} \hat{\mathbf{T}}_i^{\hat{\mathbf{n}}_i} \frac{T_i^i - T_i^l}{1 - T_i} \\ &= \frac{(\mathbb{L} - 1)T_i^i}{1 - T_i} a_i(\hat{\mathbf{T}}_i) - \frac{(\mathbb{L} - 1)a_i(\mathbf{T})}{1 - T_i}, \end{aligned}$$

which has the form as desired. It therefore follows that

$$\Phi^{-1}(a(\mathbf{T})) = a_1(\mathbf{T}) \cdot_{\mathcal{H}} \frac{\gamma_2(\mathbf{T})}{1 - T_2} \cdot_{\mathcal{H}} \cdots \cdot_{\mathcal{H}} \frac{\gamma_r(\mathbf{T})}{1 - T_r},$$

which is obviously integrable due to Lemma 5.1. By the decomposition (5.3), the morphism Φ^{-1} may be extended to $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$. It is easily checked that $\Phi \circ \Phi^{-1} = \text{id}_{\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}}$. The lemma is thus proved. \square

Corollary 5.9. *Let $\mathbf{f} = (f_1, \dots, f_r)$ be an ordered family of regular functions on X_1, \dots, X_r . Then the multiple motivic zeta function $\zeta_{\mathbf{f}}(\mathbf{T})$ is an integrable series, i.e., $\zeta_{\mathbf{f}}(\mathbf{T}) \in \overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$.*

Proof. Let $a(\mathbf{T}) = Z_{\mathbf{f}}^{\mathbb{N}_{>0}^r}(\mathbf{T})$ be the motivic zeta function with respect to the trivial cone $\mathbb{N}_{>0}^r$ defined as in Definition 2.2. Then we have

$$a(\mathbf{T}) = Z_{f_1}(T_1) \times_{\mathcal{H}} \cdots \times_{\mathcal{H}} Z_{f_r}(T_r),$$

it is therefore integrable due to Lemma 5.3. On the other hand, we deduce from Lemma 5.6 that the series $a_{\Delta_{<}}(\mathbf{T}) = Z_{\mathbf{f}}^{\Delta_{<}}(\mathbf{T})$ is integrable. Since the identity

$\Phi^{-1} \left(Z_{\mathbf{f}}^{\Delta_{<}}(\mathbf{T}) \right) = \zeta_{\mathbf{f}}(\mathbf{T})$ holds in $\overline{\mathcal{M}}_X^{\Delta_{<}}[[\mathbf{T}]]$, Lemma 5.8 gives us the integrability of the series $\zeta_{\mathbf{f}}(\mathbf{T})$. \square

5.2. Generalized \boxtimes -product. Let X_i and Y_j , $1 \leq i \leq r$, $1 \leq j \leq s$, be smooth algebraic k -varieties, and let

$$(5.5) \quad X := X_1 \times \cdots \times X_r \quad \text{and} \quad Y := Y_1 \times \cdots \times Y_s.$$

As usual we also use the multivariates $\mathbf{T} = (T_1, \dots, T_r)$ and $\mathbf{U} = (U_1, \dots, U_s)$. Now for tuples $\mathbf{n} = (n_1, \dots, n_r)$ and $\mathbf{m} = (m_1, \dots, m_s)$ having the $\Delta_{<}$ -property, we let

$$I := I_{\mathbf{n}, \mathbf{m}} := \{(i, j) \in \mathbb{N}^2 \mid n_i = m_j\},$$

and let I_1 (resp. I_2) be the image of I under the projection on the first component (resp. the second component). Then, to define the \boxtimes -product of a series in $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]$ and a series in $\overline{\mathcal{M}}_Y^{\hat{\mu}}[[\mathbf{U}]]$ it suffices to define the \boxtimes -product of a series in $\overline{\mathcal{M}}_X^{\Delta_{<}}[[\mathbf{T}]]$ and a series in $\overline{\mathcal{M}}_Y^{\Delta_{<}}[[\mathbf{U}]]$.

Definition 5.10. Let $a(\mathbf{T}) = \sum a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$ and $b(\mathbf{U}) = \sum b_{\mathbf{m}} \mathbf{U}^{\mathbf{m}}$ be formal series in $\overline{\mathcal{M}}_X^{\Delta_{<}}[[\mathbf{T}]]$ and $\overline{\mathcal{M}}_Y^{\Delta_{<}}[[\mathbf{U}]]$, respectively. We define the product $a(\mathbf{T}) \boxtimes b(\mathbf{U})$ in two steps as follows.

(i) Put

$$a(\mathbf{T}) \boxtimes_0 b(\mathbf{U}) := \sum_{\mathbf{n} \in \Delta_{<}, \mathbf{m} \in \Delta_{<}} c_{\mathbf{n}, \mathbf{m}} \mathbf{T}^{\mathbf{n}} \mathbf{U}^{\mathbf{m}},$$

where

$$c_{\mathbf{n}, \mathbf{m}} = \prod_{i \notin I_1} a_{\mathbf{n}}^{(i)} \times \prod_{j \notin I_2} b_{\mathbf{m}}^{(j)} \times \prod_{(i, j) \in I} \tilde{c}_{\mathbf{n}, \mathbf{m}}^{(i, j)},$$

and, for any $(i, j) \in I$, the quantity $\tilde{c}_{\mathbf{n}, \mathbf{m}}^{(i, j)}$ is defined to be

$$-a_{\mathbf{n}}^{(i)} * b_{\mathbf{m}}^{(j)} + \sum_{0 \leq l < n_i} \mathbb{L}^{-l} a_{\mathbf{n} - l e_i}^{(i)} *_0 b_{\mathbf{m} - l e_j}^{(j)} + (\mathbb{L} - 1) \sum_{l > 0} \left(a_{\mathbf{n}}^{(i)} \times (b_{\mathbf{m} + l e_j}^{(j)})' + (a_{\mathbf{n} + l e_i}^{(i)})' \times b_{\mathbf{m}}^{(j)} \right)$$

with z' the image of z under the augmentation map.

(ii) Put

$$a(\mathbf{T}) \boxtimes b(\mathbf{U}) := \Phi^{-1} \left(\Phi(a(\mathbf{T})) \boxtimes_0 \Phi(b(\mathbf{U})) \right),$$

where Φ is defined previously in (5.4).

It is clear that the \boxtimes -product in Definition 5.10 is well defined since Φ is well defined. Moreover, when reduced to the univariate case, i.e., $r = s = 1$, this product is nothing else than the one defined in Definition 3.1.

Theorem 5.11. *With previous notation and hypotheses, if $a(\mathbf{T})$ is in $\overline{\mathcal{M}}_X^{\hat{\mu}}[[\mathbf{T}]]_{\text{int}}$ and $b(\mathbf{U})$ is in $\overline{\mathcal{M}}_Y^{\hat{\mu}}[[\mathbf{U}]]_{\text{int}}$, then $a(\mathbf{T}) \boxtimes b(\mathbf{U})$ is in $\overline{\mathcal{M}}_{X \times Y}^{\hat{\mu}}[[\mathbf{T}, \mathbf{U}]]_{\text{int}}$.*

Proof. We first consider $a(\mathbf{T})$ and $b(\mathbf{U})$ in $\overline{\mathcal{M}}_X^{\Delta_{<}}[[\mathbf{T}]]_{\text{int}}$ and $\overline{\mathcal{M}}_Y^{\Delta_{<}}[[\mathbf{U}]]_{\text{int}}$, respectively. It follows from the proof of Lemma 5.8 that

$$(\mathbb{L} - 1) \sum_{\mathbf{n} \in \Delta_{<}} \sum_{l > 0} \mathbf{a}_{\mathbf{n} + l e_i}^{(i)} \mathbf{T}^{\mathbf{n}} = \frac{\alpha_i(\mathbf{T})}{1 - T_i},$$

and that

$$(\mathbb{L} - 1) \sum_{\mathbf{m} \in \Delta_{<}} \sum_{l > 0} \mathbf{b}_{\mathbf{m} + l e_j}^{(j)} \mathbf{U}^{\mathbf{m}} = \frac{\beta_j(\mathbf{U})}{1 - U_j},$$

for some integrable series $\alpha_i(\mathbf{T})$ and $\beta_j(\mathbf{U})$. Then, by simple computation, we deduce that

$$a(\mathbf{T}) \boxtimes_0 b(\mathbf{U}) = \prod_{i \notin I_1} a_i(\mathbf{T}) \times_{\mathcal{H}} \prod_{j \notin I_2} b_j(\mathbf{U}) \times_{\mathcal{H}} \prod_{(i,j) \in I} c_{ij}(\mathbf{T}, \mathbf{U}),$$

where for each $(i, j) \in I$, the series $c_{ij}(\mathbf{T}, \mathbf{U})$ is equal to

$$-a_i(\mathbf{T}_i) *_{\mathcal{H}} b_j(\mathbf{U}_j) + \frac{a_i(\mathbf{T}_i) *_{0\mathcal{H}} b_j(\mathbf{U}_j)}{1 - \mathbb{L}^{-1} T_i U_j} + a_i(\mathbf{T}_i) \times_{\mathcal{H}} \frac{\beta_j(\mathbf{U}_j)}{1 - U_j} + \frac{\alpha_i(\mathbf{T}_i)}{1 - T_i} \times_{\mathcal{H}} b_j(\mathbf{U}_j),$$

where

$$\mathbf{T}_i := (T_1, \dots, T_{i-1}, T_i U_j, T_{i+1}, \dots, T_r)$$

and

$$\mathbf{U}_j := (U_1, \dots, U_{j-1}, T_i U_j, U_{j+1}, \dots, U_s).$$

By using Lemma 5.3, we get the integrability of the series $a(\mathbf{T}) \boxtimes_0 b(\mathbf{U})$. The theorem is then follows from Lemma 5.6 and 5.8. \square

5.3. Motivic reflexion formulas. In this paragraph, we formulate the motivic reflexion formulas for the multivariate case that generalize the motivic Euler reflexion formula. As a consequence, we show that the \boxtimes -product is associative in the class of motivic multiple zeta functions defined in Definition 2.2. A corollary of the associativity will be also given.

Theorem 5.12. *Let $\mathbf{f} = (f_1, \dots, f_r)$ and $\mathbf{g} = (g_1, \dots, g_s)$ be ordered families of regular functions on smooth algebraic k -varieties X_1, \dots, X_r and Y_1, \dots, Y_s , respectively. Then*

$$\zeta_{\mathbf{f}}(\mathbf{T}) \boxtimes \zeta_{\mathbf{g}}(\mathbf{U}) = \sum \iota^* \zeta_{p_1, \dots, p_\eta}(T_{\alpha_1}^{a_1} U_{\beta_1}^{b_1}, \dots, T_{\alpha_\eta}^{a_\eta} U_{\beta_\eta}^{b_\eta}),$$

where the sum is taken over all the ordered families of regular functions (p_1, \dots, p_η) satisfying

$$p_i = a_i f_{\alpha_i} \oplus b_i g_{\beta_i}, \quad 1 \leq i \leq \eta,$$

with $(a_i, b_i) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, $\sum (a_i + b_i) = r + s$, and $\{\alpha_i\}_{a_i=1}$ and $\{\beta_i\}_{b_i=1}$ being strictly monotonic increasing sequences; ι is the inclusion of $X_0 \times Y_0$ in $X \times Y$ (cf. (5.5)).

Proof. First, we note that $\zeta_{\mathbf{f}}(\mathbf{T})$ and $\zeta_{\mathbf{g}}(\mathbf{U})$ are elements of $\overline{\mathcal{M}}_{X_0}^<[[\mathbf{T}]]$ and $\overline{\mathcal{M}}_{Y_0}^<[[\mathbf{U}]]$, respectively. By definition, it suffices to show that

$$(5.6) \quad \Phi(\zeta_{\mathbf{f}}(\mathbf{T})) \boxtimes_0 \Phi(\zeta_{\mathbf{g}}(\mathbf{U})) = \sum_{\mathbf{p}} \Phi(\iota^* \tilde{\zeta}_{\mathbf{p}}),$$

where $\mathbf{p} = (p_1, \dots, p_\eta)$, $\tilde{\zeta}_{\mathbf{p}} = \zeta_{\mathbf{p}}(T_{\alpha_1}^{a_1} U_{\beta_1}^{b_1}, \dots, T_{\alpha_\eta}^{a_\eta} U_{\beta_\eta}^{b_\eta})$, and the sum is taken over all the \mathbf{p} in the theorem. Writing $\Phi(\zeta_{\mathbf{f}}(\mathbf{T})) = \sum_{\mathbf{n} \in \Delta_{<}} a_{\mathbf{n}} \mathbf{T}^{\mathbf{n}}$ and $\Phi(\zeta_{\mathbf{g}}(\mathbf{U})) = \sum_{\mathbf{m} \in \Delta_{<}} b_{\mathbf{m}} \mathbf{U}^{\mathbf{m}}$ we get

$$\begin{aligned} a_{\mathbf{n}}^{(i)} &= [\{\varphi \in \mathcal{L}_{n_i}(X_i) \mid f_i(\varphi) = t^{n_i} \pmod{t^{n_i+1}} \rightarrow X_{i,0}\} \mathbb{L}^{-d_i n_i}, \\ b_{\mathbf{m}}^{(j)} &= [\{\psi \in \mathcal{L}_{m_j}(Y_j) \mid g_j(\psi) = t^{m_j} \pmod{t^{m_j+1}} \rightarrow Y_{j,0}\} \mathbb{L}^{-e_j m_j}, \end{aligned}$$

with $d_i = \dim_k X_i$ and $e_j = \dim_k Y_j$.

Observe that the coefficients of $\mathbf{T}^{\mathbf{n}} \mathbf{U}^{\mathbf{m}}$ in both sides of (5.6) are zero for $\mathbf{n} \notin \Delta_{<}$ or $\mathbf{m} \notin \Delta_{<}$. In this case, indeed, the statement for the left hand side comes from Definition 5.10 (i), and that for the right hand side is due to the hypothesis

that the sequences $\{\alpha_i\}_{a_i=1}$ and $\{\beta_i\}_{b_i=1}$ are strictly monotonic increasing. For $\mathbf{n} \in \Delta_{<}$ and $\mathbf{m} \in \Delta_{<}$, since the supports of the $\tilde{\zeta}_{\mathbf{p}}$ are distinct, it suffices to show that there exists \mathbf{p} such that the coefficient of $\mathbf{T}^{\mathbf{n}}\mathbf{U}^{\mathbf{m}}$ in $\Phi(\iota^*\tilde{\zeta}_{\mathbf{p}})$ equals the one in $\Phi(\zeta_{\mathbf{f}}(\mathbf{T})) \boxtimes_0 \Phi(\zeta_{\mathbf{g}}(\mathbf{U}))$. To prove this, we set

$$\{l_1 < \dots < l_\eta\} := \{\mathbf{n}\} \cup \{\mathbf{m}\} = \{n_1, \dots, n_r, m_1, \dots, m_s\}$$

and set

$$p_i = a_i f_{\alpha_i} \oplus b_i g_{\beta_i}, \quad 1 \leq i \leq \eta,$$

with $a_i = 1$ (resp. $b_i = 1$) if $l_i = n_{\alpha_i} \in \{\mathbf{n}\}$ (resp. $l_i = m_{\beta_i} \in \{\mathbf{m}\}$), otherwise $a_i = 0$ (resp. $b_i = 0$). Define $\mathbf{l} := (l_1, \dots, l_\eta)$. It is easily checked that the coefficient $c_{\mathbf{l}}$ of $\mathbf{T}^{\mathbf{n}}\mathbf{U}^{\mathbf{m}}$ in $\Phi(\iota^*\tilde{\zeta}_{\mathbf{p}})$ equals $c_{\mathbf{l}}^{(1)} \times \dots \times c_{\mathbf{l}}^{(\eta)}$, where

$$c_{\mathbf{l}}^{(i)} := [\{\omega \in \mathcal{L}_{l_i}(Z_i) \mid p_i(\omega) = t^{l_i} \pmod{t^{l_i+1}}\} \rightarrow Z_{i,0}] \mathbb{L}^{-\delta_i l_i}$$

with $Z_i := (X_{\alpha_i})^{a_i} \times (Y_{\beta_i})^{b_i}$ and $\delta_i = \dim_k Z_i$. It follows from the proof of Theorem 4.1 and direct calculations that

$$c_{\mathbf{l}}^{(i)} = \begin{cases} a_{\mathbf{n}}^{(\alpha_i)} & \text{if } b_i = 0, \\ b_{\mathbf{m}}^{(\beta_i)} & \text{if } a_i = 0, \\ a_{\mathbf{n}}^{(\alpha_i)} * b_{\mathbf{m}}^{(\beta_i)} & \text{if } a_i = b_i = 1. \end{cases}$$

This proves the theorem. \square

The following corollaries are direct consequences of Theorem 5.12.

Corollary 5.13. *Let $\mathbf{f} = (f_1, \dots, f_r)$, $\mathbf{g} = (g_1, \dots, g_s)$ and $\mathbf{h} = (h_1, \dots, h_\tau)$ be ordered families of regular functions on smooth algebraic k -varieties X_1, \dots, X_r , Y_1, \dots, Y_s and Z_1, \dots, Z_τ , respectively. Then*

$$(\zeta_{\mathbf{f}}(\mathbf{T}) \boxtimes \zeta_{\mathbf{g}}(\mathbf{U})) \boxtimes \zeta_{\mathbf{h}}(\mathbf{V}) = \sum \iota^* \zeta_{p_1, \dots, p_\tau} (T_{\alpha_1}^{a_1} S_{\beta_1}^{b_1} U_{\gamma_1}^{c_1}, \dots, T_{\alpha_\tau}^{a_\tau} S_{\beta_\tau}^{b_\tau} U_{\gamma_\tau}^{c_\tau}),$$

where the sum is taken over all the ordered families of regular functions (p_1, \dots, p_τ) satisfying

$$p_i = a_i f_{\alpha_i} \oplus b_i g_{\beta_i} \oplus c_i h_{\gamma_i}, \quad 1 \leq i \leq \tau,$$

with $(a_i, b_i, c_i) \in \{0, 1\}^3 \setminus \{(0, 0, 0)\}$, $\sum (a_i + b_i + c_i) = r + s + \tau$, and $\{\alpha_i\}_{a_i=1}$, $\{\beta_i\}_{b_i=1}$ and $\{\gamma_i\}_{c_i=1}$ being strictly monotonic increasing sequences.

In particular, the \boxtimes -product is associative in the class of motivic multiple zeta functions.

Corollary 5.14. *Let f , g and h be regular functions on smooth algebraic k -varieties X , Y and Z , respectively. Then, up to the pullback of an inclusion of $X_0 \times Y_0 \times Z_0$ in a Zariski closed subset of $X \times Y \times Z$, the following identity holds in $\mathcal{M}_{X_0 \times Y_0 \times Z_0}^{\hat{\mu}}[[T, U, V]]$:*

$$\begin{aligned} \zeta_f(T) \boxtimes \zeta_g(U) \boxtimes \zeta_h(V) &= \zeta_{f,g,h}(T, U, V) + \zeta_{f,h,g}(T, V, U) + \zeta_{g,h,f}(U, V, T) \\ &\quad + \zeta_{g,f,h}(U, T, V) + \zeta_{h,f,g}(V, T, U) + \zeta_{h,g,f}(V, U, T) \\ &\quad + \zeta_{f \oplus g, h}(TU, V) + \zeta_{h, f \oplus g}(V, TU) + \zeta_{f, g \oplus h}(T, UV) \\ &\quad + \zeta_{g \oplus h, f}(UV, T) + \zeta_{g, f \oplus h}(U, TV) + \zeta_{f \oplus h, g}(TV, U) \\ &\quad + \zeta_{f \oplus g \oplus h}(TUV). \end{aligned}$$

Remark 5.15. After many attempts we still do not know whether the \boxtimes -product is associative in the class of integrable series over monodromic Grothendieck rings of algebraic varieties.

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