

Israel P. Rivera-Ríos*

A quantitative approach to weighted Carleson condition

Abstract: Quantitative versions of weighted estimates obtained by F. Ruiz and J.L. Torrea [30, 31] for the operator

$$\mathcal{M}f(x, t) = \sup_{x \in Q, l(Q) \geq t} \frac{1}{|Q|} \int_Q |f(x)| dx \quad x \in \mathbb{R}^n, t \geq 0$$

are obtained. As a consequence, some sufficient conditions for the boundedness of \mathcal{M} in the two weight setting in the spirit of the results obtained by C. Pérez and E. Rela [26] and very recently by M. Lacey and S. Spencer [17] for the Hardy-Littlewood maximal operator are derived. As a byproduct some new quantitative estimates for the Poisson integral are obtained.

Keywords: weighted Carleson condition, Maximal operator, weights, Poisson integral, bumps, entropy

DOI

Received ; revised ; accepted .

1 Introduction

In 1962 L. Carleson [3] studied a variation of the Hardy-Littlewood maximal operator defined on $\mathbb{R}_+^{n+1} = \{(x, t) : x \in \mathbb{R}^n, t \geq 0\}$ as follows

$$\mathcal{M}f(x, t) = \sup_{x \in Q, l(Q) \geq t} \frac{1}{|Q|} \int_Q |f(x)| dx.$$

The interest of the study of this operator stems from the fact that it controls pointwise the Poisson integral. Indeed, if we call

$$P(x, t) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}$$

the Poisson integral is defined as

$$Pf(x, t) = \int_{\mathbb{R}^n} f(y) P(x - y, t) dy \quad x \in \mathbb{R}^n, t \geq 0$$

and it is not hard to check that (cf. [10] Chapter 2)

$$Pf(x, t) \leq c_n \mathcal{M}f(x, t). \quad (1.1)$$

L. Carleson characterized the positive Borel measures on \mathbb{R}_+^{n+1} such that \mathcal{M} is of strong type (p, p) for $p > 1$ and of weak type $(1, 1)$. The precise statement of his result is the following

***Corresponding Author: Israel P. Rivera-Ríos:**

(1) Universidad del País Vasco/Euskal Herriko Unibertsitatea, Departamento de Matematicas/Matematika saila, Apdo. 644, 48080 Bilbao, Basque Country, Spain

(2) BCAM, Basque Center for Applied Mathematics, Mazarredo, 14. 48009 Bilbao, Basque Country, Spain

E-mail: petnapet@gmail.com

Theorem I. Let μ be a positive Borel measure on \mathbb{R}_+^{n+1} and let $1 < p < \infty$. Then, the following statements are equivalent,

1. $\mathcal{M} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)$ or $\mathcal{M} : L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}_+^{n+1}, \mu)$
2. There is a constant C such that for each cube Q in \mathbb{R}^n

$$\mu(\tilde{Q}) \leq C|Q|$$

where $\tilde{Q} = \{(x, t) : x \in Q, 0 \leq t \leq l(Q)\}$.

In fact, it can be established that exactly the same result holds for the Poisson integral (cf. [10] Chapter 2). Condition (2) in the preceding Theorem is the so called ‘‘Carleson Condition’’. Later on, in the 70’s C. Fefferman and E. Stein [8] obtained the following weighted result

Theorem II. Let μ be a positive Borel measure on \mathbb{R}_+^{n+1} and let w be a weight in \mathbb{R}^n such that

$$M\mu(x) = \sup_{x \in Q} \frac{\mu(\tilde{Q})}{|Q|} \leq Cw(x).$$

Then we have both $\mathcal{M} : L^p(\mathbb{R}^n, w) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)$, and $\mathcal{M} : L^1(\mathbb{R}^n, w) \rightarrow L^{1,\infty}(\mathbb{R}_+^{n+1}, \mu)$

This result was extended in the 80’s by F. Ruiz and J. L. Torrea [30, 31]. The conditions obtained by these authors are in the spirit of the ones obtained by B. Muckenhoupt [22] and E. Sawyer [32] for the Hardy-Littlewood maximal function. The main results are the following.

Theorem III. Let μ be a positive Borel measure on \mathbb{R}_+^{n+1} and let w be a weight in \mathbb{R}^n . Let also $1 < p < \infty$, then the following statements are equivalent:

1. $\mathcal{M} : L^p(\mathbb{R}^n, w) \rightarrow L^{p,\infty}(\mathbb{R}_+^{n+1}, \mu)$
2. $\sup_Q \frac{\mu(\tilde{Q})}{|Q|} \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty$.

Theorem IV. Let μ be a positive Borel measure on \mathbb{R}_+^{n+1} and let v be a weight in \mathbb{R}^n . Let also $1 < p < \infty$, then the following conditions are equivalent:

1. $\mathcal{M} : L^p(\mathbb{R}^n, v) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)$
2. There is a constant $c > 0$ such that for each cube Q

$$\int_{\tilde{Q}} \mathcal{M}(v^{1-p'} \chi_Q)(x, t)^p d\mu(x, t) \leq c \int_Q v^{1-p'}(x) dx < \infty$$

Theorems III and IV characterize qualitatively the weak and the strong type L^p boundedness of \mathcal{M} in the sense that they characterize the L^p boundedness but they don’t provide a quantitative relationship between the operator norm and the relevant constants associated to the couple (μ, v) . Quantitative estimates for the main operators in Harmonic Analysis have been widely considered by many authors in the last years. The first result can be traced back to the work of S. Buckley [2] where a quantitative version of the classical Muckenhoupt theorem is obtained, namely

$$\|Mf\|_{L^p(w)} \leq c_{n,p} [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)},$$

where M is the classical Hardy-Littlewood maximal function. Later on and motivated by the A_2 conjecture for the Ahlfors-Beurling transform formulated in [1], the sharp dependence of weighted inequalities on the A_p constant has been studied thoroughly for operators such as Calderón-Zygmund operators (see for instance [11, 19, 27–29] or the more recent works [5, 16, 20, 21]), rough singular integrals (c.f. [15]), commutators (c.f. [4]) or the square function (for example [12]).

Our aim in this paper is to obtain quantitative versions of Theorems III and IV which, as a consequence, will provide two new quantitative sufficient conditions for the boundedness of \mathcal{M} . Since \mathcal{M}

controls pointwise P , as a direct consequence of those quantitative estimates for \mathcal{M} we will also obtain corresponding quantitative estimates for the Poisson integral.

This paper is organized as follows. In section 2 we state our main results. In section 3 we introduce some definitions needed to understand in full detail the main results. Finally in section 4 we prove our main results.

2 Main Results

Before we state our main results we would like to note that the precise definitions of the operators and the constants involved are summarized and commented in section 3. Our first of result is a quantitative characterization of the weak-type (p, p) boundedness.

Theorem 1. *Let μ be a positive Borel measure on \mathbb{R}_+^{n+1} and let σ be a weight in \mathbb{R}^n . Let also $1 < p < \infty$, then*

$$\|\mathcal{M}(\sigma \cdot)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^{p, \infty}(\mathbb{R}_+^{n+1}, \mu)} \simeq [\mu, \sigma]_{A'_p}^{\frac{1}{p}}$$

where

$$[\mu, \sigma]_{A'_p} := \sup_Q \frac{\mu(\tilde{Q})}{|Q|} \left(\frac{1}{|Q|} \int_Q \sigma(x) dx \right)^{p-1}.$$

Theorem 2. *Let μ , σ and p as in the previous theorem. Then*

$$\|\mathcal{M}(\sigma \cdot)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \simeq [\mu, \sigma]_{S'_p}$$

where

$$[\mu, \sigma]_{S'_p} = \sup_Q \left(\frac{\int_{\tilde{Q}} \mathcal{M}(\sigma \chi_Q)(x, t)^p d\mu(x, t)}{\int_Q \sigma(x) dx} \right)^{\frac{1}{p}}.$$

Relying on the preceding result and following the proof for the Hardy-Littlewood maximal operator obtained by C. Pérez and E. Rela [26] we can obtain the following quantitative sufficient condition for the boundedness of the operator in the weighted setting.

Theorem 3. *Let μ , σ and p as before. Also, let Φ be a Young function with conjugate function $\bar{\Phi}$. Then*

$$\|\mathcal{M}(\cdot \sigma)\|_{L^p(\sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \lesssim \left([\mu, \sigma, \Phi]_{A'_p} [\sigma, \bar{\Phi}]_{W_p} \right)^{\frac{1}{p}}$$

where

$$[\mu, \sigma, \Phi]_{A'_p} = \sup_Q \frac{\mu(\tilde{Q})}{|Q|} \left\| \sigma^{\frac{1}{p'}} \right\|_{\Phi, Q}^p \quad \text{and} \quad [\sigma, \bar{\Phi}]_{W_p} = \sup_Q \frac{1}{\sigma(Q)} \int_Q M_{\Phi} \left(\sigma^{\frac{1}{p}} \chi_Q \right)^p dx.$$

Conditions considered like the ones considered in Theorem 3 are often called “bump” conditions. They were introduced in the 90’s in [24] and considered again to study sharp two weighted estimates for Singular Integrals [25] and have been often used in recent literature.

A very interesting recent result obtained by L. Slavíková is that this condition is not necessary. We remit to [33] for details.

The following result generalizes the result due to M. Lacey and S. Spencer [17] which is based on the very recent idea of entropy introduced in [34].

Theorem 4. *Let μ , σ and p as before. Then*

$$\|\mathcal{M}(\cdot \sigma)\|_{L^p(\sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \lesssim [\mu, \sigma]_{p, \varepsilon}^{\frac{1}{p}}$$

where

$$[\mu, \sigma]_{p, \varepsilon} = \sup_Q \rho_{\sigma, \varepsilon}(Q) \left(\frac{\sigma(Q)}{|Q|} \right)^{p-1} \frac{\mu(\tilde{Q})}{|Q|}$$

with ε a monotonic increasing function such that $\int_{\frac{1}{2}}^{\infty} \frac{1}{\varepsilon(t)} \frac{dt}{t} = 1$ and $\rho_{\sigma, \varepsilon}(Q) = \frac{\int_Q M(\sigma \chi_Q) dx}{\sigma(Q)} \varepsilon\left(\frac{\int_Q M(\sigma \chi_Q) dx}{\sigma(Q)}\right)$.

It is clear that if we replace \mathcal{M} by the standard Hardy-Littlewood maximal operator and if we take $\mu(x, t) = w(x)\delta_0(t)$, where δ_0 denotes Dirac's delta, we recover all the results in their classical setting.

As a consequence of the preceding results and (1.1), we obtain also analogue estimates for the Poisson integral. We compile all those estimates in the following

Corollary 1. *Let σ be a weight in \mathbb{R}^n and let μ be a Borel measure on \mathbb{R}_+^{n+1} and $1 < p < \infty$. Then the Poisson integral satisfies the following estimates*

- $\|P(\cdot\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^{p, \infty}(\mathbb{R}_+^{n+1}, \mu)} \lesssim [\mu, \sigma]_{A'_p}^{\frac{1}{p}}$
- $\|P(\cdot\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \lesssim [\mu, \sigma]_{S'_p}$
- $\|P(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \lesssim ([\mu, \sigma, \Phi]_{A_p} [\sigma, \bar{\Phi}]_{W^p})^{\frac{1}{p}}$
- $\|P(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \lesssim [\sigma, \mu]_{p, \varepsilon}^{\frac{1}{p}}$.

3 Preliminaries and definitions

3.1 Basic definitions

In this section we recall some basic facts that play a main role in this paper. We also give the precise definitions of the operators and the constants used in our results.

Definition 1. Let f a locally integrable function in \mathbb{R}^n , we define the function $\mathcal{M}f$ on $\mathbb{R}_+^{n+1} = \{(x, t) : x \in \mathbb{R}^n, t \geq 0\}$ as

$$\mathcal{M}f(x, t) = \sup_{x \in Q, l(Q) \geq t} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

We also consider dyadic versions of this operator. First we recall the definition of dyadic grid (c.f. [18]).

Definition 2. We say that a family of cubes \mathcal{D} is a dyadic grid if it satisfies the following properties

1. For any $Q \in \mathcal{D}$ its sidelength l_Q is of the form 2^k , $k \in \mathbb{Z}$
2. If $Q_1, Q_2 \in \mathcal{D}$ then $Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\}$
3. The cubes of a fixed sidelength 2^k form a partition of \mathbb{R}^n

We will also use following

Definition 3. Let \mathcal{D} a dyadic grid. Given a cube $Q \in \mathcal{D}$ we call $\mathcal{D}(Q)$ the family of all the cubes of \mathcal{D} that are contained in Q .

Definition 4. Let f a locally integrable function in \mathbb{R}^n and \mathcal{D} a dyadic grid. We define the function $\mathcal{N}^{\mathcal{D}}f$ on $\mathbb{R}_+^{n+1} = \{(x, t) : x \in \mathbb{R}^n, t \geq 0\}$ as

$$\mathcal{N}^{\mathcal{D}}f(x, t) = \sup_{\substack{x \in Q \in \mathcal{D} \\ l(Q) \geq t}} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

We shall drop the superscript \mathcal{D} when we work with just one dyadic grid.

We denote by \tilde{Q} the cube built from a cube Q as follows

$$\tilde{Q} = \{(x, t) \in \mathbb{R}_+^{n+1} : x \in Q, \text{ and } 0 \leq t < l(Q)\},$$

in other words, \tilde{Q} is the cube in \mathbb{R}_+^{n+1} having Q as a face.

Using the argument given in [6, Lemma 5.38 pg. 111], we can obtain the following Lemma that we will use in the sequel.

Lemma 1. *For every Q and every $(x, t) \in \tilde{Q}$ we have that*

$$\mathcal{M}(f\chi_Q)(x, t) = \sup_{\substack{x \in P \subseteq Q \\ l(P) \geq t}} \frac{1}{|P|} \int_P |f(y)| dy.$$

3.2 Orlicz averages

We recall that Φ is a Young function if it is a continuous, nonnegative, strictly increasing and convex function defined on $[0, \infty)$ such that $\Phi(0) = 0$ and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$. The localized Luxembourg norm of a function f with respect to a Young function Φ can be defined as follows

$$\|f\|_{\Phi, Q} = \|f\|_{\Phi(L), Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}$$

We note that the case $\Phi(t) = t$ corresponds to the usual average. We can see these localized norms as a “different” way of taking averages. We can also define the maximal function associated to Φ as

$$M_\Phi f(x) = \sup_{x \in Q} \|f\|_{\Phi, Q}.$$

For each Young function there exists an associated complementary Young function $\bar{\Phi}$ that satisfies the following inequalities

$$t \leq \Phi^{-1}(t)\bar{\Phi}^{-1}(t) \leq 2t.$$

A basic example of this is $\Phi(t) = t^p$. For that Young function $\bar{\Phi}(t) = t^{p'}$. Some basic facts are that

$$\frac{1}{|Q|} \int_Q |fg| \leq 2\|f\|_{\Phi, Q} \|g\|_{\bar{\Phi}, Q}$$

and that

$$\|M_\Phi\|_{L^p(\mathbb{R}^n)} \lesssim \alpha_p(\Phi)$$

where $\alpha_p(\Phi) = \left(\int_1^\infty \frac{\Phi(t)}{t^p} \frac{dt}{t} \right)^{\frac{1}{p}}$ (cf. [24]).

3.3 A_p and bump type conditions

In this section we give precise definitions of all the “ A_p type” conditions that appear in this paper. All of them resemble in some way their classical counterparts as A_p or bump type conditions. We begin with the constant involved in the weak type (p, p) inequality.

Definition 5. Let $1 < p < \infty$. Given a weight σ and a Borel measure μ on \mathbb{R}_+^{n+1} we define

$$[\mu, \sigma]_{A'_p} := \sup_Q \frac{\mu(\tilde{Q})}{|Q|} \left(\frac{1}{|Q|} \int_Q \sigma(x) dx \right)^{p-1} < \infty$$

We define now the constants involved in the characterization of the L^p boundedness.

Definition 6. Let $1 < p < \infty$. Given a weight $\sigma \geq 0$ and a Borel measure μ on \mathbb{R}^{n+1} we define

$$[\mu, \sigma]_{S'_p} = \sup_Q \left(\frac{\int_{\tilde{Q}} \mathcal{M}(\sigma \chi_Q)(x, t)^p d\mu(x, t)}{\int_Q \sigma(x) dx} \right)^{\frac{1}{p}}$$

where the supremum is taken over all cubes of \mathbb{R}^n .

The definition of the dyadic variant of this constant is almost the same

Definition 7. Let \mathcal{D} a dyadic grid. Let $1 < p < \infty$. Given a weight $\sigma \geq 0$ and a Borel measure μ on \mathbb{R}^{n+1} we define

$$[\mu, \sigma]_{S'_{p, \mathcal{D}}} = \sup_{Q \in \mathcal{D}} \left(\frac{\int_{\tilde{Q}} \mathcal{M}(\sigma \chi_Q)(x, t)^p d\mu(x, t)}{\int_Q \sigma(x) dx} \right)^{\frac{1}{p}}$$

We observe that the constants we have just defined are quite natural. Indeed, if we consider $\mu(x, t) = w(x)\delta(t)$ where w is a weight and δ is Dirac's delta we recover the quantitative result obtained in terms of the S_p constant by K. Moen in [23].

To end this section we give precise definitions of the constants involved in the quantitative sufficient conditions provided in our main results. First we focus on the constants involved in Theorem 3.

Definition 8. Given a weight σ , a Borel measure μ on \mathbb{R}_+^{n+1} and a Young function Φ we define the quantity

$$[\mu, \sigma, \Phi]_{A'_p} = \sup_Q \frac{\mu(\tilde{Q})}{|Q|} \left\| \sigma^{\frac{1}{p}} \right\|_{\Phi, Q}^p.$$

We say that μ, σ belong to the $A'_{p, \Phi}$ class if $[\mu, \sigma, \Phi]_{A'_p} < \infty$.

We observe that if we choose $\Phi(t) = t^{p'}$ we recover the $[\mu, \sigma]_{A'_p}$ constant.

Definition 9. Given a weight σ , a Borel measure μ on \mathbb{R}_+^{n+1} and a Young function Φ we define the quantity

$$[\sigma, \Phi]_{W_p} = \sup_Q \frac{1}{\sigma(Q)} \int_Q M_{\Phi} \left(\sigma^{\frac{1}{p}} \chi_Q \right)^p dx$$

If we choose $\Phi_p(t) = t^p$ we obtain

$$[\sigma, \Phi]_{W_p} = \sup_Q \frac{1}{\sigma(Q)} \int_Q M(\sigma \chi_Q) dx$$

which is the well known A_{∞} constant that was discovered by N. Fujii in [9], rediscovered by M. Wilson in [35] and shown to be the most suitable one in [13] (see also [14]).

Let us now turn our attention to the constant involved in Theorem 4.

Definition 10. Let ε be a monotonic increasing function that satisfies

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{\varepsilon(t)} \frac{dt}{t} = 1.$$

We define the quantity

$$[\mu, \sigma]_{p, \varepsilon} = \sup_Q \rho_{\sigma, \varepsilon}(Q) \left(\frac{\sigma(Q)}{|Q|} \right)^{p-1} \frac{\mu(\tilde{Q})}{|Q|}$$

where $\rho_{\sigma, \varepsilon}(Q) = \rho(Q)\varepsilon(\rho(Q))$ and $\rho(Q) = \frac{\int_Q M(\sigma \chi_Q) dx}{\sigma(Q)}$,

It is worth comparing the constants involved in Theorems 4 and 3 when we choose $\Phi(t) = t^{p'}$. In that case the constant in Theorem 3 is

$$\left([\mu, \sigma]_{A'_p} [\sigma]_{A_\infty}\right)^{\frac{1}{p}}.$$

In contrast with that couple of supremums, the definition of $[\mu, \sigma]_{p, \varepsilon}^{\frac{1}{p}}$ “includes in some way” the A'_p constant and a “bumped” A_∞ constant. It is not clear which of those quantities is larger.

Observe that since $\rho(Q) \geq 1$

$$[\mu, \sigma]_{p, \varepsilon} \geq \varepsilon(1) \sup_Q \left(\frac{\sigma(Q)}{|Q|}\right)^{p-1} \frac{\mu(\tilde{Q})}{|Q|} = \varepsilon(1) [\mu, \sigma]_{A'_p}$$

which is a not unexpected fact since the strong-type (p, p) implies the weak-type (p, p) . Trivially, we have also that

$$[\mu, \sigma]_{A'_p} [\sigma]_{A_\infty} \geq [\mu, \sigma]_{A'_p}.$$

4 Proofs of the main results

4.1 Proof of Theorem 1

We first prove that

$$[\mu, \sigma]_{A'_p} \lesssim \|\mathcal{M}(\cdot\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^{p, \infty}(\mathbb{R}_+^{n+1}, \mu)}$$

holds for any $1 < p < \infty$. Let us assume that $\|\mathcal{M}(\cdot\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^{p, \infty}(\mathbb{R}_+^{n+1}, \mu)} < \infty$ since otherwise there's nothing to prove. We fix a cube Q in \mathbb{R}^n . For all $(x, t) \in \tilde{Q}$ by the definition of \mathcal{M} it is clear that

$$\frac{1}{|Q|} \int_Q |f(x)|\sigma(x)dx \leq \mathcal{M}(f\sigma)(x, t).$$

This yields

$$\tilde{Q} \subseteq \left\{ (x, t) \in \mathbb{R}_+^{n+1} : \mathcal{M}(f\sigma)(x, t) > \frac{1}{|Q|} \int_Q |f(x)|\sigma(x)dx \right\}$$

Then

$$\mu(\tilde{Q}) \leq \mu \left\{ (x, t) \in \mathbb{R}_+^{n+1} : \mathcal{M}(f\sigma)(x, t) > \frac{1}{|Q|} \int_Q |f(x)|\sigma(x)dx \right\}$$

and recalling that $\|\mathcal{M}(\cdot\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^{p, \infty}(\mathbb{R}_+^{n+1}, \mu)} < \infty$ we have that

$$\mu(\tilde{Q}) \leq \frac{\|\mathcal{M}(\cdot\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^{p, \infty}(\mathbb{R}_+^{n+1}, \mu)}^p}{\left(\frac{1}{|Q|} \int_Q |f(x)|\sigma(x)dx\right)^p} \int_{\mathbb{R}^n} |f|^p \sigma.$$

If we choose $f = \chi_Q$ we obtain the desired conclusion.

Now we want to prove that

$$\|\mathcal{M}(\cdot\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^{p, \infty}(\mathbb{R}_+^{n+1}, \mu)} \lesssim [\mu, \sigma]_{A'_p}^{\frac{1}{p}}.$$

We assume that $[\mu, \sigma]_{A'_p} < \infty$ since otherwise there's nothing to prove.

Let us call

$$E_\alpha = \left\{ (x, t) \in \mathbb{R}_+^{n+1} : \mathcal{M}(f\sigma)(x, t) > \alpha \right\}$$

We may assume that f is bounded with compact support. Fix $\alpha > 0$. If $(x, t) \in E_\alpha$ then we can find a cube R containing x with $l(R) \geq t$ and such that

$$\frac{1}{|R|} \int_R |f(y)\sigma(y)|dy > \alpha.$$

Let k be the only integer such that

$$\frac{1}{2^{n(k+1)}} < |R| \leq \frac{1}{2^{kn}}.$$

Then there exists some dyadic cube Q with sidelength 2^k such that $\overset{\circ}{R} \cap Q \neq \emptyset$ and

$$\int_{R \cap Q} |\sigma(y)f(y)| dy > \frac{\alpha|R|}{2^n} > \frac{\alpha|Q|}{4^n}.$$

From this it readily follows that

$$\frac{1}{|Q|} \int_Q |\sigma(y)f(y)| > \frac{\alpha}{4^n}.$$

Taking into account that f is integrable we can consider $C_{\frac{\alpha}{4^n}}$ the family of maximal dyadic cubes P such that

$$\frac{1}{|P|} \int_P |\sigma(y)f(y)| > \frac{\alpha}{4^n}.$$

Then we have that there exists some $Q_j \in C_{\frac{\alpha}{4^n}}$ such that $Q \subseteq Q_j$ and also $x \in R \subseteq 3Q \subseteq 3Q_j$. This yields $t \leq l(R) \leq l(3Q_j)$ and then $(x, t) \in 3\tilde{Q}_j$. Thus we have that

$$E_\alpha \subseteq \bigcup_j 3\tilde{Q}_j.$$

Then

$$\begin{aligned} \mu(E_\alpha) &\leq \sum_j \mu(3\tilde{Q}_j) = 3^{np} \sum_j \frac{\mu(3\tilde{Q}_j)}{|3Q_j|^p} |Q_j|^p \\ &\leq \frac{3^{np}}{\alpha^p} \sum_j \frac{\mu(3\tilde{Q}_j)}{|3Q_j|^p} \left(\int_{Q_j} |\sigma(x)f(x)| dx \right)^p \\ &\leq \frac{3^{np}}{\alpha^p} \sum_j \frac{\mu(3\tilde{Q}_j)}{|3Q_j|^p} \left(\int_{Q_j} \sigma(x) dx \right)^{p-1} \int_{Q_j} |f(x)|^p \sigma(x) dx \\ &\leq \frac{3^{np}}{\alpha^p} \sum_j \frac{\mu(3\tilde{Q}_j)}{|3Q_j|^p} \left(\int_{Q_j} \sigma(x) dx \right)^{p-1} \int_{Q_j} |f(x)|^p \sigma(x) dx \\ &\leq \frac{3^{np}}{\alpha^p} [\mu, \sigma]_{A'_p} \sum_j \int_{Q_j} |f(x)|^p \sigma(x) dx \\ &\leq \frac{3^{np}}{\alpha^p} [\mu, \sigma]_{A'_p} \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) dx. \end{aligned}$$

4.2 Proof of Theorem 2

The proof of Theorem 2 is based on the corresponding dyadic version, namely,

Theorem 5. *Let μ be a positive Borel measure on \mathbb{R}_+^{n+1} and let σ be a weight in \mathbb{R}^n . Let \mathcal{D} a dyadic grid and let $1 < p < \infty$. Then*

$$\|\mathcal{N}^{\mathcal{D}}(\sigma \cdot)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \simeq [\mu, \sigma]_{S'_p}^{\mathcal{D}}$$

The proof of Theorem 5 consists essentially in tracking the constants in the proof of this result in [31].

Proof of Theorem 5. The proof is the same for every choice of dyadic grid \mathcal{D} so for simplicity we denote $\mathcal{N}^{\mathcal{D}}$ by \mathcal{N} and $[\mu, \sigma]_{S'_p, \mathcal{D}}$ by $[\mu, \sigma]_{S'_p}$.

The fact that $[\mu, \sigma]_{S'_p} \lesssim \|\mathcal{N}(\sigma \cdot)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)}$ is straightforward. It suffices to choose $f = \chi_Q$. We focus now on the converse. As always we may assume that f is bounded with compact support. We need to introduce the following truncations of the dyadic maximal operator:

$$\mathcal{N}^R(\sigma f)(x, t) = \sup_{x \in Q, t \leq l(Q) \leq R} \frac{1}{|Q|} \int_Q |f(x)| dx$$

We observe that $\mathcal{N}^R f(x, t) = 0$ for $t > R$ and that

$$\lim_{R \rightarrow \infty} \mathcal{N}^R(\sigma f)(x, t) \uparrow \mathcal{N}(\sigma f)(x, t).$$

Let

$$\Omega_k = \{(x, t) \in \mathbb{R}_+^{n+1} : \mathcal{N}^R(\sigma f)(x, t) > 2^k\} \quad k \in \mathbb{Z}.$$

We will also need a Calderón-Zygmund decomposition suited to that truncations. The following Lemma contains that decomposition.

Lemma 2 ([31]). *Let $g \in L^1$. For each $k \in \mathbb{Z}$ we can choose a family $\{Q_j^k\}_{j \in J_k} \subseteq \mathcal{D}$ such that*

1. $\frac{1}{|Q_j^k|} \int_{Q_j^k} |g(x)| dx > 2^k$.
2. *The interiors of \tilde{Q}_j^k are disjoint.*
3. $\Omega_k = \{(x, t) \in \mathbb{R}_+^{n+1} : \mathcal{N}^R(g)(x, t) > 2^k\} = \bigcup_{j \in J_k} \tilde{Q}_j^k$

The proof of that lemma is the same as the one contained [31] with minor modifications.

Applying Lemma with $g = \sigma f$, we can consider the sets

$$E_j^k = \tilde{Q}_j^k \setminus \{(x, t) : \mathcal{N}^R(\sigma f)(x, t) \geq 2^{k+1}\}.$$

The sets E_j^k have disjoint interiors and

$$\begin{aligned} \int_{\mathbb{R}_+^{n+1}} \mathcal{N}^R(\sigma f)(x, t)^p d\mu(x, t) &\leq \sum_{j, k} \int_{E_j^k} \mathcal{N}^R(\sigma f)(x, t)^p d\mu(x, t) \\ &\leq \sum_{j, k} 2^{(k+1)p} \mu(E_j^k) \leq 2^p \sum_{j, k} \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |\sigma(x)f(x)| dx \right)^p \mu(E_j^k) \end{aligned}$$

Following ideas from E. Sawyer and B. Jawerth we introduce the following notation

$$\gamma_{jk} = \mu(E_j^k) \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^p \quad g_{jk} = \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |\sigma(x)f(x)| dx \right)^p$$

$X = \{(j, k) : k \in \mathbb{Z}, j \in J_k\}$ with atomic measure γ_{jk}

$$\Gamma(\lambda) = \{(j, k) \in X : g_{jk} > \lambda\}$$

Using this notation we have that

$$\begin{aligned} \int_{\mathbb{R}_+^{n+1}} \mathcal{N}^R(\sigma f)(x, t)^p d\mu(x, t) &\leq \sum_{j, k} \gamma_{jk} g_{jk} = \\ &= 2^p \int_0^\infty \gamma(\{(j, k) : g_{jk} > \lambda\}) d\lambda = 2^p \int_0^\infty \left(\sum_{(j, k) \in \Gamma(\lambda)} \gamma_{jk} \right) d\lambda \\ &= 2^p \int_0^\infty \left(\sum_{(j, k) \in \Gamma(\lambda)} \mu(E_j^k) \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^p \right) d\lambda \\ &= 2^p \int_0^\infty \left(\sum_{(j, k) \in \Gamma(\lambda)} \int_{E_j^k} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^p d\mu(x, t) \right) d\lambda \end{aligned}$$

If we call Q_i the maximal cubes of the family $\{Q_j^k : (j, k) \in \Gamma(\lambda)\}$ we can rewrite the preceding sum as follows

$$\begin{aligned} & 2^p \int_0^\infty \left(\sum_i \sum_{\substack{(j,k) \in \Gamma(\lambda) \\ Q_j^k \subset Q_i}} \int_{E_j^k} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^p d\mu(x, t) \right) d\lambda \\ & \leq 2^p \int_0^\infty \left(\sum_i \sum_{\substack{(j,k) \in \Gamma(\lambda) \\ Q_j^k \subset Q_i}} \int_{E_j^k} \mathcal{N}^R(\sigma \chi_{Q_i})(x, t)^p d\mu(x, t) \right) d\lambda \\ & \leq 2^p \int_0^\infty \left(\sum_i \int_{\tilde{Q}_i} \mathcal{N}^R(\sigma \chi_{Q_i})(x, t)^p d\mu(x, t) \right) d\lambda \end{aligned}$$

Now we use 2 and we obtain

$$\begin{aligned} & \leq [\mu, v]_{S'_p}^p 2^p \int_0^\infty \left(\sum_i \int_{Q_i} \sigma(x) dx \right) d\lambda = 2^p \int_0^\infty \sigma \left(\bigcup_i Q_i \right) d\lambda \\ & = [\mu, v]_{S'_p}^p 2^p \int_0^\infty \sigma \left(\bigcup_{(j,k) \in \Gamma(\lambda)} Q_j^k \right) d\lambda \end{aligned}$$

Now from the definition of $\Gamma(\lambda)$ and the g'_{jk} s it is clear that

$$\bigcup_{(j,k) \in \Gamma(\lambda)} Q_j^k \subseteq \{x \in \mathbb{R}^n : N_\sigma(f)(x)^p > \lambda\}$$

Then

$$\begin{aligned} & \leq [\mu, v]_{S'_p}^p 2^p \int_0^\infty \sigma \left(\bigcup_{(j,k) \in \Gamma(\lambda)} Q_j^k \right) d\lambda \\ & \leq [\mu, v]_{S'_p}^p 2^p \int_0^\infty \sigma(\{x \in \mathbb{R}^n : N_\sigma(f)(x)^p > \lambda\}) d\lambda \\ & = [\mu, v]_{S'_p}^p 2^p \int_{\mathbb{R}^n} N_\sigma(f)(x)^p \sigma(x) dx \\ & \lesssim_{n,p} [\mu, v]_{S'_p}^p \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) dx \end{aligned}$$

We have obtained that for every $R > 0$

$$\int_{\mathbb{R}^{n+1}} \mathcal{N}^R(\sigma f)(x, t)^p d\mu(x, t) \leq c_{n,p} [\mu, v]_{S'_p}^p \int_{\mathbb{R}^n} |f(x)|^p \sigma(x) dx.$$

Since $\mathcal{N}^R f(x, t) \uparrow \mathcal{N} f(x, t)$ by monotone convergence theorem we're done. \square

Theorem 2 will be proved reducing it to the dyadic case. To do that we will show next that \mathcal{M} is controlled pointwise by a sum of $2^n \mathcal{N}^{\mathcal{D}_j}$ operators.

Lemma 3. *We can build 2^n dyadic grids such that*

$$\mathcal{M}f(x, t) \leq 6^n \sum_{j=1}^{2^n} \mathcal{N}^{\mathcal{D}_j} f(x, t).$$

Proof. To prove this lemma we need to use the fact that we can find a finite family of dyadic grids such that every cube is “well controlled” by a cube in one these families

Lemma ([18, Prop. 5.1]). *There are 2^n dyadic grids \mathcal{D}_j such that for any cube Q , there exists a cube $Q_j \in \mathcal{D}_j$ such that $Q \subseteq Q_j$ and $l(Q_j) \leq 6l(Q)$.*

We note that this result appears implicitly in [10, p. 136-137].

Armed with that lemma we can establish our result. Let us take $(x, t) \in \mathbb{R}_+^{n+1}$ and consider

$$\mathcal{M}f(x, t) = \sup_{x \in Q, l(Q) \geq t} \frac{1}{|Q|} \int_Q |f(x)| dx.$$

If we choose any of the averages

$$\frac{1}{|Q|} \int_Q |f(x)| dx$$

involved in that supremum we can take $Q_j \in \mathcal{D}_j$ such that $Q \subseteq Q_j$ and $l(Q) \leq l(Q_j) \leq 6l(Q)$. This yields

$$\frac{1}{|Q|} \int_Q |f(x)| dx \leq 6^n \frac{1}{|Q_j|} \int_{Q_j} |f(x)| dx \leq 6^n \mathcal{N}^{\mathcal{D}_j} f(x, t)$$

since $l(Q_j) \geq l(Q) \geq t$. Consequently

$$\mathcal{M}f(x, t) \leq 6^n \sum_{j=1}^{2^n} \mathcal{N}^{\mathcal{D}_j} f(x, t).$$

□

To end this section we give the proof of Theorem 2.

Proof of Theorem 2. The proof of $[\mu, \sigma]_{S'_p} \lesssim \|\mathcal{M}(\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)}$ is exactly the same of the dyadic version Theorem 5. It is simply testing with cubes.

Now suppose that $[\mu, \sigma]_{S'_p, \mathcal{D}_j} < \infty$, since otherwise there’s nothing to prove. It is clear that for every dyadic grid \mathcal{D}_j

$$[\mu, \sigma]_{S'_p, \mathcal{D}_j} \leq [\mu, \sigma]_{S'_p}. \quad (4.1)$$

Consequently Theorem 5 applies for every dyadic grid, that is,

$$\left(\int_{\mathbb{R}_+^{n+1}} \mathcal{N}^{\mathcal{D}_j} f(x, t)^p d\mu(x, t) \right)^{\frac{1}{p}} \leq c_{n,p} [\mu, \sigma]_{S'_p, \mathcal{D}_j} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{\frac{1}{p}}.$$

Now we apply Lemma 3 and use (4.1) to obtain

$$\begin{aligned} \left(\int_{\mathbb{R}_+^{n+1}} \mathcal{M}f\sigma(x, t)^p d\mu(x, t) \right)^{\frac{1}{p}} &\leq \left(\int_{\mathbb{R}_+^{n+1}} \left[\sum_{j=1}^{2^n} \mathcal{N}^{\mathcal{D}_j} (f\sigma)(x, t) \right]^p d\mu(x, t) \right)^{\frac{1}{p}} \\ &\leq \sum_{j=1}^{2^n} \left(\int_{\mathbb{R}_+^{n+1}} \mathcal{N}^{\mathcal{D}_j} (f\sigma)(x, t)^p d\mu(x, t) \right)^{\frac{1}{p}} \\ &\leq \sum_{j=1}^{2^n} c_{n,p} [\mu, \sigma]_{S_p, \mathcal{D}_j} \left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x) dx \right)^{\frac{1}{p}} \\ &\leq c_{n,p} [\mu, \sigma]_{S_p} \left(\int_{\mathbb{R}^n} |f(x)|^p \sigma(x) dx \right)^{\frac{1}{p}} \end{aligned}$$

as we wanted to prove.

□

4.3 Proof of Theorems 3 and 4

To proof both theorems we need a Calderón-Zygmund type decomposition suited to our purposes. We obtain that decomposition in the following

Lemma 4. *Let \mathcal{D} be a dyadic grid. Let f be a locally integrable function such that $\text{supp } f \subseteq Q_0 \in \mathcal{D}$. Let $k_0 \in \mathbb{Z}$ be the smallest integer such that*

$$\frac{1}{|Q_0|} \int_{Q_0} |f| \leq 2^{k_0}$$

For each $k \in \mathbb{Z}$ such that $k \geq 0$ we can choose a family $\{Q_j^k\}_{j \in J_k} \subseteq \mathcal{D}(Q_0)$ such that

1. $2^{k(n+1)+k_0} < \frac{1}{|Q_j^k|} \int_{Q_j^k} |f(x)| dx \leq 2^{k(n+1)+k_0+n}$.
2. The interiors of \tilde{Q}_j^k with $j \in J_k$ are pairwise disjoint.
3. $\Omega_k = \{(x, t) \in \tilde{Q}_0 : \mathcal{N}^{\mathcal{D}} f(x, t) > 2^{k(n+1)+k_0}\} = \bigcup_{j \in J_k} \tilde{Q}_j^k$

Furthermore, the family of cubes $\{Q_j^k\}_{j \in \cup J_k}$ is sparse, that is, for each Q_j^k we can take a measurable subset E_j^k such that

$$\frac{1}{2} |Q_j^k| \leq |E_j^k|$$

and the sets E_j^k are pairwise disjoint.

Proof. Let us fix $k \geq 0$. First of all, a dyadic version of Lemma 1 allows us to say that for each $(x, t) \in \tilde{Q}_0$ we have that

$$\mathcal{N}^{\mathcal{D}} f(x, t) = \sup_{x \in Q \in \mathcal{D}, Q \subseteq Q_0, l(Q) \geq t} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Then we observe that for each $(x, t) \in \Omega_k$ we can find a cube $Q_{(x,t)} \in \mathcal{D}$ such that $x \in Q_{(x,t)}$, $Q_{(x,t)} \subseteq Q_0$, $l(Q_{(x,t)}) \geq t$ and $2^{k(n+1)+k_0} < \frac{1}{|Q_{(x,t)}|} \int_{Q_{(x,t)}} |f(x)| dx$. Since all those cubes are contained in Q_0 we can choose among them the maximal ones and call them $\{Q_j^k\}_{j \in J_k}$. It is clear that this collection of cubes satisfies conditions (1), (2) and (3). To end the proof we have to prove that the family $\{Q_j^k\}_{\cup_{k \geq k_0} J_k}$ is sparse.

Let us denote for each $k \geq k_0$

$$H_k = \left\{ x \in Q_0 : \frac{1}{|P|} \int_P |f(y)| dy > 2^{k(n+1)+k_0}, \text{ for some } P \in \mathcal{D}, P \subseteq Q \right\}$$

We observe that

$$|Q_j^k \cap H_{k+1}| = \sum_m |Q_j^k \cap Q_m^{k+1}|$$

Now since $Q_j^k, Q_m^{k+1} \in \mathcal{D}$, we have that $Q_j^k \cap Q_m^{k+1} \neq \emptyset$ implies that either $Q_j^k \subseteq Q_m^{k+1}$ or $Q_m^{k+1} \subseteq Q_j^k$. Now we observe that from the definition of H_k it follows that $Q_m^{k+1} \subseteq H_k$. Consequently $Q_m^{k+1} \subseteq Q_j^k$ by maximality. Taking that into account and using (1) and (2)

$$\begin{aligned} |Q_j^k \cap H_{k+1}| &= \sum_m |Q_j^k \cap Q_m^{k+1}| = \sum_{Q_m^{k+1} \subseteq Q_j^k} |Q_m^{k+1}| \leq \sum_{Q_m^{k+1} \subseteq Q_j^k} \frac{1}{2^{(k+1)(n+1)+k_0}} \int_{Q_m^{k+1}} |f(y)| dy \\ &\leq \frac{1}{2^{(k+1)(n+1)+k_0}} \int_{Q_j^k} |f(y)| dy \leq \frac{2^{k(n+1)+k_0+n}}{2^{(k+1)(n+1)+k_0}} |Q_j^k| = \frac{1}{2} |Q_j^k| \end{aligned}$$

Consequently we have that

$$|Q_j^k| = |Q_j^k \setminus H_{k+1}| + |Q_j^k \cap H_{k+1}| \leq |Q_j^k \setminus H_{k+1}| + \frac{1}{2} |Q_j^k|$$

and we obtain the desired conclusion taking $E_j^k = Q_j^k \setminus H_{k+1}$. \square

To give the proof of Theorem 3 we will adapt the argument in Pérez-Rela [26].

Proof of Theorem 3. As in the proof of Theorem 2 it is enough to consider $\mathcal{N}^{\mathcal{D}_j}$,

$$\|\mathcal{N}^{\mathcal{D}_j}(\sigma)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \simeq [\mu, \sigma]_{S'_p, \mathcal{D}_j} \quad j = 1, \dots, 2^n$$

where

$$[\mu, \sigma]_{S'_p, \mathcal{D}_j} = \sup_{Q \in \mathcal{D}_j} \left(\frac{\int_{\tilde{Q}} \mathcal{N}^{\mathcal{D}_j}(\sigma \chi_Q)(x, t)^p d\mu(x, t)}{\int_Q \sigma(x) dx} \right)^{\frac{1}{p}}.$$

Hence, it is enough to prove that

$$[\mu, \sigma]_{S'_p, \mathcal{D}_j} \lesssim \left([\mu, \sigma, \Phi]_{A'_p} [\sigma, \bar{\Phi}]_{W^p} \right)^{\frac{1}{p}}.$$

Let us fix a cube Q . Consider $k_0 \in \mathbb{Z}$ such that

$$2^{k_0-1} < \frac{\sigma(Q)}{|Q|} \leq 2^{k_0}.$$

Now we call

$$A = \left\{ (x, t) \in \tilde{Q}; \mathcal{M}(\sigma \chi_Q)(x, t) \leq 2 \frac{\sigma(Q)}{|Q|} \right\}.$$

If $(x, t) \in \tilde{Q} \setminus A$ then

$$\mathcal{M}(\sigma \chi_Q)(x, t) > 2 \frac{\sigma(Q)}{|Q|} > 2^{k_0} \geq \frac{\sigma(Q)}{|Q|}.$$

We call

$$\Omega_k = \left\{ (x, t) \in \tilde{Q}; \mathcal{M}(\sigma \chi_Q)(x, t) > 2^{k(n+1)+k_0} \right\}.$$

Now we see that

$$\begin{aligned} \int_{\tilde{Q}} \mathcal{M}(\sigma \chi_Q)(x, t)^p d\mu(x, t) &= \int_A \mathcal{M}(\sigma \chi_Q)(x, t)^p d\mu(x, t) + \int_{\tilde{Q} \setminus A} \mathcal{M}(\sigma \chi_Q)(x, t)^p d\mu(x, t) \\ &\leq 2^p \left(\frac{\sigma(Q)}{|Q|} \right)^p \mu(\tilde{Q}) + \sum_{k \geq 0} \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}(\sigma \chi_Q)(x, t)^p d\mu(x, t) \\ &= I + II \end{aligned}$$

We first obtain an estimate for I . Using the generalized Hölder inequality

$$\begin{aligned} I &= 2^p \left(\frac{1}{|Q|} \int_Q \sigma^{\frac{1}{p}}(x) \sigma^{\frac{1}{p'}}(x) dx \right)^p \mu(\tilde{Q}) \leq 2^p 2^p \left\| \sigma^{\frac{1}{p}} \right\|_{\bar{\Phi}, Q}^p \left\| \sigma^{\frac{1}{p'}} \right\|_{\Phi, Q}^p \mu(\tilde{Q}) \\ &= 2^p 2^p \left\| \sigma^{\frac{1}{p}} \right\|_{\bar{\Phi}, Q}^p \left\| \sigma^{\frac{1}{p'}} \right\|_{\Phi, Q}^p \frac{\mu(\tilde{Q})}{|Q|} \leq 2^p 2^p [\mu, \sigma, \Phi]_{A_p} \left\| \sigma^{\frac{1}{p}} \right\|_{\bar{\Phi}, Q}^p |Q| \\ &\leq 2^p 2^p [\mu, \sigma, \Phi]_{A_p} \int_Q M_{\bar{\Phi}} \left(\sigma^{\frac{1}{p}} \chi_Q \right)^p(x) dx \end{aligned}$$

We now focus on II

$$\begin{aligned} II &\leq \sum_{k \geq 0} 2^{((k+1)(n+1)+k_0)p} \mu(\Omega_k) \\ &= 2^{(n+1)p} \sum_{k \geq 0} 2^{(k(n+1)+k_0)p} \mu(\Omega_k) \end{aligned}$$

Using now Lemma 4 we have that

$$\begin{aligned} II &\leq 2^{(n+1)p} \sum_{k \geq 0} 2^{(k(n+1)+k_0)p} \mu(\Omega_k) \\ &\leq 2^{(n+1)p} \sum_{k, i} \left(\frac{1}{|Q_i^k|} \int_{Q_i^k} \sigma(x) \chi_Q(x) dx \right)^p \mu(\tilde{Q}_i^k) \\ &\leq 2^{(n+1)p} \sum_{k, i} \left(\frac{1}{|Q_i^k|} \int_{Q_i^k} \sigma^{\frac{1}{p}}(x) \sigma^{\frac{1}{p'}}(x) \chi_Q(x) dx \right)^p \frac{\mu(\tilde{Q}_i^k)}{|Q_i^k|} |Q_i^k| \end{aligned}$$

Now using generalized Hölder inequality

$$\begin{aligned}
& 2^{(n+1)p} \sum_{k,i} \left(\frac{1}{|Q_i^k|} \int_{Q_i^k} \sigma^{\frac{1}{p}}(x) \sigma^{\frac{1}{p'}}(x) \chi_Q(x) dx \right)^p \frac{\mu(\widetilde{Q}_i^k)}{|Q_i^k|} |Q_i^k| \\
& \leq 2^{(n+2)p} \sum_{k,i} \left\| \sigma^{\frac{1}{p}} \chi_Q \right\|_{\bar{\Phi}, Q_i^k}^p \left\| \sigma^{\frac{1}{p'}} \right\|_{\Phi, Q_i^k}^p \frac{\mu(\widetilde{Q}_i^k)}{|Q_i^k|} |Q_i^k| \\
& \leq 2^{(n+2)p} [\mu, \sigma, \Phi]_{A'_p} \sum_{k,i} \left\| \sigma^{\frac{1}{p}} \chi_Q \right\|_{\bar{\Phi}, Q_i^k}^p |Q_i^k|
\end{aligned}$$

Since Lemma 4 allows us choose $E_i^k \subseteq Q_i^k$ pairwise disjoint and such that $|Q_i^k| \leq 2|E_i^k|$, we have that

$$\begin{aligned}
& 2^{(n+2)p} [\mu, \sigma, \Phi]_{A'_p} \sum_{k,i} \left\| \sigma^{\frac{1}{p}} \chi_Q \right\|_{\bar{\Phi}, Q_i^k}^p |Q_i^k| \\
& \leq 2^{(n+2)p+1} [\mu, \sigma, \Phi]_{A_p} \sum_{k,i} \left\| \sigma^{\frac{1}{p}} \chi_Q \right\|_{\bar{\Phi}, Q_i^k}^p |E_i^k| \\
& \leq 2^{(n+2)p+1} [\mu, \sigma, \Phi]_{A_p} \sum_{k,i} \int_{E_i^k} M_{\bar{\Phi}}(\sigma^{\frac{1}{p}} \chi_Q)^p(x) dx \\
& \leq 2^{(n+2)p+1} [\mu, \sigma, \Phi]_{A_p} \int_Q M_{\bar{\Phi}}(\sigma^{\frac{1}{p}} \chi_Q)^p(x) dx.
\end{aligned}$$

Finally, combining estimates

$$\int_{\tilde{Q}} \mathcal{N}^{\mathcal{D}}(\sigma \chi_Q)(x, t)^p d\mu(x, t) \leq c_{n,p} [\mu, \sigma, \Phi]_{A'_p} \int_Q M_{\bar{\Phi}}(\sigma^{\frac{1}{p}} \chi_Q)^p(x) dx,$$

dividing by $\sigma(Q)$ and raising to the power $\frac{1}{p}$

$$[\mu, \sigma]_{S'_p, \mathcal{D}} \leq c_{n,p} \left([\mu, \sigma, \Phi]_{A'_p} [\sigma, \bar{\Phi}]_{W_p} \right)^{\frac{1}{p}}.$$

□

To end this section we give the proof of Theorem 4. We will adapt the proof of [17].

Proof. As in the proof of Theorem 2 it is enough to consider $\mathcal{N}^{\mathcal{D}_j}$,

$$\|\mathcal{N}^{\mathcal{D}_j}(\sigma \cdot)\|_{L^p(\mathbb{R}^n, \sigma) \rightarrow L^p(\mathbb{R}_+^{n+1}, \mu)} \simeq [\mu, \sigma]_{S_p, \mathcal{D}_j} \quad j = 1, \dots, 2^n$$

where

$$[\mu, \sigma]_{S_p, \mathcal{D}_j} = \sup_{Q \in \mathcal{D}_j} \left(\frac{\int_{\tilde{Q}} \mathcal{N}^{\mathcal{D}_j}(\sigma \chi_Q)(x, t)^p d\mu(x, t)}{\int_Q \sigma(x) dx} \right)^{\frac{1}{p}}.$$

Hence, it is enough to prove that

$$[\mu, \sigma]_{S_p, \mathcal{D}_j} \lesssim [\mu, \sigma]_{p, \varepsilon}.$$

We recall that the quantity $[\mu, \sigma]_{p, \varepsilon}$ is defined by

$$[\mu, \sigma]_{p, \varepsilon} = \sup_Q \rho_{\sigma, \varepsilon}(Q) \left(\frac{\sigma(Q)}{|Q|} \right)^{p-1} \frac{\mu(\tilde{Q})}{|Q|}$$

with $\rho_{\sigma, \varepsilon}(Q) = \rho(Q) \varepsilon(\rho(Q))$ and $\rho(Q) = \frac{\int_Q M(\sigma \chi_Q) dx}{\sigma(Q)}$, where ε is a monotonic increasing function that satisfies

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{\varepsilon(t)} \frac{dt}{t} = 1. \tag{4.2}$$

To simplify the notation we fix one of these j and denote \mathcal{D}_j and $\mathcal{N}^{\mathcal{D}_j}$ by \mathcal{D} and \mathcal{N} . Let us fix $Q \in \mathcal{D}$. Arguing as we did in the proof of Theorem 3, we can write

$$\begin{aligned} & \int_{\tilde{Q}} \mathcal{N}(\sigma\chi_Q)(x,t)^p d\mu(x,t) \\ & \leq 2^p \left(\frac{\sigma(Q)}{|Q|} \right)^p \mu(\tilde{Q}) + 2^{(n+1)p} \sum_{k,i} \left(\frac{1}{|Q_i^k|} \int_{Q_i^k} \sigma(x)\chi_Q(x) dx \right)^p \mu(\tilde{Q}_i^k) \\ & = I + II. \end{aligned}$$

First we observe that

$$\begin{aligned} I & = 2^p \left(\frac{\sigma(Q)}{|Q|} \right)^{p-1} \frac{\mu(\tilde{Q})}{|Q|} \sigma(Q) \leq 2^p \left(\frac{\sigma(Q)}{|Q|} \right)^{p-1} \frac{\mu(\tilde{Q})}{|Q|} \rho(Q) \sigma(Q) \\ & \leq \frac{2^p}{\varepsilon(1)} 2^p \left(\frac{\sigma(Q)}{|Q|} \right)^{p-1} \frac{\mu(\tilde{Q})}{|Q|} \rho_{\sigma,\varepsilon}(Q) \sigma(Q) \leq \frac{2^p}{\varepsilon(1)} [\mu, \sigma]_{p,\varepsilon} \sigma(Q) \end{aligned}$$

To end the proof we have to control II . To simplify the notation we denote by \mathcal{S} the family of cubes $\{Q_i^k\}$ and everything is left is to prove that

$$II \leq 2^{(n+1)p} \sum_{S \in \mathcal{S}} \sigma_S^p \mu(\tilde{S}) \leq c [\mu, \sigma]_{p,\varepsilon} \sigma(Q)$$

where $\sigma_S = \frac{\sigma(S)}{|S|}$. We divide now the collection \mathcal{S} into subcollections $\mathcal{S}_{a,r}$ as follows. $S \in \mathcal{S}_{a,r}$ for some $a \in \mathbb{Z}$ and $r \in \{0, 1, 2, \dots\}$ if and only if

$$2^{a-1} \leq \sigma_S^{p-1} \frac{\mu(\tilde{S})}{|S|} \rho_{\sigma,\varepsilon}(S) \leq 2^a \quad \text{and} \quad 2^r \leq \rho(S) \leq 2^{r+1}.$$

Note that $\mathcal{S}_{a,r}$ is empty if $[\mu, \sigma]_{p,\varepsilon} < 2^{a-1}$. Now for these collections we have that

$$\begin{aligned} \sum_{S \in \mathcal{S}_{a,r}} \sigma_S^p \mu(\tilde{S}) & = \sum_{S \in \mathcal{S}_{a,r}} \sigma_S^{p-1} \sigma(S) \frac{\mu(\tilde{S})}{|S|} = \sum_{S \in \mathcal{S}_{a,r}} \frac{\sigma_S^{p-1} \frac{\mu(\tilde{S})}{|S|} \rho_{\sigma,\varepsilon}(S) \sigma(S)}{\rho_{\sigma,\varepsilon}(S)} \\ & \leq 2^a \sum_{S \in \mathcal{S}_{a,r}} \frac{\sigma(S)}{\rho_{\sigma,\varepsilon}(S)}. \end{aligned}$$

Now we observe that $\frac{1}{\rho(S)} \leq \frac{1}{2^r}$ and since ε is increasing $\varepsilon(2^r) \leq \varepsilon(\rho(S))$. Then

$$\frac{1}{\rho_{\sigma,\varepsilon}(S)} \leq \frac{1}{2^r \varepsilon(2^r)}.$$

Now we recall that, by Lemma 4, \mathcal{S} is a sparse family, that is, for each S there is a measurable set $E(S) \subset S$ such that $|S| \leq 2|E(S)|$ for each S and such that the family $\{E(S)\}_S$ is pairwise disjoint. Then,

$$\begin{aligned} \sum_{S \in \mathcal{S}_{a,r}} \sigma_S^p \mu(\tilde{S}) & \leq 2^a \sum_{S \in \mathcal{S}_{a,r}} \frac{\sigma(S)}{\rho_{\sigma,\varepsilon}(S)} \leq 2^a \sum_{S \in \mathcal{S}_{a,r}} \frac{\sigma(S)}{2^r \varepsilon(2^r)} = \frac{2^a}{2^r \varepsilon(2^r)} \sum_{S \in \mathcal{S}_{a,r}} \sigma(S) \\ & = \frac{2^a}{2^r \varepsilon(2^r)} \sum_{\substack{S' \in \mathcal{S}_{a,r} \\ S' \text{ maximal}}} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \subseteq S'}} \sigma(S) = \frac{2^a}{2^r \varepsilon(2^r)} \sum_{\substack{S' \in \mathcal{S}_{a,r} \\ S' \text{ maximal}}} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \subseteq S'}} \frac{\sigma(S)}{|S|} |S| \\ \mathcal{S} \text{ is sparse} & \leq c \frac{2^a}{2^r \varepsilon(2^r)} \sum_{\substack{S' \in \mathcal{S}_{a,r} \\ S' \text{ maximal}}} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \subseteq S'}} \frac{\sigma(S)}{|S|} |E(S)| \end{aligned}$$

$$\begin{aligned}
&\leq c \frac{2^a}{2^r \varepsilon(2^r)} \sum_{\substack{S' \in \mathcal{S}_{a,r} \\ S' \text{ maximal}}} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \subseteq S'}} \left(\inf_{z \in S} M(\sigma \chi_{S'})(z) \right) |E(S)| \\
&\leq c \frac{2^a}{2^r \varepsilon(2^r)} \sum_{\substack{S' \in \mathcal{S}_{a,r} \\ S' \text{ maximal}}} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \subseteq S'}} \int_{E(S)} M(\sigma \chi_{S'})(x) dx \\
&\leq c \frac{2^a}{2^r \varepsilon(2^r)} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \text{ maximal}}} \int_S M(\sigma \chi_S) dx \\
&\leq 2c \frac{2^a}{\varepsilon(2^r)} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \text{ maximal}}} \frac{\int_S M(\sigma \chi_S) dx}{2^{r+1}} \leq 2c \frac{2^a}{\varepsilon(2^r)} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \text{ maximal}}} \frac{\int_S M(\sigma \chi_S) dx}{\rho(S)} \\
&= 2c \frac{2^a}{\varepsilon(2^r)} \sum_{\substack{S \in \mathcal{S}_{a,r} \\ S \text{ maximal}}} \sigma(S) \leq 2c \frac{2^a}{\varepsilon(2^r)} \sigma(Q).
\end{aligned}$$

Taking this estimate into account we have that

$$\begin{aligned}
\sum_{S \in \mathcal{S}} \sigma_S^p \mu(\tilde{S}) &= \sum_{\substack{a \in \mathbb{Z} \\ [\mu, \sigma]_{p, \varepsilon} \geq 2^{a-1}}} \sum_{r \in \{0, 1, \dots\}} \sum_{S \in \mathcal{S}_{a,r}} \sigma_S^p \mu(\tilde{S}) \leq \sum_{\substack{a \in \mathbb{Z} \\ [\mu, \sigma]_{p, \varepsilon} \geq 2^{a-1}}} \sum_{r \in \{0, 1, \dots\}} 2c \frac{2^a}{\varepsilon(2^r)} \sigma(Q) \\
&= 2c \left(\sum_{\substack{a \in \mathbb{Z} \\ [\mu, \sigma]_{p, \varepsilon} \geq 2^{a-1}}} 2^a \right) \left(\sum_{r \in \{0, 1, \dots\}} \frac{1}{\varepsilon(2^r)} \right) \sigma(Q) \doteq 2cAB\sigma(Q)
\end{aligned}$$

and we are left with controlling both sums.

It is a straightforward computation that

$$A \leq 2 [\mu, \sigma]_{p, \varepsilon}.$$

For B , using (4.2), it is clear that

$$\sum_{r \in \{0, 1, \dots\}} \frac{1}{\varepsilon(2^r)} = \frac{1}{\log(2)} \sum_{r \in \{0, 1, \dots\}} \frac{\log(2^r) - \log(2^{r-1})}{\varepsilon(2^r)} \approx \int_{\frac{1}{2}}^{\infty} \frac{1}{\varepsilon(t)} \frac{dt}{t} = 1.$$

Combining these estimates we conclude that

$$\sum_{S \in \mathcal{S}} \sigma_S^p \mu(\tilde{S}) \leq c [\mu, \sigma]_{p, \varepsilon} \sigma(Q).$$

This ends the proof. \square

Acknowledgements

This research was supported by the Basque Government through the BERC 2014-2017 program and by the Spanish Ministry of Economy and Competitiveness MINECO through BCAM Severo Ochoa excellence accreditation SEV-2013-0323 and also through the project MTM2014-53850-P. The author would like to thank his advisor Carlos Pérez for turning his attention to this problem and for his guidance and support during the elaboration of this paper and the referee for some interesting remarks about the paper.

References

- [1] Astala, K.; Tadeusz Iwaniec, T.; Saksman, E. *Beltrami operators in the plane*. Duke Math. J., 107(1) (2001), 27–56, 10.1215/S0012-7094-01-10713-8
- [2] Buckley, S. M. *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. 340(1993), 253–272, 10.1090/S0002-9947-1993-1124164-0
- [3] Carleson, L. *Interpolation by bounded analytic functions and the corona problem*, Annals of Math, 76 (1962), 547–559, 10.2307/1970375
- [4] Chung, D. ; Pereyra, M. C.; Pérez, C. *Sharp bounds for general commutators on weighted Lebesgue spaces* Trans. Amer. Math. Soc. V. 364, (2012), 1163–1177, 10.1090/S0002-9947-2011-05534-0
- [5] Conde-Alonso, J. M.; Rey, G. *A pointwise estimate for positive dyadic shifts and some applications*, Math. Annalen, 10.1007/s00208-015-1320-y
- [6] Cruz-Uribe, D.; Martell, J. M.; Pérez, C. *Weights, extrapolation and the theory of Rubio de Francia*, Operator Theory: Advances and Applications, 215. Birkhäuser/Springer Basel AG, Basel, 2011. xiv+280 pp. ISBN: 978-3-0348-0071-6, 10.1007/978-3-0348-0072-3
- [7] Cruz-Uribe, D.; Martell, J.M.; Pérez, C. *Sharp weighted estimates for classical operators* Advances in Mathematics, V. 229, (2012), 408–441, 10.1016/j.aim.2011.08.013
- [8] Fefferman, C.; Stein, E. M. *Some maximal inequalities*, Amer. J. Math, 93 (1971), 107–115, 10.2307/2373450
- [9] Fujii, N. *Weighted bounded mean oscillation and singular integrals*, Math. Japon. 22 (1977/78), no. 5, 529–534.
- [10] García-Cuerva, J.; Rubio de Francia, J. L. *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, 116. Notas de Matemática [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985. x+604 pp. ISBN: 0-444-87804-1
- [11] Hytönen, T. *The sharp weighted bound for general Calderón-Zygmund operators*, Ann. of Math. (2), 175(3) (2012) 1473–1506, 10.4007/annals.2012.175.3.9
- [12] Hytönen, T.; Li, K. *Weak and strong A_p - A_∞ estimates for square functions and related operators* (2015) available at <http://arxiv.org/abs/1509.00273>
- [13] Hytönen, T.; Pérez, C. *Sharp weighted bounds involving A_∞* , Analysis and PDE, 6 (2013), 777–818, 10.2140/apde.2013.6.777
- [14] Hytönen, T.; Pérez, C.; Rela, E. *Sharp Reverse Hölder property for A_∞ weights on spaces of homogeneous type*, Journal of Functional Analysis V. 263, (2012) 3883–3899, 10.1016/j.jfa.2012.09.013
- [15] Hytönen, T.; Roncal, L.; Tapiola, O. *Quantitative weighted estimates for rough homogeneous singular integrals* Israel J. Math. 218 (2017) , no. 1, 133 –164, 10.1007/s11856-017-1462-6
- [16] Lacey, M. T.; *An elementary proof of the A_2 bound*. Israel J. Math. 217 (2017), no. 1, 181 –195. 42 10.1007/s11856-017-1442-x
- [17] Lacey, M. T.; Spencer, S. *On Entropy Bumps for Calderón-Zygmund Operators*, Concr. Oper. 2 (2015), 47 –52, 10.1515/conop-2015-0003
- [18] Lerner, A. K. *On an estimate of Calderón-Zygmund operators by dyadic positive operators*. J. Anal. Math. 121 (2013), 141–161, 10.1007/s11854-013-0030-1
- [19] Lerner, A. K. *A simple proof of the A_2 conjecture*. Int. Math. Res. Not. IMRN, 14(2013), 3159–3170, 10.1093/imrn/rns145
- [20] Lerner, A. K. *On pointwise estimates involving sparse operators* New York J. Math. (2016), 22 341 –349.
- [21] Lerner, A. K.; Nazarov, F. *Intuitive dyadic calculus: the basics* (2015) available at <http://arxiv.org/abs/1508.05639>
- [22] Muckenhoupt, B. *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc, 165 (1972), 115–121, 10.2307/1995882
- [23] Moen, K. *Sharp one-weight and two-weight bounds for maximal operators*. Studia Math. 194 (2009), no. 2, 163–180, 10.4064/sm194-2-4
- [24] Pérez, C. *On sufficient conditions for the boundedness of the Hardy–Littlewood maximal operator between weighted L^p -spaces with different weights*, Proc. London Math. Soc. 71 (1995), 135–157, 10.1112/plms/s3-71.1.135
- [25] Pérez, C. *Weighted norm inequalities for singular integral operators*, J. London Math. Soc. 49 (1994), 296–308.10.1112/jlms/49.2.296
- [26] Pérez, C.; Rela, E. *A new quantitative two weight theorem for the Hardy-Littlewood maximal operator*, Proc. Amer. Math. Soc. 143 (2015), no. 2, 641–655, 10.1090/S0002-9939-2014-12353-7
- [27] Petermichl, S. *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic*, Amer. J. Math. 129 (2007), 1355–1375, 10.1353/ajm.2007.0036
- [28] Petermichl, S. *The sharp weighted bound for the Riesz transforms*, Proc. Amer. Math. Soc. 136 (4) (2008), 1237–1249, 10.1215/S0012-9074-02-11223-X
- [29] Petermichl, S. and Volberg, A. *Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. 112 (2002), no. 2, 281–305, 10.1215/S0012-9074-02-11223-X

- [30] Ruiz, F. J. *A unified approach to Carleson measures and A_p weights*, Pacific J. Math. 117 (1985), no. 2, 397–404, URL: <http://projecteuclid.org/euclid.pjm/1102706791>
- [31] Ruiz, F. J.; Torrea, J. L. *A unified approach to Carleson measures and A_p weights II*, Pacific J. Math. 120 (1985), no. 1, 189–197, URL: <http://projecteuclid.org/euclid.pjm/1102703893>
- [32] Sawyer, E. T. *A characterization of a two-weight norm inequality for maximal operators*, Studia Math. 75 (1982), no. 1, 1–11.
- [33] Slavíková L., *On the necessity of bump conditions for the two-weighted maximal inequality*. Proc. Amer. Math. Soc. 145 (2017), no. 1, 109–118.
- [34] Treil, S.; Volberg, A. Treil, Sergei; Volberg, Alexander *Entropy conditions in two weight inequalities for singular integral operators*. Adv. Math. 301 (2016), 4997548. 10.1090/proc/13355
- [35] Wilson, J. M. *Weighted inequalities for the dyadic square function without dyadic A_∞* . Duke Math. J., 55 (1987), no. 1, 19–50, 0.1215/S0012-7094-87-05502-5