# LOGARITHMIC CONNECTIONS ON PRINCIPAL BUNDLES OVER A RIEMANN SURFACE

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Abstract. Let $E_G$ be a holomorphic principal $G$–bundle on a compact connected Riemann surface $X$, where $G$ is a connected reductive complex affine algebraic group. Fix a finite subset $D \subset X$, and for each $x \in D$ fix $w_x \in \text{ad}(E_G)_x$. Let $T$ be a maximal torus in the group of all holomorphic automorphisms of $E_G$. We give a necessary and sufficient condition for the existence of a $T$–invariant logarithmic connection on $E_G$ singular over $D$ such that the residue over each $x \in D$ is $w_x$. We also give a necessary and sufficient condition for the existence of a logarithmic connection on $E_G$ singular over $D$ such that the residue over each $x \in D$ is $w_x$, under the assumption that each $w_x$ is $T$–rigid.

## 1. Introduction

Let $X$ be a compact connected Riemann surface. Given a holomorphic vector bundle $E$ on $X$, a theorem of Weil and Atiyah says that $E$ admits a holomorphic connection if and only if the degree of every indecomposable component of $E$ is zero (see [We], [At]). Now let $G$ be a connected complex affine algebraic group and $E_G$ a holomorphic principal $G$–bundle on $X$. Then $E_G$ admits a holomorphic connection if and only if for every holomorphic reduction of structure group $E_H \subset E_G$, where $H$ is a Levi factor of some parabolic subgroup of $G$, and for every holomorphic character $\chi$ of $H$, the degree of the associated line bundle

$$E_H(\chi) = E_H \times^{\chi} \mathbb{C} \longrightarrow X \tag{1.1}$$

is zero [AB]. Our aim here is to investigate the logarithmic connections on $E_G$ with fixed residues, where $(G, E_G)$ is as above. More precisely, fix a finite subset $D \subset X$ and also fix

$$w_x \in \text{ad}(E_G)_x$$

for each $x \in D$, where $\text{ad}(E_G)$ is the adjoint vector bundle for $E_G$. We investigate the existence of logarithmic connections on $E_G$ singular over $D$ such that residue is $w_x$ for every $x \in D$.

Let $\text{Aut}(E_G)$ denote the group of all holomorphic automorphisms of $E_G$; it is a complex affine algebraic group. Fix a maximal torus

$$T \subset \text{Aut}(E_G)\,.$$

This choice produces a Levi factor $H$ of a parabolic subgroup of $G$ as well as a holomorphic reduction of structure group $E_H \subset E_G$ to $H$ [BBN]. This pair $(H, E_H)$ is determined by $T$ uniquely up to a holomorphic automorphism of $E_G$.

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The group $\text{Aut}(E_G)$ acts on the vector bundle $\text{ad}(E_G)$. An element of $\text{ad}(E_G)$ will be called $T$–rigid if it is fixed by the action of $T$; some examples are given in Section 4.2.

We prove the following (see Theorem 5.3):

**Theorem 1.1.** *The following two are equivalent:*

1. *There is a $T$–invariant logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$.*
2. *The element $w_x$ is $T$–rigid for each $x \in D$, and*
$$\text{degree}(E_H(\chi)) + \sum_{x\in D} d\chi(w_x) \,=\, 0$$
*for every holomorphic character $\chi$ of $H$, where $E_H(\chi)$ is the line bundle in* (1.1), *and $d\chi\,:\, \text{Lie}(H) \longrightarrow \mathbb{C}$ is the homomorphism of Lie algebras corresponding to $\chi$.*

The Lie algebra $\mathbb{C}$ being abelian the homomorphism $d\chi$ factors through the conjugacy classes in $\text{Lie}(H)$, so it can be evaluated on the elements of $\text{ad}(E_H)$.

We also the prove the following (see Theorem 5.1):

**Theorem 1.2.** *Assume that each $w_x$ is $T$–rigid. Then there is a logarithmic connection on $E_G$ singular over $D$, with residue $w_x$ at every $x \in D$, if and only if*
$$\text{degree}(E_H(\chi)) + \sum_{x\in D} d\chi(w_x) \,=\, 0\,,$$
*where $E_H(\chi)$ and $d\chi(w_x)$ are as in Theorem 1.1.*

## 2. Logarithmic connections and residue

2.1. **Preliminaries.** Let $G$ be a connected reductive affine algebraic group defined over $\mathbb{C}$. A Zariski closed connected subgroup $P \subset G$ is called a parabolic subgroup if $G/P$ is a projective variety [Bo, 11.2], [Hu]. The unipotent radical of a parabolic subgroup $P \subset G$ will be denoted by $R_u(P)$. The quotient group $P/R_u(P)$ is called the *Levi quotient* of $P$. A *Levi factor* of $P$ is a Zariski closed connected subgroup $L \subset P$ such that the composition $L \hookrightarrow P \longrightarrow P/R_u(P)$ is an isomorphism [Hu, p. 184]. We note that $P$ admits Levi factors, and any two Levi factors of $P$ are conjugate by an element of $R_u(P)$ [Hu, § 30.2, p. 185, Theorem].

The multiplicative group $\mathbb{C}\setminus\{0\}$ will be denoted by $\mathbb{G}_m$. A torus is a product of copies of $\mathbb{G}_m$. Any two maximal tori in a complex algebraic group are conjugate [Bo, p. 158, Proposition 11.23(ii)].

By a homomorphism between algebraic groups or by a character we will always mean a holomorphic homomorphism or a holomorphic character.

2.2. **Logarithmic connections.** Let $X$ be a compact connected Riemann surface. Fix a finite subset
$$D \,:=\, \{x_1,\, \cdots\, ,\, x_n\} \,\subset\, X\,.$$
The reduced effective divisor $x_1 + \ldots + x_n$ will also be denoted by $D$.

Let

$$p\,:\, E_H\,\longrightarrow\, X \tag{2.1}$$

be a holomorphic principal $H$–bundle on $X$, where $H$ is a connected affine algebraic group defined over $\mathbb{C}$. The Lie algebra of $H$ will be denoted by $\mathfrak{h}$. Let

$$\mathrm{d}p\,:\, \mathrm{T}E_H\,\longrightarrow\, p^*\mathrm{T}X \tag{2.2}$$

be the differential of the map $p$ in (2.1), where $\mathrm{T}E_H$ and $\mathrm{T}X$ are the holomorphic tangent bundles of $E_H$ and $X$ respectively; note that $\mathrm{d}p$ is surjective. The action of $H$ on $E_H$ produces an action of $H$ on $\mathrm{T}E_H$. This action on $\mathrm{T}E_H$ clearly preserves the subbundle kernel($\mathrm{d}p$). Define

$$\mathrm{ad}(E_H)\,:=\, \mathrm{kernel}(\mathrm{d}p)/H\,\longrightarrow\, X\,,$$

which is a holomorphic vector bundle on $X$; it is called the adjoint vector bundle for $E_H$. We note that $\mathrm{ad}(E_H)$ is identified with the vector bundle $E_H\times^H\mathfrak{h}\,\longrightarrow\, X$ associated to $E_H$ for the adjoint action of $H$ on its Lie algebra $\mathfrak{h}$. So the fibers of $\mathrm{ad}(E_H)$ are Lie algebras isomorphic to $\mathfrak{h}$. Define the Atiyah bundle for $E_H$

$$\mathrm{At}(E_H)\,:=\, (\mathrm{T}E_H)/H\,\longrightarrow\, X\,.$$

The action of $H$ on $\mathrm{T}E_H$ produces an action of $H$ on the direct image $p_*\mathrm{T}E_H$. We note that

$$\mathrm{At}(E_H)\,=\, (p_*\mathrm{T}E_H)^H\,\subset\, p_*\mathrm{T}E_H$$

(see [At]). Taking quotient by $H$, the homomorphism $\mathrm{d}p$ in (2.2) produces a short exact sequence

$$0\,\longrightarrow\,\mathrm{ad}(E_H)\,\longrightarrow\,\mathrm{At}(E_H)\,\stackrel{\mathrm{d}'p}{\longrightarrow}\,\mathrm{T}X\,\longrightarrow\, 0\,, \tag{2.3}$$

where $\mathrm{d}'p$ is constructed from $\mathrm{d}p$; this is known as the Atiyah exact sequence for $E_H$.

The subsheaf $\mathrm{T}X\otimes\mathcal{O}_X(-D)$ of $\mathrm{T}X$ will be denoted by $\mathrm{T}X(-D)$. Now define

$$\mathrm{At}(E_H,\, D)\,:=\, (\mathrm{d}'p)^{-1}(\mathrm{T}X(-D))\,\subset\,\mathrm{At}(E_H)\,,$$

where $\mathrm{d}'p$ is the projection in (2.3). So from (2.3) we have the exact sequence of vector bundles on $X$

$$0\,\longrightarrow\,\mathrm{ad}(E_H)\,\stackrel{i_0}{\longrightarrow}\,\mathrm{At}(E_H,\, D)\,\stackrel{\sigma}{\longrightarrow}\,\mathrm{T}X(-D)\,\longrightarrow\, 0\,, \tag{2.4}$$

where $\sigma$ is the restriction of $\mathrm{d}'p$; this will be called the *logarithmic Atiyah exact sequence* for $E_H$.

A logarithmic connection on $E_H$ singular over $D$ is a holomorphic homomorphism

$$\theta\,:\, \mathrm{T}X(-D)\,\longrightarrow\,\mathrm{At}(E_H,\, D) \tag{2.5}$$

such that $\sigma\circ\theta\,=\,\mathrm{Id}_{\mathrm{T}X(-D)}$, where $\sigma$ is the homomorphism in (2.4). Note that giving such a homomorphism $\theta$ is equivalent to giving a homomorphism $\varpi\,:\,\mathrm{At}(E_H,\, D)\,\longrightarrow\,\mathrm{ad}(E_H)$ such that $\varpi\circ i_0\,=\,\mathrm{Id}_{\mathrm{ad}(E_H)}$, where $i_0$ is the homomorphism in (2.4).

2.3. **Residue of a logarithmic connection.** Given a vector bundle $W$ on $X$, the fiber of $W$ over any point $x \in X$ will be denoted by $W_x$. For any $\mathcal{O}_X$–linear homomorphism $f : W \longrightarrow V$ of holomorphic vector bundles, its restriction $W_x \longrightarrow V_x$ will be denoted by $f(x)$.

From (2.3) and (2.4) we have the commutative diagram of homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_H) & \stackrel{i_0}{\longrightarrow} & \mathrm{At}(E_H,\, D) & \stackrel{\sigma}{\longrightarrow} & \mathrm{T}X(-D) & \longrightarrow & 0 \\ && \| && \Big\downarrow j && \Big\downarrow \iota && \\ 0 & \longrightarrow & \mathrm{ad}(E_H) & \stackrel{i}{\longrightarrow} & \mathrm{At}(E_H) & \stackrel{\mathrm{d}'p}{\longrightarrow} & \mathrm{T}X & \longrightarrow & 0 \end{array} \tag{2.6}$$

on $X$. So for any point $x \in X$, we have

$$\mathrm{d}'p(x) \circ j(x) \,=\, \iota(x) \circ \sigma(x) \,:\, \mathrm{At}(E_H,\, D)_x \longrightarrow (\mathrm{T}X)_x \,=\, \mathrm{T}_x X\,.$$

Note that $\iota(x) = 0$ if $x \in D$, therefore in that case $\mathrm{d}'p(x) \circ j(x) = 0$. Consequently, for every $x \in D$ there is a homomorphism

$$R_x \,:\, \mathrm{At}(E_H,\, D)_x \longrightarrow \mathrm{ad}(E_H)_x \tag{2.7}$$

uniquely defined by the identity $i(x) \circ R_x(v) \,=\, j(x)(v)$ for all $v \in \mathrm{At}(E_H,\, D)_x$. Note that

$$R_x \circ i_0(x) \,=\, \mathrm{Id}_{\mathrm{ad}(E_H)_x}\,,$$

where $i_0$ is the homomorphism in (2.6). Therefore, from (2.4) we have

$$\mathrm{At}(E_H,\, D)_x \,=\, \mathrm{ad}(E_H)_x \oplus \mathrm{kernel}(R_x) \,=\, \mathrm{ad}(E_H)_x \oplus \mathrm{T}X(-D)_x\,; \tag{2.8}$$

note that the composition $\mathrm{kernel}(R_x) \hookrightarrow \mathrm{At}(E_H,\, D)_x \stackrel{\sigma(x)}{\longrightarrow} \mathrm{T}X(-D)_x$ is an isomorphism.

For any $x \in D$, the fiber $\mathrm{T}X(-D)_x$ is identified with $\mathbb{C}$ using the Poincaré adjunction formula [GH, p. 146]. Indeed, for any holomorphic coordinate $z$ around $x$ with $z(x) = 0$, the image of $z\frac{\partial}{\partial z}$ in $\mathrm{T}X(-D)_x$ is independent of the choice of the coordinate function $z$; the above mentioned identification between $\mathrm{T}X(-D)_x$ and $\mathbb{C}$ sends this independent image to $1 \in \mathbb{C}$. Therefore, from (2.8) we have

$$\mathrm{At}(E_H,\, D)_x \,=\, \mathrm{ad}(E_H)_x \oplus \mathbb{C} \tag{2.9}$$

for all $x \in D$.

For a logarithmic connection $\theta \,:\, \mathrm{T}X(-D) \longrightarrow \mathrm{At}(E_H,\, D)$ as in (2.5), and any $x \in D$, define

$$\mathrm{Res}(\theta,\, x) \,:=\, R_x(\theta(1)) \,\in\, \mathrm{ad}(E_H)_x\,, \tag{2.10}$$

where $R_x$ is the homomorphism in (2.7); in the above definition 1 is considered as an element of $\mathrm{T}X(-D)_x$ using the identification of $\mathbb{C}$ with $\mathrm{T}X(-D)_x$ mentioned earlier.

The element $\mathrm{Res}(\theta,\, x)$ in (2.10) is called the *residue*, at $x$, of the logarithmic connection $\theta$.

2.4. **Extension of structure group.** Let $M$ be a complex affine algebraic group and

$$\rho\, :\, H\,\longrightarrow\, M$$

a homomorphism. As before, $E_H$ is a holomorphic principal $H$–bundle on $X$. Let

$$E_M\, :=\, E_H\times^\rho M\,\longrightarrow\, X$$

be the holomorphic principal $M$–bundle obtained by extending the structure group of $E_H$ using $\rho$. So $E_M$ is the quotient of $E_H\times M$ obtained by identifying $(y,\, m)$ and $(yh^{-1},\, \rho(h)m)$, where $y$, $m$ and $h$ run over $E_H$, $M$ and $H$ respectively. Therefore, we have a morphism

$$\widehat{\rho}\, :\, E_H\,\longrightarrow\, E_M\, ,\ \ y\,\longmapsto\, \widetilde{(y,\, e_M)}\, ,$$

where $\widetilde{(y,\, e_M)}$ is the equivalence class of $(y,\, e_M)$ with $e_M$ being the identity element of $M$. The homomorphism of Lie algebras $d\rho\, :\, \mathfrak{h}\,\longrightarrow\, \mathfrak{m}\, :=\, \text{Lie}(M)$ associated to $\rho$ produces a homomorphism of vector bundles

$$\alpha\, :\, \text{ad}(E_H)\,\longrightarrow\, \text{ad}(E_M)\, . \tag{2.11}$$

The maps $\widehat{\rho}$ and $d\rho$ together produce a homomorphism of vector bundles

$$\widetilde{A}\, :\, \text{At}(E_H)\,\longrightarrow\, \text{At}(E_M)\, .$$

This map $\widetilde{A}$ produces a homomorphism

$$A\, :\, \text{At}(E_H,\, D)\,\longrightarrow\, \text{At}(E_M,\, D)\, , \tag{2.12}$$

which fits in the following commutative diagram of homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{ad}(E_H) & \stackrel{i_0}{\longrightarrow} & \text{At}(E_H,\, D) & \stackrel{\sigma}{\longrightarrow} & \text{T}X(-D) & \longrightarrow & 0\\ && \Big\downarrow\alpha && \Big\downarrow A && \Vert &&\\ 0 & \longrightarrow & \text{ad}(E_M) & \longrightarrow & \text{At}(E_M,\, D) & \longrightarrow & \text{T}X(-D) & \longrightarrow & 0 \end{array} \tag{2.13}$$

where the top exact sequence is the one in (2.4) and the bottom one is the corresponding sequence for $E_M$.

If $\theta\, :\, \text{T}X(-D)\,\longrightarrow\, \text{At}(E_H,\, D)$ is a logarithmic connection on $E_H$ as in (2.5), then

$$A\circ\theta\, :\, \text{T}X(-D)\,\longrightarrow\, \text{At}(E_M,\, D) \tag{2.14}$$

is a logarithmic connection on $E_M$ singular over $D$. From the definition of residue in (2.10) it follows immediately that

$$\alpha(\text{Res}(\theta,\, x))\,=\, \text{Res}(A\circ\theta,\, x) \tag{2.15}$$

for all $x\,\in\, D$. This proves the following:

**Lemma 2.1.** *With the above notation, if $E_H$ admits a logarithmic connection $\theta$ singular over $D$ with residue $w_x\in\text{ad}(E_H)_x$ at each $x\,\in\, D$, then $E_M$ admits a logarithmic connection $\theta'\,=\,A\circ\theta$ singular over $D$ with residue $\alpha(w_x)$ at each $x\,\in\, D$.*

## 3. Connections with given residues

3.1. **Formulation of residue condition.** Fix a holomorphic principal $H$–bundle $E_H$ on $X$, and fix elements

$$w_x \,\in\, \mathrm{ad}(E_H)_x$$

for all $x \,\in\, D$. Consider the decomposition of $\mathrm{At}(E_H,\, D)_x$ in (2.9). For any $x \,\in\, D$, let

$$\ell_x \,:=\, \mathbb{C}\cdot(w_x,\, 1) \,\subset\, \mathrm{ad}(E_H)_x\oplus\mathbb{C} \,=\, \mathrm{At}(E_H,\, D)_x$$

be the line in the fiber $\mathrm{At}(E_H,\, D)_x$. Let

$$\mathcal{A} \,\subset\, \mathrm{At}(E_H,\, D)$$

be the subsheaf that fits in the short exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathrm{At}(E_H,\, D) \longrightarrow \bigoplus_{x\in D}\mathrm{At}(E_H,\, D)_x/\ell_x \longrightarrow 0\,. \tag{3.1}$$

Note that the composition

$$\mathrm{ad}(E_H)_x \xrightarrow{i_0(x)} \mathrm{At}(E_H,\, D)_x \longrightarrow \mathrm{At}(E_H,\, D)_x/\ell_x$$

is injective, hence it is an isomorphism, where $i_0$ is the homomorphism in (2.13); this composition will be denoted by $\phi_x$. Therefore, from (2.4) and (3.1) we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{ad}(E_H)\otimes\mathcal{O}_X(-D) & \longrightarrow & \mathcal{A} & \xrightarrow{\sigma_1} & \mathrm{T}X(-D) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow\nu & & \downarrow\mathrm{id} & & \\
0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{i_0} & \mathrm{At}(E_H,\, D) & \xrightarrow{\sigma} & \mathrm{T}X(-D) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{x\in D}\mathrm{ad}(E_H)_x & \xrightarrow{\bigoplus_{x\in D}\phi_x} & \bigoplus_{x\in D}\mathrm{At}(E_H,\, D)_x/\ell_x & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.2}$$

where all the rows and columns are exact; the restriction of $\sigma$ to the subsheaf $\mathcal{A}$ is denoted by $\sigma_1$.

**Lemma 3.1.** *Consider the space of all logarithmic connections on $E_H$ singular over $D$ such that the residue over every $x \,\in\, D$ is $w_x$. It is in bijection with the space of all holomorphic splittings of the short exact sequence of vector bundles*

$$0 \longrightarrow \mathrm{ad}(E_H)\otimes\mathcal{O}_X(-D) \longrightarrow \mathcal{A} \xrightarrow{\sigma_1} \mathrm{T}X(-D) \longrightarrow 0$$

*on $X$ in* (3.2).

*Proof.* Let $\theta \,:\, \mathrm{T}X(-D) \longrightarrow \mathrm{At}(E_H,\, D)$ be a logarithmic connection on $E_H$ singular over $D$ such that the residue over every $x \in D$ is $w_x$. From the definition of residue and the construction of $\mathcal{A}$ it follows that

$$\theta(\mathrm{T}X(-D)) \subset \mathcal{A} \subset \mathrm{At}(E_H,\, D)\,.$$

Therefore, $\theta$ defines a holomorphic homomorphism

$$\theta' \,:\, \mathrm{T}X(-D) \longrightarrow \mathcal{A}\,.$$

Evidently, we have $\sigma_1 \circ \theta' = \mathrm{Id}_{\mathrm{T}X(-D)}$. So $\theta'$ is a holomorphic splitting of the exact sequence in the lemma.

To prove the converse, let

$$\theta_1 \,:\, \mathrm{T}X(-D) \longrightarrow \mathcal{A}$$

be a holomorphic homomorphism such that $\sigma_1 \circ \theta_1 = \mathrm{Id}_{\mathrm{T}X(-D)}$. Consider the composition

$$\nu \circ \theta_1 \,:\, \mathrm{T}X(-D) \longrightarrow \mathrm{At}(E_H,\, D)\,,$$

where $\nu$ is the homomorphism in (3.2). This defines a logarithmic connection on $E_H$ singular over $D$, because $\sigma \circ \nu \circ \theta_1 = \sigma_1 \circ \theta_1 = \mathrm{Id}_{\mathrm{T}X(-D)}$ by the commutativity of (3.2). From (3.2) it follows immediately that $\nu \circ \theta_1(\mathrm{T}X(-D)_x) = \ell_x \subset \mathrm{At}(E_H,\, D)_x$ for every $x \in D$. Now from the definition of residue it follows that the residue of the connection $\nu \circ \theta_1$ at any $x \in D$ is $w_x$. $\square$

3.2. **Extension class.** The short exact sequence in Lemma 3.1 determines a cohomology class

$$\beta \in H^1(X,\, \mathrm{Hom}(\mathrm{T}X(-D),\, \mathrm{ad}(E_H) \otimes \mathcal{O}_X(-D))) = H^1(X,\, \mathrm{ad}(E_H) \otimes K_X)\,, \qquad (3.3)$$

where $K_X = (\mathrm{T}X)^*$ is the holomorphic cotangent bundle of $X$. Therefore, $E_H$ admits a logarithmic connection singular over $D$ with residue $w_x \in \mathrm{ad}(E_H)_x$ at each $x \in D$ if and only if the cohomology class $\beta$ in (3.3) vanishes.

Let $\rho : H \longrightarrow M$ be a homomorphism of affine algebraic groups. Let $E_M := E_H \times^\rho M$ be the principal $M$–bundle over $X$ obtained by extending the structure group of $E_H$ to $M$ by $\rho$. Consider the homomorphism $\alpha$ in (2.11). It produces a homomorphism

$$\overline{\rho} \,:\, H^1(X,\, \mathrm{ad}(E_H) \otimes K_X) \longrightarrow H^1(X,\, \mathrm{ad}(E_M) \otimes K_X)\,.$$

From (2.13) we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_H) \otimes \mathcal{O}_X(-D) & \longrightarrow & \mathcal{A} & \xrightarrow{\sigma_1} & \mathrm{T}X(-D) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{ad}(E_M) \otimes \mathcal{O}_X(-D) & \longrightarrow & \mathcal{A}(E_M) & \longrightarrow & \mathrm{T}X(-D) & \longrightarrow & 0
\end{array}$$

where the top exact sequence is the one in Lemma 3.1 and the bottom one is the same sequence for $E_M$. From this diagram it follows that the cohomology class in $H^1(X,\, \mathrm{ad}(E_M) \otimes K_X)$ for the short exact sequence in Lemma 3.1 for $E_M$ coincides with $\overline{\rho}(\beta)$, where $\beta$ is the cohomology class in (3.3).

**Corollary 3.2.** *Assume that $E_H$ admits a holomorphic reduction of structure group $E_J \subset E_H$ to a complex algebraic subgroup $J \subset H$. The cohomology class $\beta$ in* (3.3) *is contained in the image of the natural homomorphism $H^1(X,\, \text{ad}(E_J)\otimes K_X) \hookrightarrow H^1(X,\, \text{ad}(E_H)\otimes K_X)$*

3.3. **A necessary condition for logarithmic connections with given residue.** Let $\theta$ be a logarithmic connection on $E_H$ singular over $D$. Take any character

$$\chi\,:\, H\,\longrightarrow\, \mathbb{G}_m\,.$$

The group $H$ acts on $\mathbb{C}$; the action of $h\,\in\, H$ sends any $c\,\in\,\mathbb{C}$ to $\chi(h)\cdot c$. Let

$$E_H(\chi)\,:=\, E_H\times^{\chi}\mathbb{C}\,\longrightarrow\, X$$

be the holomorphic line bundle over $X$ associated to $E_H$ for this action of $H$ on $\mathbb{C}$. Since $\mathbb{G}_m$ is abelian, the adjoint vector bundle for $E_H(\chi)$ is the trivial holomorphic line bundle $\mathcal{O}_X$ over $X$. The above logarithmic connection $\theta$ induces a logarithmic connection on $E_H(\chi)$ (see (2.14)); this induced logarithmic connection on $E_H(\chi)$ will be denoted by $\theta^{\chi}$.

For any $x\,\in\, D$, let

$$\text{Res}(\theta^{\chi},\, x)\,\in\,\mathbb{C}$$

be the residue of $\theta^{\chi}$. The residue $\text{Res}(\theta,\, x)$ defines a conjugacy class in the Lie algebra $\mathfrak{h}$, because any fiber of $\text{ad}(E_H)$ is identified with $\mathfrak{h}$ uniquely up to conjugation. From (2.15) it follows immediately that $\text{Res}(\theta^{\chi},\, x)$ coincides with with $d\chi(\text{Res}(\theta,\, x))$, where $d\chi\,:\,\mathfrak{h}\,\longrightarrow\,\mathbb{C}$ is the homomorphism of Lie algebras associated to $\chi$; note that since the Lie algebra $\mathbb{C}$ is abelian, the homomorphism $d\chi$ factors through the conjugacy classes in $\mathfrak{h}$.

As $\theta^{\chi}$ is a logarithmic connection on the line bundle $E_H(\chi)$ with residue $d\chi(\text{Res}(\theta,\, x))$ at each $x\,\in\, D$, using a computation in [Oh] it follows that

$$\text{degree}(E_H(\chi))+\sum_{x\in D} d\chi(\text{Res}(\theta,\, x))\,=\,0$$

(see [BDP, Lemma 2.3]).

Therefore, we have the following:

**Lemma 3.3.** *Let $E_H$ be a holomorphic principal $H$–bundle on $X$. Fix*

$$w_x\,\in\,\text{ad}(E_H)_x$$

*for every $x\,\in\, D$. If there is a logarithmic connection on $E_H$ singular over $D$ with residue $w_x$ at every $x\,\in\, D$, then*

$$\text{degree}(E_H(\chi))+\sum_{x\in D} d\chi(w_x)\,=\,0$$

*for every character $\chi$ of $H$, where $E_H(\chi)$ is the associated holomorphic line bundle, and $d\chi$ is the homomorphism of Lie algebras corresponding to $\chi$.*

Let $\rho\,:\, H\,\longrightarrow\, M$ be an injective homomorphism to a connected complex algebraic group $M$ and $E_M\,:=\, E_H\times^{\rho} M$ the holomorphic principal $M$–bundle on $X$ obtained by extending the structure group of $E_H$ using $\rho$. As before, the Lie algebras of $H$ and $M$ will be denoted by $\mathfrak{h}$ and $\mathfrak{m}$ respectively. Using the injective homomorphism of Lie algebras

$$d\rho \,:\, \mathfrak{h} \longrightarrow \mathfrak{m} \tag{3.4}$$

associated to $\rho$, we have an injective homomorphism $\alpha$ as in (2.11). For every $x \in D$, fix an element $w_x \in \text{ad}(E_H)_x$.

**Lemma 3.4.** *Assume that $H$ is reductive. There is a logarithmic connection on $E_H$ singular over $D$ with residue $w_x$ at each $x \in D$ if there is a logarithmic connection on $E_M$ singular over $D$ with residue $\alpha(x)(w_x)$ at each $x \in D$, where $\alpha$ is the homomorphism in* (2.11).

*Proof.* The adjoint action of $H$ on $\mathfrak{h}$ makes it an $H$–module. On the other hand, the homomorphism $\rho$ composed with the adjoint action of $M$ on $\mathfrak{m}$ produces an action of $H$ on $\mathfrak{m}$. The injective homomorphism $d\rho$ in (3.4) is a homomorphism of $H$–modules. Since $H$ is reductive, there is an $H$–submodule $V \subset \mathfrak{m}$ which is a complement of $d\rho(\mathfrak{h})$, meaning

$$\mathfrak{m} = d\rho(\mathfrak{h}) \oplus V\,.$$

Let $\eta \,:\, \mathfrak{m} \longrightarrow \mathfrak{h}$ be the projection constructed from this decomposition of $\mathfrak{m}$; in particular, we have $\eta \circ d\rho = \text{Id}_{\mathfrak{h}}$.

Since the above $\eta$ is a homomorphism of $H$–modules, it produces a projection

$$\widehat{\eta} \,:\, \text{ad}(E_M) \longrightarrow \text{ad}(E_H)$$

such that $\widehat{\eta} \circ \alpha = \text{Id}_{\text{ad}(E_H)}$, where $\alpha$ is the homomorphism in (2.11).

Now if $\theta \,:\, \text{At}(E_M,\, D) \longrightarrow \text{ad}(E_M)$ is a logarithmic connection on $E_M$ singular over $D$ with residue $\alpha(x)(w_x)$ at each $x \in D$, consider the composition

$$\widehat{\eta} \circ \theta \circ A \,:\, \text{At}(E_H,\, D) \longrightarrow \text{ad}(E_H)\,,$$

where $A$ is constructed in (2.12). Evidently, it is a logarithmic connection on $E_H$ singular over $D$ with residue $w_x$ at each $x \in D$. □

## 4. $T$–rigid elements of adjoint bundle

4.1. **Definition.** As before, $H$ is a complex affine algebraic group and $p \,:\, E_H \longrightarrow X$ a holomorphic principal $H$–bundle on $X$. An automorphism of $E_H$ is a holomorphic map $F \,:\, E_H \longrightarrow E_H$ such that

- $p \circ F = p$, and
- $F(zh) = F(z)h$ for all $z \in E_H$ and $h \in H$.

Let $\text{Aut}(E_H)$ be the group of all automorphisms of $E_H$. We will show that $\text{Aut}(E_H)$ is a complex affine algebraic group.

First consider the case of $H = \text{GL}(r, \mathbb{C})$. For a holomorphic principal $\text{GL}(r, \mathbb{C})$–bundle $E_{\text{GL}}$ on $X$, let $E := E_{\text{GL}} \times^{\text{GL}(r,\mathbb{C})} \mathbb{C}^r$ be the holomorphic vector bundle of rank $r$ on $X$ associated to $E_{\text{GL}}$ for the standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$. Then $\text{Aut}(E_{\text{GL}})$ is identified with the group of all holomorphic automorphisms $\text{Aut}(E)$ of the vector bundle $E$ over the identity map of $X$. Note that $\text{Aut}(E)$ is the Zariski open subset of the complex affine space $H^0(X,\, \text{End}(E))$ consisting of all global endomorphisms $f$ of $E$ such that $\det(f(x_0)) \neq 0$ for a fixed point $x_0 \in X$; since $x \longmapsto \det(f(x))$ is a holomorphic function on $X$, it is in fact a constant function. Therefore, $\text{Aut}(E_{\text{GL}})$ is an affine algebraic variety over $\mathbb{C}$.

For a general $H$, fix an algebraic embedding $\rho : H \hookrightarrow \text{GL}(r, \mathbb{C})$ for some $r$. For a holomorphic principal $H$–bundle $E_H$ on $X$, let $E_{\text{GL}} := E_H \times^{\rho} \text{GL}(r, \mathbb{C})$ be the holomorphic principal $\text{GL}(r, \mathbb{C})$–bundle on $X$ obtained by extending the structure group of $E_H$ using $\rho$. The injective homomorphism $\rho$ produces an injective homomorphism

$$\rho' : \text{Aut}(E_H) \longrightarrow \text{Aut}(E_{\text{GL}}) .$$

The image of $\rho'$ is Zariski closed in the algebraic group $\text{Aut}(E_{\text{GL}})$. Hence $\rho'$ produces the structure of a complex affine algebraic group on $\text{Aut}(E_H)$. This structure of a complex algebraic group is independent of the choices of $r$, $\rho$. Therefore, $\text{Aut}(E_H)$ is an affine algebraic group. Note that $\text{Aut}(E_H)$ need not be connected, although the automorphism group of a holomorphic vector bundle is always connected (as it is a Zariski open subset of a complex affine space).

The Lie algebra of $\text{Aut}(E_H)$ is $H^0(X,\, \text{ad}(E_H))$. The group $\text{Aut}(E_H)$ acts on any fiber bundle associated to $E_H$. In particular, $\text{Aut}(E_H)$ acts on the adjoint vector bundle $\text{ad}(E_H)$. This action evidently preserves the Lie algebra structure on the fibers of $\text{ad}(E_H)$.

Let $\text{Aut}(E_H)^0 \subset \text{Aut}(E_H)$ be the connected component containing the identity element. Fix a maximal torus

$$T \subset \text{Aut}(E_H)^0 .$$

An element $w \in \text{ad}(E_H)_x$, where $x \in X$, will be called *$T$–rigid* if the action of $T$ on $\text{ad}(E_H)_x$ fixes $w$.

Consider the adjoint action of $H$ on itself. Let

$$\text{Ad}(E_H) := E_H \times^H H \longrightarrow X \tag{4.1}$$

be the associated holomorphic fiber bundle. Since this adjoint action preserves the group structure of $H$, the fibers of $\text{Ad}(E_H)$ are complex algebraic groups isomorphic to $H$. More precisely, each fiber of $\text{Ad}(E_H)$ is identified with $H$ uniquely up to an inner automorphism of $H$. The corresponding Lie algebra bundle on $X$ is $\text{ad}(E_H)$.

The group $\text{Aut}(E_H)$ is the space of all holomorphic sections of $\text{Ad}(E_H)$. For any $x \in X$, the action of $\text{Aut}(E_H)$ on the fiber $\text{ad}(E_H)_x$ coincides with the one obtained via the composition

$$\text{Aut}(E_H) \xrightarrow{\text{ev}_x} \text{Ad}(E_H)_x \xrightarrow{\text{ad}} \text{Aut}(\text{ad}(E_H)_x) ,$$

where $\text{ev}_x$ is the evaluation map that sends a section of $\text{Ad}(E_H)$ to its evaluation at $x$, and ad is the adjoint action of the group $\text{Ad}(E_H)_x$ on its Lie algebra $\text{ad}(E_H)_x$.

Therefore, an element $w \in \text{ad}(E_H)_x$ is $T$–rigid if and only if the adjoint action of $\text{ev}_x(T) \subset \text{Ad}(E_H)_x$ on $\text{ad}(E_H)_x$ fixes $w$.

4.2. **Examples.** The center of $H$ will be denoted by $Z_H$. Let $E_H$ be a holomorphic principal $H$–bundle on $X$. Since $Z_H$ commutes with $H$, for any $t \in Z_H$, the map $E_H \longrightarrow E_H$, $z \longmapsto zt$ is $H$–equivariant. Therefore, we have $Z_H \subset \text{Aut}(E_H)$. The principal $H$–bundle $E_H$ is called *simple* if $Z_H = \text{Aut}(E_H)$. Note that if $E_H$ is simple then every element of $\text{ad}(E_H)$ is $T$–rigid, where $T$ is any maximal torus in $\text{Aut}(E_H)^0$.

Let $H$ be connected reductive, and let $E_H$ be stable. Then $Z_H$ is a finite index subgroup of $\text{Aut}(E_H)$. Therefore, any maximal torus of $\text{Aut}(E_H)^0$ is contained in $Z_H$. This implies that every element of $\text{ad}(E_H)$ is $T$–rigid, where $T$ is any maximal torus in $\text{Aut}(E_H)^0$.

Take $E_H$ to be the trivial holomorphic principal $H$–bundle $X \times H$. Then the left–translation action of $H$ identifies $H$ with $\text{Aut}(E_H)$. Also, $\text{ad}(E_H)$ is the trivial vector $X \times \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of $H$. Let $T$ be a maximal torus of $H = \text{Aut}(E_H)$. Then an element $v \in \mathfrak{h} = \text{ad}(E_H)_x$ is $T$–rigid if and only if $v \in \text{Lie}(T)$.

## 5. A criterion for logarithmic connections with given residue

5.1. **Logarithmic connections with $T$–rigid residue.** As in Section 2.1, $G$ is a connected reductive affine algebraic group defined over $\mathbb{C}$. Let $E_G$ be a holomorphic principal $G$–bundle over $X$. Fix a maximal torus

$$T \subset \text{Aut}(E_G)^0 \, ,$$

where $\text{Aut}(E_G)^0$ as before is the connected component containing the identity element of the group of automorphisms of $E_G$.

We now recall some results from [BBN], [BP].

As in (4.1), define the adjoint bundle $\text{Ad}(E_G) = E_G \times^G G$. For any point $y \in X$, consider the evaluation homomorphism

$$\varphi_y \,:\, T \longrightarrow \text{Ad}(E_G)_y \, , \quad s \longmapsto s(y) \, .$$

Then $\varphi_y$ is injective and its image is a torus in $G$ [BBN, p. 230, Section 3]. Since $G$ is identified with $\text{Ad}(E_G)_y$ uniquely up to an inner automorphism, the image $\varphi_y(T)$ determines a conjugacy class of tori in $G$; this conjugacy class is independent of the choice of $y$ [BBN, p. 230, Section 3], [BP, p. 63, Theorem 4.1]. Fix a torus

$$T_G \subset G \tag{5.1}$$

in this conjugacy class of tori. The centralizer

$$H := C_G(T_G) \subset G \tag{5.2}$$

of $T_G$ in $G$ is a Levi factor of a parabolic subgroup of $G$ [BBN, p. 230, Section 3], [BP, p. 63, Theorem 4.1]. The principal $G$–bundle $E_G$ admits a holomorphic reduction of structure group

$$E_H \subset E_G \tag{5.3}$$

to the above subgroup $H$ [BBN, p. 230, Theorem 3.2], [BP, p. 63, Theorem 4.1]. Since $T_G$ is in the center of $H$, the action of $T_G$ on $E_H$ commutes with the action of $H$, so $T_G \subset \text{Aut}^0(E_H)$ (this was noted in Section 4.2). The image of $T_G$ in $\text{Aut}^0(E_H)$ coincides with $T$. This reduction $E_H$ is minimal in the sense that there is no Levi factor $L$ of some parabolic subgroup of $G$ such that

- $L \subsetneq H$, and
- $E_G$ admits a holomorphic reduction of structure group to $L$.

(See [BBN, p. 230, Theorem 3.2].)

The above reduction $E_H$ is unique in the following sense. Let $L$ be a Levi factor of a parabolic subgroup of $G$ and $E_L \subset E_G$ a holomorphic reduction of structure group to $L$ satisfying the condition that $E_G$ does not admit any holomorphic reduction of structure group to a Levi factor $L'$ of some parabolic subgroup of $G$ such that $L' \subsetneq L$. Then there is an automorphism $\varphi \in \mathrm{Aut}(E_G)^0$ such that $E_L = \varphi(E_H)$ [BP, p. 63, Theorem 4.1]. In particular, the subgroup $L \subset G$ is conjugate to $H$.

The Lie algebras of $G$ and $H$ will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The inclusion of $\mathfrak{h}$ in $\mathfrak{g}$ and the reduction in (5.3) together produce an inclusion $\mathrm{ad}(E_H) \hookrightarrow \mathrm{ad}(E_G)$. This subbundle $\mathrm{ad}(E_H)$ of $\mathrm{ad}(E_G)$ coincides with the invariant subbundle $\mathrm{ad}(E_G)^T$ for the action of $T$ on $\mathrm{ad}(E_G)$ [BBN, p. 230, Theorem 3.2], [BP, p. 61, Proposition 3.3] (this action is explained in Section 4.1), in other words,

$$\mathrm{ad}(E_H) \,=\, \mathrm{ad}(E_G)^T \,\subset\, \mathrm{ad}(E_G)\,. \tag{5.4}$$

For every $x \in D$ fix a $T$–rigid element

$$w_x \,\in\, \mathrm{ad}(E_G)_x \tag{5.5}$$

(see Section 4.1). Since each $w_x$ is $T$–rigid, from (5.4) we conclude that

$$w_x \,\in\, \mathrm{ad}(E_H)_x \quad \forall\ x \,\in\, D\,. \tag{5.6}$$

So $w_x$ determines a conjugacy class in $\mathfrak{h}$. For any character $\chi$ of $H$, the corresponding homomorphism of Lie algebras $d\chi \,:\, \mathfrak{h} \,\longrightarrow\, \mathbb{C}$ factors through the conjugacy classes in $\mathfrak{h}$, because $\mathbb{C}$ is abelian. Therefore, we have $d\chi(w_x) \,\in\, \mathbb{C}$.

**Theorem 5.1.** *There is a logarithmic connection on $E_G$ singular over $D$, and with $T$–rigid residue $w_x$ at every $x \in D$ (see* (5.5)*), if and only if*

$$\mathrm{degree}(E_H(\chi)) + \sum_{x\in D} d\chi(w_x) \,=\, 0 \tag{5.7}$$

*for every character $\chi$ of $H$, where $E_H(\chi)$ is the holomorphic line bundle on $X$ associated to $E_H$ for $\chi$, and $d\chi$ is the homomorphism of Lie algebras corresponding to $\chi$.*

*Proof.* Assume that there is a logarithmic connection on $E_G$ singular over $D$ such that the residue at each $x \in D$ is $w_x$. Since the group $H$ is reductive, from Lemma 3.4 it follows that $E_H$ admits a logarithmic connection singular over $D$ such that the residue at each $x \in D$ is $w_x$ (see (5.6)). Now from Lemma 3.3 we know that (5.7) holds for every character $\chi$ of $G$.

To prove the converse, assume that (5.7) holds for every character $\chi$ of $H$. We will show that $E_G$ admits a logarithmic connection singular over $D$ such that the residue at each $x \in D$ is $w_x$.

Since $E_G$ is the extension of structure group of $E_H$ using the inclusion of $H$ in $G$ (see (5.3)), a logarithmic connection on $E_H$ induces a logarithmic connection on $E_G$ (this is explained in Section 2.4). Therefore, in view of (2.15) and (5.6), the following proposition completes the proof of the theorem.

**Proposition 5.2.** *There is a logarithmic connection on $E_H$ singular over $D$ such that the residue over any $x \in D$ is $w_x \in \text{ad}(E_H)_x$.*

*Proof.* The connected component of the center of $H$ containing the identity element coincides with $T_G$ in (5.1). Define the quotient groups

$$S := H/T_G\,, \quad Z := H/[H,\, H]\,.$$

So $S$ is semisimple, and $Z$ is a torus. The projections of $H$ to $S$ and $Z$ will be denoted by $p_S$ and $p_Z$ respectively. Let $E_S$ (respectively, $E_Z$) be the principal $S$–bundle (respectively, $Z$–bundle) on $X$ obtained by extending the structure group of $E_H$ using $p_S$ (respectively, $p_Z$). Consider the homomorphism

$$\varphi : H \longrightarrow S \times Z\,, \quad h \longmapsto (p_S(h),\, p_Z(h))\,. \tag{5.8}$$

It is surjective with finite kernel, hence it induces an isomorphism of Lie algebras. Let $E_{S\times Z}$ be the principal $S \times Z$–bundle on $X$ obtained by extending the structure group of $E_H$ using $\varphi$. Note that $E_{S\times Z} \cong E_S \times_X E_Z$. Since $\varphi$ induces an isomorphism of Lie algebras, we have

$$\text{ad}(E_H) = \text{ad}(E_{S\times Z}) = \text{ad}(E_S) \oplus \text{ad}(E_Z) \tag{5.9}$$

and

$$\text{At}(E_H) = \text{At}(E_{S\times Z})\,, \quad \mathcal{A} = \mathcal{A}_{E_{S\times Z}}\,,$$

where $\mathcal{A}_{E_{S\times Z}}$ is constructed as in (3.1) for $(E_{S\times Z},\, \{w_x\}_{x\in D})$ (see (5.9)). Consequently, $E_H$ admits a logarithmic connection singular over $D$, with residue $w_x$ for all $x \in D$, if and only if $E_{S\times Z}$ admits a logarithmic connection singular over $D$ with residue $w_x$ for all $x \in D$.

For $x \in D$, let

$$w_x = w^s_x \oplus w^z_x\,, \quad w^s_x \in \text{ad}(E_S)_x\,, \quad w^z_x \in \text{ad}(E_Z)_x$$

be the decomposition given by (5.9). Consider the short exact sequences

$$0 \longrightarrow \text{ad}(E_S) \otimes \mathcal{O}_X(-D) \longrightarrow \mathcal{A}_{E_S} \stackrel{\sigma_{1,S}}{\longrightarrow} \text{T}X(-D) \longrightarrow 0 \tag{5.10}$$

and

$$0 \longrightarrow \text{ad}(E_Z) \otimes \mathcal{O}_X(-D) \longrightarrow \mathcal{A}_{E_Z} \stackrel{\sigma_{1,Z}}{\longrightarrow} \text{T}X(-D) \longrightarrow 0\,, \tag{5.11}$$

as in Lemma 3.1 for the data $(E_S,\, \{w^s_x\}_{x\in D})$ and $(E_Z,\, \{w^z_x\}_{x\in D})$ respectively. Let

$$q_S : \mathcal{A}_{E_S} \oplus \mathcal{A}_{E_Z} \longrightarrow \mathcal{A}_{E_S}\,, \quad q_Z : \mathcal{A}_{E_S} \oplus \mathcal{A}_{E_Z} \longrightarrow \mathcal{A}_{E_Z}$$

be the projections. Note that

$$\mathcal{A}_{E_S} \oplus \mathcal{A}_{E_Z} \supset \text{kernel}(\sigma_{1,S} \circ q_S - \sigma_{1,Z} \circ q_Z) = \mathcal{A}_{E_S} \times_{TX(-D)} \mathcal{A}_{E_Z} = \mathcal{A}_{E_{S\times Z}}\,.$$

Therefore, giving a holomorphic splitting of the exact sequence

$$0 \longrightarrow \text{ad}(E_{S\times Z}) \otimes \mathcal{O}_X(-D) \longrightarrow \mathcal{A}_{E_{S\times Z}} \longrightarrow TX(-D) \longrightarrow 0$$

(see Lemma 3.1 for it) is equivalent to giving holomorphic splittings of both (5.10) and (5.11). Consequently, $E_{S\times Z}$ admits a logarithmic connection singular over $D$ with residue $w_x$ over every $x \in D$ if and only if both $E_S$ and $E_Z$ admit logarithmic connections singular over $D$ such that their residues over any $x \in D$ are $w^s_x$ and $w^z_x$ respectively.

Consider the homomorphism of character groups $\text{Hom}(Z, \mathbb{G}_m) \longrightarrow \text{Hom}(H, \mathbb{G}_m)$ given by the projection $p_Z$. It is an isomorphism because being semisimple $[H, H]$ does not admit any nontrivial character. Since $E_Z = E_H \times^{p_Z} Z$, for any character $\chi \in \text{Hom}(Z, \mathbb{G}_m)$, the holomorphic line bundle $E_Z(\chi) = E_Z \times^{\chi} \mathbb{C}$ is identified with the holomorphic line bundle $E_H(\chi \circ p_Z)$.

A holomorphic line bundle $L$ on $X$ admits a logarithmic connection singular over $D$ with residue $\lambda_x \in \mathbb{C}$ for every $x \in D$ if and only if

$$\text{degree}(L) + \sum_{x\in D} \lambda_x = 0$$

(see [BDP, Lemma 2.3]). Since $Z = H/[H, H] = (\mathbb{G}_m)^d$ for some $d$, it follows that $E_Z$ admits a logarithmic connection singular over $D$ with residue $w^z_x$ at each $x \in D$ if and only if for each $1 \leq i \leq d$, the line bundle $E_Z(\pi_i)$ admits a logarithmic connection singular over $D$ with residue $\text{d}\pi_i(w^z_x)$ at each $x \in D$, where $\pi_i : Z = (\mathbb{G}_m)^d \longrightarrow \mathbb{G}_m$ is the projection to the $i$-th factor. From this and the given condition in (5.7) we conclude that $E_Z$ admits a logarithmic connection singular over $D$ with residue $w^z_x$ for all $x \in D$.

To complete the proof of the proposition we need to show that $E_S$ admits logarithmic connection singular over $D$ such that the residues over each $x \in D$ is $w^s_x$. We will show that (5.10) splits holomorphically.

Let

$$\beta \in H^1(X, \text{Hom}(\text{T}X(-D), \text{ad}(E_S) \otimes \mathcal{O}_X(-D))) = H^1(X, \text{ad}(E_S) \otimes K_X) \tag{5.12}$$

be the extension class for (5.10) as in (3.3). The exact sequence in (5.10) splits holomorphically if and only if

$$\beta = 0\,. \tag{5.13}$$

The Lie algebra of $S$ will be denoted by $\mathfrak{s}$. Consider $\mathfrak{s}$ as a $S$–module using the adjoint action of $S$ on $\mathfrak{s}$. Since $S$ is semisimple, the Killing form

$$\kappa : \mathfrak{s} \times \mathfrak{s} \longrightarrow \mathbb{C}\,, \quad (v, w) \longmapsto \text{trace}(\text{ad}_v \circ \text{ad}_w)\,,$$

is nondegenerate, where $\text{ad}_u(u') := [u, u']$. Therefore, the Killing form induces an isomorphism $\mathfrak{s} \xrightarrow{\sim} \mathfrak{s}^*$ of $S$–modules. This isomorphism produces a holomorphic isomorphism of $\text{ad}(E_S)$ with the dual vector bundle $\text{ad}(E_S)^*$. Now Serre duality gives

$$H^1(X, \text{ad}(E_S) \otimes K_X) = H^0(X, \text{ad}(E_S)^*)^* = H^0(X, \text{ad}(E_S))^*\,.$$

Let

$$\beta' \in H^0(X, \text{ad}(E_S))^*$$

be the element corresponding to $\beta$ (defined in (5.12)) by the above isomorphism. Then

$$\beta'(\gamma) = \int_X \kappa(\widehat{\beta}, \gamma)\,, \quad \forall\, \gamma \in H^0(X, \text{ad}(E_S))\,, \tag{5.14}$$

where $\widehat{\beta}$ is an $\text{ad}(E_S)$–valued (1, 1)–form on $X$ which represents the cohomology class $\beta$ using the Dolbeault isomorphism.

As before, $\text{Aut}(E_H)^0 \subset \text{Aut}(E_H)$ (respectively, $\text{Aut}(E_G)^0 \subset \text{Aut}(E_G)$) is the connected component containing the identity element, and $T \subset \text{Aut}(E_G)^0$ is the fixed maximal torus. Since $T$ is abelian, from (5.4) it follows immediately that

$$T \subset \text{Aut}(E_H)^0 \subset \text{Aut}(E_G)^0 .$$

Therefore, the maximal torus $T \subset \text{Aut}(E_G)^0$ (see (5.1)) is also a maximal torus of $\text{Aut}(E_H)^0$. Since $T_G$ is the connected component, containing the identity element, of the center of $H$, and $T$ is the image of $T_G$ in $\text{Aut}(E_H)^0$, it now follows that the maximal torus of $\text{Aut}(E_S)^0$ is trivial. Hence every holomorphic section of $\text{ad}(E_S)$ is nilpotent.

Take any nonzero element $\gamma \in H^0(X,\, \text{ad}(E_S))$. Following the proof of [AB, Proposition 3.9], using $\gamma$ we construct a holomorphic reduction of the structure group of $E_S$ to a parabolic subgroup of $S$ as follows. For each $x \in X$, since $\gamma(x) \in \text{ad}(E_S)_x$ is nilpotent, there is a parabolic Lie subalgebra $\mathfrak{p}_x \subset \text{ad}(E_S)_x$ canonically associated to $\gamma(x)$ [AB, p. 340, Lemma 3.7]. Exponentiating $\mathfrak{p}_x$ we get a proper parabolic subgroup $P_x \subset \text{Ad}(E_S)_x$ associated to $\gamma(x)$. Since there are only finitely many conjugacy classes of nilpotent elements of $\mathfrak{s}$, and the algebraic subvariety of $\mathfrak{s}$ defined by the nilpotent elements has a natural filtration defined using the adjoint action of $S$, there is a finite subset $C \subset X$ such that the conjugacy classes of $P_x$, $x \in X \setminus C$, coincide. Fix a parabolic subgroup $P \subset S$ in this conjugacy class.

For any $x \in X \setminus C$, consider the projection map

$$\xi_x \,:\, (E_S)_x \times S \longrightarrow \text{Ad}(E_S)_x\,, \quad (z,\, s) \longmapsto \widetilde{(z,\, s)}\,,$$

where $\widetilde{(z,\, s)}$ is the equivalence class of $(z,\, s)$. Define

$$(E_P)_x := \{z \in (E_S)_x \mid \xi_x(z,\, g) \in P_x\,,\ \forall\, g \in P\}\,.$$

For the natural action of $S$ on $(E_S)_x$, the action of $P \subset S$ preserves $(E_P)_x$. Since $P$ is a parabolic subgroup of $S$, its normalizer $N_S(P)$ is $P$ itself [Hu, p. 143, Corollary B]. So the action of $P$ on $(E_P)_x$ is transitive (and also free, since the $G$–action on $(E_S)_x$ is free). Therefore, we have a holomorphic reduction of structure group $E_P \subset E_S$ to $P \subset S$ over $X \setminus C$. This holomorphic reduction defines a holomorphic section $\eta \,:\, X \setminus C \longrightarrow E_S/P$ which is meromorphic over $X$. Since $S/P$ is a projective variety, the above section $\eta$ extends holomorphically to a section $\widetilde{\eta} \,:\, X \longrightarrow E_S/P$. This defines a holomorphic reduction of structure group $E_P \subset E_S$ to $P$.

From Corollary 3.2 it follows that the cohomology class $\beta$ in (5.12) lies in the image of the natural homomorphism $H^1(X,\, \text{ad}(E_P)\otimes K_X) \longrightarrow H^1(X,\, \text{ad}(E_S)\otimes K_X)$. Therefore, $\beta$ is represented by an $\text{ad}(E_P)$–valued (1, 1)–form $\widehat{\beta}$ on $X$.

Now for all $x \in X \setminus C$, the element $\gamma(x) \in \text{ad}(E_S)_x$ lies in the Lie algebra $\mathfrak{u}_x$ of the unipotent radical of $\text{ad}(E_P)_x$ [Hu, p. 186, Corollary A], and $\widehat{\beta}(x) \in \mathfrak{p}_x$. Therefore, $\kappa(\widehat{\beta}(x),\, \gamma(x)) = 0$, since $\mathfrak{u}_x$ is the orthogonal complement $(\text{ad}(E_P)_x)^{\perp}$ with respect to the Killing form on $\text{ad}(E_S)_x$. Hence from (5.14) we have $\beta'(\gamma) = 0$. This proves (5.13), and completes the proof of the proposition. □

As noted before, Proposition 5.2 completes the proof of Theorem 5.1. □

5.2. **$T$–invariant logarithmic connections with given residue.** The automorphism group $\text{Aut}(E_G)$ has a natural action on the space of all logarithmic connections on $E_G$ singular over $D$. Given a maximal torus $T \subset \text{Aut}(E_G)^0$, by a $T$–invariant logarithmic connection we mean a logarithmic connection on $E_G$ singular over $D$ which is fixed by the action of $T$.

**Theorem 5.3.** *Let $E_G$ be a holomorphic principal $G$–bundle on $X$, where $G$ is reductive. Fix $w_x \in \text{ad}(E_G)_x$ for each $x \in D$. Fix a maximal torus $T \subset \text{Aut}(E_G)^0$. The following two are equivalent:*

1. *There is a $T$–invariant logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$.*
2. *The element $w_x$ is $T$–rigid for each $x \in D$, and* (5.7) *holds for every character $\chi$ of $H$.*

*Proof.* Let $\theta$ be a $T$–invariant logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$. Since $\theta$ is $T$–invariant, its residues are also $T$–invariant. Hence $w_x$ is $T$–rigid for each $x \in D$. From Theorem 5.1 we know that (5.7) holds for every character $\chi$ of $H$.

Now assume that the second statement in the theorem holds. From Theorem 5.1 we know that there is a logarithmic connection on $E_G$ singular over $D$ with residue $w_x$ at every $x \in D$.

As noted in Section 4.2, for a holomorphic principal $M$–bundle $E_M$ on $X$, the center $Z_M$ of $M$ is contained in the automorphism group $\text{Aut}(E_M)$. It is straight-forward to check that the action of $Z_M \subset \text{Aut}(E_M)$ on the space of all logarithmic connections on $E_M$ is trivial.

Since $T_G$ is contained in the center of $H$ (see (5.2)), and $T$ is the image of $T_G$ in $\text{Aut}(E_H)$, every logarithmic connection on the principal $H$–bundle $E_H$ in Proposition 5.2 is $T$–invariant. Consequently, from Proposition 5.2 it follows that $E_G$ admits a $T$–invariant logarithmic connection singular over $D$ with residue $w_x$ at every $x \in D$. □

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