

Local nonobtuse tetrahedral refinements around an edge

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September 21, 2010

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Abstract: In this note we show how to generate and conformly refine nonobtuse tetrahedral meshes locally around and towards an edge so that all dihedral angles of all resulting tetrahedra remain nonobtuse. The proposed technique can be used e.g. for a numerical treatment of solution singularities, and also for various mesh adaptivity procedures, near the reentrant corners of cylindrical-type 3D domains.

Keywords: finite element method, nonobtuse tetrahedron, local refinement, discrete maximum principle, edge singularity, reentrant corner, mesh adaptivity

Mathematical Subject Classification: 65N30, 65N50

1 Introduction

A tetrahedron is said to be *nonobtuse*, if all its six dihedral angles between faces are nonobtuse. Nonobtuse simplicial (triangular, tetrahedral, etc) finite elements play an important role in the finite element analysis of boundary value problems, since they yield irreducible and diagonally dominant stiffness matrices for a sufficiently small discretization parameter and guarantee the validity of the discrete maximum principle when solving the Poisson and some other elliptic equations with various boundary conditions (see [3]). Note that just one obtuse simplex in a triangulation can completely destroy the discrete maximum principle [2]. In [4], we gave a global refinement algorithm which produces nonobtuse tetrahedra. However, various local refinements of simplicial meshes are often necessary to handle e.g. boundary or interior layers, large oscillations and singularities of the solution or its derivatives at interior interfaces, where one kind of media changes into another, or near some special edges, or points, see [8], and also for mesh adaptivity procedures. An algorithm for treating vertex (or point) singularities by nonobtuse tetrahedra

34 was first presented in [5], and later generalized in [1] to any space dimension. Edge and
 35 face singularities can also be treated by that algorithm if we select sufficiently many addi-
 36 tional nodes along edges or faces, respectively. In this note we present another algorithm
 37 for a face-to-face tetrahedral refinement around and towards an edge that produces only
 38 nonobtuse tetrahedra. The algorithm may have practical applications in solving partial
 39 differential equations by the finite element or finite volume methods in 3D.

40 In Figure 1 we observe several kinds of nonobtuse tetrahedra, namely, the path-
 41 tetrahedron, the cube corner tetrahedron, and the regular tetrahedron. Notice that legs
 42 of right-triangular faces in the cases a) and b) are not necessarily of the same length. In
 43 Figure 1a) and 1b) these legs are mutually orthogonal. In Figure 1a) they form a path,
 44 whereas in Figure 1b) they pass through one vertex.

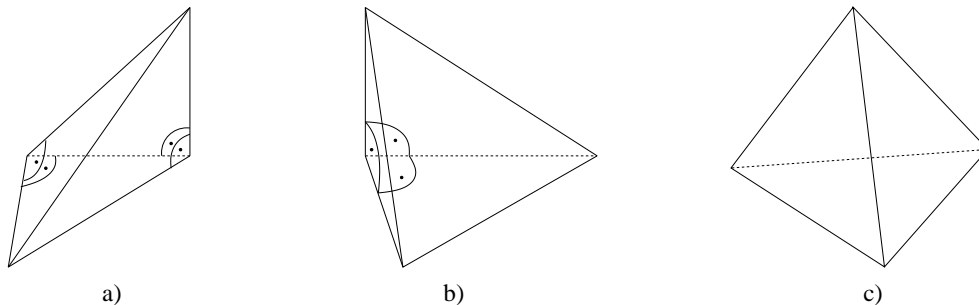


Figure 1: Examples of nonobtuse tetrahedra – a) path, b) cube corner, and c) regular.

45 Note that two well-known partitions of a cube into 5 and 6 tetrahedra (see Figure 2)
 46 are formed by nonobtuse tetrahedra from Figure 1.

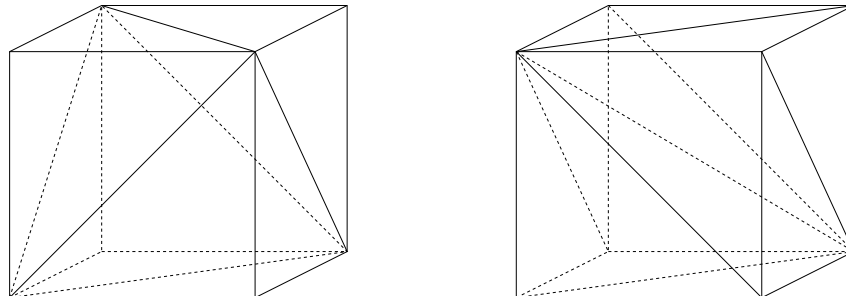


Figure 2: Two partitions of a cube into 5 and 6 nonobtuse tetrahedra. The left partition consists of four cube corners and one regular tetrahedron, and the right partition consists of six path tetrahedra.

47 2 The mesh refinement algorithm

48 In this section we first recall (see [6]) the key idea and also illustrations (see Figures 3 and
 49 4) of the nonobtuse tetrahedral refinements towards a flat face of the 3D solution domain
 50 (or towards some interface inside of it). For this purpose we take two adjacent square
 51 prisms and denote their nodes as sketched in Figure 3, where also partitions of some faces
 52 are marked.

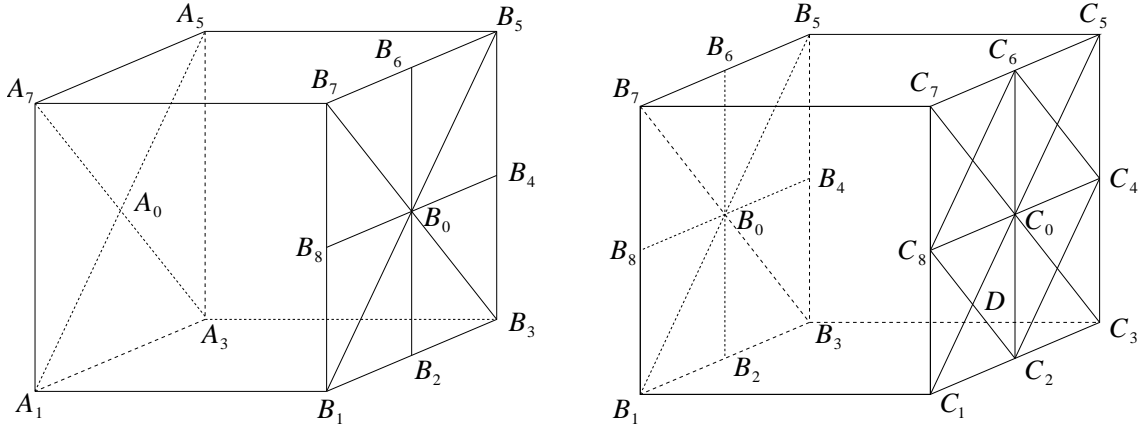


Figure 3: A sketch of a partition of two adjacent square prisms into nonobtuse tetrahedra from Step A.

53 **Step A:** Let $d = |B_1B_3| = |B_3B_5|$ denote the length of sides of the square faces of the
 54 considered two prisms, and let $l_1 = |A_0B_0|$ and $l_2 = |B_0C_0|$ be their thicknesses in the
 55 horizontal direction.

56 First we partition the left square prism $A_1A_3A_5A_7B_1B_3B_5B_7$ of Figure 3 into four
 57 triangular prisms whose common edge is A_0B_0 . Second we partition each triangular
 58 prism into four tetrahedra. For instance, the triangular prism $A_0A_1A_3B_0B_1B_3$ will be
 59 divided in the following way (see Figure 4):

60 $A_0A_1A_3B_0$ (cube corner tetrahedron), $A_1B_1B_2B_0$ (path tetrahedron),
 61 $A_3B_3B_2B_0$ (path tetrahedron), and $A_1A_3B_0B_2$.

62 The first three resulting tetrahedra are clearly nonobtuse. Further, we see that
 63 $A_1A_3B_0B_2$ is nonobtuse if and only if

$$|B_1B_3| \leq 2|A_0B_0|, \quad \text{i.e.} \quad l_1 \geq \frac{d}{2}. \quad (1)$$

64 The other three triangular prisms $A_0A_3A_5B_0B_3B_5$, $A_0A_5A_7B_0B_5B_7$, and $A_0A_1A_7B_0B_1B_7$
 65 can be subdivided similarly.

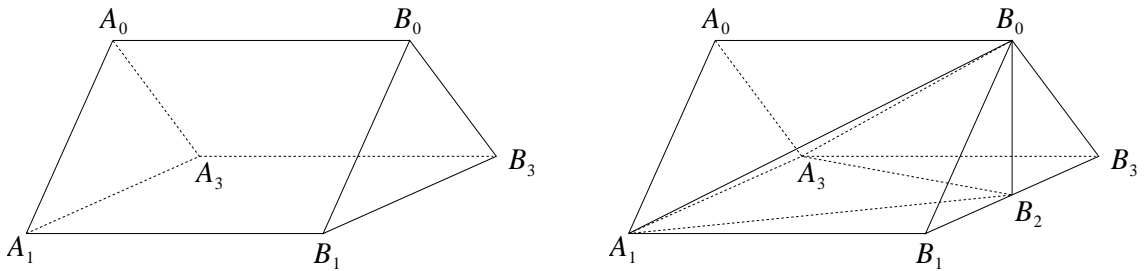


Figure 4: Partition of a triangular prism $A_0A_1A_3B_0B_1B_3$ into four tetrahedra.

66 Next, we partition the right adjacent square prisms $B_1B_3B_5B_7C_1C_3C_5C_7$ of Figure 3
 67 into eight triangular prisms whose common edge is B_0C_0 . To this end we denote by D
 68 the midpoint of C_0C_1 . Further, e.g., the triangular prism $B_0B_1B_2C_0C_1C_2$ will be divided
 69 into four tetrahedra like in the previous step:

70 $B_0B_1B_2C_2$ (cube corner tetrahedron), $B_0C_0DC_2$ (path tetrahedron),
 71 $B_1C_1DC_2$ (path tetrahedron), and $B_0B_1DC_2$.

72 The last tetrahedron is nonobtuse provided

$$|B_0B_1| \leq 2|B_0C_0|, \quad \text{i.e.} \quad l_2 \geq \frac{\sqrt{2}d}{4}. \quad (2)$$

73 This condition is necessary and sufficient to guarantee a nonobtuse partition of the trian-
 74 gular prism $B_0B_1B_2C_0C_1C_2$ into four nonobtuse tetrahedra as described above.

75 The other seven triangular prisms can be divided into nonobtuse tetrahedra similarly.
 76 In this way (i.e., under conditions (1) and (2)) we get a face-to-face nonobtuse partition of
 77 two adjacent square prisms. The left square prism of Figure 3 is thus conformly subdivided
 78 into 16 and the right prism into 32 nonobtuse tetrahedra.

79 **Step B:** Now, in the construction of Step A, we take both prisms be of the thickness $\frac{d}{2}$,
 80 i.e. $l_1 = l_2 = \frac{d}{2}$. Therefore, two square prisms in Figure 3 form a cube with edges of the
 81 length d .

82 In Figure 5, we observe several principal refinement steps towards the chosen vertical
 83 edge in the upper right corner (view from the top). The advancing (according to arrows)
 84 blocks with the shown refinement of their upper faces are always treated as in Step A
 85 (with their “own” $l_1 = l_2 = \frac{d}{2}$). The problematic zone (marked by the question sign) and
 86 its refinement which provide the conformity with the “surroundings” will be discussed
 87 further. It is enough to consider a few first steps only (as in Figure 5), since the situation
 88 repeats up to scaling. In what follows, we always consider only the upper layer (of the
 89 width d) from the initial mesh, since the (possible) other layers (under it, each of the
 90 width d , too) can be treated similarly using symmetry argument.

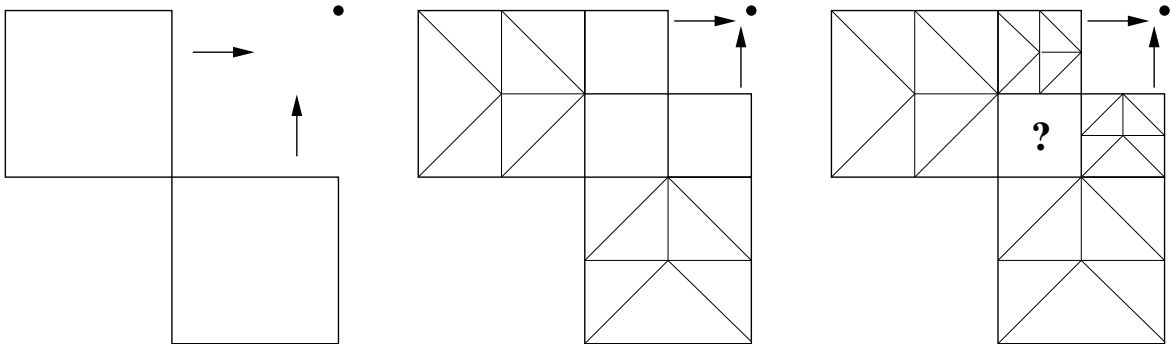


Figure 5: Several refinements towards the chosen vertical edge (marked by the bold dot, view from the top). The problematic zone is marked by the question mark.

91 The zone marked by the question mark in Figure 5 is, in fact, made of two cubes (one
 92 above the other) of the size $\frac{d}{2} \times \frac{d}{2} \times \frac{d}{2}$ each. It is enough to show how to partition, conformly
 93 with the surroundings, the upper cube, since the lower cube can be partitioned using the
 94 mirror reflection via their common face. The method for this purpose is illustrated in
 95 Figures 6 and 7. First, in Figure 6 (left) we sketch those faces whose refinement stencils
 96 are dictated by previous constructions, and further, as illustrated in Figure 6 (right) we
 97 partition the cube into nonobtuse tetrahedra taking convex hulls of the center of the cube
 98 and the marked right triangles on the faces, besides the upper and lower right subcubes,

99 which we split in a special way (into 5 nonobtuse tetrahedra each) as demonstrated in
 100 Figure 7.

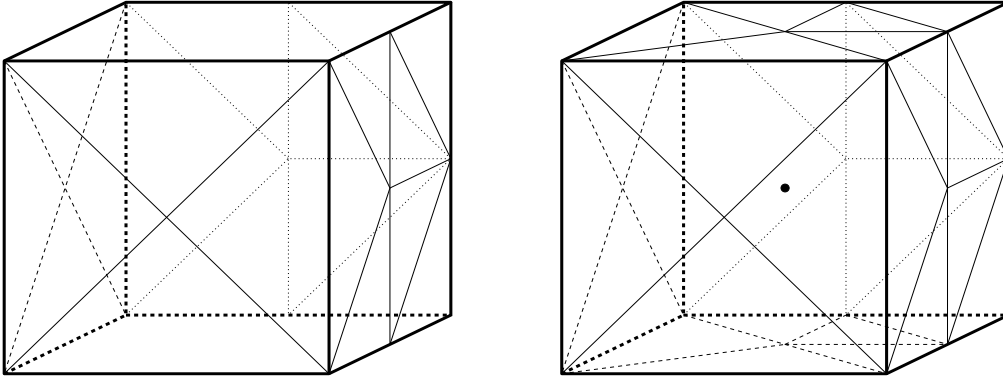


Figure 6: Refinement of one of the two cubes in the problematic zone, only forced “face refinement” lines (due to the conformity requirements) are sketched on the left. Full “face refinement” of the left cube (the bold dot is the center of the cube) is on the right.

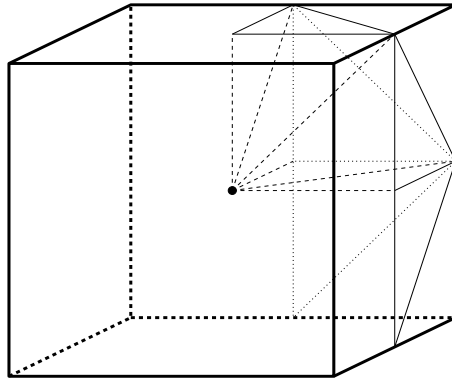


Figure 7: Refinement of one of the two sub-cubes into 5 nonobtuse tetrahedra.

101 We notice that we can do infinitely many steps in the total above construction with the
 102 choice $l_1 = l_2 = \frac{d}{2}$ and the overall conformity of any resulting meshes, obviously moving
 103 towards the chosen edge, is guaranteed.

104 3 Final remarks

105 **Remark 1** In real-life calculations we perform only a finite number of refinements. There-
 106 fore, we have to divide the “remaining” zone around the edge, whose position is illustrated
 107 by the black dot in Figure 5. Each subcube in that zone has to be divided so that it fits
 108 to the triangulation of the right face of the right cube from Figure 3. We could apply for
 109 this purpose e.g the division into 24 cube corner tetrahedra as sketched in Figure 8.

110 **Remark 2** Actually, the construction given in Figure 5 can be easily adapted to gener-
 111 ating nonobtuse local refinements e.g. inside $2d \times 2d \times 2d$ cube to one of its edges if we
 112 take precisely two layers in Figure 5 (left) and use the technique proposed in Figures 6

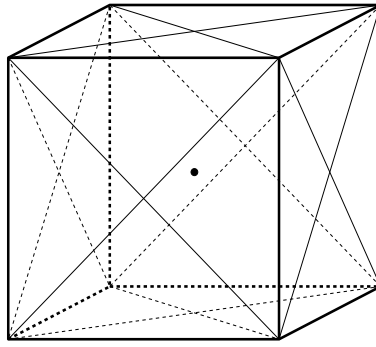


Figure 8: Partition of a cube into 24 cube corner tetrahedra. They are defined as the convex hull of the centre of the cube and a particular triangle on the surface.

113 and 7 to conformly fill in the (not considered so far) area in the “left lower corner” of
 114 Figure 5.

115 **Remark 3** Notice that nonobtuse tetrahedral meshes (whose elements have nonobtuse
 116 triangular faces [2]) satisfy the maximum angle condition [7], which is one of popular
 117 sufficient conditions for convergence proofs in the finite element analysis.

118 **Acknowledgement:** The first author was supported by Grant MTM2008-03541 of the
 119 MICINN, Spain, the ERC Advanced Grant FP7-246775 NUMERIWAVES, and Grant
 120 PI2010-04 of the Basque Government. The second author was supported by Institutional
 121 Research Plan nr. AV0Z 10190503 of the Academy of Sciences of the Czech Republic and
 122 Grant nr. IAA 100190803 of the Academy of Sciences of the Czech Republic.

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