Generalization of the Zlámal condition for simplicial finite elements in $\mathbf{R}^d$

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Abstract: The famous Zlámal's minimum angle condition is widely used for construction of a regular family of triangulations (containing nondegenerating triangles) as well as in convergence proofs for the finite element method in $2d$. In this paper we shall present and discuss its generalization to simplicial partitions in any space dimension.

Keywords: linear finite element, mesh regularity, minimum angle condition, convergence, higher-dimensional problems

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1 Introduction

The finite element method (FEM) is nowadays one of the most powerful and popular numerical techniques widely used in various software packages that solve problems in, for instance, mathematical physics and mechanics. The initial step in FEM implementations is to establish an appropriate partition (also called mesh, grid, triangulation, etc.) of the solution domain. For a number of applications and complicated geometries, simplicial partitions are prefered over the others due to their flexibility. However, from both

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theoretical and practical points of view, such partitions and their refinements cannot be constructed arbitrarily. Thus, first of all we must ensure, at least theoretically, that the finite element approximations converge to the exact (weak) solution of the mathematical model under consideration when the associated partitions become finer [5]. Mainly due to this reason the notions of regular families of partitions or shape-preserving partitions appeared. Second, the regularity is also important for real-life computations, because partitions that contain almost flat (i.e. degenerating) elements may yield ill-conditioned stiffness matrices [10].

In 1968, Miloš Zlámal [12] introduced the so-called *minimum angle condition* that ensures the convergence of the finite element approximations for solving linear elliptic boundary value problems of the second and fourth order on triangular meshes. This condition requires that there exists a constant $\alpha_0 > 0$ such that the minimal angle α_S of each triangle S in all triangulations used satisfies

$$\alpha_S \geq \alpha_0.$$

In fact, Zlámal used the equivalent condition $\sin \alpha_S \geq \sin \alpha_0$. The same condition was also introduced by Alexander Ženíšek [11] for the finite element method applied to a system of linear elasticity equations of second order published in 1969. However, this paper was submitted already on April 3, 1968, whereas Zlámal's paper on April 17, 1968.

Later, the so-called inscribed ball condition was introduced, see, e.g. [5, p. 124], which uses a ball contained in a given element (cf. (2)). Thus, it can also be used for nonsimplicial elements in any dimension. This condition has an elegant geometrical interpretation: the ratio of the radius of the inscribed ball of any element and the diameter of this element must be bounded from below by a positive constant over all partitions. Roughly speaking, no element of no partitions should degenerate to a hyperplane as the *discretization parameter* h (which is the maximal diameter of all elements in the corresponding partition) tends to zero. This property is called in [5] the *regularity of a family of partitions*. For triangular elements it is, obviously, equivalent to Zlámal's condition. This condition has a large number of applications, see e.g. [5, 7, 11].

In [8] the inscribed ball condition (cf. (2)) was replaced by a simpler equivalent condition on the volume of every element (cf. (1)). Another equivalent circumscribed ball condition for simplices (cf. (3)) was first introduced in [4].

In the present paper Zlámal's condition will be generalized into $\mathbf{R}^d$ for any $d \in \{2, 3, \ldots\}$ so that the d-dimensional sinus of all angles at vertices of all simplices (as defined in [6]) is bounded from below by a positive constant. This will be proved to be equivalent with the inscribed ball condition.

2 On mesh regularity conditions

A *simplex* S in $\mathbf{R}^d$, $d \in \{1, 2, 3, \ldots\}$, is the convex hull of $d+1$ vertices $A_1, A_2, \ldots, A_{d+1}$ that do not belong to the same $(d-1)$-dimensional hyperplane, i.e., $S = \operatorname{conv}\{A_1, A_2, \ldots, A_{d+1}\}$. We denote by h_S the length of the longest edge of S.

Let

$$F_i = \operatorname{conv}\{A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{d+1}\}$$

be the facet of S opposite to vertex A_i for $i \in \{1, \ldots, d+1\}$. The dihedral angle α between two such facets is defined by means of the inner product of their outward unit normals

n_1 and n_2,

$$\cos\alpha = -n_1 \cdot n_2,$$

see also [6, p. 74].

Let $\Omega \subset \mathbf{R}^d$ be a bounded domain. Assume that $\overline{\Omega}$ is *polytopic*. By this we mean that $\overline{\Omega}$ is the closure of a domain whose boundary $\partial\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes.

Next we define a simplicial partition of a bounded closed polytopic domain $\overline{\Omega} \subset \mathbf{R}^d$ as follows. We subdivide $\overline{\Omega}$ into a finite number of simplices (called *elements* and denoted by S), so that their union is $\overline{\Omega}$, any two distinct simplices have disjoint interiors, and any facet of any simplex is either a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$. The set of such simplices will be called *simplicial partition* and denoted by $\mathcal{T}_h$, where $h = \max\limits_{S\in\mathcal{T}_h} h_S$.

The sequence of partitions $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of $\overline{\Omega}$ is called a *family of partitions* if for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

In what follows, all constants C_i are independent of S and h, but can depend on the dimension d. By vol_p we denote the p-dimensional volume ($p \le d$).

The equivalence of the following three regularity conditions for simplicial partitions in $\mathbf{R}^d$ was proved in [3] for any $d \ge 2$. They guarantee the optimal order of the interpolation error of simplicial finite elements, which is employed in various convergence proofs of the finite element method [5].

Condition 1: There exists $C_1 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\mathrm{vol}_d\, S \ge C_1 h_S^d\,. \tag{1}$$

Condition 2: There exists $C_2 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\mathrm{vol}_d\, b \ge C_2 h_S^d\,, \tag{2}$$

where $b \subset S$ is the inscribed ball of S.

Condition 3: There exists $C_3 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\mathrm{vol}_d\, S \ge C_3\, \mathrm{vol}_d\, B, \tag{3}$$

where $B \supset S$ is the circumscribed ball about S.

If one of the above conditions (1)–(3) is satisfied, the family $\mathcal{F}$ of simplicial partitions is called *regular*. In addition, in [2] the following *minimum angle condition* was introduced and its equivalence with the above three, (1), (2), and (3), was proved for $d = 2, 3$:

Condition 4: There exists a constant $C_4 > 0$ such that for any partition $\mathcal{T}_h \in \mathcal{F}$, any simplex $S \in \mathcal{T}_h$, and any dihedral angle α and for $d = 3$ also any angle α within any triangular face of S, we have

$$\alpha \ge C_4. \tag{4}$$

However, the problem how to formulate **Condition 4** in dimensions higher than 3 so that it is equivalent to **Conditions 1**, **2**, and **3** (or just to one of them) remained open. In order to fill this gap, in the next section we shall present a definition of d-dimensional sine used by Folke Eriksson in [6].

3 The minimum angle condition in $\mathbf{R}^d$

A convenient definition for the d-dimensional sine of angles in $\mathbf{R}^d$ was introduced in [6]. In terms of the simplex S, for any of its vertices A_i, the d-dimensional sine of the angle of S at A_i, denoted by $\hat{A}_i$, is defined as follows (see (3) in [6, p. 72]):

$$\sin_d(\hat{A}_i|A_1A_2\dots A_{d+1}) = \frac{d^{d-1}\,(\mathrm{vol}_d S)^{d-1}}{(d-1)!\,\mathbf{\Pi}_{j=1,j\neq i}^{d+1}\mathrm{vol}_{d-1}F_j}. \tag{5}$$

Remark 1 *For $d=2$, $\sin_2(\hat{A}_i|A_1A_2A_3)$ is the standard sine of the angle $\hat{A}_i$ in the triangle $A_1A_2A_3$, due to the following well-known formula, e.g. for $i=1$,*

$$\mathrm{vol}_2(A_1A_2A_3) = \frac{1}{2}|A_1A_2||A_1A_3|\sin\hat{A}_1. \tag{6}$$

In fact, one can similarly define a sine for any k-dimensional (vertex) angle of any k-dimensional facet of S for $k\in\{2,\dots,d\}$. Namely, let us denote the k-dimensional facet of S spanned by the $k+1$ (distinct) vertices $A_{i_1}, A_{i_2},\dots,A_{i_{k+1}}$ by $F_{i_1,i_2,\dots,i_{k+1}}$. Then for any index $i_\ell\in\{i_1,\dots,i_{k+1}\}$ we set

$$\sin_k(\hat{A}_{i_\ell}|A_{i_1}A_{i_2}\dots A_{i_{k+1}}) = \frac{k^{k-1}\,(\mathrm{vol}_k F_{i_1,i_2,\dots,i_{k+1}})^{k-1}}{(k-1)!\,\mathbf{\Pi}_{i_j\in\{\{i_1,\dots,i_{k+1}\}\setminus\{i_\ell\}\}}\mathrm{vol}_{k-1}F^{i_j}_{i_1,i_2,\dots,i_{k+1}}}, \tag{7}$$

where $F^{i_j}_{i_1,i_2,\dots,i_{k+1}}$ denotes $(k-1)$-dimensional facet of $F_{i_1,i_2,\dots,i_{k+1}}$ (which is clearly itself a $(k-1)$-dimensional simplex) lying against the vertex A_{i_j}.

Remark 2 *Notice that in the above denotation we have $S\equiv F_{1,2,\dots,d+1}$ and $F_j\equiv F^j_{1,2,\dots,d+1}$ for $j=1,2,\dots,d+1$.*

The above definition of sine inspires the following:

Generalized Condition 4: There exists $C_4'>0$ such that for any $\mathcal{T}_h\in\mathcal{F}$ and any $S=\mathrm{conv}\{A_1,\dots,A_{d+1}\}\in\mathcal{T}_h$ we have

$$\sin_d(\hat{A}_i|A_1A_2\dots A_{d+1}) \geq C_4' > 0 \qquad \forall\, i\in\{1,2,\dots,d+1\}, \tag{8}$$

where $\sin_d$ is defined in (5).

From Remark 1 we observe that (8) really presents a generalization of Zlámal's condition to higher dimensions, i.e. for $d\geq 3$.

Remark 3 *The angle $\hat{A}_i$ at vertex A_i defined in (5) is, in general, not equal to the solid angle as it could be guessed. For instance, for the regular tetrahedron the solid angle is $3\arccos 1/3-\pi\approx 0.5513$ steradians (see [9, p. 83]), whereas $\sin_3(A_i|A_1A_2A_3A_4)=4\sqrt{3}/9$ and $\arcsin 4\sqrt{3}/9\approx 0.8785$.*

4 Equivalence of Condition 1 and Generalized Condition 4 in any dimension

First, we present some useful results.

Proposition 1 *For any simplex* S $(\equiv F_{1,2,\dots,d+1})$ *and any its* k*-dimensional facet* $F_{i_1,i_2,\dots,i_{k+1}}$ *we have*

$$\mathrm{vol}_k\, F_{i_1,i_2,\dots,i_{k+1}} \leq h_S^k. \tag{9}$$

P r o o f : Relation (9) follows from the fact that the distance between any two points of S is not larger than h_S. Thus, S and its facets are contained in hypercubes (of the corresponding dimensions) with edges of length h_S. □

Proposition 2 *Let condition (1) hold for the family* $\mathcal{F}$*. Then there exists* $C_5 > 0$ *such that for any lower dimensional facet* $F_{i_1,i_2,\dots,i_{k+1}}$ *of any simplex* S *from any* $\mathcal{T}_h \in \mathcal{F}$ *we have the following lower estimate*

$$\mathrm{vol}_k\, F_{i_1,i_2,\dots,i_{k+1}} \geq C_5 h_S^k. \tag{10}$$

P r o o f : First, we demonstrate how to prove (10) for $(d-1)$-dimensional facets F_j. Write v_j for the altitude of the vertex A_j to the face F_j. Then, since $v_j \leq h_S$, we observe that (i.e. for $k = d-1$)

$$C_1 h_S^d \leq \mathrm{vol}_d S = \frac{1}{d} v_j \mathrm{vol}_{d-1} F_j \leq \frac{1}{d} h_S \mathrm{vol}_{d-1} F_j,$$

from which (10) follows immediately. By induction, we can easily prove (10) for the other (lower-dimensional) facets of S in the same manner. □

Theorem 1 *Conditions (1) and (8) are equivalent in* $\mathbf{R}^d$ *for any* $d \geq 2$.

P r o o f : Condition 1 $\Longrightarrow$ **Generalized Condition 4:** Let $i \in \{1, \dots, d+1\}$ be given. Then from (5), (1), and (9) we immediately observe that

$$\sin_d(\hat{A}_i | A_1 A_2 \dots A_{d+1}) = \frac{d^{d-1} \left(\mathrm{vol}_d S\right)^{d-1}}{(d-1)!\, \mathbf{\Pi}_{j=1, j\neq i}^{d+1} \mathrm{vol}_{d-1} F_j} \geq$$

$$\geq \frac{d^{d-1} \left(C_1 h_S^d\right)^{d-1}}{(d-1)!\, h_S^{(d-1)d}} = \frac{d^{d-1}}{(d-1)!} C_1^{d-1} = C_4' > 0.$$

Generalized Condition 4 $\Longrightarrow$ **Condition 1:** By [6, p. 76] and (8), we get for any $i_\ell \in \{i_1, i_2, i_3\}$ that

$$\sin_2(\hat{A}_{i_\ell} | A_{i_1} A_{i_2} A_{i_3}) \geq \sin_3(\hat{A}_{i_\ell} | A_{i_1} A_{i_2} A_{i_3} A_{i_4}) \geq \dots$$

$$\geq \sin_d(\hat{A}_{i_\ell} | A_1 \dots A_{d+1}) \geq C_4', \tag{11}$$

where $i_1, i_2, i_3, i_4, \dots$ are distinct indices from the set $\{1, 2, \dots, d+1\}$. From this and (7) we have for $k = 2, 3, \dots, d$ the following relation

$$(\mathrm{vol}_k F_{i_1,i_2,\dots,i_{k+1}})^{k-1} =$$

$$= \frac{(k-1)!}{k^{k-1}} \left(\mathbf{\Pi}_{i_j \in \{\{i_1,\dots,i_{k+1}\}\setminus\{i_\ell\}\}} \mathrm{vol}_{k-1} F^{i_j}_{i_1,i_2,\dots,i_{k+1}}\right) \sin_k(\hat{A}_{i_\ell} | A_{i_1} A_{i_2} \dots A_{i_{k+1}}) \geq \qquad (12)$$

$$\geq \frac{(k-1)!}{k^{k-1}} C'_4 \, \mathbf{\Pi}_{i_j \in \{\{i_1,\dots,i_{k+1}\}\setminus\{i_\ell\}\}} \mathrm{vol}_{k-1} F^{i_j}_{i_1,i_2,\dots,i_{k+1}}$$

for any $i_\ell \in \{i_1, \dots, i_{k+1}\}$. Now we apply (12) several times in order to estimate $\mathrm{vol}_d S$ from below by a product of some constant which only depends on k and the given constant C'_4 from (8) times lengths of some d edges of S.

Further, in order to finally prove (1), it remains to show that there exists a constant $C_6 > 0$ such that for any simplex S from any $\mathcal{T}_h \in \mathcal{F}$ and any edge $A_m A_n$ of S we have $|A_m A_n| \geq C_6 h_S$. In order to show that let us take any edge in S. Without loss of generality, let it be $A_1 A_2$ and assume that $h_S = |A_i A_j|$ for some i and j. We consider first the triangle $A_1 A_2 A_3$, where by the law of sines and (11) one has

$$|A_1 A_2| \geq |A_1 A_2| \sin \hat{A}_1 = |A_2 A_3| \sin \hat{A}_3 \geq C'_4 |A_2 A_3|.$$

Further, we consider the triangle $A_2 A_3 A_4$, where we show that $|A_2 A_3| \geq C'_4 |A_3 A_4|$, etc. Finally, using all the inequalities obtained, we get that $|A_1 A_2| \geq C_6 |A_i A_j| = C_6 h_S$, where C_6 depends only on the dimension d and the given constant C'_4. □

Corollary 1 *Conditions 1, 2, 3, and Generalized Condition 4 are equivalent for any* $d \geq 2$.

Remark 4 *Condition* (4) *is clearly equivalent to Generalized Condition 4 when* $d \in \{2, 3\}$*, since both are equivalent, e.g., to the inscribed ball condition* (2)*. In* [6, Sect. 5]*, one can find useful relations between, e.g.,* 3*-dimensional sine and dihedral angles and angles between edges of a tetrahedron.*

5 Final remarks

As an eye visualization is almost impossible in dimensions higher than three, the concept of "measuring" angles due to F. Eriksson is found to be very convenient (uniform and compact) for any dimension.

Moreover, the original estimates of Zlámal [12, p. 397] from which his condition is derived are, in fact, using the sine of two-dimensional angles themselves.

Zlámal's minimum angle can be generalized to the maximum angle condition (mostly for $d = 2, 3$) which represents a weaker sufficient condition guaranteeing the convergence of the finite element method. The maximum angle condition enables us to treat anisotropic partitions (see e.g. [1]) containing flat elements. They can yield ill-conditioned stiffness matrices, but the optimal interpolation order may be preserved.

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