

Spherically symmetric solutions to a model for phase transitions driven by configurational forces

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Abstract

We prove the global in time existence of spherically symmetric solutions to an initial-boundary value problem for a system of partial differential equations, which consists of the equations of linear elasticity and a nonlinear, non-uniformly parabolic equation of second order. The problem models the behavior in time of materials in which martensitic phase transitions, driven by configurational forces, take place, and can be considered to be a regularization of the corresponding sharp interface model. By assuming that the solutions are spherically symmetric, we reduce the original multidimensional problem to the one in one space dimension, then prove the existence of spherically symmetric solutions. Our proof is valid due to the essential feature that the reduced problem is one space dimensional.

1 Introduction

Many inhomogeneous systems can be characterized by domains of different phases separated by a distinct interface. When driven out of equilibrium, their dynamics result in the evolution of those interfaces, and the systems might develop into structures (compositional and structural inhomogeneities) with characteristic length scales at the nano-, micro- or meso-scale. To a large extent, the material properties of such systems are determined by those structures of small-scale. Thus it is important to understand precisely the mechanisms that drive the evolution of those structures. Materials microstructures may consist of spatially distributed phases of different compositions and/or crystal structures, grains of different orientations, domains of different structural variants, domains of different electrical or magnetic polarizations, and structural defects. These structural features usually have an intermediate mesoscopic length scale in the range of nanometers to microns. The size, shape, and spatial arrangement of the local structural features in a microstructure play a critical role in determining the physical properties of a material. Because of the complex and nonlinear nature of microstructure evolution, numerical approaches are often employed. For more details, see e.g. [13, 14, 22].

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In this article we are interested in a model for the evolution, driven by configurational forces, of microstructures in elastically deformable solids. There are two main types of modeling for the evolution of microstructures. In the conventional approach, the regions separating the domains are treated as mathematically sharp interfaces. The local interfacial velocity is then determined as part of the boundary conditions, or is calculated from the driving force for interface motion and the interfacial mobility. This approach requires the explicit tracking of the interface positions. Such an interface-tracking approach can be successful in one-dimensional systems, however it will be impractical for complicated three-dimensional microstructures. Therefore, during the past decades, another approach has been invented, namely, the phase-field approach in which the interface is not of zero thickness, instead an interfacial region with thickness of certain order of a small regularization parameter. Though it is still a young discipline in condensed matter physics, this approach has emerged to be one of the most powerful methods for modeling the evolution of microstructures. It can be traced back the theory of diffuse-interface description, which is developed, independently, more than a century ago by van der Waals [26] and some half century ago by Cahn and Hilliard [11].

The two well-known models for temporal evolution of microstructures are the Cahn-Hilliard/Allen-Cahn equations corresponding, respectively, to the case that the order parameter is conserved and not conserved. These phase field models describe microstructure phenomena at the mesoscale (see e.g. [22]), and one suitable limit of it may be the corresponding sharp- or thin-interface descriptions. In this article we study a model for the behavior in time of materials with diffusionless phase transitions. The model has diffusive interfaces and consists of the partial differential equations of linear elasticity coupled to a quasilinear, non-uniformly parabolic equation of second order that differs from the Allen-Cahn equation (the Cahn-Hilliard equation in the case that the order parameter is conserved) by a gradient term. It is derived in [2, 4] from a sharp interface model for diffusionless phase transitions and can be considered to be a regularization of that model. To verify the validity of the new model, mathematical analysis has been carried out for the existence/regularity of weak solutions to initial boundary value problems in one space dimension, [3, 5, 7, 27, 28], the motion of interfaces [6], and the existence of traveling waves [19]. In the present article, the existence of spherically symmetric solutions to an initial-boundary value problem will be studied. We first formulate this initial-boundary value problem in the three-dimensional case, then reduce it, by assuming that the solution is spherically symmetric, to the one-dimensional case. The existence of weak solutions to this one dimensional problem is proved.

Let $\Omega \subset \mathbb{R}^3$ be an open set. It represents the material points of a solid body. The different phases are characterized by the order parameter $S(t, x) \in \mathbb{R}$. A value of $S(t, x)$ close to zero indicates that the material is in the matrix phase at the point $x \in \Omega$ at time t , a value close to one indicates that the material is in the second phase. The other unknowns are the displacement $u(t, x) \in \mathbb{R}^3$ of the material point x at time t and the Cauchy stress tensor $T(t, x) \in \mathcal{S}^3$, where \mathcal{S}^3 denotes the set of symmetric 3×3 -matrices. The unknowns must satisfy the quasi-static equations

$$-\operatorname{div}_x T(t, x) = b(t, x), \quad (1.1)$$

$$T(t, x) = D(\varepsilon(\nabla_x u(t, x)) - \bar{\varepsilon}S(t, x)), \quad (1.2)$$

$$S_t(t, x) = -c\left(\psi_S(\varepsilon(\nabla_x u(t, x)), S(t, x)) - \nu \Delta_x S(t, x)\right) |\nabla_x S(t, x)| \quad (1.3)$$

for $(t, x) \in (0, \infty) \times \Omega$. The boundary and initial conditions are

$$u(t, x) = \gamma(t, x), \quad S(t, x) = 0, \quad (t, x) \in [0, \infty) \times \partial\Omega, \quad (1.4)$$

$$S(0, x) = S_0(x), \quad x \in \Omega. \quad (1.5)$$

Here $\nabla_x u$ denotes the 3×3 -matrix of first order derivatives of u , the deformation gradient, $(\nabla_x u)^T$ denotes the transposed matrix and

$$\varepsilon(\nabla_x u) = \frac{1}{2} (\nabla_x u + (\nabla_x u)^T)$$

is the strain tensor. $\bar{\varepsilon} \in \mathcal{S}^3$ is a given matrix, the misfit strain, and $D : \mathcal{S}^3 \rightarrow \mathcal{S}^3$ is the elasticity tensor, a linear, symmetric, positive definite mapping. In the free energy

$$\psi(\varepsilon, S) = \frac{1}{2} (D(\varepsilon - \bar{\varepsilon}S)) \cdot (\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S), \quad (1.6)$$

we assume that $\hat{\psi} \in C^2(\mathbb{R}, [0, \infty))$, choose $\hat{\psi}$ as a double well potential with minima at $S = 0$ and $S = 1$. ψ_S is the partial derivative. The scalar product of two matrices A and B is denoted by $A \cdot B = \sum a_{ij} b_{ij}$. $c > 0$ is a constant and ν is a small positive constant. Given are the volume force $b : [0, \infty) \times \Omega \rightarrow \mathbb{R}^3$ and the data $\gamma : [0, \infty) \times \partial\Omega \rightarrow \mathbb{R}^3$, $S_0 : \Omega \rightarrow \mathbb{R}$.

This completes the formulation of the initial-boundary value problem. Equations (1.1) and (1.2) differ from the system of linear elasticity only by the term $\bar{\varepsilon}S$. The evolution equation (1.3) for the order parameter S is non-uniformly parabolic because of the term $\nu \Delta S |\nabla_x S|$. Since this initial-boundary value problem is derived from a sharp interface model, to verify that it is indeed a diffusive interface model regularizing the sharp interface model, it must be shown that the equations (1.1) – (1.5) with positive ν have solutions which exist globally in time, and that these solutions tend to solutions of the sharp interface model for $\nu \rightarrow 0$. This would also be a method to prove existence of solutions to the original sharp interface model.

We only contribute to the first part of this program in this work and show that there exist some special solutions to the initial-boundary value problem that is essentially in one space dimension. Up to now we still can't solve the following problem: either solutions in three space dimensions exist or these solutions converge to a solution of the sharp interface model for $\nu \rightarrow 0$. We shall see later that the existence result of spherically symmetric solutions is of interest because the problem has a stronger nonlinear term (compared with the problem by assuming all unknowns depend on only one component of space variable x which is studied in [3]), despite it is essentially one space dimensional.

Related to our investigations is the model for diffusion dominated phase transformations obtained by coupling the elasticity equations (1.1), (1.2) with the Allen-Cahn/Cahn-Hilliard equations. They have recently been studied in [9, 12, 16].

Statement of the main result. Since we shall look for solutions, which are spherically symmetric, to problem (1.1) – (1.5), the problem can be reduced to the one which is one space dimensional. To this end we now assume that the body force boundary and initial data and the unknowns, which are defined in the domain $\Omega \times (0, T_e)$, have the following form

$$b(t, x) = \hat{b}(t, r) \frac{x}{r}, \quad \gamma(t, x) = \hat{\gamma}(t, r), \quad S_0(x) = \hat{S}_0(r)$$

and

$$u(t, x) = \hat{u}(t, r) \frac{x}{r}, \quad S(t, x) = \hat{S}(t, r),$$

respectively, where T_e is a positive constant which denotes the life-span of weak solutions, $r = |x|$, $\Omega = \{x \in \mathbb{R}^3 \mid a < r < d\}$ for two positive constant a, d satisfying $a < d$, and $\hat{b}, \hat{\gamma}, \hat{S}_0$ are given functions and \hat{u}, \hat{S} are scalar functions to be determined, which depend only on t, r . We write

$$x = (x_i), \quad u = (u_i), \quad T = (T_{ij}), \quad D = (D_{kl}^{ij}).$$

Here and hereafter, $i, j, k, l = 1, 2, 3$, and we assume that D satisfies the properties of symmetry:

$$D_{kl}^{ij} = D_{ij}^{kl} = D_{lk}^{ij} = D_{kl}^{ji}. \quad (1.7)$$

Moreover we assume that D satisfies

$$D_{ij}^{kl} = 0, \quad \text{if } k \neq j; \quad (1.8)$$

$$D_{ij}^{jl} = 0, \quad \text{if } i \neq l, \quad \text{for any fixed } j. \quad \text{Assume that } D_{ij}^{jl} \text{ is independent of } j; \quad (1.9)$$

$$C_{il} := D_{ij}^{jl}. \quad \text{Assume that } C_{il} \text{ is independent of } l \text{ and is equal to } \mu. \quad (1.10)$$

$$E_{kl} := \sum_{i,j=1}^3 D_{ij}^{kl} \bar{\varepsilon}_{ij} = 0, \quad \text{if } k \neq l, \quad \text{and} \quad (1.11)$$

$$E_{kk} \text{ is independent of } k \text{ and is equal to } \lambda. \quad (1.12)$$

Under these assumptions equations (1.1) – (1.3) are reduced to

$$\frac{\partial^2}{\partial r^2} \hat{u} + \frac{2}{r} \frac{\partial}{\partial r} \hat{u} - \frac{2}{r^2} \hat{u} = \mathcal{G}, \quad (1.13)$$

$$\frac{\partial}{\partial t} \hat{S} + \left(-c\nu \frac{\partial^2}{\partial r^2} \hat{S} + \mathcal{F} \right) \Big| \frac{\partial}{\partial r} \hat{S} \Big| = 0. \quad (1.14)$$

Here \mathcal{F}, \mathcal{G} are nonlinear functions defined by

$$\mathcal{G} = \mathcal{G}\left(\frac{\partial}{\partial r} \hat{S}, \hat{b}\right) = \frac{\lambda}{\mu} \frac{\partial}{\partial r} \hat{S} + \frac{\hat{b}}{\mu}, \quad (1.15)$$

$$\begin{aligned} \mathcal{F}_1 &= \mathcal{F}_1\left(\hat{u}, \frac{\partial}{\partial r} \hat{u}, \hat{S}\right) \\ &= c \left(-\lambda \left(\frac{\partial}{\partial r} \hat{u} + \frac{2}{r} \hat{u} \right) + D\bar{\varepsilon} \cdot \bar{\varepsilon} \hat{S} + \hat{\psi}'(\hat{S}) \right), \end{aligned} \quad (1.16)$$

$$\mathcal{F} = \mathcal{F}\left(\hat{u}, \frac{\partial}{\partial r} \hat{u}, \hat{S}, \frac{\partial}{\partial r} \hat{S}\right) = \mathcal{F}_1 - \frac{2c\nu}{r} \frac{\partial}{\partial r} \hat{S}. \quad (1.17)$$

The boundary and initial conditions become

$$\hat{u}(t, r) = \hat{\gamma}(t, r), \quad \hat{S}(t, r) = 0, \quad (t, r) \in [0, T_e] \times \partial\Omega, \quad (1.18)$$

$$\hat{S}(0, r) = \hat{S}_0(r), \quad r \in \Omega, \quad (1.19)$$

where $\hat{\gamma}(t, r)$ is defined by $\gamma(t, r) = \hat{\gamma}(t, r) \frac{x}{r}$.

Remark 1. *One can easily find an example which meets the above assumptions: The media is isotropic and homogenous. These assumptions still lead to an elliptic-parabolic coupled system, thus the reduced system possesses the main difficulties in the proof of existence of weak solutions as in [3].*

To what follows, except Section 2 in which we reduce the problem to one dimensional form, we shall change the independent variable r to x , and drop the hat $\hat{\cdot}$ for all quantities (except $\hat{\psi}$), namely, $\hat{u} \rightarrow u$, $\hat{b} \rightarrow b$, etc. Denote $f_x = \frac{\partial}{\partial x} f$, $f_{xx} = \frac{\partial^2}{\partial x^2} f$, etc. The domain Ω is reduced to an interval: $\Omega = (a, d)$ is a bounded open interval with constants $a < d$. We write $Q_{T_e} := (0, T_e) \times \Omega$, where T_e is a positive constant, and define

$$(v, \varphi)_Z = \int_Z v(y) \varphi(y) dy,$$

for $Z = \Omega$ or $Z = Q_{T_e}$. If v is a function defined on Q_{T_e} we denote the mapping $x \rightarrow v(t, x)$ by $v(t)$. If no confusion is possible we sometimes drop the argument t and write $v = v(t)$. Since equation (1.13) is linear, the inhomogeneous Dirichlet boundary condition for \hat{u} can be reduced in the standard way to the homogeneous condition. For simplicity we thus assume that

$$\hat{\gamma} = 0.$$

Then with these simplifications, equations (1.13) – (1.14) can be written in the form

$$u_{xx} + \frac{2}{x} u_x - \frac{2}{x^2} u = \mathcal{G}, \quad (1.20)$$

$$\frac{\partial}{\partial t} S + (\mathcal{F} - c\nu S_{xx}) |S_x| = 0. \quad (1.21)$$

The boundary and initial conditions turn out to be

$$u(t, x) = 0, \quad S(t, x) = 0, \quad (t, x) \in [0, T_e] \times \partial\Omega, \quad (1.22)$$

$$S(0, x) = S_0(x), \quad x \in \Omega. \quad (1.23)$$

To define weak solutions of this initial-boundary value problem we note that because of $\frac{1}{2}(|y|y)' = |y|$ equation (1.21) is equivalent to

$$\frac{\partial}{\partial t} S - \frac{c\nu}{2} (S_x |S_x|)_x + \mathcal{F} |S_x| = 0. \quad (1.24)$$

Definition 1.1. *Let $b \in L^\infty(0, T_e, L^2(\Omega))$, $S_0 \in L^\infty(\Omega)$. A function (u, S) with*

$$u \in L^\infty(0, T_e; W_0^{1,\infty}(\Omega)), \quad (1.25)$$

$$S \in L^\infty(Q_{T_e}) \cap L^\infty(0, T_e, H_0^1(\Omega)), \quad (1.26)$$

is a weak solution to the problem (1.20) – (1.23), if the equation (1.20) is satisfied weakly and if for all $\varphi \in C_0^\infty((-\infty, T_e) \times \Omega)$

$$(S, \varphi_t)_{Q_{T_e}} - \frac{c\nu}{2} (|S_x| S_x, \varphi_x)_{Q_{T_e}} - (\mathcal{F} |S_x|, \varphi)_{Q_{T_e}} + (S_0, \varphi(0))_\Omega = 0. \quad (1.27)$$

The main result of this article is

Theorem 1.1 *To all $S_0 \in H_0^1(\Omega)$ and $b \in C(\overline{Q_{T_e}})$ with $b_t \in C(\overline{Q_{T_e}})$ there exists a weak solution (u, S) of problem (1.20) – (1.23), which in addition to (1.25) – (1.27) satisfies*

$$S_t \in L^{\frac{4}{3}}(Q_{T_e}), \quad S_x \in L^{\frac{8}{3}}(0, T_e; L^\infty(\Omega)), \quad (1.28)$$

and

$$(|S_x|S_x)_x \in L^{\frac{4}{3}}(Q_{T_e}), \quad S_{xt} \in L^{\frac{4}{3}}(0, T_e; W^{-1, \frac{4}{3}}(\Omega)). \quad (1.29)$$

Consequently we find spherically symmetric solution $(\hat{u}(t, r) \frac{x}{r}, \hat{S}(t, r))$ to the original problem (1.1) – (1.5).

The remaining sections are devoted to the proof of Theorem 1.1. The difficulties in the proof of existence of weak solutions to the one dimensional problem are due to the following features: The system is of elliptic-parabolic type, it consists of a linear second order elliptic equation coupled with a nonlinear equation for the order parameter equation. The nonlinearity of this nonlinear equation is stronger than the one in [3], where a one-dimensional initial boundary value problem is investigated. Moreover, this equation is degenerate and the nonlinearity depends non-smoothly on the gradient of unknown S . This can be judged easily from the fact that the coefficient $\nu|S_x|$ of the highest order derivative S_{xx} in the order parameter equation is not bounded away from zero and that it is not differentiable with respect to S_x .

The rest of this article is organized as follows: In Section 2, assuming that the domain Ω , the elasticity tensor D and the misfit stain tensor satisfy suitable conditions, and that the solutions (u, S) to problem (1.1) – (1.5) and the initial and boundary data are spherically symmetric, we reduce the original problem to the one dimensional form.

Then to prove Theorem 1.1 we first consider in Section 3 a modified initial-boundary value problem which consists of (1.20) and the equation

$$S_t - c\nu|S_x|_\kappa S_{xx} + \mathcal{F} \cdot (|S_x|_\kappa - \kappa) = 0, \quad x \in \Omega, \quad t > 0 \quad (1.30)$$

with a constant $\kappa > 0$. Here we use the notation

$$|p|_\kappa := \sqrt{\kappa^2 + p^2}. \quad (1.31)$$

Since (1.30) is a uniformly parabolic equation we can use a standard theorem to conclude that the modified initial-boundary value problem has a sufficiently smooth solution (u^κ, S^κ) . For this solution we derive in Section 4 a-priori estimates that are uniform in κ for $\kappa \in (0, 1]$. The assumption $\kappa \in (0, 1]$ is reasonable since we consider limits of approximate solutions for $\kappa \rightarrow 0$. We shall see that the selection of a function in the form (1.31) results in a simpler proof of the existence of weak solutions than that in [3].

To select a subsequence converging to a solution for $\kappa \rightarrow 0$ we need a compactness result. However, our a-priori estimates are not strong enough to show that the sequence S_x^κ is compact; instead, we can only show that the sequence $\int_0^{S_x^\kappa} |y| dy = \frac{1}{2} S_x^\kappa |S_x^\kappa|$ has bounded derivatives, with respect to both x and t , in some suitable spaces, and thus can be proved to be compact. It turns out that this is enough to prove existence of a solution. For the compactness proof in Section 5 we use the Aubin-Lions Lemma; since one of our a-priori estimates for derivatives of the approximate solutions is only valid in $L^1(0, T_e; H^{-2}(\Omega))$, we must use the generalized form of this lemma given by Roubíček [24], which is valid in L^1 .

Despite we prove the existence of spherically symmetric solutions, the existence of weak solutions to the original problem (1.1) – (1.5) is still open. The method of the proof in this article and [3] (in which problem (1.1) – (1.5) in one dimensional case is studied) is limited to one space dimension, since for the a-priori estimates it is crucial that the term $|S_x|S_{xx}$ in (1.21) can be written in the form $\frac{1}{2}(|S_x|S_x)_x$. In the higher dimensional case the corresponding term $|\nabla_x S|\Delta_x S$ cannot be rewritten in this way. Yet, we believe that these essentially one-dimensional existence results can also be helpful in an existence proof for higher space dimensions.

2 Reduction to one dimensional problem

We shall prove in this section that under suitable assumptions, the original problem can be reduced to a one dimensional problem. We now assume that the body force and the unknowns, which are defined in the domain $\Omega \times (0, T_e)$, have the following form

$$b(t, x) = \hat{b}(t, r)\frac{x}{r}, \quad (2.1)$$

and

$$u(t, x) = \hat{u}(t, r)\frac{x}{r}, \quad S(t, x) = \hat{S}(t, r), \quad (2.2)$$

respectively, where T_e is a positive constant, $r = |x|$, $\Omega = \{x \in \mathbb{R}^3 \mid a < r < d\}$ for two positive constant a , d satisfying $a < d$, and \hat{u} , \hat{S} are scalar functions to be determined, which depend only on t , r , and \hat{b} is a given function in t , r .

Theorem 2.1 *Suppose that the tensors D and $\bar{\varepsilon}$ satisfy (1.7) – (1.12). Then the following two statements are equivalent:*

1. $(u, S)(t, x)$ of the form (2.2) is a classical solution to the problem (1.1) – (1.5) with b chosen in (2.1),
2. $(\hat{u}, \hat{S})(t, r)$ solves classically the problem (1.13) – (1.19).

Proof. To simplify notations, the Einstein summation convention applies to the rest of this section: When an index variable (e.g. i, j, k, l , but with an exception r in this article, for instance, $\hat{S}_{,r}\frac{x^i}{r}$ in (2.6) does not mean that we take the sum for the index r) appears twice in a single term that is a product of two or more numbers, it implies that we are summing over all of its possible values. However we shall still use the symbol Σ to avoid some possible confusion when in a single term, an index appears more than two times. Recalling

$$x = (x_i), \quad u = (u_i), \quad T = (T_{ij}), \quad D = (D_{kl}^{ij}).$$

where $i, j, k, l = 1, 2, 3$.

For partial derivatives, we denote for a function $\hat{f} = \hat{f}(t, r)$

$$\hat{f}_{,r} = \frac{\partial \hat{f}}{\partial r}, \quad \hat{f}_{,rr} = \frac{\partial^2 \hat{f}}{\partial r^2}.$$

An index j (or i, k, l) after a comma in subscript of a quantity (for example, a function $f = f(t, x)$, a vector $u = u(t, x)$ and tensor $T = T(t, x)$, etc.) indicates the partial derivative with respect to x_j , namely

$$f_{,j} = \frac{\partial f}{\partial x_j}, \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}, \quad T_{ik,j} = \frac{\partial T_{ik}}{\partial x_j}, \dots$$

Similar convention applies to multiple indices after a comma in subscript of a quantity. We can thus write

$$r_{,i} = \frac{x_i}{r}, \quad \left(\frac{x_i}{r}\right)_{,j} = \frac{\delta_{ij}r^2 - x_ix_j}{r^3}, \quad (2.3)$$

$$u_{i,j} = \left(\hat{u}\frac{x_i}{r}\right)_{,j} = \hat{u}_{,r}\frac{x_ix_j}{r^2} + \hat{u}\frac{\delta_{ij}r^2 - x_ix_j}{r^3}, \quad (2.4)$$

$$\begin{aligned} u_{i,jk} &= \hat{u}_{,rr}\frac{x_ix_jx_k}{r^3} + \hat{u}_{,r}\frac{r^2((x_ix_j)_{,k} + \delta_{ij}x_k) - 3x_ix_jx_k}{r^4} \\ &\quad + \hat{u}\frac{r^2(-\delta_{ij}x_k - (x_ix_j)_{,k}) + 3x_ix_jx_k}{r^5}, \end{aligned} \quad (2.5)$$

$$S_{,i} = \hat{S}_{,r}\frac{x_i}{r}. \quad (2.6)$$

Here δ_{ij} is the Kronecker delta.

Hence, the first two equations (1.1) – (1.2) can be rewritten as

$$\begin{aligned} \hat{b}\frac{x_l}{r} &= \frac{1}{2}D_{ij}^{kl}u_{j,ik} + \frac{1}{2}D_{ij}^{kl}u_{i,jk} - D_{ij}^{kl}\bar{\varepsilon}_{ij}S_{,k} \\ &= \hat{u}_{,rr}D_{ij}^{kl}\frac{x_ix_jx_k}{r^3} + \hat{u}_{,r}\left(D_{ij}^{kl}\frac{(x_ix_j)_{,k}}{r^2} + D_{ij}^{kl}\frac{\delta_{ij}x_k}{r^2} - D_{ij}^{kl}\frac{3x_ix_jx_k}{r^4}\right) \\ &\quad + \frac{\hat{u}}{r^5}\left(-r^2\left(D_{ij}^{kl}\delta_{ij}x_k + D_{ij}^{kl}(x_ix_j)_{,k}\right) + 3D_{ij}^{kl}x_ix_jx_k\right) - D_{ij}^{kl}\bar{\varepsilon}_{ij}\hat{S}_{,r}\frac{x_k}{r}. \end{aligned} \quad (2.7)$$

From now on we take one l from $\{1, 2, 3\}$. Invoking assumptions (1.8) – (1.12) we obtain

$$D_{ij}^{kl}\frac{x_ix_jx_k}{r^3} = \sum_{j=1}^3 D_{ij}^{jl}\frac{x_ix_jx_j}{r^3} = C_{il}\frac{x_i}{r} = C_{il}\frac{x_l}{r} = \mu\frac{x_l}{r}. \quad (2.8)$$

Since $(x_ix_j)_{,k} = \delta_{ik}x_j + x_i\delta_{jk}$, one has

$$\begin{aligned} &D_{ij}^{kl}\frac{(x_ix_j)_{,k}}{r^2} + D_{ij}^{kl}\frac{\delta_{ij}x_k}{r^2} - D_{ij}^{kl}\frac{3x_ix_jx_k}{r^4} \\ &= \sum_{j=1}^3 \left(D_{ij}^{jl}\frac{\delta_{ij}x_j + x_i\delta_{jj}}{r^2} + D_{ij}^{jl}\frac{\delta_{ij}x_j}{r^2} - D_{ij}^{jl}\frac{3x_ix_jx_j}{r^4}\right) \\ &= \sum_{i=1}^3 C_{il}\frac{r^2(3x_i + 2x_i) - 3x_ir^2}{r^4} = C_{il}\frac{2r^2x_i}{r^4} \\ &= 2\mu\frac{x_l}{r^2}, \end{aligned} \quad (2.9)$$

and

$$\begin{aligned}
& -\frac{1}{r^3} \left(D_{ij}^{kl} \delta_{ij} x_k + D_{ij}^{kl} (x_i x_j)_{,k} \right) + D_{ij}^{kl} \frac{3x_i x_j x_k}{r^5} \\
&= \sum_{j=1}^3 D_{ij}^{jl} \frac{-r^2 (\delta_{ij} x_j + (x_i x_j)_{,j}) + 3x_i x_j x_j}{r^5} \\
&= \sum_{j=1}^3 D_{ij}^{jl} \frac{-r^2 (2\delta_{ij} x_j + x_i \delta_{jj}) + 3x_i x_j x_j}{r^5} \\
&= \sum_{i=1}^3 C_{il} \frac{-r^2 (2x_i + 3x_i) + 3r^2 x_i}{r^5} = C_{il} \frac{-2x_i}{r^3} \\
&= -2\mu \frac{x_l}{r^3}. \tag{2.10}
\end{aligned}$$

Using (2.8) – (2.10), we are in a position to rewrite equation (2.7) as

$$\hat{b} \frac{x_l}{r} = \frac{x_l}{r} \left(\mu \left(\hat{u}_{,rr} + \frac{2}{r} \hat{u}_{,r} - \frac{2}{r^2} \hat{u} \right) - \lambda \hat{S}_{,r} \right). \tag{2.11}$$

This holds, for $r \geq a > 0$, if and only if the following equation is satisfied

$$\mu \left(\hat{u}_{,rr} + \frac{2}{r} \hat{u}_{,r} - \frac{2}{r^2} \hat{u} \right) - \lambda \hat{S}_{,r} = \hat{b}, \tag{2.12}$$

which is just (1.13).

Next, to deal with the order parameter equation we make use of the following formula

$$\psi_S(\varepsilon, S) = -T \cdot \bar{\varepsilon} + \hat{\psi}'(S) = -D\varepsilon(\nabla u) \cdot \bar{\varepsilon} + D\bar{\varepsilon} \cdot \bar{\varepsilon} S + \hat{\psi}'(S).$$

We evaluate $D\varepsilon(\nabla u) \cdot \bar{\varepsilon}$. Invoking (1.12),

$$\begin{aligned}
D\varepsilon(\nabla u) \cdot \bar{\varepsilon} &= \frac{1}{2} D_{ij}^{kl} u_{i,j} \bar{\varepsilon}_{kl} + \frac{1}{2} D_{ij}^{kl} u_{j,i} \bar{\varepsilon}_{kl} \\
&= \hat{u}_{,r} D_{ij}^{kl} \frac{x_i x_j}{r^2} \bar{\varepsilon}_{kl} + \hat{u} \left(D_{ij}^{kl} \frac{\delta_{ij}}{r} \bar{\varepsilon}_{kl} - D_{ij}^{kl} \frac{x_i x_j}{r^3} \bar{\varepsilon}_{kl} \right) \\
&= \hat{u}_{,r} E_{ij} \frac{x_i x_j}{r^2} + \hat{u} \left(E_{ij} \frac{\delta_{ij}}{r} - E_{ij} \frac{x_i x_j}{r^3} \right) \\
&= \sum_{i=1}^3 E_{ii} \left(\hat{u}_{,r} \frac{x_i x_i}{r^2} + \hat{u} \frac{\delta_{ii} r^2 - x_i x_i}{r^3} \right) \\
&= \lambda \left(\hat{u}_{,r} + \frac{2}{r} \hat{u} \right), \tag{2.13}
\end{aligned}$$

where λ is a constant as in (1.12). Thus (1.3) turns out to be

$$\hat{S}_t + (-c\nu \hat{S}_{,rr} + \mathcal{F}) | \hat{S}_{,r} | = 0, \tag{2.14}$$

where \mathcal{F} is the same function as in (1.17). Thus we obtain (1.14).

To finish the reduction of the problem, we write the initial boundary conditions in the following form: $\gamma(t, x) = \hat{\gamma}(t, r) \frac{x}{r}$, $S_0(x) = \hat{S}_0(r)$. Thus we obtain the one dimensional problem and the proof of Theorem 2.1 is complete.

3 Existence of solutions to the modified problem

In this section, we study the modified initial-boundary value problem and show that it has a Hölder continuous classical solution, consequently we construct approximate solutions whose limit is a solution to the original problem (1.20) – (1.23). To formulate this problem, let $\chi \in C_0^\infty(\mathbb{R}, [0, \infty))$ satisfy $\int_{-\infty}^\infty \chi(t)dt = 1$. For $\kappa > 0$, we set

$$\chi_\kappa(t) := \frac{1}{\kappa} \chi\left(\frac{t}{\kappa}\right),$$

and for $S \in L^\infty(Q_{T_e}, \mathbb{R})$ we define

$$(\chi_\kappa * S)(t, x) = \int_0^{T_e} \chi_\kappa(t-s)S(s, x)ds. \quad (3.1)$$

The modified initial-boundary value problem consists of the equations

$$u_{xx} + \frac{2}{x}u_x - \frac{2}{x^2}u = \mathcal{G}((\chi_\kappa * S)_x, b), \quad (3.2)$$

$$S_t - c\nu|S_x|_\kappa S_{xx} = -\mathcal{F} \cdot (|S_x|_\kappa - \kappa), \quad (3.3)$$

which must hold in Q_{T_e} , and of the boundary and initial conditions

$$u(t, x) = 0, \quad S(t, x) = 0, \quad (t, x) \in (0, T_e) \times \partial\Omega, \quad (3.4)$$

$$S(0, x) = S_0(x), \quad x \in \bar{\Omega}. \quad (3.5)$$

Now we want to rewrite the system as an equation with a nonlocal term. Applying the Sturm-Liouville theory for ordinary differential equations of the form

$$\frac{d}{dx}(p(x)y_x(x)) + q(x)y(x) = 0,$$

with suitable boundary conditions at $x = a, d$, we first solve u in terms of S_x and b . To this end, we rewrite, by multiplying it by x^2 , (2.12) as

$$L[u] := \frac{d}{dx}(p(x)u_x) + q(x)u, \quad (3.6)$$

$$L[u](x) = x^2\mathcal{G}((\chi_\kappa * S)_x, b), \quad (3.7)$$

and the boundary conditions are chosen as $u(t, x) = 0$ at $x = a, d$. Here

$$p(x) = x^2, \quad q(x) = -2.$$

Consider the eigen-problem $L[u] = \sigma u$ with $\sigma = 0$ and with $u(t, x) = 0$ at $x = a, d$. It is easy to show from (3.7) that

$$0 = \int_a^d L[u] \cdot u dx = \int_a^d \left(\frac{d}{dx}(p(x)u_x)u + q(x)u^2 \right) dx = - \int_a^d (x^2u_x^2 + 2u^2) dx,$$

whence $u \equiv 0$, and 0 is not an eigenvalue of this operator. One asserts that for any fixed $t \in [0, T_e]$, there exists a unique solution u to (3.2), which can be represented by

$$u(t, x) = \int_a^d G(x, y) \left(\frac{y^2}{\mu} b(t, y) + \frac{\lambda}{\mu} y^2 (\chi_\kappa * S(t, y))_y \right) dy. \quad (3.8)$$

Here $G(x, y)$ is the Green function, associated with the operator L , such that

1. $G(x, y)$ is continuous in x and y ;
2. For $x \neq y$, $L[G(x, y)] = 0$;
3. $G(x, \cdot) = 0$ at $x = a, d$;
4. Derivative jump: $G'(y_{+0}, y) - G'(y_{-0}, y) = \frac{1}{p(y)}$;
5. Symmetry: $G(x, y) = G(y, x)$.

Recalling the boundary condition (3.4) and integrating by parts we infer from (3.8) that

$$\begin{aligned}
u(t, x) &= \frac{1}{\mu} \int_a^d G(x, y) y^2 b(t, y) dy - \frac{\lambda}{\mu} \int_a^d (G(x, y) y^2)_y \chi_\kappa * S(t, y) dy \\
&= \frac{1}{\mu} \int_a^d G(x, y) y^2 b(t, y) dy - \frac{\lambda}{\mu} \int_{\{x \neq y\}} 2G(x, y) y \chi_\kappa * S(t, y) dy \\
&\quad - \frac{\lambda}{\mu} \int_{\{x \neq y\}} G(x, y)_y y^2 \chi_\kappa * S(t, y) dy.
\end{aligned} \tag{3.9}$$

Thus $u(t, x)$ depends linearly on S and a nonlocal term of S .

To formulate an existence theorem for this problem we need some function spaces: For nonnegative integers m, n and a real number $\alpha \in (0, 1)$ we denote by $C^{m+\alpha}(\bar{\Omega})$ the space of m -times differentiable functions on $\bar{\Omega}$, whose m -th derivative is Hölder continuous with exponent α . The space $C^{\alpha, \alpha/2}(\bar{Q}_{T_e})$ consists of all functions on \bar{Q}_{T_e} , which are Hölder continuous in the parabolic distance

$$d((t, x), (s, y)) := \sqrt{|t - s| + |x - y|^2}.$$

$C^{m, n}(\bar{Q}_{T_e})$ and $C^{m+\alpha, n+\alpha/2}(\bar{Q}_{T_e})$, respectively, are the spaces of functions, whose x -derivatives up to order m and t -derivatives up to order n belong to $C(\bar{Q}_{T_e})$ or to $C^{\alpha, \alpha/2}(\bar{Q}_{T_e})$, respectively.

Theorem 3.1 *Let $\nu, \kappa > 0$, $T_e > 0$, suppose that the function $b \in C(\bar{Q}_{T_e})$ has the derivative $b_t \in C(\bar{Q}_{T_e})$ and that the initial data $S_0 \in C^{2+\alpha}(\bar{\Omega})$ satisfy $S_0|_{\partial\Omega} = S_{0,x}|_{\partial\Omega} = S_{0,xx}|_{\partial\Omega} = 0$. Then there is a solution*

$$(u, S) \in C^{2,1}(\bar{Q}_{T_e}) \times C^{2+\alpha, 1+\alpha/2}(\bar{Q}_{T_e})$$

of the modified initial-boundary value problem (3.2) – (3.5). This solution satisfies $S_{tx} \in L^2(Q_{T_e})$ and

$$\max_{\bar{Q}_{T_e}} |S| \leq \max_{\bar{\Omega}} |S_0|. \tag{3.10}$$

Proof. Making use of (3.9), we rewrite the system (3.2) – (3.3) as a single equation

$$\begin{aligned}
S_t &= a_1(S_x) S_{xx} \\
&\quad + a_2 \left(t, x, S, S_x, \tilde{S}, \int_{\{x \neq y\}} G(x, y) y \tilde{S}(t, y) dy, \int_{\{x \neq y\}} G(x, y)_y y^2 \tilde{S}(t, y) dy \right)
\end{aligned} \tag{3.11}$$

in Q_{T_e} , where $\tilde{S} = \chi_\kappa * S$,

$$a_1(p) = c\nu|p|_\kappa$$

and

$$a_2(t, x, S, p, r, s) = c \overline{\mathcal{F}}(t, x, S, p, r, s_1, s_2)(|p|_\kappa - \kappa).$$

Here $\overline{\mathcal{F}}$ is obtained by using formula (3.9) and inserting u, u_x into the formula of \mathcal{F} .

Equation (3.11) is quasilinear, uniformly parabolic equation, which contains nonlocal terms. Then we can apply, with a little modification, [20, Theorem 2.9, p.23] to (3.11) to prove the existence of classical solution S^κ , and conclude that the estimate (3.10) holds by applying the maximum principle to (3.11). We refer the reader to the paper [3] for the details. Thus we complete the proof of Theorem 3.1.

4 A priori estimates

This section is devoted to the derivation of a-priori estimates for solutions of the modified problem, which are uniform with respect to $\kappa \in (0, 1]$. We remark that the estimates in Lemma 4.1 and Corollary 3.1, though stated in the one-dimensional case, can be generalized to higher space dimensions.

In what follows we assume that

$$0 < \kappa \leq 1, \tag{4.1}$$

since we consider the limit $\kappa \rightarrow 0$. The $L^2(\Omega)$ -norm is denoted by $\|\cdot\|$, and the letter C stands for varies positive constants independent of κ . We start by constructing a family of approximate solutions to the modified problem. To this end let T_e be a fixed positive number and choose for every κ a function $S_0^\kappa \in C_0^\infty(\Omega)$ such that

$$\|S_0^\kappa - S_0\|_{H_0^1(\Omega)} \rightarrow 0, \quad \kappa \rightarrow 0, \tag{4.2}$$

where $S_0 \in H_0^1(\Omega)$ are the initial data given in Theorem 1.1. We insert for S_0 in (3.5) the function S_0^κ and choose for b in (3.2) the function given in Theorem 1.1. These functions satisfy the assumptions of Theorem 3.1, hence there exists a solution (u^κ, S^κ) of the modified problem (3.2) – (3.5), which exists in Q_{T_e} . The inequality (3.10) and Sobolev's embedding theorem yield for this solution

$$\sup_{0 < \kappa \leq 1} \|S^\kappa\|_{L^\infty(Q_{T_e})} \leq \sup_{0 < \kappa \leq 1} \|S_0^\kappa\|_{L^\infty(\Omega)} \leq C. \tag{4.3}$$

Remembering the formula (3.9) and assumptions of b , we show easily that u^κ belongs to $C^{1,1}(\overline{Q}_{T_e})$, and conclude from (3.2) that also $\|u_x^\kappa\|_{L^\infty(Q_{T_e})} \leq C$, and invoke the definition of \mathcal{F}_1 to get

$$\max_{\overline{Q}_{T_e}} |\mathcal{F}_1(u^\kappa, u_x^\kappa, S^\kappa)| \leq C. \tag{4.4}$$

With the help of this estimate, we can evaluate derivatives of S^κ .

Lemma 4.1 *There holds for any $t \in [0, T_e]$*

$$\|S_x^\kappa(t)\|^2 + c\nu \int_0^t \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx d\tau \leq C. \tag{4.5}$$

Proof. From the assertion that $S_{tx}^\kappa \in L^2(Q_{T_e})$ in Theorem 3.1, it follows that there holds for almost all t

$$\frac{1}{2} \frac{d}{dt} \|S_x^\kappa(t)\|^2 = \int_{\Omega} S_x^\kappa(t) S_{xt}^\kappa(t) dx.$$

Making use of this relation and (4.4) we obtain by multiplication of (3.3) by $-S_{xx}^\kappa$ and integration by parts with respect to x , where we take the boundary condition (3.4) into account, that for almost all t

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|S_x^\kappa\|^2 + \int_{\Omega} c\nu |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx = \int_{\Omega} \mathcal{F}(|S_x|_\kappa - \kappa) S_{xx}^\kappa dx \\ & \leq C \int_{\Omega} (1 + |S_x^\kappa|) (|S_x^\kappa|_\kappa + \kappa) |S_{xx}^\kappa| dx \\ & = C \left(\int_{\Omega} |S_x^\kappa|_\kappa |S_{xx}^\kappa| dx + \int_{\Omega} \kappa |S_{xx}^\kappa| dx + \int_{\Omega} |S_x^\kappa| |S_x|_\kappa |S_{xx}^\kappa| dx + \int_{\Omega} \kappa |S_x^\kappa| |S_{xx}^\kappa| dx \right) \\ & = C(I_1 + I_2 + I_3 + I_4) \end{aligned} \quad (4.6)$$

Now we estimate I_i ($i = 1, 2, 3, 4$). For I_1 , we have

$$\begin{aligned} I_1 & \leq C \int_{\Omega} |S_x^\kappa|^{\frac{1}{2}} (|S_x^\kappa|^{\frac{1}{2}} |S_{xx}^\kappa|) dx \\ & \leq \frac{c\nu}{8} \int_{\Omega} |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx + C_\nu \int_{\Omega} (|S_x^\kappa|_\kappa)^2 dx + C, \end{aligned} \quad (4.7)$$

where we denote C_ν a constant depending on ν . By definition, there holds

$$|S_x^\kappa|_\kappa \geq \kappa. \quad (4.8)$$

Thus we can use the second term on the left hand side of (4.6) to absorb I_2 . By the Cauchy-Schwarz and Young inequalities,

$$\begin{aligned} I_2 & \leq C \int_{\Omega} \kappa^{\frac{1}{2}} (\kappa^{\frac{1}{2}} |S_{xx}^\kappa|) dx \leq C \left(\int_{\Omega} \kappa dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} \\ & \leq \frac{c\nu\kappa}{8} \int_{\Omega} |S_{xx}^\kappa|^2 dx + C_\nu. \end{aligned} \quad (4.9)$$

Here we have used the fact that $0 < a \leq x \leq d$ which implies the term $\frac{1}{x}$ contained in \mathcal{F} is uniformly bounded from below and above, and $C_{\nu,\kappa}$ is a constant depending on ν, κ . Moreover, I_3 is evaluated by

$$I_4 \leq C \int_{\Omega} \kappa |S_x^\kappa| |S_{xx}^\kappa| dx \leq C \left(\int_{\Omega} \kappa |S_x^\kappa|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}}. \quad (4.10)$$

Using the Nirenberg inequality in the following form

$$\|f_x\| \leq C \|f_{xx}\|^{\frac{1}{3}} \|f\|_{L^\infty(\Omega)}^{\frac{2}{3}} + C' \|f\|_{L^\infty(\Omega)}, \quad (4.11)$$

one infers from (4.10) that

$$\begin{aligned}
I_4 &\leq C\kappa^{\frac{1}{2}} \left(\|S_{xx}^\kappa\|^{\frac{1}{3}} \|S^\kappa\|_{L^\infty(\Omega)}^{\frac{2}{3}} + C' \|S^\kappa\|_{L^\infty(\Omega)} \right) \left(\int_{\Omega} \kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} \\
&\leq C\kappa^{\frac{1}{2}} \left(\|S_{xx}^\kappa\|^{\frac{1}{3}} + 1 \right) \left(\int_{\Omega} \kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} \\
&= C\kappa \left(\|S_{xx}^\kappa\|^{\frac{1}{3}+1} + \|S_{xx}^\kappa\| \right) \\
&\leq \frac{C\nu\kappa}{8} \int_{\Omega} |S_{xx}^\kappa|^2 dx + C_\nu, \tag{4.12}
\end{aligned}$$

where we used the Young inequality of the form $ab \leq \delta a^{\frac{4}{3}} + C_\delta b^4$ for non-negative real numbers a, b . Here C_δ is a constant depending on δ .

I_3 is the most difficult term to deal with. Again by the Cauchy-Schwarz inequality one gets

$$\begin{aligned}
I_3 &= C \int_{\Omega} (|S_x^\kappa| |S_x^\kappa|^{\frac{1}{2}}) (|S_x^\kappa|^{\frac{1}{2}} |S_{xx}^\kappa|) dx \\
&\leq C \left(\int_{\Omega} |S_x^\kappa|^2 |S_x^\kappa| dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |S_x^\kappa| |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} \\
&= CJ^{\frac{1}{2}} \cdot \left(\int_{\Omega} |S_x^\kappa| |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} \tag{4.13}
\end{aligned}$$

To deal with J , we recall the boundary conditions for S^κ and rewrite by integration by parts

$$\begin{aligned}
J &= \int_{\Omega} (S_x^\kappa)^2 |S_x^\kappa| dx = \int_{\Omega} S_x^\kappa (S_x^\kappa |S_x^\kappa|) dx \\
&= - \int_{\Omega} S^\kappa S_{xx}^\kappa (|S_x^\kappa|_\kappa + S_x^\kappa (|y|_\kappa)'|_{y=S_x^\kappa}) dx \\
&= - \int_{\Omega} S^\kappa S_{xx}^\kappa \left(|S_x^\kappa|_\kappa + \frac{(S_x^\kappa)^2}{|S_x^\kappa|_\kappa} \right) dx \tag{4.14}
\end{aligned}$$

Applying estimate (4.3) and invoking the definition of $|y|_\kappa$, from (4.14) one obtains

$$\begin{aligned}
J &\leq C \int_{\Omega} 2|S_x^\kappa|_\kappa |S_{xx}^\kappa| dx \\
&\leq C \left(\int_{\Omega} |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |S_x^\kappa|_\kappa dx \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\Omega} |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} (\|S_x^\kappa\| + 1). \tag{4.15}
\end{aligned}$$

Therefore, (4.13) becomes

$$\begin{aligned}
I_3 &\leq C(\|S_x^\kappa\| + 1)^{\frac{1}{2}} \left(\int_{\Omega} |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{4} + \frac{1}{2}} \\
&\leq \frac{C\nu}{8} \int_{\Omega} |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx + C_\nu (\|S_x^\kappa\|^2 + 1). \tag{4.16}
\end{aligned}$$

Here we have again used the Young inequality: $ab \leq \delta a^{\frac{4}{3}} + C_\delta b^4$.

Combining estimates (4.7) – (4.16), subtracting the terms $\frac{c\nu}{4} \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx$ and $\frac{c\nu\kappa}{4} \int_\Omega |S_{xx}^\kappa|^2 dx$ on both sides of inequality (4.6), splitting the second term on the left hand side of (4.6) into two equal terms, and recalling the property (4.8), we derive

$$\frac{1}{2} \frac{d}{dt} \|S_x^\kappa\|^2 + \frac{c\nu}{4} \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx + \frac{c\nu\kappa}{4} \int_\Omega |S_{xx}^\kappa|^2 dx \leq C \|S_x^\kappa\|^2 + C_\nu. \quad (4.17)$$

Then using the Gronwall inequality, one gets (4.5), noting also (4.2). And the proof of this lemma is thus complete.

Furthermore, we obtain

Corollary 4.1 *There holds for any $t \in [0, T_e]$*

$$\int_0^t \int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^{\frac{4}{3}} dx d\tau \leq C, \quad (4.18)$$

$$\int_0^t \int_\Omega (|S_x^\kappa S_{xx}^\kappa|)^{\frac{4}{3}} dx d\tau \leq C, \quad (4.19)$$

$$\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa dy \right\|_{W^{1, \frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau \leq C, \quad (4.20)$$

$$\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa dy \right\|_{L^\infty(\Omega)}^{\frac{4}{3}} d\tau \leq C, \quad (4.21)$$

$$\| |S_x^\kappa|_\kappa |S_{xx}^\kappa| \|_{L^{\frac{4}{3}}(0, T_e; L^\infty(\Omega))} \leq C, \quad (4.22)$$

$$\int_0^t \|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac{8}{3}} d\tau \leq C. \quad (4.23)$$

Proof. For some $2 > p \geq 1$ we choose q, q' such that

$$q = \frac{2}{p}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

By Hölder's inequality, we have

$$\begin{aligned} & \int_0^t \int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^p dx d\tau \\ &= \int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{p}{2}} \left((|S_x^\kappa|_\kappa)^{\frac{p}{2}} |S_{xx}^\kappa|^p \right) dx d\tau \\ &\leq \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{pq'}{2}} dx d\tau \right)^{\frac{1}{q'}} \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{pq}{2}} |S_{xx}^\kappa|^{pq} dx d\tau \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{p}{2-p}} dx d\tau \right)^{\frac{2-p}{2}} \left(\int_0^t \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx d\tau \right)^{\frac{p}{2}}. \end{aligned} \quad (4.24)$$

Estimate (4.5) implies that if p satisfies $\frac{p}{2-p} \leq 2$ (i.e. $p \leq \frac{4}{3}$) then the right hand side of (4.24) is bounded. This yields estimate (4.18). Then (4.19) follows from (4.18) and the estimate (4.5), and the fact that $|S_x^\kappa| < |S_x^\kappa|_\kappa$.

Next we are going to prove (4.20). Writing

$$|S_x^\kappa|_\kappa S_{xx}^\kappa = \left(\int_0^{S_x^\kappa} |y|_\kappa dy \right)_x, \quad (4.25)$$

and invoking that the primitive of $|y|_\kappa$ is equal to

$$\frac{1}{2} \left(y \sqrt{y^2 + \kappa^2} + \kappa^2 \log \left(y + \sqrt{y^2 + \kappa^2} \right) \right),$$

which, thanks to $\log x \leq x - 1$ for all $x > 0$, is bounded by $C(y^2 + 1)$, we then show easily that

$$\int_\Omega \int_0^{S_x^\kappa} |y|_\kappa dy dx \leq C \int_\Omega (|S_x^\kappa|^2 + 1) dx \leq C.$$

To apply the Poincaré inequality of the form

$$\|f - \bar{f}\|_{L^p(\Omega)} \leq C \|f_x\|_{L^p(\Omega)}$$

where $\bar{f} := \frac{1}{|\Omega|} \int_\Omega f(x) dx$, we choose

$$p = \frac{4}{3}, \quad f = \int_0^{S_x^\kappa} |y|_\kappa dy,$$

and obtain

$$\begin{aligned} & \int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa dy \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau \\ & \leq C \int_0^t \left\| \left(\int_0^{S_x^\kappa} |y|_\kappa dy \right)_x \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau + C \int_0^t \left\| \overline{\int_0^{S_x^\kappa} |y|_\kappa dy} \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau \\ & \leq C \int_0^t \| |S_x^\kappa|_\kappa S_{xx}^\kappa \|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau + C \int_0^t 1 d\tau, \end{aligned} \quad (4.26)$$

which implies, by (4.18), that

$$\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa dy \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau \leq C. \quad (4.27)$$

Hence (4.20) follows, and we get $\int_0^{S_x^\kappa} |y|_\kappa dy \in L^{\frac{4}{3}}(0, T_e; W^{1, \frac{4}{3}}(\Omega))$. Making use of the Sobolev embedding theorem, we get (4.21).

It remains to prove estimate (4.22), since (4.23) is equivalent to (4.22). We rewrite $\int_0^{S_x^\kappa} |y|_\kappa dy$ as

$$\begin{aligned} \int_0^{S_x^\kappa} |y|_\kappa dy &= \int_0^{S_x^\kappa} |y| dy + \int_0^{S_x^\kappa} (|y|_\kappa - |y|) dy \\ &= \frac{1}{2} |y|_0^2 + \int_0^{S_x^\kappa} \frac{\kappa^2}{|y|_\kappa + |y|} dy \\ &= \frac{1}{2} |S_x^\kappa|_\kappa^2 + \int_0^{S_x^\kappa} \frac{\kappa^2}{|y|_\kappa + |y|} dy. \end{aligned} \quad (4.28)$$

Thus

$$\frac{1}{2}(|S_x^\kappa|S_x^\kappa)_x = \left(\int_0^{S_x^\kappa} |y|_\kappa dy \right)_x - \frac{\kappa^2 S_{xx}^\kappa}{|S_x^\kappa|_\kappa + |S_x^\kappa|}. \quad (4.29)$$

By (4.8) and the Young inequality we obtain from (4.5) and the assumption that $k \leq 1$ that

$$\begin{aligned} \left| \frac{\kappa^2 S_{xx}^\kappa}{|S_x^\kappa|_\kappa + |S_x^\kappa|} \right| &\leq |\kappa S_{xx}^\kappa|, \text{ thus} \\ \|\kappa S_{xx}^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} &\leq \left(\int_{Q_{T_e}} (\kappa^2 + \kappa |S_{xx}^\kappa|^2) dx d\tau \right)^{\frac{3}{4}} \leq C. \end{aligned} \quad (4.30)$$

Combination with (4.20) and (4.29) yields

$$\|(|S_x^\kappa|S_x^\kappa)_x\|_{L^{\frac{4}{3}}(Q_{T_e})} \leq C \left\| \left(\int_0^{S_x^\kappa} |y|_\kappa dy \right)_x \right\|_{L^{\frac{4}{3}}(Q_{T_e})} + C \|\kappa S_{xx}^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \leq C. \quad (4.31)$$

It is clear that $\overline{|S_x^\kappa|S_x^\kappa} \leq C \int_\Omega |S_x^\kappa|^2 dx \leq C$. Applying again the Poincaré inequality to the function $f = |S_x^\kappa|S_x^\kappa$, we arrive at

$$\| |S_x^\kappa|S_x^\kappa \|_{L^{\frac{4}{3}}(Q_{T_e})} \leq C.$$

Hence this, combined with (4.31), implies that

$$\| |S_x^\kappa|S_x^\kappa \|_{L^{\frac{4}{3}}(0, T_e; W^{1, \frac{4}{3}}(\Omega))} \leq C,$$

one concludes by using the Sobolev embedding theorem that

$$\| |S_x^\kappa|S_x^\kappa \|_{L^{\frac{4}{3}}(0, T_e; L^\infty(\Omega))} \leq C,$$

which is

$$\| S_x^\kappa \|_{L^{\frac{8}{3}}(0, T_e; L^\infty(\Omega))} \leq C.$$

This completes the proof of the corollary.

To apply some compactness lemma to the approximate solutions, we need estimates on the time derivative of the unknown S^κ and also $|S_x^\kappa|S_x^\kappa$.

Lemma 4.2 *The function S_t^κ belongs to $L^{\frac{4}{3}}(Q_{T_e})$ and we have the estimates*

$$\| S_t^\kappa \|_{L^{4/3}(Q_{T_e})} \leq C, \quad (4.32)$$

$$\| (|S_x^\kappa|S_x^\kappa)_t \|_{L^1(0, T_e; H^{-2}(\Omega))} \leq C. \quad (4.33)$$

Proof. From equation (3.3) and the estimates (4.18), and (4.5) we immediately see that $S_t^\kappa \in L^{\frac{4}{3}}(Q_{T_e})$ and that (4.32) holds. Therefore we only need to prove the second estimate.

To verify (4.33) we must show that there exists a constant C , which is independent of κ , such that

$$\left| \left((|S_x^\kappa| S_x^\kappa)_t, \varphi \right)_{Q_{T_e}} \right| \leq C \|\varphi\|_{L^\infty(0, T_e; H^2(\Omega))} \quad (4.34)$$

for all functions $\varphi \in L^\infty(0, T_e; H_0^2(\Omega))$. To prove (4.34), we first prove that for any $1 \geq \delta > 0$ there holds

$$\left| \left(\left(\int_0^{S_x^\kappa} |y|_\delta dy \right)_t, \varphi \right)_{Q_{T_e}} \right| \leq C \|\varphi\|_{L^\infty(0, T_e; H^2(\Omega))} \quad (4.35)$$

for all functions $\varphi \in L^\infty(0, T_e; H_0^2(\Omega))$. Here δ is independent of κ . Inequality (4.34) is obtained from this estimate as follows: From $S_x^\kappa \in L^\infty(0, T_e, L^2(\Omega)) \subset L^2(Q_{T_e})$, $S_{xt}^\kappa \in L^2(Q_{T_e})$ and $\| |y|_\delta - |y| \| \leq \delta \rightarrow 0$ as $\delta \rightarrow 0$ we infer that $\| |S_x^\kappa|_\delta - |S_x^\kappa| \|_{L^\infty(Q_{T_e})} \rightarrow 0$. A straightforward computation yields that

$$\left(\int_0^{S_x^\kappa} |y|_\delta dy \right)_t = |S_x^\kappa|_\delta S_{xt}^\kappa. \quad (4.36)$$

Therefore, $\left(\int_0^{S_x^\kappa} |y|_\delta dy \right)_t = |S_x^\kappa|_\delta S_{xt}^\kappa \rightarrow |S_x^\kappa| S_{xt}^\kappa$ strongly in $L^2(Q_{T_e})$. Whence, as $\delta \rightarrow 0$,

$$\left(\left(\int_0^{S_x^\kappa} |y|_\delta dy \right)_t, \varphi \right) \rightarrow \frac{1}{2} \left((|S_x^\kappa| S_x^\kappa)_t, \varphi \right)_{Q_{T_e}}$$

for all $\varphi \in L^\infty(0, T_e; H_0^2(\Omega)) \subset L^\infty(Q_{T_e})$. This relation together with (4.35) implies (4.34).

Thus it suffices to prove (4.35). To simplify the notations we define

$$\mathcal{R}_\kappa := c\nu |S_x^\kappa|_\kappa S_{xx}^\kappa - \mathcal{F}^\kappa(|S_x^\kappa|_\kappa - \kappa). \quad (4.37)$$

Here $\mathcal{F}^\kappa = \mathcal{F}(u^\kappa, u_x^\kappa, S_x^\kappa, S_x^\kappa)$. Recalling estimate (4.4), we have

$$|\mathcal{R}_\kappa| \leq C \left(|S_x^\kappa|_\kappa |S_{xx}^\kappa| + (1 + |S_x^\kappa|) (|S_x^\kappa|_\kappa + \kappa) \right). \quad (4.38)$$

Multiplying equation (3.3) by $(|S_x^\kappa|_\delta \varphi)_x$, integrating the resulting equation with respect to (t, x) over Q_{T_e} , using integration by parts for the term with the time derivative and noting (4.36), we obtain

$$\begin{aligned} 0 &= (S_t^\kappa - \mathcal{R}_\kappa, (|S_x^\kappa|_\delta \varphi)_x)_{Q_{T_e}} \\ &= -(S_{xt}^\kappa, |S_x^\kappa|_\delta \varphi)_{Q_{T_e}} - (\mathcal{R}_\kappa, (|S_x^\kappa|_\delta \varphi)_x)_{Q_{T_e}} \\ &= - \left(\left(\int_0^{S_x^\kappa} |y|_\delta dy \right)_t, \varphi \right)_{Q_{T_e}} - \left(\mathcal{R}_\kappa, (|y|_\delta)' \Big|_{y=S_x^\kappa} S_{xx}^\kappa \varphi \right) - (\mathcal{R}_\kappa, |S_x^\kappa|_\delta \varphi_x). \end{aligned} \quad (4.39)$$

Remembering that $S_{xt}^\kappa \in L^2(Q_{T_e})$ for any fixed κ , we see that the first term in the second equality of (4.39) is properly defined.

To estimate the last two terms on the right hand side of inequality (4.39), we note that there holds

$$|(y|_\delta)'| = \left| \frac{y}{|y|_\delta} \right| \leq 1 \quad \text{and} \quad |y|_\delta \leq |y| + 1,$$

which yields the estimates

$$\begin{aligned}
& \left| \left(\mathcal{R}_\kappa, (|y|\delta)' \Big|_{y=S_x^\kappa} S_{xx}^\kappa \varphi \right)_{Q_{T_e}} \right| \leq (|\mathcal{R}_\kappa|, |S_{xx}^\kappa \varphi|)_{Q_{T_e}} \\
& \leq (|S_x^\kappa|_\kappa |S_{xx}^\kappa|^2, |\varphi|)_{Q_{T_e}} + ((1 + |S_x^\kappa|)(|S_x^\kappa|_\kappa + \kappa), |S_{xx}^\kappa \varphi|)_{Q_{T_e}} \\
& \leq C \|\varphi\|_{L^\infty(Q_{T_e})} + I \leq C \|\varphi\|_{L^\infty(0, T_e; H^2(\Omega))} + I,
\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}
\left| (\mathcal{R}_\kappa, |S_x^\kappa|_\delta \varphi_x)_{Q_{T_e}} \right| & \leq C \int_{Q_{T_e}} (|S_x^\kappa| + 1) |S_x^\kappa|_\kappa |S_{xx}^\kappa \varphi_x| d(\tau, x) \\
& \quad + C \int_{Q_{T_e}} (1 + |S_x^\kappa|)^2 (|S_x^\kappa|_\kappa + \kappa) |\varphi_x| d(\tau, x) \\
& = J_1 + J_2.
\end{aligned} \tag{4.41}$$

We estimate I first. Write

$$\begin{aligned}
I & = C (|S_x^\kappa|_\kappa + \kappa + |S_x^\kappa| |S_x^\kappa|_\kappa + \kappa |S_x^\kappa|, |S_{xx}^\kappa \varphi|)_{Q_{T_e}} \\
& = I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.42}$$

One needs to estimate I_i ($i = 1, 2, 3, 4$). By estimate (4.5), I_1 can be treated as

$$\begin{aligned}
I_1 & = C (|S_x^\kappa|_\kappa, |S_{xx}^\kappa \varphi|)_{Q_{T_e}} \\
& \leq C \int_{Q_{T_e}} |S_x^\kappa|_\kappa^{\frac{1}{2}} |S_x^\kappa|_\kappa^{\frac{1}{2}} |S_{xx}^\kappa| |\varphi| d(t, x) \\
& \leq C \int_0^{T_e} \| |S_x^\kappa|_\kappa^{\frac{1}{2}} \|_{L^4(\Omega)} \| |S_x^\kappa|_\kappa^{\frac{1}{2}} S_{xx}^\kappa \| \|\varphi\|_{L^4(\Omega)} d\tau \\
& \leq C \left(\int_0^{T_e} \| |S_x^\kappa|_\kappa^{\frac{1}{2}} S_{xx}^\kappa \|^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^{T_e} \|\varphi\|_{L^4(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \\
& \leq C \|\varphi\|_{L^2(0, T_e; L^4(\Omega))},
\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}
I_2 & = C (\kappa, |S_{xx}^\kappa \varphi|)_{Q_{T_e}} \\
& \leq C \int_{Q_{T_e}} \kappa |S_{xx}^\kappa| |\varphi| d(t, x) \\
& \leq C \kappa \|S_{xx}^\kappa\|_{L^2(Q_{T_e})} \|\varphi\|_{L^2(Q_{T_e})} \\
& \leq C \|\varphi\|_{L^2(0, T_e; L^2(\Omega))}.
\end{aligned} \tag{4.44}$$

With the help of (4.19) and of the Cauchy-Schwarz inequality, we deal with I_4 as follows

$$\begin{aligned}
I_4 & = C (\kappa |S_x^\kappa|, |S_{xx}^\kappa \varphi|)_{Q_{T_e}} \\
& \leq C \int_0^{T_e} \left(\int_\Omega |S_x^\kappa S_{xx}^\kappa|^{\frac{4}{3}} dx \right)^{\frac{3}{4}} \left(\int_\Omega |\varphi|^4 dx \right)^{\frac{1}{4}} dt \\
& \leq C \|S_x^\kappa S_{xx}^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \|\varphi\|_{L^4(Q_{T_e})} \\
& \leq C \|\varphi\|_{L^4(0, T_e; L^4(\Omega))}.
\end{aligned} \tag{4.45}$$

The remaining term I_3 is the most difficult to evaluate. Making use of estimates (4.5) and (4.23), we have

$$\begin{aligned}
I_3 &= C (|S_x^\kappa| |S_x^\kappa|_\kappa, |S_{xx}^\kappa \varphi|)_{Q_{T_e}} \\
&\leq C \|\varphi\|_{L^\infty(Q_{T_e})} \int_0^{T_e} \left(\int_\Omega |S_x^\kappa|^2 |S_x^\kappa|_\kappa dx \right)^{\frac{1}{2}} \left(\int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq C \|\varphi\|_{L^\infty(Q_{T_e})} \int_0^{T_e} (\|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac{1}{2}} + 1) \left(\int_\Omega |S_x^\kappa|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx \right)^{\frac{1}{2}} dt \\
&\leq C \|\varphi\|_{L^\infty(Q_{T_e})} \left(\int_0^{T_e} (\|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac{1}{2}} + 1)^2 dt \right)^{\frac{1}{2}} \left(\int_0^{T_e} \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx dt \right)^{\frac{1}{2}} \\
&\leq C \|\varphi\|_{L^\infty(Q_{T_e})}. \tag{4.46}
\end{aligned}$$

Next, we consider J_1, J_2 . The term J_1 can be handled as

$$\begin{aligned}
J_1 &= C \int_{Q_{T_e}} (|S_x^\kappa| + 1) |S_x^\kappa|_\kappa |S_{xx}^\kappa \varphi_x| d(\tau, x) \\
&\leq C \|\varphi_x\|_{L^\infty(Q_{T_e})} \int_0^t \left(\int_\Omega (1 + |S_x^\kappa|)^4 dx \right)^{\frac{1}{4}} \left(\int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^{\frac{4}{3}} dx \right)^{\frac{3}{4}} d\tau \\
&\leq C \|\varphi_x\|_{L^\infty(Q_{T_e})} \int_0^t \left(1 + \|S_x^\kappa\|_{L^\infty(\Omega)}^2 \|S_x^\kappa\|^2 \right)^{\frac{1}{4}} \left(\int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^{\frac{4}{3}} dx \right)^{\frac{3}{4}} d\tau \\
&\leq C \|\varphi_x\|_{L^\infty(Q_{T_e})} \left(\int_0^t (1 + \|S_x^\kappa\|_{L^\infty(\Omega)}^2) d\tau \right)^{\frac{1}{4}} \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^{\frac{4}{3}} dx d\tau \right)^{\frac{3}{4}} \\
&\leq C \|\varphi\|_{L^\infty(0, T_e; H^2(\Omega))}. \tag{4.47}
\end{aligned}$$

Here we used the estimates in (4.5) and Corollary 4.1, which will also be used to evaluate the term J_2 . Invoking inequality (4.38), we obtain that

$$\begin{aligned}
J_2 &\leq C \int_{Q_{T_e}} (1 + |S_x^\kappa|)^2 (|S_x^\kappa|_\kappa + \kappa) |\varphi_x| d(\tau, x) \\
&\leq C \|\varphi_x\|_{L^\infty(Q_{T_e})} \int_{Q_{T_e}} (1 + |S_x^\kappa|^3) d(\tau, x) \\
&\leq C \|\varphi_x\|_{L^\infty(Q_{T_e})} \int_0^{T_e} \left(1 + \|S_x^\kappa\|_{L^\infty(\Omega)} \int_\Omega |S_x^\kappa|^2 dx \right) d\tau \\
&\leq C \|\varphi_x\|_{L^\infty(Q_{T_e})} \left(1 + \int_0^{T_e} \|S_x^\kappa\|_{L^\infty(\Omega)} d\tau \right) \\
&\leq C \|\varphi\|_{L^\infty(0, T_e; H^2(\Omega))}. \tag{4.48}
\end{aligned}$$

Combination of (4.39) – (4.48) and using the Sobolev embedding theorem yield

$$\begin{aligned}
&\left| \left(\left(\int_0^{S_x^\kappa} |y|_\delta dy \right)_t, \varphi \right)_{Q_{T_e}} \right| \\
&\leq C \left(\|\varphi\|_{L^\infty(0, T_e; H_0^2(\Omega))} + \|\varphi\|_{L^\infty(Q_{T_e})} + \|\varphi\|_{L^4(0, T_e; L^4(\Omega))} \right) \\
&\leq C \|\varphi\|_{L^\infty(0, T_e; H_0^2(\Omega))}, \tag{4.49}
\end{aligned}$$

which implies (4.35) and we complete the proof.

5 Existence of solutions to the phase field model

We shall use in this section the a priori estimates established in the previous section to study the convergence of (u^κ, S^κ) as $\kappa \rightarrow 0$. We shall show that there is a subsequence, which converges to a weak solution of the initial-boundary value problem (1.20) – (1.23), thereby proving Theorem 1.1.

Note first that the estimates in Corollary 4.1 and Lemma 4.2, the fact that Ω is bounded, and Poincaré's inequality imply

$$\|S^\kappa\|_{W^{1,4/3}(Q_{T_e})} \leq C, \quad (5.1)$$

for a constant C independent of κ . Hence, we can select a sequence $\kappa_n \rightarrow 0$ and a function $S \in W^{1,4/3}(Q_{T_e})$, such that the sequence S^{κ_n} , which we again denote by S^κ , satisfies

$$\|S^\kappa - S\|_{L^{4/3}(Q_{T_e})} \rightarrow 0, \quad S_x^\kappa \rightharpoonup S_x, \quad S_t^\kappa \rightharpoonup S_t, \quad (5.2)$$

where the weak convergence is in $L^{4/3}(Q_{T_e})$.

As usual, since equation (3.3) is nonlinear, the weak convergence of S_x^κ is not enough to prove that the limit function solves this equation. In the following lemma we therefore show that S_x^κ converges pointwise almost everywhere:

Lemma 5.1 *There exists a subsequence of S_x^κ , we still denote it by S_x^κ , such that*

$$S_x^\kappa \rightarrow S_x, \quad \text{a.e. in } Q_{T_e}, \quad (5.3)$$

$$|S_x^\kappa|_\kappa \rightarrow |S_x|, \quad \text{a.e. in } Q_{T_e}, \quad (5.4)$$

$$|S_x^\kappa|_\kappa \rightharpoonup |S_x|, \quad \text{weakly in } L^{\frac{4}{3}}(Q_{T_e}), \quad (5.5)$$

$$\int_0^{S_x^\kappa} |y| dy \rightarrow \frac{1}{2} S_x |S_x|, \quad \text{strongly in } L^{\frac{4}{3}}(0, T_e; L^2(\Omega)), \quad (5.6)$$

$$\int_0^{S_x^\kappa} |y|_\kappa dy \rightarrow \frac{1}{2} S_x |S_x|, \quad \text{strongly in } L^{\frac{4}{3}}(0, T_e; L^2(\Omega)), \quad (5.7)$$

as $\kappa \rightarrow 0$.

The proof is based on the following two results:

Theorem 5.1 *Let B_0 be a normed linear space imbedded compactly into another normed linear space B which is continuously imbedded into a Hausdorff locally convex space B_1 . Assume that $1 \leq p < +\infty$, that $v, v_i \in L^p(0, T_e; B_0)$ for all $i \in \mathbb{N}$, that the sequence $\{v_i\}_{i \in \mathbb{N}}$ converges weakly to v in $L^p(0, T_e; B_0)$ and that $\{\frac{\partial v_i}{\partial t}\}_{i \in \mathbb{N}}$ is bounded in $L^1(0, T_e; B_1)$. Then v_i converges to v strongly in $L^p(0, T_e; B)$.*

Lemma 5.2 *Let $(0, T_e) \times \Omega$ be an open set in $\mathbb{R}^+ \times \mathbb{R}^n$ and assume that $1 < q < \infty$. Suppose that the functions $g_n, g \in L^q((0, T_e) \times \Omega)$ satisfy*

$$\|g_n\|_{L^q((0, T_e) \times \Omega)} \leq C, \quad g_n \rightarrow g \text{ almost everywhere in } (0, T_e) \times \Omega.$$

Then g_n converges to g weakly in $L^q((0, T_e) \times \Omega)$.

Theorem 5.1 is a general version of Aubin-Lions lemma valid under the weak assumption $\partial_t v_i \in L^1(0, T_e; B_1)$. This version, which we need here, is proved in Simon [25] and in Roubíček [24]. A proof of Lemma 5.2 can be found in [21, p.12].

Proof of Lemma 5.1: We choose $p = \frac{4}{3}$ and

$$B_0 = W^{1, \frac{4}{3}}(\Omega), \quad B = L^2(\Omega), \quad B_1 = H^{-2}(\Omega).$$

These spaces satisfy the assumptions of the theorem. Since the estimates (4.18), (4.20) and (4.33) imply that the sequence $\int_0^{S_x^\kappa} |y| dy$ is uniformly bounded in $L^p(0, T_e; B_0)$ for $\kappa \rightarrow 0$ and $\left(\int_0^{S_x^\kappa} |y| dy\right)_t$ is uniformly bounded in $L^1(0, T_e; B_1)$, it follows from Theorem 3.1 that there is a subsequence, still denoted by $\int_0^{S_x^\kappa} |y| dy$, which converges strongly in $L^p(0, T_e; B) = L^{\frac{4}{3}}(0, T_e; L^2(\Omega))$ to a limit function $G \in L^{\frac{4}{3}}(0, T_e; L^2(\Omega))$. Consequently, from this sequence we can select another subsequence, denoted in the same way, which converges almost everywhere in Q_{T_e} . Using that the mapping $y \mapsto f(y) := \int_0^y |\xi| d\xi = \frac{1}{2}y|y|$ has a continuous inverse $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$, we infer that also the sequence $S_x^\kappa = f^{-1}\left(\int_0^{S_x^\kappa} |y| dy\right)$ converges pointwisely almost everywhere to $f^{-1}(G)$ in Q_{T_e} . From the uniqueness of the weak limit we conclude that $f^{-1}(G) = S_x$ almost everywhere in Q_{T_e} .

For the proof of (5.7) we write

$$\int_0^{S_x^\kappa} |y|_\kappa dy = \int_0^{S_x^\kappa} |y| dy + \int_0^{S_x^\kappa} (|y|_\kappa - |y|) dy = I_1 + I_2.$$

It is easy to estimate I_2 as $\|I_2\|_{L^2(Q_{T_e})} \leq \|\kappa S_x^\kappa\|_{L^2(Q_{T_e})} \leq C\kappa \|S_x^\kappa\|_{L^\infty(0, T_e; L^2(\Omega))} \leq C\kappa \rightarrow 0$. Therefore, $\int_0^{S_x^\kappa} |y|_\kappa dy \rightarrow \lim_{\kappa \rightarrow 0} I_1 = \frac{1}{2}|S_x|S_x$ strongly in $L^{\frac{4}{3}}(0, T_e; L^2(\Omega))$. This is (5.7).

To prove (5.5) we note that the estimate $|S_x^\kappa|_\kappa \leq |S_x^\kappa| + \kappa$ and the inequality (5.1) together imply that the sequence $|S_x^\kappa|_\kappa$ is uniformly bounded in $L^{\frac{4}{3}}(Q_{T_e})$. Thus, (5.5) is a consequence of (5.4) and Lemma 5.2.

Proof of Theorem 1.1: Define the function u by inserting S into (3.9) where S is the limit function of the sequence S^κ . We shall prove that (u, S) is a weak solution of problem (1.20) – (1.23).

Remember first that by Lemma 4.1 we have $S \in L^\infty(Q_{T_e})$. From this relation, from the above definition of u we immediately see that u satisfies (1.25). Observe next that $\|S^\kappa\|_{L^\infty(0, T_e; H_0^1(\Omega))} \leq C$, by Lemma 4.1 and Sobolev's embedding theorem. This implies $S \in L^\infty(0, T_e; H_0^1(\Omega))$, since we can select a subsequence of S^κ which converges weakly to S in this space. Thus, S satisfies (1.26).

Noting that from (3.1) and (5.2)

$$\begin{aligned} \|\chi_\kappa * S^\kappa - S\|_{L^{\frac{4}{3}}(Q_{T_e})} &\leq \|\chi_\kappa * (S^\kappa - S)\|_{L^{\frac{4}{3}}(Q_{T_e})} + \|(S - \chi_\kappa * S)\|_{L^{\frac{4}{3}}(Q_{T_e})} \\ &\leq \|(S - \chi_\kappa * S)\|_{L^{\frac{4}{3}}(Q_{T_e})} + \|S^\kappa - S\|_{L^{\frac{4}{3}}(Q_{T_e})} \rightarrow 0, \end{aligned} \quad (5.8)$$

for $\kappa \rightarrow 0$, we conclude easily that the function u defined in this way satisfy weakly equation (1.20). It is thus enough to prove that the equation (1.20) – (1.21) are fulfilled in the weak sense. By definition, these equation are satisfied in the weak sense if the

relation (1.27) holds. To verify (1.27) we use that by construction S^κ solves (3.3). Now we multiply equation (3.3) by a test function $\varphi \in C_0^\infty((-\infty, T_e) \times \Omega)$ and integrate the resulting equation over Q_{T_e} , then obtain

$$\begin{aligned} 0 &= (S_t^\kappa, \varphi)_{Q_{T_e}} + (-c\nu |S_x^\kappa|_\kappa S_{xx}^\kappa + \mathcal{F}^\kappa(|S_x^\kappa|_\kappa - \kappa), \varphi)_{Q_T} \\ &= -(S_0^\kappa, \varphi(0))_\Omega - (S^\kappa, \varphi_t)_{Q_{T_e}} + \left(c\nu \int_0^{S_x^\kappa} |y|_\kappa dy, \varphi_x \right)_{Q_{T_e}} \\ &\quad + (\mathcal{F}^\kappa(|S_x^\kappa|_\kappa - \kappa), \varphi)_{Q_T}. \end{aligned}$$

Equation (1.27) follows from this relation if we show that

$$(S_0^\kappa, \varphi(0))_\Omega \rightarrow (S_0, \varphi(0))_\Omega, \quad (5.9)$$

$$(S^\kappa, \varphi_t)_{Q_{T_e}} \rightarrow (S, \varphi_t)_{Q_{T_e}}, \quad (5.10)$$

$$\left(\int_0^{S_x^\kappa} |y|_\kappa dy, \varphi_x \right)_{Q_{T_e}} \rightarrow \left(\frac{1}{2} |S_x| S_x, \varphi_x \right)_{Q_{T_e}}, \quad (5.11)$$

$$(\mathcal{F}^\kappa(|S_x^\kappa|_\kappa - \kappa), \varphi)_{Q_{T_e}} \rightarrow (\mathcal{F}, \varphi)_{Q_{T_e}}, \quad (5.12)$$

for $\kappa \rightarrow 0$. Now, the relation (5.9) follows from (4.2), and the relation (5.10) is a consequence of (5.2). By (5.7), one has (5.11).

To verify (5.12) we note that (5.8), (4.23) and the definition of \mathcal{F}^κ yield

$$\|\mathcal{F}^\kappa(|S_x^\kappa|_\kappa - \kappa)\|_{L^{\frac{4}{3}}(Q_{T_e})} \leq C, \quad (5.13)$$

$$\mathcal{F}^\kappa(|S_x^\kappa|_\kappa - \kappa) \rightarrow \mathcal{F}|S_x|, \text{ almost everywhere.} \quad (5.14)$$

Then by Lemma 5.2,

$$\mathcal{F}^\kappa(|S_x^\kappa|_\kappa - \kappa) \rightharpoonup \mathcal{F}|S_x|,$$

weakly in $L^{\frac{4}{3}}(Q_{T_e})$, which implies (5.12). Consequently (1.27) holds.

It remains to prove that the solution has the regularity properties stated in (1.28) and (1.29). The relation $S_t \in L^{\frac{4}{3}}(Q_{T_e})$ is implied by (5.2). To verify the second assertion in (1.28), we use estimate (4.23) to get

$$\int_0^{T_e} \|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac{8}{3}} dt \leq C.$$

This inequality and $S_x^\kappa \rightharpoonup S_x$ in $L^{\frac{8}{3}}(0, T_e; L^\infty(\Omega))$ imply $S_x \in L^{\frac{8}{3}}(0, T_e; L^\infty(\Omega))$.

To prove (1.29), we recall that $\int_0^{S_x^\kappa} |y|_\kappa dy$ converges to $|S_x| S_x$ strongly in the space $L^{\frac{4}{3}}(0, T_e; L^2(\Omega)) \subset L^{\frac{4}{3}}(Q_{T_e})$ and that $\left(\int_0^{S_x^\kappa} |y|_\kappa dy \right)_x$ is uniformly bounded in $L^{\frac{4}{3}}(Q_{T_e})$ for $\kappa \rightarrow 0$, by (4.18). This together implies that $(|S_x| S_x)_x \in L^{\frac{4}{3}}(Q_{T_e})$. Finally, to prove the second assertion of (1.29) we choose a test function $\varphi \in L^4(0, T_e, W_0^{1,4}(\Omega))$, multiply equation (3.3) by $-\varphi_x$ and integrate the resulting equation over Q_{T_e} to obtain

$$0 = (S_t^\kappa - \mathcal{R}_\kappa, -\varphi_x)_{Q_{T_e}} = (S_{xt}^\kappa, \varphi)_{Q_{T_e}} + (\mathcal{R}_\kappa, \varphi_x)_{Q_{T_e}}, \quad (5.15)$$

with \mathcal{R}_κ defined in (4.37). Invoking the estimates (4.5) and (4.23) we deduce that

$$\|\mathcal{R}_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \leq C,$$

hence equation (5.15) yields

$$(S_{xt}^\kappa, \varphi)_{Q_{T_e}} \leq \|\mathcal{R}_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \|\varphi_x\|_{L^4(Q_{T_e})} \leq C \|\varphi\|_{L^4(0, T_e; W_0^{1,4}(\Omega))},$$

and this means that S_{xt}^κ is uniformly bounded in $L^{\frac{4}{3}}(0, T_e; W^{-1, \frac{4}{3}}(\Omega))$. From this estimate and from $S_t^\kappa \rightharpoonup S_t$ in $L^{\frac{4}{3}}(Q_{T_e})$ we deduce easily that S_{xt} belongs to the dual space of $L^4(0, T_e; W_0^{1,4}(\Omega))$, which is $L^{\frac{4}{3}}(0, T_e; W^{-1, \frac{4}{3}}(\Omega))$.

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