Low Mach number limit of viscous polytropic fluid flows

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Abstract

This paper studies the singular limit of the non-isentropic Navier-Stokes equations with zero thermal coefficient in a two dimensional bounded domain as the Mach number goes to zero. A uniform existence result is obtained in a time interval independent of the Mach number, provided that the initial data satisfy the "bounded derivative conditions", that is, the time derivatives up to order two are bounded initially, and Navier's slip boundary condition is imposed.

Keywords:
Singular limit, Navier-Stokes equations, Mach number, non-isentropic

1. Introduction and Main Results

The motion of low Mach number fluid flows are described by the following non-dimensional Navier-Stokes equations:

$$\rho_t + \text{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}\mathbb{S} + \frac{1}{\epsilon^2}\nabla p = 0, \tag{1.2}$$

$$(\rho e)_t + \text{div}(\rho u e) + p\text{div}u - \text{div}(\kappa\nabla\mathcal{T}) = \epsilon^2\mathbb{S} \cdot D(u), \tag{1.3}$$

where $\rho, u, p, e, \mathcal{T}$ stand for the density, velocity, pressure, internal energy and temperature respectively. The constants μ, λ are the viscous coefficients with $\mu > 0$, $\mu + \lambda \geq 0$ (in 2D), ϵ is the Mach number, κ is the heat conductivity coefficient, and

$$\mathbb{S} \equiv \mathbb{S}(u) = 2\mu D(u) + \lambda\text{div}u I$$

is the viscous stress tensor, with $D(u) = (\nabla u + \nabla u^t)/2$. Moreover, we assume that the fluid flows are polytropic ideal gases:

$$e = C_V\mathcal{T}, \quad p = R\rho\mathcal{T}, \tag{1.4}$$

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where $C_V > 0$ is the specific heat at constant volume, and R is the generic gases constant. The ratio of specific heats is $\gamma = 1 + R/C_V$. In this paper, we are interested in the 2-dimensional flow. In this case, the velocity field is given by $u = (u^1, \ u^2)$. Formally, as ϵ tends to zero, the solutions to (1.1)-(1.3) converge to those for the following problem:

$$\rho_t + \mathrm{div}(\rho v) = 0, \tag{1.5}$$

$$(\rho v)_t + \mathrm{div}(\rho v \otimes v) - \mathrm{div}(2\mu D(v)) + \nabla \pi = 0, \tag{1.6}$$

$$C_V \gamma \mathrm{div} v = \mathrm{div}(\kappa \nabla (\frac{1}{\rho})). \tag{1.7}$$

Specifically, in the isentropic case, or under the assumption $\kappa = 0$, which is considered in this paper, (1.7) reduces to the incompressible constraint

$$\mathrm{div} v = 0.$$

This low Mach number limit process is singular. It is not only a physically interesting problem (see [31]), but also challenging mathematically, since it is difficult to obtain uniform estimates in Mach number which is necessary to justify the convergence to the background incompressible flows.

The investigation of low Mach number limit began at seventies of last century by EBIN [16] in the isentropic regime. KLAINERMAN and MAJDA [27, 28] set up a framework for studying this singular limit in case of no physical boundary conditions for smooth and "well-prepared" initial data, which means that the initial data is nearly appropriate for the limit equation. When the fluids are isentropic, the results are rather plentiful (see e.g. [4, 5, 11, 12, 13, 14, 23, 24, 25, 29, 30, 32, 36, 39, 40, 41, 42, 44]), even though the initial data are "ill-prepared".

In the non-isentropic regime, the behavior of solutions is much more complicated because the temperature variations play an important role in creating the resonance.

For Euler equations with the solid boundary condition, SCHOCHET [38] studied the low Mach number limit under the assumptions that the initial data are "well-prepared" and the entropy is purely transported. By analyzing the acoustics, METIVIER and SCHOCHET investigated the situation of "ill-prepared" data in $\mathbb{R}^n$ by the techniques of pseudo-differential operators and wave-packet transform [34]. These methods can also be adapted to the case of exterior domains [2], since the decay of energy also implies the local strong convergence. For the case of periodic boundary, one may refer to [35] for the results in one spacial dimension.

For the Navier-Stokes equations, the way to obtain the uniform estimates is very different from the case of the Euler equations, since the viscous dissipation effects would prevent us identifying the oscillatory acoustic waves from the background incompressible flows. In the case that both the density and the temperature vary in a small range of $O(\epsilon)$, the differential operators of $O(\frac{1}{\epsilon})$ are still anti-symmetric. HAGSTROM and LORENZ [22] proved a uniform estimate independent of $\epsilon \in (0, 1]$ and $t \in [0, +\infty)$ in the whole space, provided that the background incompressible fluid flows are sufficiently smooth. We remark that in the reality, the temperature may vary in a large range, the coefficient of the heat diffusion term is $O(\frac{1}{\epsilon})$, thus the strategies in the estimates for Euler equations do not apply. KIM and LEE [26] verified the low Mach number limit of local strong solutions of Navier-Stokes

equation with zero heat-conductivity and "well-prepared" initial data in $\mathbb{R}^3$. The strategy of KLAINERMAN and MAJDA [27, 28] and the elementary energy estimates are applied in [26]. Recently, ALAZARD studied the low Mach number limit of the full system in R^n for certain "ill-prepared" initial data [3] by the technique of pseudo-differential operators. Moreover, the uniform estimates in the Mach number hold in both R^n and T^n. However, the technique of pseudo-differential operators does not apply directly to the case of bounded domain due to the restriction of the Fourier transform. FEIREISL and NOVOTNÝ [18] considered the low Mach number limit for the periodic "variational solutions" to the full Navier-Stokes-Fourier equations for "ill-prepared" initial data for certain radiative gases [18], which excludes the ideal polytropic gas. Related results on bounded domains with various boundary conditions can be found in [15, 17, 19]. Note that different gaseous laws may lead to different frameworks for the low Mach number analysis.

The study of low Mach number limit for the non-isentropic Navier-Stokes equations governing ideal polytropic gases in bounded domains is far from completed. In a recent work, OU [37] studied the incompressible limit of non-isentropic Navier-Stokes equations with zero thermal conductivity coefficient in a finite interval. If the time derivatives up to order two are bounded initially, that is,

$$\|(\rho_0, u_0, q_0)\|_{H^2} + \|(\rho_t(0), u_t(0), q_t(0))\|_{H^1} + \|(\rho_{tt}(0), u_{tt}(0), q_{tt}(0))\|_{L^2} + \|\rho_0^{-1}\|_{L^\infty} \le C,$$

with $q = (p-1)/\epsilon$ being the pressure variation, the solutions are bounded uniformly with respect to the Mach number in the same class as initial data, which implies that the limiting solution is exactly an incompressible profile. In this case, the analysis relies on the estimates for the temporal derivatives, which serve as the only tangential derivatives to the boundary.

Note that in the usual sense of "well-prepared" initial data, only the time derivatives of first order are bounded initially in some norms (see [26, 27, 28] for example), which is almost equivalent to the boundedness of divu_0/ϵ and $\nabla p_0/\epsilon$. However, in [37] and this paper, there are more restrictive bounded derivative conditions on the initial data, due to the presence of solid boundary. This is motivated by the works [8, 9] of KREISS, who first developed the method of bounded derivatives. In this paper, the boundedness of time derivatives of second order coincide with the boundedness of $\|\mathrm{div}u_0/\epsilon^2\|_{H^1}$, $\|\nabla q_0/\epsilon^2\|_{H^1}$ and $\|u_0\|_{H^4}$.

Our purpose is to verify the low Mach number limit for the zero heat-conductive Navier-Stokes equations in bounded domains of two spatial dimensions. The geometry of boundary causes much difficulty when showing the uniform estimates in the Mach number, due to the boundary effects of the acoustic waves. In contrast to [3, 26], integrating by parts is usually invalid in the boundary case, especially when estimating high order spatial derivatives. Thus the usual way to balance the singular differential operators in the whole space or periodic space is not applicable. The result of this paper also generalize the 1D result [37] in some sense. In the analysis of 1D, high order spatial estimates can nearly be controlled by the temporal estimates. But it is not true in 2D when we handle the normal derivatives near the boundary. To circumvent this trouble, we separate the uniform full-norm estimates into the estimates of vorticity and of divergence of velocity. For this purpose, the most important observation is that, Navier's slip boundary condition (see (1.13)), is equivalent to a representation of vorticity on the boundary (see Lemma 2.7), which is only valid for

dimension two. Another observation is that,

$$\Delta u = \nabla \operatorname{div} u - \overrightarrow{\operatorname{curl}}\operatorname{curl} u,$$

where $\overrightarrow{\operatorname{curl}} = (-\partial_2, \partial_1)^t$ and $\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ for any $u = (u_1, u_2)^t$, so that the separation of divergence and vorticity is reasonable (yet, this is not limited to dimension two). Note that we don't take the strategy in [45] that dealing with the boundary estimates by consider the tangential and normal components near the boundary separately, since it may lose the uniform estimates of the highest order normal derivatives to the boundary. Furthermore, the strategy applied here is more simple and clear.

We prove the local existence of the strong solution to the Navier-Stokes equations under Navier's slip boundary conditions with the uniform estimates in a small time interval. The key ingredients of this paper are the global existence of the "essentially linear" system in any finite time interval, and the uniform estimates for this system in a small time interval independent of the Mach number. Note also that in this paper, the space-time derivatives of same order can be estimated as an entity. Then due to the completeness of L^2 estimates, one can show the uniform estimates of the full norm order by order. This is also a key point of this paper.

In the following, we assume that the heat conductivity coefficient $\kappa = 0$. We introduce the pressure variation q by

$$p = 1 + \epsilon q. \tag{1.8}$$

Then the non-dimensional system (1.1)-(1.3) can be rewritten as

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.9}$$

$$\rho(u_t + u \cdot \nabla u) + \frac{1}{\epsilon}\nabla q = \operatorname{div}(2\mu D(u)) + \lambda \nabla \operatorname{div} u, \tag{1.10}$$

$$\frac{1}{\gamma}(q_t + u \cdot \nabla q) + q\operatorname{div} u + \frac{1}{\epsilon}\operatorname{div} u = \frac{\gamma - 1}{\gamma}\epsilon(2\mu|D(u)|^2 + \lambda(\operatorname{div} u)^2). \tag{1.11}$$

We impose the initial conditions

$$(\rho, u, q)|_{t=0} = (\rho_0, u_0, q_0)(x), \quad x \in \Omega, \tag{1.12}$$

where $\Omega \subset R^2$ is a simply connected, bounded domain with smooth boundary $\partial\Omega$, and Navier's slip boundary conditions are described as

$$u \cdot n = 0, \quad \tau \cdot \mathbb{S}(u) \cdot n + \alpha u \cdot \tau = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.13}$$

where $\alpha(x) \geq 0$ is a C^2 function, n and τ are the unit outer normal and unit tangential vector to $\partial\Omega$, respectively. Navier's slip boundary conditions describe an interaction between a viscous fluid and a solid wall. Various existence results of compressible Navier-Stokes equations with this kind of boundary conditions have been proved by Tani [43], Zajaczkowski [45] and among others.

We denote the norm in Sobolev space $W^{k,p}(\Omega)$ by $\|\cdot\|_{W^{k,p}}$ for $k \geq 0$, $1 \leq p \leq \infty$, the norm in $H^k(\Omega)$ by $\|\cdot\|_{H^k}$ for $k \geq 0$ ($H^0 \equiv L^2$), the norm in $L^p(\Omega)$ by $\|\cdot\|_{L^p}$ for $1 \leq p \leq +\infty$. Moreover, we denote the Sobolev norm in $L^p(0, t; H^k(\Omega))$ by $\|\cdot\|_{L^p_t H^k}$ for

$1 \le p \le \infty$, $0 \le k < \infty$ and $0 < t < \infty$, and the norm in $C([0,t], H^k(\Omega))$ by $\|\cdot\|_{C_t H^k}$ for $k \ge 0$ and $t > 0$. Throughout this paper, we denote various positive constants, various positive and continuous functions of M_0, M (to be defined in Definition 2.1), which are all independent of ϵ, by C, $F_0(\cdot)$ and $F(\cdot)$, respectively. Moreover, we shall use $F_i(\cdot)$ $(i = 0, 1, 2, \cdots)$, $F(\cdot)$, $F_i(\cdot, \cdot)$, $F(\cdot, \cdot)$, $F_i(\cdot, \cdot, \cdot)$, $F(\cdot, \cdot, \cdot)$ e.t.c. to stand for various positive and continuous functions. Furthermore, we would denote the partial derivatives by subscripts and the components of a vector by superscripts. For example, u^j means the j-th component of a vector u, and u_y stands for the partial derivative of u with respect to y.

During the estimates, *Hölder*'s inequality, *Sobolev*'s inequality and the following *interpolation* inequality will be used implicitly.

Lemma 1.1. (Interpolation inequality, see Chaper II, [21]) *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with locally Lipschitz boundary $\partial\Omega$. Then for any $u \in W^{1,q}(\Omega)$,*

$$\|u\|_{L^r} \le C\|u\|_{L^q}^{1-\lambda}\|u\|_{W^{1,q}}^{\lambda}, \tag{1.14}$$

where $r \in [q, Nq/(N-q)]$ if $q \in [1, N)$, $r \in [q, +\infty)$ if $q \ge N$, and C is independent of u and $\lambda = N(r-q)/(rq)$. □

Below is the main theorem of this paper.

Theorem 1.1. *Suppose that the initial datum $(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon)$ satisfies that $\rho_0^\epsilon \ge C_0^{-1} > 0$, and*

$$\|(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon)\|_{H^2} + \|(\rho_t^\epsilon, u_t^\epsilon, q_t^\epsilon)(0)\|_{H^1} + \|(\rho_{tt}^\epsilon, u_{tt}^\epsilon, q_{tt}^\epsilon)(0)\|_{L^2} \le C_0. \tag{1.15}$$

Moreover, we assume that $(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon) \rightharpoonup (\rho_0, u_0, q_0)$ in $(H^2)^3$. Then the initial-boundary value problem (1.9)-(1.13) admits a unique solution $(\rho^\epsilon, u^\epsilon, q^\epsilon)$ in $C([0,T]_0; H^2(\Omega))^3$, for some positive constant T_0 independent of ϵ. Moreover, $(\rho^\epsilon, u^\epsilon, q^\epsilon)$ satisfies

$$\max_{t \in [0,T]_0} (\|(\rho^\epsilon, u^\epsilon, q^\epsilon)\|_{H^2} + \epsilon\|u^\epsilon\|_{H^3} + \|(\rho_t^\epsilon, u_t^\epsilon, q_t^\epsilon)\|_{H^1} + \|(\rho^\epsilon)^{-1}\|_{L^\infty})(t)$$

$$+ \|(\rho_{tt}^\epsilon, u_{tt}^\epsilon, q_{tt}^\epsilon)\|_{L^\infty(0,T_0;L^2)} + \left(\int_0^{T_0}(\|u^\epsilon\|_{H^3}^2 + \|u_t^\epsilon\|_{H^2}^2 + \|u_{tt}^\epsilon\|_{H^1}^2)dt\right)^{\frac{1}{2}} \le C. \tag{1.16}$$

Here C_0, $C = C(\delta_0, C_0)$ are positive constants independent of $\epsilon \in (0,1]$. Furthermore, $(\rho^\epsilon, u^\epsilon)$ converges to (ρ, u) in certain Sobolev spaces as $\epsilon \to 0$, and there exists a function $P(x,t)$ such that (ρ, u, P) in $C([0,T]_0; H^2(\Omega)^2 \times H^1(\Omega))$ solves the initial-boundary value problem of the inhomogeneous incompressible Navier-Stokes equations:

$$\begin{aligned}
&\rho_t + u \cdot \nabla\rho = 0, \quad \mathrm{div}u = 0, \\
&\rho(u_t + u \cdot \nabla u) + \nabla P = \mu\Delta u \quad \text{in} \quad \Omega \times (0, T_0) \\
&u \cdot n = \tau \cdot \mathbb{S}(u) \cdot n + \alpha u \cdot \tau = 0 \quad \text{on} \quad \partial\Omega \times (0, T_0), \\
&(\rho, u)|_{t=0} = (\rho_0, u_0).
\end{aligned} \tag{1.17}$$

□

Remark 1.1. *The notation $\rho_t(0)$ is indeed a quantity signifying for $-\mathrm{div}(\rho_0 u_0)$ by the density equation (1.9). Analogously, the notations $q_t(0) u_t(0)$, $\rho_{tt}(0)$, $u_{tt}(0)$, $q_{tt}(0)$ are defined recursively by the equations (1.9)-(1.11) and the initial data ρ_0, u_0 and q_0.*

Remark 1.2. *From the assumption that $q_t^\epsilon(0)$, u_0^ϵ, q_0^ϵ are bounded, and the weak convergence of initial data of compressible Navier-Stokes equations, we have $\mathrm{div}u_0 = 0$ a.e. in Ω, for the initial data of the incompressible Navier-Stokes equations.*

2. Uniform estimates of the "essentially linear" equations

We consider the following "essentially linear" equations:

$$\rho_t + \text{div}(\rho v) = 0, \tag{2.1}$$

$$\rho(u_t + v \cdot \nabla u) + \frac{1}{\epsilon}\nabla q = 2\mu\text{div}(D(u)) + \lambda\nabla\text{div}u, \tag{2.2}$$

$$\frac{1}{\gamma}(q_t + v \cdot \nabla q) + q\text{div}v + \frac{1}{\epsilon}\text{div}u = \frac{\gamma - 1}{\gamma}\epsilon(2\mu|D(v)|^2 + \lambda(\text{div}v)^2). \tag{2.3}$$

We impose the following initial conditions

$$(\rho, u, q)|_{t=0} = (\rho_0, u_0, q_0)(x), \ x \in \Omega, \tag{2.4}$$

and Navier's slip boundary condition

$$u \cdot n = 0, \quad \tau \cdot \mathbb{S}(u) \cdot n + \alpha u \cdot \tau = 0 \quad \text{on } \partial\Omega \times (0, T) \tag{2.5}$$

where n is the unit outer normal to $\partial\Omega$ and v is a given function satisfying $v|_{t=0} = u_0$, $v \cdot n|_{\partial\Omega} = 0$, $v_t|_{t=0} = u_t(0)$, $v \in C([0, T]; H^2) \cap L^2(0, T; H^3)$, $v_t \in C([0, T]; H^1) \cap L^2(0, T; H^2)$, and $v_{tt} \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$.

Theorem 2.1. (Global existence for the "essentially linear" system) *Suppose that the initial datum $(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon)$ satisfies*

$$(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon) \in H^2(\Omega)^3, \ (\rho_t^\epsilon(0), u_t^\epsilon(0), q_t(0)) \in H^1(\Omega)^3, \ (\rho_{tt}^\epsilon(0), u_{tt}^\epsilon(0), q_{tt}(0)) \in L^2(\Omega)^3,$$

with $\rho_0^\epsilon \geq \delta_0$ for some positive constant δ_0. Assume the following compatibility conditions are satisfied:

$$u_0^\epsilon \cdot n = \tau \cdot \mathbb{S}(u_0^\epsilon) \cdot n + \alpha u_0^\epsilon \cdot \tau = u_t^\epsilon(0) \cdot n = 0 \quad \text{on} \quad \partial\Omega. \tag{2.6}$$

Then the initial boundary problem (2.1)-(2.5) admits a unique solution $(\rho^\epsilon, u^\epsilon, q^\epsilon) \in C([0, T]; H^2 \times H^3 \times H^2)$ satisfying $\rho^\epsilon > 0$ in $\Omega \times (0, T)$ and

$$(\rho_t^\epsilon, u_t^\epsilon, q_t^\epsilon) \in C([0, T]; H^1)^3, \ u_t^\epsilon \in L^2(0, T; H^2),$$
$$(\rho_{tt}^\epsilon, u_{tt}^\epsilon, q_{tt}^\epsilon) \in L^\infty(0, T; L^2)^3, \ u_{tt}^\epsilon \in L^2(0, T; H^1).$$

$\square$

The proof of this theorem will be given in the last section of this article.

Definition 2.1.

$$M_0 := \sum_{i=0}^{2}\|\partial_t^i(\rho, u, q)(t = 0)\|_{H^{2-i}} + \|\rho_0^{-1}\|_{L^\infty},$$

$$M := \max_{t \in [0,T]}\left(\sum_{i=0}^{2}\|\partial_t^i v\|_{H^{2-i}} + \epsilon\|v\|_{H^3}\right)(t) + \left(\int_0^T \sum_{i=0}^{2}\|\partial_t^i v\|_{H^{3-i}}^2 dt\right)^{\frac{1}{2}}.$$

We will obtain two sets of the uniform estimates for (2.1)-(2.5) in the following proposition, which is the key to our problem. Note that both the short-time estimates and the long-time estimates are stated in this proposition, which are necessary for the proofs of both Theorem 1.1 and Theorem 2.1, respectively. Indeed, the intermediate long-time estimates are also applied in the proof of Theorem 2.1 when we extend the local solution in $[0, T_0]$ to the global solution in $[0, T]$ in a finite number of fixed time-steps.

Proposition 2.1. *Assume that the initial datum (ρ_0, u_0, q_0) satisfies*

$$M_0 \leq C_0, \tag{2.7}$$

for some positive constant C_0 independent of $\epsilon \in (0, 1]$. Suppose that (ρ, u, q) is the unique global solution described in Theorem 2.1. Then there exist positive constants $T_0(M)$ and $C(C_0)$ independent of $\epsilon \in (0, 1]$ and M, such that

$$
\max_{t \in [0, T_0]} (\|(\rho, u, q)\|_{H^2} + \epsilon \|u\|_{H^3} + \|(\rho_t, u_t, q_t)\|_{H^1} + \|\rho^{-1}\|_{L^\infty})(t)
$$
$$
+ \operatorname*{ess\,sup}_{t \in [0, T_0]} \|(\rho_{tt}, u_{tt}, q_{tt})\|_{L^2}(t) + \left(\int_0^{T_0} (\|u\|_{H^3}^2 + \|u_t\|_{H^2}^2 + \|u_{tt}\|_{H^1}^2) dt \right)^{\frac{1}{2}} \leq C. \tag{2.8}
$$

Moreover, if M is fixed, then $T_0 = T$, while C depends on both M and C_0. □

This proposition follows from Lemma 2.1 and Lemma 2.17, since $\epsilon \|u\|_{H^3}$ can be estimated directly from equation (1.10). In the sequel, we derive the estimates of ρ and the estimates of q and u separately, since the density equation (2.1) does not contain $O(1/\epsilon)$ terms.

2.1. Estimates of ρ

The proof of this part is standard, we sketch it here for the sake of completeness. First, we derive the lower bound for ρ. For any integer $k \geq 2$, we multiply (2.1) by $-\rho^{-k}$ to derive

$$
\frac{1}{k-1} \int_\Omega (\partial_t \rho^{1-k} + u \cdot \nabla \rho^{1-k}) dx - \int_\Omega \rho^{1-k} \operatorname{div} u \, dx = 0,
$$

which gives, by integrating by parts,

$$
\frac{d}{dt} \|\rho^{-1}\|_{L^{k-1}} \leq C \frac{k}{k-1} \|\operatorname{div} u\|_{L^\infty} \|\rho^{-1}\|_{L^{k-1}},
$$

for some constant $C > 0$ independent of k. Then by Gronwall's inequality and letting $k \to +\infty$, we obtain

$$
\|\rho^{-1}\|_{L^\infty}(t) \leq \|\rho_0^{-1}\|_{L^\infty} \exp\left(C\sqrt{t} \|v\|_{L_T^2 H^3} \right)
$$
$$
\leq C M_0 \exp\left(\sqrt{t} M \right), \quad \forall t \in [0, T].
$$

Next, we shall establish the H^2 estimates of ρ. It follows from (2.1) that

$$
(D^\alpha \rho)_t + v \cdot \nabla (D^\alpha \rho) + D^\alpha (\rho \operatorname{div} v) = [v, D^\alpha] \cdot \nabla \rho, \tag{2.9}
$$

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for $0 \leq |\alpha| \leq 2$, where D^α is the spatial derivative with multi-index α and the commutator $[a, b]$ is defined by $[a, b] := ab - ba$. Multiplying (2.9) by $D^\alpha \rho$ and integrating the resulting equality over Ω to get, and applying the Gronwall inequality, we have

$$\|\rho\|_{H^2}(t) \leq \|\rho_0\|_{H^2} \exp\left(C \int_0^t \|v\|_{H^3} ds\right)$$

$$\leq M_0 \exp\left(C\sqrt{t}M\right), \qquad \forall t \in [0, T].$$

Differentiating (2.1) in temporal variables once and calculating as above, we obtain

$$\|\rho_t\|_{L^2}(t) \leq \left(\|\rho_t(0)\|_{L^2} + \int_0^t \|v_t\|_{H^1} \|\rho\|_{H^2} ds\right) \exp\left(C \int_0^t \|v\|_{H^3} ds\right)$$

$$\leq (M_0 + \sqrt{t}M\|\rho\|_{C_t H^2}) \exp\left(C\sqrt{t}M\right), \qquad \forall t \in [0, T].$$

On account of the lower order estimates, one shows

$$\|\nabla \rho_t\|_{L^2}(t) \leq \exp\left(C \int_0^t \|v\|_{H^3} ds\right) \left(\|\rho_t(0)\|_{H^1} + \int_0^t (\|\rho_t\|_{L^2}\|v\|_{H^3} + \|\rho\|_{H^2}\|v_t\|_{H^2}) ds\right)$$

$$\leq \exp\left(C\sqrt{t}M\right) \left(M_0 + \sqrt{t}M(\|\rho_t\|_{C_t L^2} + \|\rho\|_{C_t H^2})\right), \qquad \forall t \in [0, T].$$

Finally, we estimate ρ_{tt} to complete the estimates of ρ. Differentiating (2.1) twice in time variables, we have

$$\rho_{ttt} + v \cdot \nabla \rho_{tt} + \operatorname{div} v \rho_{tt} + 2 v_t \cdot \nabla \rho_t + v_{tt} \cdot \nabla \rho + 2 \rho_t \operatorname{div} v_t + \rho \operatorname{div} v_{tt} = 0. \qquad (2.10)$$

Similar to the previous computations, we deduce

$$\frac{d}{dt} \|\rho_{tt}\|_{L^2}^2 \leq C(\|\operatorname{div} v\|_{L^\infty} \|\rho_{tt}\|_{L^2}^2 + C\|\rho_{tt}\|_{L^2}(\|v_t\|_{H^2}\|\nabla \rho_t\|_{L^2} +$$

$$+ \|v_{tt}\|_{H^1}\|\nabla \rho\|_{H^1} + \|\operatorname{div} v_t\|_{H^1}\|\rho_t\|_{H^1} + \|\operatorname{div} v_{tt}\|_{L^2}\|\rho\|_{H^2}),$$

and thus from the assumptions on the initial data, $\forall t \in [0, T]$,

$$\|\rho_{tt}\|_{L^2}(t) \leq \exp\left(C\sqrt{t}M\right) (M_0 + \sqrt{t}M(\|\rho_t\|_{C_t H^1} + \|\rho\|_{C_t H^2})).$$

Collecting all the estimates above and applying Gronwall's inequality, we obtain, by observing that T and M are fixed numbers in the long-time estimates in $[0, T]$,

Lemma 2.1. *There exist positive constants $T_1(M) := \min(T, (1 + M^2)^{-1})$ and $\tilde{C}$ such that for any $t \in [0, T_1]$,*

$$(\|\rho\|_{H^2} + \|\rho_t\|_{H^1} + \|\rho_{tt}\|_{L^2} + \|\rho^{-1}\|_{L^\infty})(t) \leq \tilde{C} M_0.$$

Moreover, we have, for any $t \in [0, T]$,

$$(\|\rho\|_{H^2} + \|\rho_t\|_{H^1} + \|\rho_{tt}\|_{L^2} + \|\rho^{-1}\|_{L^\infty})(t) \leq F(M_0, M),$$

for some positive continuous function $F(\cdot, \cdot)$. $\qquad \square$

The later estimate, which is a part of the long-time estimate in Proposition 2.1, can be obtain at the same time with a bound depending on both M and M_0.

In the following steps, we will estimate (u, q) by the anti-symmetric property of the singular operators.

2.2. L^2-estimates of (u, q)

Lemma 2.2. (Korn's inequality, see [45]) *Let $\Omega \subset \mathbb{R}^2$ be a simply connected open bounded domain with C^2 boundary $\partial\Omega$. Then for any $u \in H^1(\Omega)$, with $u \cdot n|_{\partial\Omega} = 0$, one has*

$$\|u\|_{H^1} \leq C\|D(u)\|_{L^2}. \tag{2.11}$$

□

From Lemma 2.2, one gets

$$-\int_\Omega (2\mu\mathrm{div}(D(u)) + \lambda\nabla\mathrm{div}u) \cdot u dx$$
$$= \int_\Omega (2\mu|D(u)|^2 + \lambda(\mathrm{div}u)^2)dx + \int_{\partial\Omega} \alpha(u \cdot \tau)^2 dS \geq \gamma_0\|u\|_{H^1}^2,$$

for some constant $\gamma_0 > 0$. Therefore we integrate the inner product of both sides of (2.2) and u over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \gamma_0\|u\|_{H^1}^2 - \frac{1}{\epsilon}\int_\Omega q\mathrm{div}u dx \leq 0. \tag{2.12}$$

Meanwhile, direct computations show that

$$\frac{1}{2\gamma}\frac{d}{dt}\|q\|_{L^2}^2 + \frac{1}{\epsilon}\int_\Omega q\mathrm{div}u dx \leq C\|\mathrm{div}v\|_{L^\infty}\|q\|_{L^2}^2 + C\epsilon\|\nabla v\|_{L^4}^2\|q\|_{L^2}.$$

It follows from the previous two inequalities, the un-signed large terms of $O(\frac{1}{\epsilon})$ are canceled:

$$\|\sqrt{\rho}u\|_{L^2}(t) + \|q\|_{L^2}(t)$$
$$\leq \exp\left(\sqrt{t}\|v\|_{L_T^2 H^3}\right)\left(\|(\sqrt{\rho_0}u_0, q_0)\|_{L^2} + \int_0^t \epsilon\|v\|_{H^2}^2 ds\right)$$
$$\leq C(M_0 + \epsilon t M)\exp(\sqrt{t}M), \quad \forall\, t \in [0, T].$$

Thus we conclude that

Lemma 2.3. *There exists a positive continuous function $F_1(\cdot)$, such that for any $t \in [0, T_1]$,*

$$\|(u, q)\|_{L^2}(t) + \|u\|_{L_t^2 H^1} \leq F_1(M_0).$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_1(M_0)$ replaced by $F(M_0, M)$.

□

2.3. Estimates of first order derivatives of (u, q)

In the following, we will estimate the time and spacial derivatives of u and q of the same order as an entity. That is one of our main strategies. First, we will estimate $\|\nabla u\|_{C_t L^2}$ and

$\|q_t\|_{L^2_t L^2}$. Differentiating (2.2) with respect to t and multiplying the resulting equation by u, we obtain, by the boundary condition in (2.5) and the immersion $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$,

$$\left(\mu\|D(u)\|^2_{L^2} + \frac{\lambda}{2}\|\mathrm{div} u\|^2_{L^2} + \int_{\partial\Omega} \frac{\alpha}{2}|u|^2 dS \right)(t) - \frac{1}{\epsilon} \int_0^t \int_\Omega \nabla q_t \cdot u dx ds$$

$$\leq C\left(\|u_0\|^2_{H^1} + \int_{\partial\Omega} |u_0|^2 dS \right) - \int_0^t \int_\Omega (\rho u_t)_t \cdot u dx ds$$

$$+ \int_0^t \int_\Omega \left((\rho_t v \cdot \nabla u + \rho(v_t \cdot \nabla u + v \cdot \nabla u_t)) \right) \cdot u dx ds \equiv C M_0^2 + I_1 + I_2,$$

where the estimates of u_{tt} are transferred into the estimates of lower order derivatives:

$$|I_1| \leq \left| \int_\Omega \rho u_t \cdot u dx \right|_{s=0}^{t} + \|\sqrt{\rho} u_t\|^2_{L^2_t L^2}$$

$$\leq \frac{1}{4}\|u_t\|^2_{L^2}(t) + \|\rho\|^2_{L^\infty_{x,t}}\|u\|^2_{L^2}(t) + M_0^3 + \|\sqrt{\rho} u_t\|^2_{L^2_t L^2}$$

and

$$|I_2| \leq \int_0^t \left(\|\rho_t\|_{H^1}\|v\|_{H^2}\|u\|^2_{H^1} + \|\rho\|_{H^2}\|u\|_{H^1}(\|v_t\|_{H^1}\|\nabla u\|_{L^2} + \|v\|_{H^2}\|\nabla u_t\|_{L^2}) \right) ds$$

$$\leq \frac{1}{4}\|\nabla u_t\|^2_{L^2_t L^2} + \int_0^t ((\|\rho_t\|_{H^1} + \|\rho\|_{H^2})M + \|\rho\|^2_{H^2}M^2)\|u\|^2_{H^1} ds.$$

Then we reach that, by Korn's inequality,

$$\|u\|^2_{H^1}(t) - \frac{1}{\epsilon} \int_0^t \int_\Omega \nabla q_t \cdot u dx ds$$

$$\leq F_0(M_0) + \frac{1}{4}(\|u_t\|^2_{L^2_t H^1} + \|u_t\|^2_{L^2}(t)) + \int_0^t \|\rho\|_{L^\infty}\|u_t\|^2_{L^2} ds$$

$$+ \int_0^t ((\|\rho_t\|_{H^1} + \|\rho\|_{H^2})M + \|\rho\|^2_{H^2}M^2)\|u\|^2_{H^1} ds, \quad \forall t \in [0, T].$$

On the other hand, we multiply (2.3) by q_t and integrate the resulting equation over $\Omega \times (0, t)$ to get

$$\frac{1}{\gamma}\|q_t\|^2_{L^2_t L^2} + \frac{1}{\epsilon} \int_0^t \int_\Omega \mathrm{div} u q_t dx ds$$

$$\leq \eta\|q_t\|^2_{L^2_t L^2} + C\eta^{-1} \int_0^t (\|v\|^2_{H^2}\|q\|^2_{H^1} + \epsilon^2\|\nabla v\|^4_{L^4}) ds$$

$$\leq \eta\|q_t\|^2_{L^2_t L^2} + C\eta^{-1} \int_0^t M^2(\|\nabla q\|^2_{L^2} + \|q\|^2_{L^2}) ds + CtM^4.$$

Summarizing the above two inequalities and choosing η sufficiently small, we obtain the following lemma by the L^2-estimates and the estimates of ρ.

Lemma 2.4. *There exist a positive constant $T_2(M) := \min\left(T, (1 + M^4)^{-1}\right)$ and a positive continuous function $F_0(\cdot)$, such that for any $t \in [0, T_2]$,*

$$\|u\|_{H^1}^2(t) + \frac{1}{2\gamma}\|q_t\|_{L_t^2 L^2}^2 \leq F_0(M_0) + \frac{1}{4}(\|u_t\|_{L_t^2 H^1}^2 + \|u_t\|_{L^2}^2(t))$$

$$+ \int_0^t F_0(M_0)(1 + M^2)(\|(u_t, \nabla q)\|_{L^2}^2 + \|u\|_{H^1}^2)ds.$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. $\square$

Next, we estimate $\|\nabla q\|_{C_t L^2}$ and $\|\nabla \text{div} u\|_{L_t^2 L^2}$. Note that the right-hand side of (2.2) can be written as

$$(2\mu + \lambda)\nabla \text{div} u - \mu \overrightarrow{\text{curl}} \text{curl} u,$$

where $\overrightarrow{\text{curl}} = (\partial_2, -\partial_1)^t$. Thus we multiply (2.2) by $\nabla \text{div} u$ and integrate to get, for any $t \in [0, T]$,

$$(2\mu + \lambda)\|\nabla \text{div} u\|_{L_t^2 L^2}^2 - \mu \int_0^t \int_\Omega \overrightarrow{\text{curl}} \text{curl} u \cdot \nabla \text{div} u \, dx \, ds - \frac{1}{\epsilon}\int_0^t \int_\Omega \nabla q \cdot \nabla \text{div} u \, dx \, ds$$

$$\leq \delta \|\nabla \text{div} u\|_{L_t^2 L^2}^2 + C\delta^{-1}\int_0^t \|\rho\|_{H^2}^2(\|u_t\|_{L^2}^2 + \|v\|_{H^2}^2\|\nabla u\|_{L^2}^2)ds.$$

In order to estimate the second term of left-hand side of the above inequality, we introduce a lemma concerning the boundary condition $u \cdot n|_{\partial\Omega} = 0$.

Lemma 2.5. *(see [10]) Suppose that Ω is a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and $v \in H^2(\Omega)$ with $v \cdot n = 0$ on $\partial\Omega$. Then we have*

$$2\tau \cdot D(v) \cdot n - \text{curl} v + 2\kappa v \cdot \tau = 0 \quad \text{on } \partial\Omega,$$

where $\text{curl} v = \partial_1 v_2 - \partial_2 v_1$ and κ is the curvature of $\partial\Omega$ given in a standard way by $\frac{d\tau}{ds} = -\kappa n$. $\square$

Due to Navier's slip boundary condition (2.5), Lemma 2.5 and the trace theorem, we have

$$\int_\Omega \overrightarrow{\text{curl}} \text{curl} u \cdot \nabla \text{div} u \, dx = \int_{\partial\Omega} \text{curl} u \frac{\partial}{\partial\tau} \text{div} u \, dS$$

$$= \int_{\partial\Omega} (2\kappa - \frac{\alpha}{\mu})(u \cdot \tau)\frac{\partial}{\partial\tau} \text{div} u \, dS$$

$$\leq \delta \|\text{div} u\|_{H^2(\Omega)}^2 + C\delta^{-1}\|u\|_{H^1(\Omega)}^2.$$

For $\delta < \mu/4$, we have, for any $t \in [0, T]$,

$$\frac{\mu}{2}\|\nabla \text{div} u\|_{L_t^2 L^2}^2 - \frac{1}{\epsilon}\int_0^t \int_\Omega \nabla q \cdot \nabla \text{div} u \, dx \, ds$$

$$\leq \delta \|\nabla^2 \text{div} u\|_{L_t^2 L^2}^2 + (\mu/4 + C\delta^{-1})\|u\|_{L_t^2 H^1}^2 \qquad (2.13)$$

$$+ C\delta^{-1}\int_0^t \|\rho\|_{H^2}^2(1 + M^2)\|(u_t, \nabla u)\|_{L^2}^2 ds.$$

In order to cancel the unsigned large integral in the above inequality, we apply the operator ∇ to (2.3) to get

$$\frac{1}{\gamma}(\nabla q_t + v \cdot \nabla^2 q + \nabla v \nabla q) + \nabla q \mathrm{div} v + q \nabla \mathrm{div} v + \frac{1}{\epsilon} \nabla \mathrm{div} u$$

$$= \frac{\gamma - 1}{\gamma} \epsilon \nabla (2\mu |D(v)|^2 + \lambda (\mathrm{div} v)^2).$$

Then we integrate the inner product of both sides of this equality and ∇q over Ω to get

$$\frac{1}{2\gamma} \|\nabla q\|_{L^2}^2(t) + \frac{1}{\epsilon} \int_0^t \int_\Omega \nabla \mathrm{div} u \cdot \nabla q dx ds$$

$$\leq CM_0^2 + \int_0^t (\|v\|_{H^3} \|q\|_{H^1}^2 + \epsilon^2 \|v\|_{H^3} \|v\|_{H^2}^2) ds$$

$$\leq CM_0^2 + \epsilon \sqrt{t} M^3 + \sqrt{t} M \|q\|_{L^2}^2(t) + \int_0^t \|v\|_{H^3} \|\nabla q\|_{L^2}^2 ds, \quad \forall t \in [0, T],$$

Combining (2.13) with the above inequality, and applying Lemma 2.3, we obtain

Lemma 2.6. *There exist a positive constant $T_3(M) := \min(T, (1 + M^6)^{-1})$ and a positive continuous function $F_0(\cdot)$, such that for any $t \in [0, T_3]$ and any $\delta \in (0, \mu/4]$,*

$$\|\nabla q\|_{L^2}^2(t) + \|\nabla \mathrm{div} u\|_{L_t^2 L^2}^2 \leq (1 + \delta^{-1}) F_0(M_0) + \delta \|\nabla^2 \mathrm{div} u\|_{L_t^2 L^2}^2$$

$$+ \int_0^t F_0(M_0) \delta^{-1} (1 + M^2 + \|v\|_{H^3}) \|(u_t, \nabla u, \nabla q)\|_{L^2}^2 ds.$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$.

$\square$

Next we estimate $\|(u_t, q_t)\|_{C_t L^2}$ and $\|u_t\|_{L_t^2 H^1}$. From (2.2) and (2.3), we have respectively

$$\rho(u_{tt} + v \cdot \nabla u_t) - \mathrm{div}(2\mu D(u_t)) - \lambda \nabla \mathrm{div} u_t + \frac{1}{\epsilon} \nabla q_t$$

$$= -\rho_t u_t - \rho_t v \cdot \nabla u - \rho v_t \cdot \nabla u, \tag{2.14}$$

and

$$\frac{1}{\gamma}(q_{tt} + v \cdot \nabla q_t + v_t \cdot \nabla q) + \frac{1}{\epsilon} \mathrm{div} u_t + q_t \mathrm{div} v + q \mathrm{div} v_t$$

$$= \frac{\gamma - 1}{\gamma} \epsilon \partial_t (2\mu |D(v)|^2 + \lambda (\mathrm{div} v)^2). \tag{2.15}$$

Then by (2.1) and (2.5), the integration of the inner product of (2.14) and u_t yields, for any

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$t \in [0, T]$,

$$\frac{1}{2}\|\sqrt{\rho}u_t\|_{L^2}^2(t) + 2\mu\|D(u_t)\|_{L_t^2 L^2}^2 + \lambda\|\text{div}u_t\|_{L_t^2 L^2}^2$$

$$+ \int_0^t \int_{\partial\Omega} \alpha|u_t|^2 dSds + \frac{1}{\epsilon}\int_0^t \int_\Omega \nabla q_t \cdot u_t dxds$$

$$\leq F_0(M_0) + \int_0^t (\|\rho_t\|_{H^1}\|u_t\|_{H^1}(\|u_t\|_{L^2} + \|v\|_{H^2}\|u\|_{H^1})$$

$$+ \|\rho\|_{H^2}\|v_t\|_{H^1}\|u\|_{H^1}\|u_t\|_{H^1})ds$$

$$\leq \eta\|u_t\|_{L_t^2 H^1}^2 + F_0(M_0) + \frac{C}{\eta}\int_0^t (\|\rho_t\|_{H^1} + \|\rho\|_{H^2})(1 + M)(\|u_t\|_{L^2}^2 + \|u\|_{H^1}^2)ds.$$

Similarly, we derive from (2.15) that, for any $t \in [0, T]$,

$$\frac{1}{2\gamma}\|q_t\|_{L^2}^2(t) + \frac{1}{\epsilon}\int_0^t \int_\Omega \text{div}u_t q_t dxds$$

$$\leq CM_0^2 + \sqrt{t}M^3 + \int_0^t (\|v\|_{H^3} + \|v_t\|_{H^2})\|(q_t, \nabla q)\|_{L^2}^2 ds.$$

By use of Korn's inequality, we combine the above two inequalities and choose η small enough to get

Lemma 2.7. *There exists a positive continuous function $F_0(\cdot)$, such that for any $t \in [0, T_3]$,*

$$\|(u_t, q_t)\|_{L^2}^2(t) + \|u_t\|_{L_t^2 H^1}^2$$

$$\leq F_0(M_0) + \int_0^t F_0(M_0)(1 + M + \|v\|_{H^3} + \|v_t\|_{H^2})(\|(u_t, \nabla q, q_t)\|_{L^2}^2 + \|u\|_{H^1}^2)ds.$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. □

Finally, we will estimate $\|\text{curl}u\|_{C_t L^2}$ and $\|\nabla\text{curl}u\|_{L_t^2 L^2}$ to close the estimates of the first order derivatives. Recall that $\text{curl}u$ and $\text{div}u$ are in different scales, thus it is convenient to estimate them separately. Let

$$w = \text{curl}u,$$

then the momentum equation becomes the equation for the vorticity w:

$$\rho(w_t + v \cdot \nabla w) - \mu\Delta w = g, \tag{2.16}$$

where $g := (\partial_2 \rho u_{1t} - \partial_1 \rho u_{2t}) + (\partial_2(\rho v) \cdot \nabla u_1 - \partial_1(\rho v) \cdot \nabla u_2)$. The boundary condition for w is

$$w|_{\partial\Omega} = (2\kappa - \frac{\alpha}{\mu})u \cdot \tau. \tag{2.17}$$

Direct calculations show that

$$\|\sqrt{\rho}w\|_{L^2}^2(t) + \mu\|\nabla w\|_{L_t^2 L^2}^2$$

$$\leq CM_0^4 + \int_0^t \int_\Omega gwdxds + \int_0^t \int_{\partial\Omega} w\frac{\partial w}{\partial n}dxds := CM_0^4 + J_1 + J_2. \tag{2.18}$$

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For any $t \in [0, T]$, one shows

$$|J_1| \leq \int_0^t \|w\|_{H^1}(\|\nabla\rho\|_{H^1}\|u_t\|_{L^2} + \|\rho\|_{H^2}\|v\|_{H^2}\|\nabla u\|_{L^2})ds$$

$$\leq \delta\|w\|_{L_t^2 H^1}^2 + C\delta^{-1}\int_0^t \|\rho\|_{H^2}^2(1 + M^2)\|(u_t, \nabla u)\|_{L^2}^2 ds.$$

Moreover, we deduce by the trace theorem that

$$|J_2| \leq C\int_0^t \int_{\partial\Omega} |\nabla w||u \cdot \tau|dSds \leq \delta\|\nabla w\|_{L_t^2 H^1}^2 + C\delta^{-1}\|u\|_{L_t^2 H^1}^2.$$

Then we derive the following lemma by (2.18), the estimates of J_1 and J_2, and the L^2 estimates, and by choosing $\delta \in (0, \delta_0]$ (for example), for some small positive constant δ_0.

Lemma 2.8. *There exists a positive continuous function $F_0(\cdot)$, such that for any $t \in [0, T]$ and any $\delta \in (0, \delta_0]$,*

$$\|w\|_{L^2}^2(t) + \|w\|_{L_t^2 H^1}^2 \leq (1 + \delta^{-1})F_0(M_0) + \delta\|(\nabla^2\mathrm{div}u, \nabla^2 w)\|_{L_t^2 L^2}^2$$

$$+ \int_0^t F_0(M_0)\delta^{-1}(1 + M^2)\|(u_t, \nabla u)\|_{L^2}^2 ds.$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ being replaced by $F(M_0, M)$.

$\square$

We are ready to derive the uniform estimates of first order derivatives. We define the following notations for convenience.

Definition 2.2.
$$\Phi_1(t) := \|(\nabla q, u_t, q_t)\|_{L^2}^2(t) + \|u\|_{H^1}^2(t)$$

$$+ \int_0^t (\|(u_t, w)\|_{H^1}^2 + \|(\nabla\mathrm{div}u, q_t)\|_{L^2}^2)ds.$$

We conclude from Lemmas 2.4, 2.6, 2.7 and 2.8, by choosing δ small enough and using the Gronwall inequality that

Lemma 2.9. *There exists a positive continuous function $F_0(\cdot)$, such that for any $t \in [0, T_3]$, and any positive constant η_1 which will be chosen sufficiently small later,*

$$\Phi_1(t) \leq (1 + \eta_1^{-1})F_0(M_0) + \eta_1\|(\nabla^2\mathrm{div}u, \nabla^2 w)\|_{L_t^2 L^2}^2. \tag{2.19}$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$.

$\square$

2.4. Estimates of second order derivatives

In order to obtain the lower bound of the density ρ, one needs to obtain the uniform high-norm estimates of u as well. We proceed the same procedure as in the estimates of the first order derivatives.

First, one needs to estimate $\|\nabla \mathrm{div} u\|_{C_t L^2}$ and $\|\nabla q_t\|_{L_t^2 L^2}$. We integrate by parts and use Lemma 2.5 to get

$$
\int_\Omega \overrightarrow{\mathrm{curl}}\mathrm{curl} u_t \cdot \nabla \mathrm{div} u \, dx
$$

$$
= \int_{\partial\Omega} (2\kappa - \frac{\alpha}{\mu}) u_t \cdot \tau(-n_2 \partial_1 \mathrm{div} u + n_1 \partial_2 \mathrm{div} u) ds
$$

$$
\leq \frac{1}{8}\|\nabla \mathrm{div} u\|_{H^1}^2 + C_1\|u_t\|_{H^1}^2.
$$

Multiplying both sides of (2.14) by $\nabla \mathrm{div} u$ and integrating the resulting product, one has

$$
\frac{2\mu+\lambda}{2}\|\nabla\mathrm{div} u\|_{L^2}^2(t) - \frac{1}{\epsilon}\int_0^t\int_\Omega \nabla q_t \cdot \nabla \mathrm{div} u \, dx ds
$$

$$
\leq \frac{1}{8}\|\nabla\mathrm{div} u\|_{L_t^2 H^1}^2 + C_1\|u_t\|_{L_t^2 H^1}^2 + \left|\int_0^t\int_\Omega \rho u_{tt}\cdot\nabla\mathrm{div} u \, dx ds\right|
$$

$$
+ \int_0^t \|\nabla\mathrm{div} u\|_{L^2}\Big(\|\rho_t\|_{H^1}(\|u_t\|_{H^1} + \|v\|_{H^2}\|\nabla u\|_{H^1})
$$

$$
+ \|\rho\|_{H^2}(\|v_t\|_{H^2}\|\nabla u\|_{L^2} + \|v\|_{H^2}\|\nabla u_t\|_{L^2})\Big) ds + CM_0^2 \qquad (2.20)
$$

$$
\leq \frac{1}{8}\|\nabla\mathrm{div} u\|_{L_t^2 H^1}^2 + C_1\|u_t\|_{L_t^2 H^1}^2 + F_0(M_0) + \left|\int_0^t\int_\Omega \rho u_{tt}\cdot\nabla\mathrm{div} u \, dx ds\right|
$$

$$
+ C\int_0^t (\|\rho_t\|_{H^1}^2 + \|\rho\|_{H^2}^2)(1+M^2)\|(u_t,\nabla u)\|_{H^1}^2 ds.
$$

Next, we convert the estimates concerning u_{tt} in terms of the estimates for other derivatives.

$$
\int_0^t\int_\Omega \rho u_{tt}\cdot\nabla\mathrm{div} u \, dx ds = \int_0^t\int_\Omega (\nabla\rho\cdot u_{tt} + \rho\,\mathrm{div} u_{tt})\mathrm{div} u \, dx ds
$$

$$
= \left(\int_\Omega \nabla\rho\cdot u_t \mathrm{div} u \, dx(s)\Big|_{s=0}^t + \int_0^t\int_\Omega u_t\cdot(\nabla\rho_t\mathrm{div} u + \nabla\rho\,\mathrm{div} u_t) dx ds\right)
$$

$$
+ \left(\int_\Omega \rho\,\mathrm{div} u_t \mathrm{div} u \, dx(s)\Big|_{s=0}^t + \int_0^t\int_\Omega \mathrm{div} u_t(\rho_t\mathrm{div} u + \rho\,\mathrm{div} u_t) dx ds\right)
$$

$$
:= K_1 + K_2.
$$

Note that for any $t\in[0,T]$,

$$
|K_1| \leq \|\rho_0\|_{H^2}\|u_t(0)\|_{L^2}\|\mathrm{div} u_0\|_{H^1} + \|\nabla\rho\|_{C_t H^1}\|u_t\|_{L^2}(t)\|\mathrm{div} u\|_{L^4}(t)
$$

$$
+ \int_0^t \|u_t\|_{H^1}(\|\nabla\rho_t\|_{L^2}\|\mathrm{div} u\|_{H^1} + \|\rho\|_{H^2}\|\mathrm{div} u_t\|_{L^2}) ds
$$

$$
\leq F_0(M_0) + C(1 + \|\rho\|_{C_t H^2}^2)(\|u_t\|_{L^2}^2(t) + \|\mathrm{div} u\|_{L^2}^2(t))
$$

$$
+ \frac{\mu+\lambda}{2}\|\nabla\mathrm{div} u\|_{L^2}^2(t) + C\int_0^t (\|\rho\|_{C_t H^2}^2 + \|\rho_t\|_{C_t H^1}^2)\|(u_t,\mathrm{div} u)\|_{H^1}^2 ds,
$$

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and similarly,

$$|K_2| \leq F_0(M_0) + \frac{1}{8}\|\mathrm{div}u_t\|_{L^2}^2(t) + C\|\rho\|_{C_tH^2}^2\|\mathrm{div}u\|_{L^2}^2(t)$$

$$+ C\int_0^t (\|\rho\|_{C_tH^2}^2 + \|\rho_t\|_{C_tH^1}^2)\|(u_t,\mathrm{div}u)\|_{H^1}^2 ds.$$

Then from (2.20), Lemma 2.9 and the estimates for K_1 and K_2, we obtain

$$\mu\|\nabla\mathrm{div}u\|_{L^2}^2(t) - \frac{1}{\epsilon}\int_0^t\int_\Omega \nabla q_t \cdot \nabla\mathrm{div}u\,dxds$$

$$\leq F_0(M_0) + \frac{1}{8}(\|\mathrm{div}u_t\|_{L^2}^2(t) + \|\nabla^2\mathrm{div}u\|_{L_t^2L^2}^2)$$

$$+ C_1\|u_t\|_{L_t^2H^1}^2 + C(1 + \|\rho\|_{C_tH^2}^2)(\|u_t\|_{L^2}^2(t) + \|\mathrm{div}u\|_{L^2}^2(t)) \qquad (2.21)$$

$$+ \int_0^t F_0(M_0)(1 + \|\rho\|_{H^2}^2 + \|\rho_t\|_{H^1}^2)\|(u_t,\mathrm{div}u)\|_{H^1}^2 ds.$$

To cancel the large quantity appearing in (2.21), we apply the operator ∇ to (2.3) and integrate the inner product of the resulting identity with ∇q_t to get

$$\frac{1}{\gamma}\|\nabla q_t\|_{L_t^2L^2}^2 + \frac{1}{\epsilon}\int_0^t\int_\Omega \nabla\mathrm{div}v \cdot \nabla q_t\,dxds$$

$$\leq \frac{1}{2\gamma}\|\nabla q_t\|_{L_t^2L^2}^2 + C\int_0^t (\|v\|_{H^2}^2\|q\|_{H^2}^2 + \epsilon^2\|\nabla^2 v\|_{L^2}^2\|\nabla v\|_{L^\infty}^2)ds \qquad (2.22)$$

$$\leq \frac{1}{2\gamma}\|\nabla q_t\|_{L_t^2L^2}^2 + t^{\frac{2}{3}}M^4 + C\int_0^t M^2\|q\|_{H^2}^2 ds.$$

By (2.21), (2.22), Lemmas 2.3 and 2.9, we obtain

Lemma 2.10. *There exist positive continuous functions $F_0(\cdot)$ and $F_2(\cdot)$ such that, for any $t \in [0, T_3]$, and any positive constant η_1 which will be chosen sufficiently small later,*

$$\mu\|\nabla\mathrm{div}u\|_{L^2}^2(t) + \frac{1}{2\gamma}\|\nabla q_t\|_{L_t^2L^2}^2$$

$$\leq \frac{1}{8}(\|\mathrm{div}u_t\|_{L^2}^2(t) + \|\nabla^2\mathrm{div}u\|_{L_t^2L^2}^2) + F_2(M_0)\eta_1\|(\nabla^2\mathrm{div}u, \nabla^2 w)\|_{L_t^2L^2}^2$$

$$+ (1 + \eta_1^{-1})F_0(M_0) + (1 + \eta_1^{-1})F(M_0)\int_0^t (1 + M^2)\|(u_t,\mathrm{div}u,\nabla q)\|_{H^1}^2 ds.$$

Moreover, there exist positive continuous functions $F(\cdot,\cdot)$ and $F_2(\cdot,\cdot)$ s.t. for any $t \in [0,T]$, the above estimate holds with $F_0(M_0)$, $F_2(M_0)$ being replaced by $F(M_0, M)$ and $F_2(M_0, M)$, respectively. $\square$

Second, we shall estimate $\|\nabla^2 q\|_{C_tL^2}$ and $\|\nabla^2\mathrm{div}u\|_{L_t^2L^2}$. Applying $\partial_{x_i}\partial_{x_j}$ (denoted by ∂_{ij}) to (2.3), and multiplying the resulting equation by $\partial_{ij}q$, then integrating this product

over $\Omega \times (0,T)$, we obtain by applying the interpolation inequality,

$$
\frac{1}{2\gamma}\|\partial_{ij}q\|_{L^2}^2(t) + \frac{1}{\epsilon}\int_0^t\int_\Omega \partial_{ijk}u_k\partial_{ij}q\,dxds
$$

$$
\leq M_0^2 + \int_0^t \|\partial_{ij}q\|_{L^2}(\|v\|_{H^3}\|q\|_{H^2} + \epsilon\|\nabla v\|_{L^\infty}\|\nabla v\|_{H^2} + \epsilon\|\nabla^2 v\|_{L^4}^2)ds
$$

$$
\leq M_0^2 + \int_0^t \|q\|_{H^2}\Big((\|v\|_{H^3} + \|v\|_{H^3}^{\frac{4}{3}})\|q\|_{H^2} + \|v\|_{H^3}^{\frac{4}{3}}\|v\|_{H^2}^{\frac{4}{3}} + \|v\|_{H^3}\|v\|_{H^2}^2\Big)ds \qquad (2.23)
$$

$$
\leq M_0^2 + C\int_0^t (1 + \|v\|_{H^3}^{\frac{4}{3}})\|q\|_{H^2}^2 ds + t^{\frac{1}{3}}M^{\frac{8}{3}} + t^{\frac{1}{2}}M^3.
$$

On the other hand, we deduce from (2.2) that

$$
(2\mu + \lambda)\|\partial_{ijk}u_k\|_{L_t^2 L^2}^2 - \frac{1}{\epsilon}\int_0^t\int_\Omega \partial_{ij}q\partial_{ijk}u_k\,dxds
$$

$$
\leq (\mu + \lambda)\|\partial_{ijk}u_k\|_{L_t^2 L^2}^2 + C\|\nabla^2 \mathrm{curl}u\|_{L_t^2 L^2}^2 \qquad (2.24)
$$

$$
+ \int_0^t \|\rho\|_{H^2}^2(\|u_t\|_{H^1}^2 + \|v\|_{H^2}^2\|\nabla u\|_{H^1}^2)ds.
$$

Thus we conclude from (2.23) and (2.24) that

Lemma 2.11. *There exist positive constants C_2 and $T_4(M) := \min(T, (1 + M^8)^{-1})$, and a positive continuous function $F_0(\cdot)$ such that, for any $t \in [0,T]_4$,*

$$
\frac{1}{2\gamma}\|\nabla^2 q\|_{L^2}^2(t) + \mu\|\nabla^2\mathrm{div}u\|_{L_t^2 L^2}^2
$$

$$
\leq F_0(M_0) + C_2\|\nabla^2\mathrm{curl}u\|_{L_t^2 L^2}^2 + \int_0^t F_0(M_0)(1 + M^2 + \|v\|_{H^3}^{\frac{4}{3}})\|(\nabla q, u_t, \nabla u)\|_{H^1}^2 ds.
$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0,T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. $\qquad\square$

Third, we need to estimate $\|(\mathrm{div}u_t, \nabla q_t)\|_{C_t L^2}$ and $\|\nabla\mathrm{div}u_t\|_{L_t^2 L^2}$. Regarding the time direction as a tangential direction to $\partial\Omega$, we have the boundary condition $u_{tt}\cdot n|_{\partial\Omega} = 0$, and thus

$$
\int_\Omega u_{tt}\cdot\nabla\mathrm{div}u_t dx = -\frac{1}{2}\frac{d}{dt}\|\mathrm{div}u_t\|_{L^2}^2. \qquad (2.25)
$$

We multiply (2.14) by $\nabla\mathrm{div}u_t/\rho$ and integrate to get, for any $t \in [0,T]$,

$$
\frac{1}{2}\|\mathrm{div}u_t\|_{L^2}^2(t) + (2\mu + \lambda)\|\sqrt{\rho^{-1}}\nabla\mathrm{div}u_t\|_{L_t^2 L^2}^2 - \frac{1}{\epsilon}\int_0^t\int_\Omega \frac{\nabla q_t}{\rho}\cdot\nabla\mathrm{div}u_t dxds
$$

$$
\leq CM_0^2 + C\|\nabla\mathrm{curl}u_t\|_{L_t^2 L^2}^2\|\rho^{-1}\|_{L_{t,x}^\infty} + C\int_0^t \Big(\|v\|_{H^2}^2\|\nabla u_t\|_{L^2}^2 \qquad (2.26)
$$

$$
+ \|v_t\|_{H^1}^2\|\nabla u\|_{H^1}^2 + \|\rho^{-1}\|_{L^\infty}^2\|\rho_t\|_{H^1}^2(\|u_t\|_{H^1}^2 + \|v\|_{H^2}^2\|\nabla u\|_{H^1}^2)\Big)ds.
$$

Next, we apply $\partial_t \nabla$ to (2.3) to get

$$
\frac{1}{\gamma}(\nabla q_{tt} + v \cdot \nabla^2 q_t + \nabla v \nabla q_t + v_t \cdot \nabla^2 q + \nabla v_t \nabla q) + \frac{1}{\epsilon}\nabla \mathrm{div} u_t + \nabla q_t \mathrm{div} v
$$
$$
+ q_t \nabla \mathrm{div} v + \nabla q \mathrm{div} v_t + q \nabla \mathrm{div} v_t = \frac{\gamma - 1}{\gamma}\epsilon \partial_t \nabla (2\mu |\mathrm{D}(v)|^2 + \lambda (\mathrm{div} v)^2). \tag{2.27}
$$

It follows from the density equation (2.1) that

$$
\int_\Omega \rho^{-1} \nabla q_t \cdot (\nabla q_{tt} + v \cdot \nabla^2 q_t) dx
$$
$$
= \frac{1}{2}\int_\Omega \rho^{-1}\left(\partial_t(|\nabla q_t|^2) + v \cdot \nabla |\nabla q_t|^2\right)dx
$$
$$
= \frac{1}{2}\frac{d}{dt}\|\sqrt{\rho^{-1}}\nabla q_t\|_{L^2}^2 - \int_\Omega \rho^{-1}\mathrm{div} v |\nabla q_t|^2 dx.
$$

Moreover, the right hand side of (2.27) can be estimated as

$$
\|\partial_t \nabla (2\mu |\mathrm{D}(v)|^2 + \lambda(\mathrm{div} v)^2)\|_{L^2}
$$
$$
\leq C(\|v_t\|_{H^2}\|\nabla v\|_{L^\infty} + \|\nabla^2 v\|_{L^4}\|\nabla v_t\|_{L^4})
$$
$$
\leq C(\|v_t\|_{H^2}\|v\|_{H^3}^{\frac{1}{3}}\|v\|_{H^2}^{\frac{2}{3}} + \|v\|_{H^3}^{\frac{1}{2}}\|v\|_{H^2}^{\frac{1}{2}}\|v_t\|_{H^2}^{\frac{1}{2}}\|v_t\|_{H^1}^{\frac{1}{2}})
$$
$$
\leq C(\|v_t\|_{H^2}^{\frac{3}{2}} + \|v\|_{H^3}\|v\|_{H^2}^2 + \|v\|_{H^3}\|v\|_{H^2} + \|v_t\|_{H^2}\|v_t\|_{H^1}).
$$

Then it yields from (2.27) and the above calculations, for any $t \in [0, T]$,

$$
\frac{1}{2\gamma}\|\sqrt{\rho^{-1}}\nabla q_t\|_{L^2}^2(t) + \frac{1}{\epsilon}\int_0^t \int_\Omega \rho^{-1}\nabla q_t \cdot \nabla \mathrm{div} u_t dx ds
$$
$$
\leq F_0(M_0) + \int_0^t \|\rho^{-1}\|_{L^\infty}((1 + M^2)\|v\|_{H^3} + \|v_t\|_{H^2} + \|v_t\|_{H^2}^{\frac{3}{2}})\|(\nabla q, q_t)\|_{H^1}^2 ds
$$
$$
+ \int_0^t \|\rho^{-1}\|_{L^\infty}\|v_t\|_{H^2}\|q\|_{L^2}^2 ds + \int_0^t \|\rho^{-1}\|_{L^\infty}(\|v_t\|_{H^2}^{\frac{3}{2}} + (1 + M^2)(\|v\|_{H^3} + \|v_t\|_{H^2}))ds.
$$

Thus we summarize the above inequality and (2.26) to get

Lemma 2.12. *There exist a positive constant C_3 and a positive continuous function $F_0(\cdot)$ such that, for any $t \in [0, T_4]$,*

$$
\|(\mathrm{div} u_t, \sqrt{\rho^{-1}}\nabla q_t)\|_{L^2}^2(t) + \|\sqrt{\rho^{-1}}\nabla \mathrm{div} u_t\|_{L_t^2 L^2}^2
$$
$$
\leq C_3 M_0 \|\nabla \mathrm{curl} u_t\|_{L_t^2 L^2}^2 + F_0(M_0)
$$
$$
+ F_0(M_0)\int_0^t (1 + (1 + M^2)\|v\|_{H^3} + \|v_t\|_{H^2}^{\frac{3}{2}})\|(u_t, \nabla u, q_t, \nabla q)\|_{H^1}^2 ds.
$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. □

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Similar to the estimates for the first order derivatives, we have to estimate $\|\mathrm{curl} u_t\|_{C_t L^2}$ and $\|\nabla \mathrm{curl} u_t\|_{L_t^2 L^2}$ to close the estimates. Differentiating (2.16) in time variable, we get

$$\rho(w_{tt} + v \cdot \nabla w_t) - \mu \Delta w_t = g_t - \rho_t w_t - (\rho_t v + \rho v_t) \cdot \nabla w, \tag{2.28}$$

where

$$|g_t| \leq C \Big(|\nabla \rho_t||u_t| + |\nabla \rho||u_{tt}| + (\rho|\nabla v| + |\nabla \rho||v|)|\nabla u_t|$$
$$+ (\rho|\nabla v_t| + |\rho_t||\nabla v| + |\nabla \rho_t||v| + |\nabla \rho||v_t|)|\nabla u| \Big).$$

The boundary condition for w_t is

$$w_t|_{\partial \Omega} = (2\kappa - \frac{\alpha}{\mu}) u_t \cdot \tau. \tag{2.29}$$

Then we derive the estimate for w_t by taking the inner product of (2.28) and w_t:

$$\|\sqrt{\rho} w_t\|_{L^2}^2(t) + 2\mu \|\nabla w_t\|_{L_t^2 L^2}^2$$
$$\leq F_0(M_0) + \int_0^t \int_{\partial \Omega} |u_t| dS ds + \left| \int_0^t \int_\Omega g_t w_t dx ds \right|$$
$$+ \left| \int_0^t \int_\Omega w_t (\rho_t w_t + (\rho_t v + \rho v_t) \cdot \nabla w) dx ds \right| \tag{2.30}$$
$$:= F_0(M_0) + C \|u_t\|_{L_t^2 H^1}^2 + L_1 + L_2.$$

By direct calculations, one shows, $\forall\, t \in [0, T]$,

$$|L_1| \leq C \int_0^t \|w_t\|_{H^1} \Big(\|u_t\|_{H^1} (\|\rho_t\|_{H^1} + \|\rho\|_{H^2} \|v\|_{H^2}) + \|\rho\|_{H^2} \|u_{tt}\|_{L^2}$$
$$+ \|\nabla u\|_{H^1} (\|\rho\|_{H^2} \|v_t\|_{H^1} + \|\rho_t\|_{H^1} \|v\|_{H^2}) \Big) ds$$
$$\leq \delta \|w_t\|_{L_t^2 H^1}^2 + C \delta^{-1} \int_0^t (\|\rho_t\|_{H^1} + \|\rho\|_{H^2}) \Big((1 + M^2) \|(u_t, \nabla u)\|_{H^1}^2 + \|u_{tt}\|_{L^2}^2 \Big) ds.$$

Similarly, for any $t \in [0, T]$, we have

$$|L_2| \leq \delta \|w_t\|_{L_t^2 H^1}^2 + C \delta^{-1} \int_0^t F_0(M_0)(1 + M^2) \|(w_t, \nabla w)\|_{L^2}^2 ds.$$

As a consequence of (2.30), the estimates of L_1 and L_2, one obtains

Lemma 2.13. *There exists a positive continuous function $F_0(\cdot)$ such that, for any $t \in [0, T_4]$,*

$$\|\sqrt{\rho} w_t\|_{L^2}^2(t) + \|w_t\|_{L_t^2 H^1}^2$$
$$\leq F_0(M_0) + \int_0^t F_0(M_0)(1 + M^2)(\|(u_t, \nabla u)\|_{H^1}^2 + \|u_{tt}\|_{L^2}^2) ds.$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, to complete the estimates for the vorticity, one needs to estimate $\|\nabla \mathrm{curl} u\|_{C_t L^2}$ and $\|\mathrm{curl} u\|_{L_t^2 H^2}$. Note that from the boundary conditions for w and w_t,

$$-\int_{\Omega} w_t \triangle w dx = \frac{1}{2}\frac{d}{dt}\|\nabla w\|_{L^2}^2 - \int_{\partial\Omega}(2\kappa - \frac{\alpha}{\mu})u_t \cdot \tau \frac{\partial w}{\partial n}dS.$$

Thus we multiply the vorticity equation (2.16) by $\rho^{-1}\triangle w$ and integrate the resulting product to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|_{L^2}^2 + \mu\|\sqrt{\rho^{-1}}\triangle w\|_{L^2}^2$$
$$\leq C\|w\|_{H^2}^2 + C\|u_t\|_{H^1}^2 + C\delta^{-1}(\|v\|_{H^2}^2\|\nabla w\|_{L^2}^2 + \|\rho^{-1}\|_{L^\infty}\|g\|_{L^2}^2),$$

by the trace theorem. It follows that, for any $t \in [0, T]$,

$$\|\nabla w\|_{L^2}^2(t) + \mu\|\sqrt{\rho^{-1}}\triangle w\|_{L_t^2 L^2}^2$$
$$\leq CM_0^2 + C\|w\|_{L_t^2 H^2}^2 + \int_0^t (1 + \|\rho\|_{H^2}^2)(1 + M^2)\|(u_t, \nabla u)\|_{H^1}^2 ds.$$

By (2.16) and the usual H^2 estimates for the elliptic equations, we have

$$\|w\|_{H^2}^2 \leq C(\|g\|_{L^2}^2 + \|\rho\|_{H^2}^2(\|w_t\|_{L^2}^2 + \|v\|_{H^2}^2\|\nabla w\|_{L^2}^2) + \|u \cdot \tau\|_{H^{3/2}(\partial\Omega)}^2),$$

and thus by the estimates obtained so far, we have, for any $t \in [0, T]$,

$$\int_0^t \|w\|_{H^2}^2 ds \leq \int_0^t (1 + \|\rho\|_{H^2}^2)(1 + M^2)\|(u_t, \nabla u)\|_{H^1}^2 ds.$$

Therefore the following lemma is shown.

Lemma 2.14. *There exists a positive continuous function $F_0(\cdot)$ such that, for any $t \in [0, T_4]$,*

$$\|\nabla w\|_{L^2}^2(t) + \|w\|_{L_t^2 H^2}^2 \leq F_0(M_0) + \int_0^t F_0(M_0)(1 + M^2)\|(u_t, \nabla u)\|_{H^1}^2 ds.$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. $\qquad\qquad\qquad\qquad\square$

Finally, one needs to estimate $\|(u_{tt}, q_{tt})\|_{C_t L^2}$ and $\|\nabla u_{tt}\|_{L_t^2 L^2}$ to close the energy estimates. Differentiating (2.2) twice in temporal variable, we obtain

$$\rho(u_{ttt} + v \cdot \nabla u_{tt}) + \frac{1}{\epsilon}\nabla q_{tt} - 2\mu \mathrm{div}(\mathrm{D}(u_{tt})) - \lambda\nabla \mathrm{div} u_{tt}$$
$$= -\rho_{tt}u_t - 2\rho_t u_{tt} - (\rho_{tt}v + 2\rho_t v_t + \rho v_{tt}) \cdot \nabla u - 2(\rho_t v + \rho v_t) \cdot \nabla u_t. \qquad (2.31)$$

We multiply this equation by u_{tt} and integrate to get that, $\forall\, t \in [0, T]$,

$$\frac{1}{2}\|\sqrt{\rho}u_{tt}\|_{L^2}^2(t) + \gamma\|u_{tt}\|_{L_t^2 H^1}^2 - \frac{1}{\epsilon}\int_0^t \int_\Omega \nabla q_{tt} \cdot u_{tt} dx ds$$
$$\leq F_0(M_0) + \int_0^t (\|\rho_{tt}\|_{L^2}^2 + \|\rho_t\|_{H^1}^2 + \|\rho\|_{H^2}^2)(1 + M^2)(\|(u_t, \nabla u)\|_{H^1}^2 + \|u_{tt}\|_{L^2}^2) ds, \tag{2.32}$$

by using Korn's inequality, for some constant $\gamma > 0$. Meanwhile, a similar calculation yields

$$\frac{1}{2\gamma}\frac{d}{dt}\|q_{tt}\|_{L^2}^2 + \frac{1}{\epsilon}\int_\Omega \mathrm{div}u_{tt}q_{tt} dx$$
$$\leq \|v\|_{H^2}^2\|q_{tt}\|_{L^2}^2 + \|q_{tt}\|_{L^2}(\|v_t\|_{L^\infty}\|\nabla q_t\|_{L^2} + \|v_{tt}\|_{H^1}\|\nabla q\|_{H^1}$$
$$+ \|\mathrm{div}v\|_{L^\infty}\|q_{tt}\|_{L^2} + \|\mathrm{div}v_t\|_{H^1}\|q_t\|_{H^1} + \|q\|_{H^2}\|\mathrm{div}v_{tt}\|_{L^2}$$
$$+ \|v_t\|_{H^2}\|v_t\|_{H^1} + \|\nabla v_{tt}\|_{L^2}\|v\|_{H^3}^{\frac{1}{3}}\|v\|_{H^2}^{\frac{2}{3}}),$$

by the interpolation inequality. Note that by Young's inequality, we have

$$\|\nabla v_{tt}\|_{L^2}\|v\|_{H^3}^{\frac{1}{3}}\|v\|_{H^2}^{\frac{2}{3}} \leq C(\|v_{tt}\|^{\frac{3}{2}} + \|v\|_{H^3}\|v\|_{H^2}^2).$$

It follows that for any $t \in [0, T]_4$,

$$\frac{1}{2\gamma}\|q_{tt}\|_{L^2}^2(t) + \frac{1}{\epsilon}\int_0^t \int_\Omega \mathrm{div}u_{tt}q_{tt} dx$$
$$\leq CM_0^2 + C\int_0^t (1 + M^2 + \|v\|_{H^3} + \|v_t\|_{H^2} + \|v_{tt}\|_{H^1}^{\frac{3}{2}})(\|q_{tt}\|_{L^2}^2 + \|(q_t, \nabla q)\|_{H^1}^2) ds. \tag{2.33}$$

From (2.32) and (2.33), one concludes the following lemma.

Lemma 2.15. *There exists a positive continuous function $F_0(\cdot)$, such that for any $t \in [0, T_4]$,*

$$(\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + \|q_{tt}\|_{L^2}^2)(t) + \|u_{tt}\|_{L_t^2 H^1}^2$$
$$\leq F_0(M_0) + \int_0^t F_0(M_0)(1 + M^2 + \|v\|_{H^3} + \|v_t\|_{H^2} + \|v_{tt}\|_{H^1}^{\frac{3}{2}})$$
$$\times \left(\|(u_{tt}, q_{tt})\|_{L^2}^2 + \|(u_t, \nabla u, q_t, \nabla q)\|_{H^1}^2\right) ds.$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. □

Definition 2.3.

$$\Phi_2(t) := \|(\nabla\mathrm{div}u, \nabla^2 q, \mathrm{div}u_t, \sqrt{\rho^{-1}}\nabla q_t, \sqrt{\rho}\mathrm{curl}u_t, \nabla\mathrm{curl}u, \sqrt{\rho}u_{tt}, q_{tt})\|_{L^2}^2$$
$$+ \int_0^t (\|(\nabla q_t, \nabla^2\mathrm{div}u, \sqrt{\rho^{-1}}\nabla\mathrm{div}u_t, \nabla\mathrm{curl}u_t)\|_{L^2}^2 + \|\mathrm{curl}u\|_{H^2}^2 + \|u_{tt}\|_{H^1}^2) ds.$$

Noting that $u_t \cdot n = 0$ on $\partial\Omega$, we have

$$\|u_t\|_{H^1}^2 \leq C(\|\mathrm{div}u_t\|_{L^2}^2 + \|\mathrm{curl}u_t\|_{L^2}^2). \tag{2.34}$$

It thus follows from Lemmas 2.10-2.15 that,

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Lemma 2.16. *There exists a positive continuous function $F_0(\cdot)$, such that for all $t \in [0, T_4]$,*

$$\Phi_2(t) \leq F_0(M_0) + F_2(M_0)\eta_1 \|(\nabla^2 \text{div} u, \nabla^2 w)\|_{L_t^2 L^2}^2$$
$$+ \int_0^t F_0(M_0)(1 + M^2 + \|v\|_{H^3} + \|v_t\|_{H^2} + \|v_{tt}\|_{H^1}^{\frac{3}{2}})(\Phi_1(s) + \Phi_2(s))ds. \tag{2.35}$$

Moreover, there exists a positive continuous function $F(\cdot, \cdot)$, s.t. for any $t \in [0, T]$, the above estimate holds with $F_0(M_0)$ replaced by $F(M_0, M)$. □

Summarizing the inequalities in Lemmas 2.9 and 2.16, and inserting the estimate of $\|\nabla^2 \text{div} u\|_{L_t^2 L^2}$, then the estimate of $\|\nabla^2 w\|_{L_t^2 L^2}$ in Lemma 2.14, into the right-hand side of the resulting inequality, we conclude the following uniform estimates by use of Lemma 2.3 and Gronwall's inequality.

Lemma 2.17. *There exists a positive continuous function $F_3(\cdot)$, such that for any $t \in [0, T_4]$,*

$$\|(u, q)\|_{L^2}(t) + \|u\|_{L_t^2 H^1} + \Phi_1(t) + \Phi_2(t) \leq F_3(M_0).$$

Furthermore, there exists a positive continuous function $F(\cdot, \cdot)$ such that, the left-hand side of the above inequality is less than $F(M, M_0)$, for any $t \in [0, T]$. □

3. Proof of the main theorem

In the following, we will prove Theorem 1.1 by fix-point arguments, given the global existence of the "essentially linear" equations in Theorem 2.1 and the uniform estimates in Proposition 2.1.

Proof of Theorem 1.1. Let $M := \tilde{C}M_0 + F_3(M_0)$ in Proposition 2.1, where $\tilde{C}$ and $F_3(\cdot)$ are defined in Lemma 2.1 and Lemma 2.17, respectively. Then we choose $T_0 > 0$, s.t. $T_0 < T_4 := T_4(M(M_0))$. Define the mapping $L: v \mapsto u$, where (ρ, q, u) is the unique solution to the problem (2.1)-(2.5). Moreover, we define

$$R_{M,T_0} := \Big\{ u \ \Big| \ u \in C([0, T_0], H^2) \cap L^2(0, T_0; H^3),$$
$$u_t \in C([0, T_0], H^1) \cap L^2(0, T_0; H^2),$$
$$u_{tt} \in L^\infty(0, T_0; L^2) \cap L^2(0, T_0; H^1);$$
$$\max_{t \in [0, T_0]} \Big(\|u\|_{H^2} + \|u_t\|_{H^1} \Big)(t) + \operatorname*{ess\,sup}_{t \in [0, T_0]} \|u_{tt}(t)\|_{L^2}$$
$$+ \Big(\int_0^{T_0} (\|u\|_{H^3}^2 + \|u_t\|_{H^2}^2 + \|u_{tt}\|_{H^1}^2)dt \Big)^{\frac{1}{2}} \leq M \Big\},$$

where M depends on M_0, but not on $\epsilon \in (0, 1]$.

Standard results on closeness of Sobolev spaces in L^p-norms ensure that R_{M,T_0} is a closed subset of $X := C([0, T_0], L^2)$. The compactness of $R_{M,T_0} \subset X$ follows directly from the celebrated Arzela-Ascoli theorem. And it is easy to check that R_{M,T_0} is also non-empty and convex. Given M (defined at the beginning of the proof) and for $T_0 < T_4$, L maps R_{M,T_0}

into itself by the uniform estimates in Proposition 2.1. Note that $\epsilon \in (0,1]$ is arbitrary and T_0 is independent of $\epsilon \in (0,1]$. The continuity of L in X can be shown in a routine manner. Assuming that $u^n = L(v^n)$, $u = L(v)$, and $v^n \to v$ in X, we can show easily that $u^n \to u$ in X by the Gronwall inequality.

Schauder's fixed-point theorem says that, if $K \subset X$ is compact and convex, and the mapping $A: K \to K$ is continuous, then A has a fixed point in K. Applying Schauder's fixed-point theorem, we conclude that L has a fixed point u in R_{M,T_0}. Thus there exists $(\rho^\epsilon, u^\epsilon, q^\epsilon)$ solving the nonlinear problem (1.9)-(1.13) in $\Omega \times [0, T_0]$, and satisfying (1.16) for all $\epsilon \in (0,1]$. Uniqueness in X can be shown by the Gronwall inequality. Then the uniform existence results in Theorem 1.1 is shown. It follows from the above uniform bounds and the Aubin-Lions Lemma that, there exists a subsequence of $(\rho^\epsilon, u^\epsilon, q^\epsilon)$, still denoted by $(\rho^\epsilon, u^\epsilon, q^\epsilon)$, such that

$$(\rho^\epsilon, u^\epsilon, q^\epsilon) \to (\rho, u, q) \quad \text{in} \quad L^\infty(0, T_0; H^s), \quad \forall s < 2,$$

$$u^\epsilon \to u \quad \text{in} \quad L^2(0, T_0; H^2)$$

and

$$(\rho_t^\epsilon, u_t^\epsilon, q_t^\epsilon) \to (\rho_t, u_t, q_t) \quad \text{in} \quad L^2(0, T_0; L^2)$$

as $\epsilon \to 0$. Therefore we can apply "curl" to (1.9)-(1.13), then pass to the limit to show that (ρ, u, q) is indeed a solution of the incompressible Navier-Stokes equations (1.17). □

In order to establish the global existence of the problem (2.1)-(2.5) in $\Omega \times [0, T]$ for any given constant $T > 0$, we should show the local existence theorem to the problem (2.2)-(2.5) first. In this step, we take $\epsilon = 1$ without loss of generality.

Lemma 3.1. (Local existence) *Assume that the initial datum* $(u_0, q_0) \in (H^2)^2$ *satisfies*

$$(u_t(0), q_t(0)) \in (H^1)^2, \ (u_{tt}(0), q_{tt}(0)) \in (L^2)^2$$

and the compatibility conditions

$$u_0 \cdot n = \tau \cdot \mathbb{S}(u_0) \cdot n + \alpha u_0 \cdot \tau = u_t(0) \cdot n = 0 \quad \text{on} \quad \partial\Omega,$$

while ρ, v *are known functions satisfying* $\rho > 0$ *in* $\Omega \times (0, T)$,

$$(\rho, v) \in C([0,T]; H^2)^2, \ (\rho_t, v_t) \in C([0,T]; H^1)^2, \ (\rho_{tt}, v_{tt}) \in L^\infty(0, T; L^2)^2,$$
$$v \in L^2(0, T; H^3), \ v_t \in C([0,T]; H^2), \ v_{tt} \in L^2(0, T; H^1),$$

$(v, \rho)|_{t=0} = (u_0, \rho_0)$ *and* $(v_t, \rho_t)|_{t=0} = (u_t(0), \rho_t(0))$. *Then there exists a positive constant* T', *such that the problem (2.2)-(2.5) admits a unique local solution* (u, q) *in* $\Omega \times (0, T')$, *satisfying*

$$\begin{aligned} &q \in C([0,T']; H^2), \ u \in C([0,T']; H_0^1 \cap H^2) \cap L^2(0, T'; H^3), \\ &q_t \in C([0,T']; H^1), \ u_t \in C([0,T']; H_0^1) \cap L^2(0, T'; H^2), \quad\quad (3.1) \\ &q_{tt} \in L^\infty(0, T'; L^2), \ u_{tt} \in L^\infty(0, T'; L^2) \cap L^2(0, T'; H^1). \end{aligned}$$

Proof. Since (2.2) contains ∇q and (2.3) contains $\mathrm{div} u$, we need to decouple the system so that the standard theory of linear equations applies. First, we show the existence of the approximate system:

$$\rho u_t - 2\mu \mathrm{div}(D(u)) - \lambda \nabla \mathrm{div} u = -\nabla \tilde{q} - \rho v \cdot \nabla \tilde{u} := f_1, \tag{3.2}$$

$$\frac{1}{\gamma}(q_t + v \cdot \nabla q) + q \mathrm{div} v = \frac{\gamma - 1}{\gamma}\epsilon(2\mu|D(v)|^2 + \lambda(\mathrm{div} v)^2) - \mathrm{div}\tilde{u} := f_2. \tag{3.3}$$

$$(u, q)|_{t=0} = (u_0, q_0)(x), \ \ x \in \Omega, \tag{3.4}$$

$$u \cdot n = 0, \quad \tau \cdot \mathbb{S}(u) \cdot n + \alpha u \cdot \tau = 0 \quad \text{on } \partial\Omega \tag{3.5}$$

where $\tilde{q}$, $\tilde{u}$ are known functions with $\tilde{u} \cdot n = 0$, $\tau \cdot \mathbb{S}(\tilde{u}) \cdot n + \alpha\tilde{u} \cdot \tau = 0$ on $\partial\Omega$. Moreover, $\tilde{q}$, $\tilde{u}$ satisfy the same regularities as ρ, v, respectively.

Since (3.2) and (3.3) are uncoupled, we are able to solve them by different methods: (3.3) by the method of characteristics, and (3.2) by Galerkin's method. Then we improve the regularities by the energy estimates on the solutions themselves and their temporal derivatives. Finally, we use Schauder's fixed-point theorem to show the existence of solutions to (2.2)-(2.3). That is, we define the mapping $\tilde{L}$: $(\tilde{q}, \tilde{u}) \rightarrow (q, u)$, and then verify that $\tilde{L}$ is continuous and has a fixed point (q, u) in a non-empty, convex, compact subset $\tilde{R}_{M,T_*} \subset \tilde{X} = C([0, T_*]; L^2)^2$, where the definition of $\tilde{R}_{M,T_*}$ is similar to R_{M,T_0}.

First, we show the existence of (3.3)-(3.4). Define $\chi(x, s; t)$ the solution to the following problem

$$\begin{cases} \frac{d}{dt}\chi(x, s; t) = v(\chi(x, s; t), t), & t, s \in [0, T], x \in \bar{\Omega}, \\ \chi(x, s; s) = x. \end{cases} \tag{3.6}$$

Then q can be expressed explicitly as

$$\begin{aligned} q(x, t) = & q_0(\chi(x, t; 0)) \exp(\int_0^t \mathrm{div} v(\chi(x, t; \tau), \tau) d\tau) \\ & + \gamma \int_0^t f_2(\chi(x, t; s), s) \exp(\int_t^s \mathrm{div} v(\chi(x, s; \tau), \tau) d\tau) ds \end{aligned}$$

along the particle path $\chi(x, s; t)$. Since $v, \tilde{u} \in L^2(0, T; H^3)$ and $f_2 \in L^2(0, T; H^2)$, we deduce that $q \in C([0, T]; H^2)$ as the estimates for ρ in the previous section. Thus we also have $\sigma_t \in C([0, T]; H^1)$ and $\sigma_{tt} \in C([0, T]; L^2)$. Additionally, we can show the energy estimates of q and its temporal derivatives which depend only on the initial data (possibly also on ϵ).

Next, we show the local existence of solutions to (3.2), (3.4)-(3.5) by the Galerkin method. As in [45], we define the weak solution to the problem (3.2) and (3.4)-(3.5) as the function u satisfying

$$\int_\Omega (\tilde{\rho} u_t \cdot \phi + \mu \mathrm{curl} u \mathrm{curl} \phi + (\mu + \lambda)\mathrm{div} u \mathrm{div}\phi - f_1 \cdot \phi) dx = 0. \tag{3.7}$$

for any $\phi \in C^\infty(\Omega)^2$ with $\phi \cdot n|_{\partial\Omega} = 0$. Then we can prove, by small modifications to [45], that $u \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3)$, $u_t \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$ and $u_{tt} \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, by Galerkin's approximations. Due to the regularity theory, we

obtain $u \in C([0, T]; H^2)$ and $u_t \in C([0, T]; H^1)$. Note that the boundary condition is a "complementing" boundary condition in the sense of Agmon-Douglis-Nirenberg [1].

One can verify it in an algebraic way by applying the arguments in pages 38-43 of [1]. However, we can also check it in a non-algebraic way by using the formulation in the paragraph near the top of page 36 in [1]. Since the equations are invariant under translation and rotation, it suffices to check the complementing boundary condition in the hyperplane $\{y = 0\}$ in the following way. For any $\xi \in \mathbb{R}$ with $\xi \neq 0$, we consider the solution

$$u = e^{ix\xi}v(y) \tag{3.8}$$

of the homogeneous problem

$$\begin{cases} -\mu \Delta u - \nu \nabla \mathrm{div} u = 0 & \text{in} \quad \mathbb{R} \times (0, +\infty), \\ u \cdot n = 0, \quad \tau \cdot \mathbb{S}(u) \cdot n + \alpha u \cdot \tau = 0 & \text{on} \quad \{y = 0\}, \\ u \to 0 & \text{as} \quad y \to +\infty, \end{cases} \tag{3.9}$$

where $v = (v_1(y), v_2(y))^t$ and $\nu = \mu + \lambda > 0$. The complementing boundary condition requires that $v \equiv 0$ for every $\xi \neq 0$, if v is bounded on the semi-axis $y \geq 0$. Substituting (3.8) into (3.9), we obtain the boundary value problem of ordinary differential equations for $v(y)$ as follows:

$$\begin{cases} (\mu + \nu)(v'' - \xi^2 v)e^{ix\xi} + 2i\nu\xi v'e^{ix\xi} = 0, \quad y \geq 0, \\ v_2 e^{ix\xi} = 0, \ (i\xi v_2 + \alpha v_1)e^{ix\xi} = 0 \quad \text{on } \{y = 0\}, \\ v_i \to 0 \ (i = 1, 2) \quad \text{as} \quad y \to +\infty. \end{cases} \tag{3.10}$$

Indeed, calculating separately the real part and the imaginary part of the first equation in (3.10), we obtain

$$(\mu + \nu)(v'' - \xi^2 v) + 2\nu\xi \sin(2x\xi)v' = 0.$$

And the boundary conditions on $\{y = 0\}$ reduce to

$$v_1 = 0, \ v_2 = 0.$$

Thus we obtain that $v_i(y) = C_1(e^{q_1 y} - e^{q_2 y})$, for any $C_1 \in \mathbb{R}$, where $q_i \in \mathbb{R}(i = 1, 2)$ are the solutions to the corresponding characteristic equation, with

$$q_1 > q_2 \text{ and } q_1 = \frac{-\nu\xi \sin 2x\xi + \sqrt{\nu^2\xi^2 \sin^2 2x\xi + \xi^2(\mu + \nu)^2}}{\mu + \nu} > 0.$$

Since v should decay at far field, we have $C_1 = 0$, and thus $v \equiv 0$. This verifies that Navier's slip boundary condition in (3.9) is a complementing boundary condition.

Thus we can apply the regularity results therein to deduce that $u \in C([0, T]; H^3)$ by regarding $f_1 - \tilde{\rho}u_t$ as source term. The energy estimates depending only on ϵ and initial data are standard, as in [45]. $\qquad\square$

From the local existence theorem proved above, we are ready to show the global existence theorem by the standard continuity arguments.

Proof of Theorem 2.1. Define the particle path $\chi(x, s; t)$ through (x, s), to be the solution of

$$\begin{cases} \frac{d}{dt}\chi(x, s; t) = v(\chi(x, s; t), t), & t, s \in [0, T], x \in \bar{\Omega}, \\ \chi(x, s; s) = x. \end{cases} \tag{3.11}$$

Then ρ can be expressed explicitly as

$$\rho(x, t) = \rho_0(\chi(x, t; 0)) \exp\left(-\int_0^t \mathrm{div}v(\chi(x, t; s), s)ds\right), \tag{3.12}$$

on the particle path $\dot{\chi} = v(\chi, t)$.

Lemma 3.2. *see* ([6]) *Assume* $G \in C([0, T]; W^{k,q}(\Omega; R^N))$ *with* $1 \leq q < +\infty$, *and* $k > N/q + 1$. *Then the problem*

$$\frac{d\chi}{dt}(x, t) = G(\chi(x, t), t), \quad \chi(x, 0) = x,$$

has a solution $\chi \in C^1([0, T]; D^{k,q}(\Omega))$, *where*

$$D^{k,q}(\Omega) = \left\{\eta \in W^{k,q}(\Omega) \mid \eta : \bar{\Omega} \to \bar{\Omega} \text{ is bijective, } \eta^{-1} \in W^{k,q}(\Omega)\right\}.$$

□

Lemma 3.3. *see* ([6]) *Let* $k \geq 2$ *be an integer, and let* $1 \leq p \leq q \leq +\infty$ *be such that* $p < +\infty$ *and* $k > \frac{N}{q} + 1$. *If* $F \in W^{k,p}(\Omega)$, *then the mapping* $G \mapsto F \circ G$ *is continuous from* $D^{k,q}(\Omega)$ *into* $W^{k,p}(\Omega)$. □

Since $v \in C([0, T]; H^3)$ and thus $v \in C([0, T]; W^{2,q})$ for any $2 \leq q < \infty$ by Sobolev's embeddings, we have $\chi \in C^1([0, T]^2; D^{2,q})$ by use of Lemma 3.2. From the formula of ρ in (3.12) and Lemma 3.3, we obtain that $\rho \in C([0, T]; H^2)$ since $\rho_0 \in H^2$. It follows immediately from the density equation in (2.1) that $\rho_t \in C([0, T]; H^1)$, $\rho_{tt} \in C([0, T]; L^2)$, and $\rho > 0$ provided that $\inf_{x \in \Omega} \rho_0(x) > 0$.

Since the global existence of ρ is available, it suffices to show the global existence of (u, q) to the problem (2.2)-(2.5). Assume that (u, q) is the local solution obtained in Lemma 3.1 with T' being replaced by $T_1 := T_1(M_0, M, \epsilon)$. Upon redefining the values of (u, q) at $t = T_1$, if necessary, we have

$$\left(\|(u, q)\|_{H^2} + \|(u_t, q_t)\|_{H^1} + \|(u_{tt}, q_{tt}\|_{L^2}\right)(T_1) \leq \tilde{F}(M_0, M, \epsilon),$$

with $u(\cdot, T_1) \cdot n = \tau \cdot \mathbb{S}(u(\cdot, T_1)) \cdot n + \alpha u(\cdot, T_1) \cdot \tau = 0$ on $\partial\Omega$. Assume that $T_1 < T$ without the loss of generality. By the uniform estimates proved in Proposition 2.1,

$$\left(\|(u, q)\|_{H^2} + \|(u_t, q_t)\|_{H^1} + \|(u_{tt}, q_{tt}\|_{L^2}\right)(t = T_1) \leq F(M_0, M),$$

for any $0 < \epsilon \leq 1$. Again, we use the local existence results obtained in Lemma 3.1 with the new initial time $t = T_1$. Then there exists $T_2 := T_2(M_0, M, \epsilon)$ such that (u, q) solves (2.2)-(2.5) in $\Omega \times [0, T_1 + T_2]$. If $T_1 + T_2 \geq T$, we are done; otherwise, we continue the extension of solution. Since $T_1 + T_2 < T$, by the uniform bounds $F(M_0, M)$ in $[0, T]$, we may

extend (u, q) to $\Omega \times [0, T_1 + 2T_2]$ as in the previous step. Applying the local existence lemma repeatedly, there must exist $m \in \mathbb{N}$ such that $T_1 + mT_2 > T$. Therefore the global existence of (u, q) in Theorem 2.1 is proved. $\qquad\square$

[1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions ii. *Comm. Pure Appl. Math.* 17 (1964) 35–92.

[2] T. Alazard. Incompressible limit of the nonisentropic Euler equations with solid wall boundary conditions. *Adv. in Differential Equations* 10 (2004) 19–44.

[3] T. Alazard. Low Mach number limit of the full Navier-Stokes equations. *Arch. Ration. Mech. Anal.* 180 (2006) 1–73.

[4] H. Beirao Da Veiga. Singular limits in compressible fluid dynamics. *Arch. Ration. Mech. Anal.*, 128 (1994), 313–327.

[5] H. Beirao Da Veiga. On the singular limit for slightly compressible fluids. *Calc. Var. Partial Differential Equations,* 2 (1994), 205–218.

[6] J. Bourguignon and H. Brezis. Remarks on the Euler equation. *J. Functional Analysis* 15 (1974) 341–363.

[7] D. Bresch, B. Desjardins, E. Grenier, and C.-K. Lin. Low Mach number limit of viscous polytropic flows: formal asymptotics in the periodic case. *stud. Appl. Math.* 109 (2002) 125–149.

[8] G. Browning, A. Kasahara and H.-O. Kreiss, Initialization of the primitive equations by the bounded derivative method, *J. Atmospheric Sci.* 37 (1980) 1424–1436.

[9] G. Browning and H.-O. Kreiss, Problems with different time scales for nonlinear partial differential equations, *SIAM J. Appl. Math.* 42 (1982) 704–718.

[10] T. Clopeau, A. Mikelic, and R. Robert. On the vanishing viscosity limit for the 2d incompressible Navier-Stokes equations with the friction type boundary condition. *Nonlinearity* 11 (1998) 1625–1636.

[11] R. Danchin. Zero Mach number limit for compressible flows with periodic boundary conditions. *Amer. J. Math.* 124 (2002) 1153–1219.

[12] R. Danchin. Zero Mach number limit in critical spaces for compressible Navier-Stokes equations. *Ann. Sci. cole Norm. Sup.* 35 (2002) 27–75.

[13] B. Desjardins and E. Grenier. Low Mach number limit of viscous compressible flows in the whole space. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 455 (1999) 2271–2279.

[14] B. Desjardins, E. Grenier, P.-L. Lions, and N. Masmoudi. Incompressible limit for solutions of the isentropic Navier-Stokes equations with dirichlet boundary conditions. *J. Math. Pures Appl. (9)* 78 (1999) 461–471.

[15] B. Dorin and E. Feireisl. The incompressible limit of the full Navier-Stokes-Fourier system on domains with rough boundaries. *Nonlinear Anal. Real World Appl.* 10 (2009) 3203–3229.

[16] D. G. Ebin. Motion of a slightly compressible fluid. *Proc. Nat. Acad. Sci. USA* 72 (1975) 539–542.

[17] E. Feireisl, J. Malek and A. Novotný. Navier's slip and incompressible limits in domains with variable bottoms. *Discrete Contin. Dyn. Syst. Ser. S* 1 (2008) 427–460.

[18] E. Feireisl and A. Novotný. The low Mach number limit for the full Navier-Stokes-fourier system. *Arch. Ration. Mech. Anal.* 186 (2007) 77–107.

[19] E. Feireisl, A. Novotný and H. Petzeltová. On the incompressible limit for the Navier-Stokes-Fourier system in domains with wavy bottoms. *Math. Models Methods Appl. Sci.* 18 (2008) 291–324.

[20] A. Friedman. *Partial Differential Equations.* Dover Publications, 2008.

[21] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems.* Springer-Verlag: New York, 1994.

[22] T. Hagstrom and J. Lorenz. On the stability of approximate solutions of hyperbolic-parabolic systems and the all-time existence of smooth, slightly compressible flows. *Indiana Univ. Math. J.* 51 (2002) 1339–1387.

[23] D. Hoff. The zero-Mach limit of compressible flows. *Comm. Math. Phys.* 192 (1998) 543–554.

[24] H. Isozaki. Singular limits for the compressible Euler equation in an exterior domain. *J. Reine Angew. Math.* 381 (1987) 1–36.

[25] H. Isozaki. Singular limits for the compressible Euler equation in an exterior domain. ii. bodies in a uniform flow. *Osaka J. Math.* 26 (1989) 399–410.

[26] H. Kim and J. Lee. The incompressible limits of viscous polytropic fluids with zero thermal conductivity coefficient. *Comm. Partial Differential Equations* 30 (2005) 1169–1189.

[27] S. KLAINERMAN and A. MAJDA. Singular perturbations of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.* 34 (1981) 481–524.

[28] S. KLAINERMAN and A. MAJDA. Compressible and incompressible fluids. *Comm. Pure Appl. Math.* 35 (1982) 629–653.

[29] C.-K. LIN. On the incompressible limit of the compressible Navier-Stokes equations. *Comm. Partial Differential Equations* 20 (1995) 677–707.

[30] C.-K. LIN. Potential flow for the compressible viscous fluid and its incompressible limit. *Indiana Univ. Math. J.* 49 (2000) 1539–1561.

[31] P.-L. LIONS. *Mathematical Topics in Fluid Dynamics, Vol.1. Incompressible models.* Oxford: London, 1996.

[32] P. L. LIONS and N. MASMOUDI. Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl.* 77 (1998) 585–627.

[33] A. MATSUMURA and T. NISHIDA. Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.* 89 (1983) 445–464.

[34] G. MÉTIVIER and S. SCHOCHET. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.* 158 (2001) 61–90.

[35] G. MÉTIVIER and S. SCHOCHET. Averaging theorems for conservative systems and the weakly compressible Euler equations. *J. Differential Equations* 187 (2003) 106–183.

[36] Y. OU. Incompressible limits of the Navier-Stokes equations for all time. *J. Differential Equations* 247 (2009) 3295-3314.

[37] Y. OU. Low Mach number limit of the non-isentropic Navier-Stokes equations. *J. Differential Equations* 246 (2009) 4441-4465.

[38] S. SCHOCHET. The compressible euler equations in a bounded domain: existence of solutions and the incompressible limit. *Comm. Math. Phys.* 104 (1986) 49–75.

[39] S. SCHOCHET. Fast singular limits of hyperbolic PDEs. *J. Differential Equations* 114 (1994) 476–512.

[40] P. SECCHI. On the singular incompressible limit of inviscid compressible fluids. *J. Math. Fluid Mech.* 2 (2000) 107–125.

[41] P. SECCHI. On slightly compressible ideal flow in the half-plane. *Arch. Ration. Mech. Anal.* 161 (2002) 231–255.

[42] P. SECCHI. 2D slightly compressible ideal flow in an exterior domain. *J. Math. Fluid Mech.* 8 (2006) 564–590.

[43] A. TANI. The initial value problem for the equations of the motion of general fluid with general slip boundary condition. *Kôkyuroku, RIMS, Kyoto Univ.* 734 (1990) 123–142.

[44] S. UKAI. The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.* 26 (1986) 323–331.

[45] W. M. ZAJACZKOWSKI. On nonstationary motion of a compressible barotropic viscous fluid with boundary slip condition. *J. Appl. Anal.* 4 (1998) 167–204.