

Solvability via viscosity solutions for a model of phase transitions driven by configurational forces

PEICHENG ZHU^{1,2*}

¹ Basque Center for Applied Mathematics (BCAM)

Building 500, Bizkaia Technology Park

E-48160 Derio, Spain

² IKERBASQUE, Basque Foundation for Science

E-48011 Bilbao, Spain

Abstract

This article is concerned with an initial boundary value problem for an elliptic-parabolic coupled system arising in martensitic phase transition theory of elastically deformable solid materials, e.g., steel. This model was proposed in [4], and investigated in [3] the existence of weak solutions which are defined in a standard way, however the key technique used in [3] is not applicable to multi-dimensional problem. One of the motivations of this study is to solve this multi-dimensional problem, and another is to investigate the sharp interface limits. Thus we define weak solutions in a way, which is different from [3], by using the notion of viscosity solution. We do prove successfully the existence of weak solutions in this sense for one dimensional problem, yet the multi-dimensional problem is still open.

1 Introduction

In this article we shall investigate an initial-boundary value problem of a new model which describes martensitic phase transitions in elastically deformable solid materials, and such phase transitions are driven by configurational forces. To formulate this problem, we firstly introduce some notations. Let Ω be an open bounded domain in \mathbb{R}^3 with smooth boundary $\partial\Omega$. It represents the points of a material body. Define $Q_t = (0, t) \times \Omega$. We use unknown functions: $u = u(t, x)$ is the displacement at time t and position x , T is the Cauchy stress tensor, and S is an order parameter which means that if S takes the values that are approximately equal to 0 and 1, then the material is in two different phases, say γ and γ' , respectively. Then the system reads

$$-\operatorname{div}_x T(t, x) = b(t, x), \quad (1.1)$$

$$T(t, x) = D(\varepsilon(\nabla_x u) - \bar{\varepsilon}S)(t, x), \quad (1.2)$$

$$S_t(t, x) = -c \left(-T \cdot \bar{\varepsilon} + \hat{\psi}'(S) - \nu \Delta_x S \right) |\nabla_x S|(t, x) \quad (1.3)$$

*AMS Classification: Primary 35D30; 35M13; Secondary 74N20. E-mail: zhu@bcamath.org

which must be satisfied in Q_t . We prescribe the following Dirichlet boundary and initial conditions

$$u|_{[0,t] \times \partial\Omega} = 0, \quad (1.4)$$

$$S|_{[0,t] \times \partial\Omega} = 0, \quad (1.5)$$

$$S|_{\{t=0\} \times \bar{\Omega}} = S_0. \quad (1.6)$$

In this model, c , ν are positive constants, D is the linear, positive definite symmetric elasticity tensor. We have chosen the free energy $\psi = \psi(\varepsilon, S, \nabla_x S)$ given by

$$\psi(\varepsilon, S, \nabla_x S) = \frac{1}{2} D(\varepsilon - \bar{\varepsilon} S) \cdot (\varepsilon - \bar{\varepsilon} S) + \hat{\psi}(S) + \frac{\nu}{2} |\nabla_x S|^2, \quad (1.7)$$

and ψ_S is the derivative, with respect to S , of ψ . The scalar product of two matrices σ , τ is denoted by $\sigma \cdot \tau = \sum_{i,j=1}^3 \sigma_{ij} \tau_{ij}$. There holds the relation $\psi_S(\varepsilon, S) = -T \cdot \bar{\varepsilon} + \hat{\psi}'(S)$. ε is the strain tensor defined by $\varepsilon = \varepsilon(\nabla_x u) = \frac{1}{2} (\nabla_x u + {}^t(\nabla_x u))$, and the upper-script t denotes the transpose of a matrix. $\bar{\varepsilon}$ is called the misfit strain. The function $\hat{\psi}(S)$ is chosen as a double-well potential for which we assume that

$$\begin{aligned} \hat{\psi}(S) \text{ is smooth and has two minima at } S = 0 \text{ and } S = 1, \text{ and} \\ \text{one maximum at } \hat{S} \text{ between } 0 \text{ and } 1, \\ \hat{\psi}'(S) > 0, \text{ if } S \in (0, \hat{S}) \cup (1, \infty); \hat{\psi}'(S) < 0, \text{ if } S \in (\hat{S}, 1) \cup (-\infty, 0). \end{aligned} \quad (1.8)$$

Finally, $b = b(t, x)$ is a given volume force.

This model was formulated in [4] by employing the second law of thermodynamics and a formula (see e.g. [1, 25, 35]) of configurational forces. Our model differs from the celebrated Allen-Cahn model (which is also called Ginzburg-Landau) by the gradient term $|\nabla_x S|$. The reason is that in the Allen-Cahn model, the driving force for the motion of interfaces is the mean curvature, while the motion of interfaces considered in this paper is driven by configurational forces. We mention the key ideas of the derivation. There are two main types of phase transition models: sharp interface model and phase field model. Our model is derived from a sharp interface model: Assuming that the jump of S , across the interface of two phases, becomes smaller and smaller, we see that the equation governing the interface approaches to a Hamilton-Jacobi equation $S_t = -c \psi_S |\nabla_x S|$ which is a fully nonlinear equation, thus is difficult to deal with and its solution may develop singularities. A usual way for regularizing it is to add an artificial term (for instance, $\nu \Delta_x S$) as in the theory of conservation laws, but this technique does not work in our case. We then think of another type of models, i.e. phase field model, to regularize such an equation. To formulate a phase field model, we choose the free energy (1.7), and also need a suitable flux which can be chosen in the form

$$q = q(u_t, T, \nabla_x S, S_t) = T \cdot u_t + \nu S_t \nabla_x S. \quad (1.9)$$

Then by straightforward computations, we see that if the equations (1.1) – (1.3) are satisfied, then the following Clausius-Duhem inequality is satisfied

$$\frac{d}{dt} \psi(\varepsilon, S, \nabla_x S) - \operatorname{div}_x q - b \cdot u_t \leq 0. \quad (1.10)$$

Hence, we assert the validity of the second law of thermodynamics. For the details of the formulation of this model, we refer to the appendix of this paper, or the articles [2, 3, 4].

The aim of this article is to propose a suitable concept of weak solutions that works for multi-dimensional problem and that makes the investigation of sharp interface limit (as ν goes to 0) easier, then to prove the existence of such defined weak solutions for problem (1.1) – (1.6). There are two most well-known concepts of weak solutions to partial differential equations: The first one is the notion of usual weak solutions that are defined by employing test functions and the technique of integration by parts, and the second one is the conception of viscosity solutions developed by Crandall and Lions in 1983, see [15], etc. In this article, we define a weak solution by combining these two notions of weak solutions. To understand why we need two concepts of weak solutions, we first investigate the features of this model. Our model consists of a subsystem of linear elasticity and a partial differential equation that is degenerate and has strong nonlinearity and non-smooth coefficients. The one space dimensional initial-boundary value problem for this model has been studied in [3], in which we define a weak solution in a usual way by using a simple technique that makes us possible to rewrite the principle part of the equation of the order parameter in a divergence form, i.e. $\nu S_{xx}|S_x| = \frac{\nu}{2}(S_x|S_x|)_x$.

However such a technique fails for the corresponding multi-dimensional problem of this model, namely $\nu \Delta_x S |\nabla_x S|$ can't be rewritten in a divergence form. Thus the notion of usual weak solutions is not suitable for this problem because we can not reduce the order of weak derivatives of the unknown by integration by parts. This is one of the difficulties in solving our model. Another one is that the maximum principle, which plays a crucial role in the theory of viscosity solutions, is not valid for the whole system of equations considered here. So it is not suitable to define weak solutions by using the notion of viscosity solutions only. Therefore one of two purposes of this article is to propose a suitable notion of weak solutions to this multi-dimensional problem. The second purpose is that we shall use our new notion of weak solutions to study, in the future, a very interesting problem, i.e. the sharp interface limit of our model. Such a problem however may be difficult under the framework of the standard weak solution, since the sharp interface problem has a fully nonlinear equation of the order parameter.

The above consideration leads us to propose a suitable notion of generalized solutions to our system by using both notions of weak solutions: we define weak solutions in the usual sense for the subsystem of elasticity, and use viscosity solutions to define weak solutions to the order parameter equation. Then we construct a sequence of solutions to an approximate initial boundary value problem of the system. Applying some compactness lemma we can show that the limit of the approximate solutions is just weak solutions in our sense. Though only the one space dimensional problem is solved up to now, we believe this technique works for the multi-dimensional case too. The other interesting open problems in this field include: The sharp interface limit of our model, and the relationship between weak solutions defined in this article and the ones in [3], respectively.

We are now going to study the definition and existence of weak solutions in a suitable sense to problem (1.1) – (1.6) in one space dimension, though the definition and some *a-priori* estimates are still valid for multi-dimensional problem. We shall see later on that the proof of the existence of weak solutions in this article is significantly simpler than that in [3].

Statement of the main result. From now on we assume that all functions only depend on the variables x_1 and t , and, to simplify the notation, denote x_1 by x . The set $\Omega = (a, d)$ is a bounded open interval with constants $a < d$. We write $Q_{t_e} := (0, t_e) \times \Omega$, where t_e is a positive constant, and define

$$(v, \varphi)_Z = \int_Z v(y)\varphi(y) dy,$$

for $Z = \Omega$ or $Z = Q_{t_e}$. If v is a function defined on Q_{t_e} we denote the mapping $x \mapsto v(t, x)$ by $v(t)$. If no confusion is possible we sometimes drop the argument t and write $v = v(t)$. We still allow that the material points can be displaced in three directions, hence $u(t, x) \in \mathbb{R}^3$, $T(t, x) \in \mathcal{S}^3$ and $S(t, x) \in \mathbb{R}$, where \mathcal{S}^3 is the set of 3×3 symmetric matrices. If we denote the first column of the matrix $T(t, x)$ by $T^1(t, x)$ and set

$$\varepsilon(u_x) = \frac{1}{2} ((u_x, 0, 0) + {}^t(u_x, 0, 0)) \in \mathcal{S}^3,$$

then with these definitions the equations (1.1) – (1.3) in the case of one space dimension can be written in the form

$$-T_x^1 = b, \tag{1.11}$$

$$T = D(\varepsilon(u_x) - \bar{\varepsilon}S), \tag{1.12}$$

$$S_t = c \left(T \cdot \bar{\varepsilon} - \hat{\psi}'(S) + \nu S_{xx} \right) |S_x|, \tag{1.13}$$

which must be satisfied in Q_{t_e} . The boundary and initial conditions therefore are

$$u(t, x) = 0, \quad (t, x) \in [0, t_e] \times \partial\Omega, \tag{1.14}$$

$$S(t, x) = 0, \quad (t, x) \in [0, t_e] \times \partial\Omega, \tag{1.15}$$

$$S(0, x) = S_0(x), \quad x \in \Omega. \tag{1.16}$$

To define weak solutions to problem (1.11) – (1.16), we first introduce some definitions on semi-continuous functions. Let $f = f(x)$ be a real function defined in $U \subset \mathbb{R}^N$ with $N \in \{1, 2, 5\}$. We denote the so-called upper semi-continuous envelope of f by

$$f^*(x) : U \rightarrow \mathbb{R} \cup \{-\infty, +\infty\} \tag{1.17}$$

which is defined by

$$f^*(x) := \lim_{r \downarrow 0} \text{ess sup}_y \{f(y) \mid y \in U, \quad |x - y| \leq r\}. \tag{1.18}$$

Obviously, $f^*(x)$ is upper semi-continuous. And $f_*(x) := -(-f)^*(x)$ is called lower semi-continuous envelope of f .

We define the Hamiltonian H_T which depends on the unknown T by

$$H_T(t, x, p, q, r) = c \left(T(t, x) \cdot \bar{\varepsilon} - \hat{\psi}'(p) + \nu r \right) |q|, \tag{1.19}$$

where, $(t, x) \in Q_{t_e}$, $p, q, r \in \mathbb{R}$, so $(t, x, p, q, r) \in \mathbb{R}^5$. It is easy to show that if T is a continuous function in (t, x) and $\hat{\psi}'$ is continuous in S , then we have that H_T is continuous in (t, x, p, q, r) , thus

$$(H_T)^*(t, x, p, q, r) = (H_T)_*(t, x, p, q, r) = H_T(t, x, p, q, r). \tag{1.20}$$

We now can introduce the notion of weak solutions for our problem. In what follows we shall assume that p is a real number such that

$$p > 1. \quad (1.21)$$

Definition 1.1 A function (u, T, S) which satisfies that

$$(u, T, S) \in L^\infty(0, t_e; H_0^1(\Omega)) \times L^\infty(0, t_e; L^2(\Omega)) \times L^\infty(\bar{Q}_{t_e}), \quad (1.22)$$

is called a weak solution to system (1.11) – (1.16) if

I) for almost every $t \in [0, t_e]$, equations (1.11), (1.12) and the boundary condition (1.14) are satisfied weakly.

II) S is a viscosity solution to equation (1.13), if S satisfies both i) and ii) below:

i) S is a sub-viscosity solution to equation (1.13), i.e. for any function $\phi(t, x)$ in $C^{1,2}(\bar{Q}_{t_e})$, if $S^* - \phi$ attains its local maximum at (τ, y) , then

$$\phi_t(\tau, y) \leq (H_T)_*(\tau, y, S^*(\tau, y), \phi_x(\tau, y), \phi_{xx}(\tau, y)), \quad (1.23)$$

and $S^*(0, x) \leq S_0(x)$;

ii) S is a super-viscosity solution to Eq. (1.3), i.e. for any function $\phi(t, x)$ in $C^{1,2}(\bar{Q}_{t_e})$, if $S_* - \phi$ attains its local minimum at (τ, y) , then

$$\phi_t(\tau, y) \geq (H_T)^*(\tau, y, S_*(\tau, y), \phi_x(\tau, y), \phi_{xx}(\tau, y)), \quad (1.24)$$

and $S_*(0, x) \geq S_0(x)$.

Now we are able to state our main result as follows.

Theorem 1.2 Suppose that $b, b_t \in C([0, t_e]; L^2(\Omega))$ for any given positive constant t_e , and that $S_0 \in H_0^1(\Omega)$. Furthermore, we assume that the function $\hat{\psi}$ satisfies the assumption (1.8).

Then there exists a weak solution (u, T, S) to problem (1.11) – (1.16) in the sense of Definition 1.1, and in addition to (1.22), we have that the solution satisfies

$$S \in C(\bar{Q}_{t_e}).$$

Our notion of generalized solutions is a combination of the concept of usual weak solutions and the notion of viscosity solutions. This idea comes partly from some discussions with Prof. Alber and partly from the paper by Giga, Goto and Ishii [24] which is concerned with the global existence of weak solutions, however without uniqueness, to the system consisting of a semi-linear diffusion equation in two disjoint open sub-domains denoted by $\Omega_\pm(t)$ of one simply connected domain Ω (The complement of union of these two parts is so-called the interface $\Gamma(t)$), and a nonlinear interface equation. The system is composed of the interface equation

$$V = W(v) - c\kappa, \quad \text{on } \Gamma(t) \quad (1.25)$$

and the diffusion equations

$$v_t = \nu \Delta v + g_{\pm}(v), \quad \text{for } x \in \Omega_{\pm}(t), \quad t > 0.$$

Here, $V = V(t, x)$ is the speed of $\Gamma(t)$ at $x \in \Gamma(t)$ in the normal direction of n from $\Omega_+(t)$ to $\Omega_-(t)$. κ is the mean curvature of $\Gamma(t)$ at $x \in \Gamma(t)$, v is the density. And c, ν are positive constants, W, g_{\pm} are given bounded continuous functions over \mathbb{R} . Note that in the work [24], the driving force for the motion of an interface is due to the mean curvature (see formula (1.25)), while the motion of an interface considered in this article is driven by configurational forces and the motion is governed by $V[S] = cn \cdot [E]n$ (the sharp interface case), where E is the Eshelby tensor, an energy-momentum tensor, see [19, pp. 753-767].

We recall the literature related to our results. There have been many papers on the theory of viscosity solutions since the notion of viscosity solution was proposed in 1983 by Crandall and Lions [15]. This notion is applicable to the scalar equations or the weakly coupled systems, for which the maximum principle holds. Hence, the comparison theorem is valid, this plays an important role in the proof of uniqueness of viscosity solution. For an overview of the theory, we refer for instance to Capuzzo Dolcetta and Lions[12], Ishii and Lions[29], Crandall, Ishii and Lions[14], Jensen[31], Crandall and Lions [16], Ishii[27], Souganidis[40] for the scalar equation case, and to Engler and Lenhart[18], Ishii and Koike[28], etc. for the system case, and the references are cited therein. For the background of our model and mathematical results related this article, we refer the reader to work by Alber and/or Zhu [2, 3, 4, 5, 6, 7], Kawashima and Zhu [32].

The main difficulties and our strategies in the proof of Theorem 1.2 are as follows: Firstly, the definition of weak solutions is a new problem since our system comprises of a linear elliptic system of u and a nonlinear equation of S which can not be rewritten in the divergence form. Secondly, the equation for the order parameter is degenerate and its coefficients is not smooth. To overcome these difficulties, we make a suitable smooth approximation of the non-smooth term which leads the equation of the order parameter to a uniformly parabolic equation with smooth coefficients. We employ the energy estimates to discuss the limits of approximate solutions.

The remaining of this article is organized as follows. In Section 2 we state an approximate initial boundary value problem, and apply the existence theorem in the book by Ladyzenskaya et al. [33] to prove existence of classical solution to this approximate problem. Then we derive in Section 3 the uniform *a priori* estimates which are independent of a small parameter κ for the approximate solutions. Then we apply the *a priori* estimates, a lemma of the Aubin-Lions type and a theorem on the stability of viscosity solutions to discuss the limits and prove the existence of weak solutions in the sense of Definition 1.1. Finally Section 4 we present briefly in the appendix the derivation of our model.

2 Existence of solutions to the modified problem

In this section, we are going to study an approximate initial-boundary value problem and show that it has a classical solution for any fixed positive constant κ . Since we shall let

κ go to zero, we may assume, without loss of generality, that

$$0 < \kappa < 1.$$

Let $\chi \in C_0^\infty(\mathbb{R}^2, [0, \infty))$ be a function satisfying $\int_{-\infty}^\infty \chi(t, x) dt dx = 1$. We set

$$\chi_\kappa(t, x) := \frac{1}{\kappa^2} \chi\left(\frac{t}{\kappa}, \frac{x}{\kappa}\right),$$

and for $b \in L^\infty(Q_{t_e}, \mathbb{R})$ we define

$$(\chi_\kappa * b)(t, x) = \int_0^{t_e} \chi_\kappa(t - s, x - y) b(s, y) ds dy. \quad (2.1)$$

We smooth the term $|S_x|$ as follows

$$|S_x|_\kappa = \sqrt{|S_x|^2 + \kappa^2}, \quad (2.2)$$

and choose a sequence S_0^κ such that

$$S_0^\kappa \in C_0^\infty(\Omega), \quad \|S_0^\kappa - S_0\|_{H^1(\Omega)} \rightarrow 0 \quad (2.3)$$

as $\kappa \rightarrow 0$ since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$.

Then the smoothed initial boundary value problem of (1.11) – (1.16) turns out to be

$$-T_x^1 = \chi_\kappa * b, \quad (2.4)$$

$$T = D(\varepsilon(u_x) - \bar{\varepsilon}S), \quad (2.5)$$

$$S_t = c\nu |S_x|_\kappa S_{xx} + c(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) (|S_x|_\kappa - \kappa). \quad (2.6)$$

and the boundary and initial conditions become

$$u|_{[0, t_e] \times \partial\Omega} = 0, \quad (2.7)$$

$$S|_{[0, t_e] \times \partial\Omega} = 0, \quad (2.8)$$

$$S|_{\{0\} \times \bar{\Omega}} = S_0^\kappa. \quad (2.9)$$

By the choice of S_0^κ , we see that the compatibility condition $S_0^\kappa|_{\partial\Omega} = 0$ is met.

Remark 2.1. *There are some other ways, which are different from (2.2), to smooth the function $|p|$. We need only to require that the smoothed equation (2.6) for the order parameter meets the assumptions of the maximum principle.*

To prove the existence of classical solution to the approximate problem (2.4) – (2.9), we employ the Leray-Schauder fixed-point theorem (see, e.g. [33]) and define for any $\hat{S} \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\bar{Q}_{t_e})$ (here $0 < \alpha < 1$) a mapping $P : [0, 1] \times B \rightarrow B$; $\hat{S} \mapsto S$ where S is a solution obtained by the following procedure:

i) For any fixed \hat{S} , it is easy to find a unique solution (u, T) which depends on \hat{S} , to the following boundary value problem for almost every given t

$$-T_x^1 = \chi_\kappa * b, \quad (2.10)$$

$$T = D(\varepsilon(u_x) - \bar{\varepsilon}\hat{S}), \quad (2.11)$$

$$u|_{\partial\Omega} = 0. \quad (2.12)$$

ii) Then inserting this T into equation (2.6) we can obtain a unique classical solution S to problem (2.6), (2.8) and (2.9).

Therefore we conclude that

Theorem 2.1 *Suppose that all the assumptions in Theorem 1.2 are met, and the compatibility conditions $S_0 = S_{0,x} = S_{0,xx} = 0$ at $x = a, d$ are satisfied.*

Then for any fixed $\kappa > 0$, there exists a unique classical solution $(u, T, S) \in C^{2,1}(\bar{Q}_{t_e}) \times C^{1,1}(\bar{Q}_{t_e}) \times C^{2+\alpha, 1+\alpha/2}(\bar{Q}_{t_e})$ to problem (2.4) – (2.9) which satisfies

$$S_{tx} \in L^2(Q_{t_e}). \quad (2.13)$$

Remark 2.2. *The compatibility conditions in Theorem 2.1 are different from usual ones and they are derived as follows: From the system and initial data, there must hold*

$$T(0, x)|_{x=a,d} - D\varepsilon(u_x(0, x))|_{x=a,d} = 0, \quad (2.14)$$

$$\nu|S_{0,x}|_\kappa S_{0,xx} + T(0, x) \cdot \bar{\varepsilon}(|S_{0,x}|_\kappa - \kappa)|_{x=a,d} = 0.$$

Note that the values of $u_x(0, x)$ at boundary can be arbitrary, so is $T(0, x)$. Thus from the definition of the function $|\cdot|_\kappa$ we see that the second term of (2.14) is satisfied provided that $S_{0,x} = S_{0,xx} = 0$ at $x = a, d$.

We need the following estimates, stated in Lemma 2.2 – Lemma 2.5 and Lemma 2.7, to prove this theorem. To derive the a priori estimates, we assume that there exists a classical solution $(u, T, S) \in C^{2,1}(\bar{Q}_{t_e}) \times C^{1,1}(\bar{Q}_{t_e}) \times C^{2+\alpha, 1+\alpha/2}(\bar{Q}_{t_e})$ to problem (2.4) – (2.9) such that $S_{tx} \in L^2(Q_{t_e})$.

Firstly, applying the maximum principle to (2.6) to obtain

Lemma 2.2 *There holds for $t_e > 0$*

$$\|S\|_{L^\infty(Q_{t_e})} \leq \bar{C}. \quad (2.15)$$

In this lemma and the follows context, we denote by \bar{C} a constant which is independent of κ , but may depend on ν , while a constant C may depend on both κ and ν .

Proof. To make use of the maximum principle, we solve (u, T) in terms of S from the first two equations, provided that S is given. Then the whole system can be reduced into a single equation, but with a nonlocal term. We need some notations as used in [4]. Let $\hat{\mathcal{S}}^3$ be the subspace of all matrices $A \in \mathcal{S}^3$ with $A_{ij} = 0$ for $i, j = 2, 3$. The orthogonal space to $\hat{\mathcal{S}}^3$ is denoted by $\tilde{\mathcal{S}}^3$. It consists of $A \in \mathcal{S}^3$ satisfying $A_{i1} = A_{1i} = 0$ for all $i = 1, 2, 3$. Note that $\varepsilon(u_x) \in \tilde{\mathcal{S}}^3$. Let \hat{P} be the canonical projection of \mathcal{S}^3 into $\hat{\mathcal{S}}^3$. Since $D : \mathcal{S}^3 \rightarrow \mathcal{S}^3$ is a positive definite linear mapping, $\langle \sigma, \tau \rangle = D\sigma \cdot \tau$ defines a scalar product on \mathcal{S}^3 . The projection of \mathcal{S}^3 onto $\hat{\mathcal{S}}^3$, which is orthogonal with respect to this scalar product is denoted by \hat{Q} . These definitions imply that

$$\ker \hat{Q} = D^{-1}\tilde{\mathcal{S}}^3 = D^{-1}\ker \hat{P}.$$

Define further that

$$\varepsilon^* = \hat{Q}\bar{\varepsilon}, \quad u^* = (\varepsilon_{11}^*, 2\varepsilon_{21}^*, 2\varepsilon_{31}^*),$$

we then obtain

$$u(t, x) = u^* \left(\int_a^x S(t, y) dy - \frac{x-a}{d-a} \int_a^d S(t, y) dy \right) + w(t, x), \quad (2.16)$$

$$T(t, x) = D(\varepsilon^* - \bar{\varepsilon})S(t, x) \left(\int_a^x S(t, y) dy - \frac{x-a}{d-a} \int_a^d S(t, y) dy \right) + \sigma(t, x), \quad (2.17)$$

where the function $(w(t, \cdot), \sigma(t, \cdot))$ (here t is regarded as a parameter) is the unique solution of the following boundary value problem

$$-\sigma_{1x}(x) = \hat{b}(x) \text{ in } \Omega, \quad (2.18)$$

$$\sigma(x) = D\varepsilon(w_x(x)) \text{ in } \Omega, \quad (2.19)$$

$$w(a) = \hat{f}(a), \quad w(d) = \hat{f}(d) \quad (2.20)$$

and $\hat{b} = b(t)$, $\hat{f} \equiv 0$. Note that $u^* \in \mathbb{R}^3$, $\varepsilon^* \in \mathcal{S}^3$ depend only on the misfit strain $\bar{\varepsilon}$. Inserting the formula of u , T into equation (2.6) yields that system (2.4) – (2.6) is reduced into a single equation for S with a nonlocal term. Invoking the definition of $|p|_\kappa$, we see that the assumptions required by the maximum principle are satisfied. Thus we can apply the maximum principle to this single equation and the proof of this lemma is complete.

Next we can derive the following estimates for the derivatives of S .

Lemma 2.3 *There holds for any $t \in [0, t_e]$ that*

$$\|S_x(t)\|^2 + \int_0^t \int_\Omega |S_x|_\kappa |S_{xx}|^2 dx d\tau \leq \bar{C}, \quad (2.21)$$

$$\int_0^t \int_\Omega \left((|S_x|_\kappa |S_{xx}|)^{\frac{4}{3}} + |S_t|^{\frac{4}{3}} \right) dx d\tau \leq \bar{C}. \quad (2.22)$$

Here and hereafter, we denote the L^2 -norm over Ω by $\|\cdot\|$.

Proof. By definition we have the property $|p|_\kappa \geq \kappa$, from which we obtain

$$0 \leq |p|_\kappa - \kappa \leq |p|_\kappa + \kappa \leq 2|p|_\kappa.$$

Using estimate (2.15) and formula (2.17), recalling the assumptions on b , one concludes that

$$\|T\|_{L^\infty(Q_{t_e})} \leq \bar{C}. \quad (2.23)$$

Note that $S_{tx} \in L^2(Q_{t_e})$, for any fixed κ , implies that

$$\frac{1}{2} \frac{d}{dt} \|S_x\|^2 = (S_x, S_{xt}).$$

Multiplying (2.6) by $-S_{xx}$ and integrating the resulting equation with respect to x , using integration by parts, and invoking the estimates (2.15) and (2.23) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|S_x\|^2 + c\nu (|S_x|_\kappa S_{xx}, S_{xx}) &= c \left((T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) (|S_x|_\kappa - \kappa), -S_{xx} \right) \\ &\leq \bar{C} \left(|S_x|_\kappa^{\frac{1}{2}}, |S_x|_\kappa^{\frac{1}{2}} |S_{xx}| \right), \end{aligned} \quad (2.24)$$

where we used the notation $(f, g) = \int_\Omega f(x)g(x)dx$. Applying the Cauchy-Schwarz inequality, we infer from (2.24) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|S_x\|^2 + c\nu (|S_x|_\kappa S_{xx}, S_{xx}) &\leq \bar{C} \| |S_x|_\kappa^{\frac{1}{2}} \|_{L^1(\Omega)} \| |S_x|_\kappa^{\frac{1}{2}} S_{xx} \| \\ &\leq \bar{C} (\|S_x\|^{\frac{1}{2}} + 1) \| |S_x|_\kappa^{\frac{1}{2}} S_{xx} \| \end{aligned} \quad (2.25)$$

By the Young inequality and the property that $|p|_\kappa \leq |p| + \kappa$ for any $\kappa \geq 0$, we derive from (2.25) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|S_x\|^2 + c\nu (|S_x|_\kappa S_{xx}, S_{xx}) &\leq \frac{c\nu}{2} \| |S_x|_\kappa^{\frac{1}{2}} S_{xx} \|^2 + \bar{C}_\nu (\|S_x\| + 1) \\ &\leq \frac{c\nu}{2} \int_\Omega |S_x|_\kappa |S_{xx}|^2 dx + \bar{C}_\nu \|S_x\|^2 + \bar{C}. \end{aligned} \quad (2.26)$$

Thus we arrive at

$$\frac{d}{dt} \|S_x\|^2 + c\nu \int_\Omega |S_x|_\kappa |S_{xx}|^2 dx \leq \bar{C}_\nu \|S_x\|^2 + \bar{C}. \quad (2.27)$$

Using the Gronwall inequality to (2.27) one can easily obtain (2.21).

By the interpolation technique and (2.21), we have that for some $2 > p \geq 1$, $q = \frac{2}{p}$ and $\frac{1}{q} + \frac{1}{q'} = 1$ that

$$\begin{aligned} &\int_0^t \int_\Omega (|S_x|_\kappa |S_{xx}|)^p dx d\tau \\ &= \int_0^t \int_\Omega (|S_x|_\kappa)^{\frac{p}{2}} \left((|S_x|_\kappa)^{\frac{p}{2}} |S_{xx}|^p \right) dx d\tau \\ &\leq \left(\int_0^t \int_\Omega (|S_x|_\kappa)^{\frac{pq'}{2}} dx d\tau \right)^{\frac{1}{q'}} \left(\int_0^t \int_\Omega (|S_x|_\kappa)^{\frac{pq}{2}} |S_{xx}|^{pq} dx d\tau \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^t \int_\Omega (|S_x|_\kappa)^{\frac{p}{2-p}} dx d\tau \right)^{\frac{2-p}{2}} \left(\int_0^t \int_\Omega |S_x|_\kappa |S_{xx}|^2 dx d\tau \right)^{\frac{p}{2}}. \end{aligned} \quad (2.28)$$

Invoking the property that $|p|_\kappa \leq |p| + \kappa$ and inequality (2.21) yield that for $\frac{p}{2-p} \leq 2$, i.e. $p \leq \frac{4}{3}$, the right hand side of (2.28) is bounded.

Making use of (2.28) (with $p = \frac{4}{3}$) and equation (2.6) we have for any test function $\varphi \in L^4(Q_{t_e})$

$$\begin{aligned} |(S_t, \varphi)_{Q_{t_e}}| &= c \left| \left(\nu |S_x|_\kappa S_{xx} + (T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) (|S_x|_\kappa - \kappa), \varphi \right)_{Q_{t_e}} \right| \\ &\leq \bar{C} \| |S_x|_\kappa S_{xx} \|_{L^{\frac{4}{3}}(Q_{t_e})} \|\varphi\|_{L^4(Q_{t_e})} \\ &\quad + \bar{C} \left\| (T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) \right\|_{L^4(Q_{t_e})} (\|S_x\| + 1) \|\varphi\|_{L^4(Q_{t_e})} \\ &\leq \bar{C} \left(\| |S_x|_\kappa S_{xx} \|_{L^{\frac{4}{3}}(Q_{t_e})} + \|S_x\| + 1 \right) \|\varphi\|_{L^4(Q_{t_e})} \\ &\leq \bar{C} \|\varphi\|_{L^4(Q_{t_e})}, \end{aligned} \quad (2.29)$$

where we applied the Hölder and Young inequalities. Thus we arrive at $\|S_t\|_{L^{\frac{4}{3}}(Q_{t_e})} \leq \bar{C}$, thus prove (2.22). And the proof of this lemma is complete.

For the solution to the elliptic part of the system, i.e. (2.10) – (2.12), we have

Lemma 2.4 *There hold for almost every $t \in [0, t_e]$ that*

$$\|u(t)\|_{W^{1,p}(\Omega)} + \|T(t)\|_{L^p(\Omega)} \leq \bar{C}, \quad (2.30)$$

$$\|u(t)\|_{H^2(\Omega)} + \|T(t)\|_{H^1(\Omega)} \leq \bar{C}. \quad (2.31)$$

Proof. Using the estimate (2.15), we get $S(t) \in L^p(\Omega)$ for almost every $t \in [0, t_e]$ since the domain Ω is bounded. Recalling estimate (2.21), we obtain easily (2.30) – (2.31), by the regularity theory of elliptic systems (or just using the formula (2.17) since our problem is one dimensional). This completes the proof of the lemma.

Now we differentiate (2.4) once formally with respect to t and recall the assumption on b_t , then use again the theory of the elliptic system to get

Lemma 2.5 *There hold for almost every $t \in [0, t_e]$ that*

$$\|T_t\|_{L^p(\Omega)} \leq \bar{C} (1 + \|S_t\|_{L^p(\Omega)}), \quad \|T_t\|_{L^{\frac{4}{3}}(Q_{t_e})} \leq \bar{C}, \quad (2.32)$$

$$T \in C([0, t_e]; C^\alpha(\bar{\Omega})), \quad \text{and} \quad \|T\|_{C(\bar{Q}_{t_e})} \leq \bar{C}. \quad (2.33)$$

To prove the above lemma, we shall make use of the following lemma which is of Aubin-Lions type, see, for instance, Lions [34], and for the case $r = 1$, see Simon [38], Roubíček [37].

Lemma 2.6 *Let B_0, B, B_1 be Banach spaces which satisfy that B_0, B_1 are reflexive and that*

$$B_0 \subset\subset B \subset B_1.$$

Here, by $\subset\subset$ we denote the compact imbedding. Define

$$W = \left\{ f \mid f \in L^\infty(0, t_e; B_0), \quad \frac{df}{dt} \in L^r(0, t_e; B_1) \right\}$$

with t_e being a given positive number and $1 < r < \infty$.

Then the embedding of W in $C([0, t_e]; B)$ is compact.

Proof of Lemma 2.5. We need only to prove (2.33). From (2.32) and Lemma 2.4, we assert that

$$T \in L^\infty(0, t_e; H^1(\Omega)), \quad \text{and} \quad T_t \in L^{\frac{4}{3}}(0, t_e; L^{\frac{4}{3}}(\Omega)). \quad (2.34)$$

Thus we can choose

$$B_0 = H^1(\Omega), \quad B = C^\alpha(\bar{\Omega}), \quad B_1 = L^{\frac{4}{3}}(\Omega), \quad r = \frac{4}{3},$$

which meet the requirements of Lemma 2.6, and $\alpha \in (0, \frac{1}{2}]$. Whence (2.33) holds. And the proof of Lemma 2.5 is complete.

Furthermore, for any fixed κ , we have

Lemma 2.7 *There hold for any $t \in [0, t_e]$ that*

$$\|S_t(t)\|^2 + \int_0^t \int_\Omega (|S_x|_\kappa + 1) |S_{xt}|^2 dx d\tau \leq C, \quad (2.35)$$

$$\|S_{xx}(t)\| \leq C. \quad (2.36)$$

Proof. We prove firstly (2.35). To this end, we differentiate equation (2.6) formally with respect to t , then multiply the resulting equation by S_t and integrate it with respect to x to get

$$\frac{1}{2} \frac{d}{dt} \|S_t\|^2 - c\nu \int_{\Omega} (|S_x|_{\kappa} S_{xx})_t S_t dx - c \int_{\Omega} \left((T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) (|S_x|_{\kappa} - \kappa) \right)_t S_t dx = 0. \quad (2.37)$$

It is easy to see that

$$(|S_x|_{\kappa} S_{xx})_t = \left(\int^S_{x_t} |\xi|_{\kappa} d\xi \right)_{xt}, \quad (2.38)$$

$$\int_{\Omega} \left(\int^S_{x_t} |\xi|_{\kappa} d\xi \right)_t S_{xt} dx = \int_{\Omega} |S_x|_{\kappa} |S_{xt}|^2 dx. \quad (2.39)$$

Thus (2.37) becomes

$$\frac{1}{2} \frac{d}{dt} \|S_t\|^2 + c\nu \int_{\Omega} |S_x|_{\kappa} |S_{xt}|^2 dx - c \int_{\Omega} \left((T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) (|S_x|_{\kappa} - \kappa) \right)_t S_t dx = 0. \quad (2.40)$$

We now handle the last term on the left-hand side of (2.40) as

$$\begin{aligned} & \left| c \int_{\Omega} \left((T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) (|S_x|_{\kappa} - \kappa) \right)_t S_t dx \right| \\ & \leq C \int_{\Omega} \left| T_t \cdot \bar{\varepsilon} - \hat{\psi}''(S) S_t \right| (|S_x|_{\kappa} + 1) |S_t| dx + C \int_{\Omega} \left| T \cdot \bar{\varepsilon} - \hat{\psi}'(S) \right| |(|S_x|_{\kappa})' S_{xt} S_t| dx \\ & \leq C (\|T_t\| + \|S_t\|) (\| |S_x|_{\kappa} \|_{L^{\infty}(\Omega)} + 1) \|S_t\| + C \|S_t\| \|S_{xt}\| \\ & \leq C \left((1 + \| |S_x|_{\kappa} \|_{L^{\infty}(\Omega)}) \|S_t\|^2 + (\| |S_x|_{\kappa} \|_{L^{\infty}(\Omega)} + 1) \|T_t\| \|S_t\| \right) + \frac{\kappa}{2} \|S_{xt}\|^2. \end{aligned} \quad (2.41)$$

Here we used the estimates $\|(T, S)\|_{L^{\infty}(Q_{t_e})} \leq C$, $\|T_t\| \leq C(1 + \|S_t\|)$, $|(|y|_{\kappa})'| \leq C$, and the Hölder, Young inequalities. By the Sobolev imbedding theorem, we have

$$\| |S_x(t)|_{\kappa} \|_{L^{\infty}(\Omega)} \leq C(\|S_x(t)\|_{L^{\infty}(\Omega)} + 1) \leq C(\|S_x(t)\|_{H^1(\Omega)} + 1). \quad (2.42)$$

Applying the estimate (2.32), which is valid for $p = 2$, combining (2.41) and (2.40), we arrive at

$$\frac{d}{dt} \|S_t(t)\|^2 \leq C (1 + \|S_x(t)\|_{H^1(\Omega)}) \|S_t(t)\|^2 + C \left(\|S_x(t)\|_{H^1(\Omega)}^2 + 1 \right). \quad (2.43)$$

We shall make use of the Gronwall inequality in the following form

$$y'(t) \leq A(t)y(t) + B(t) \quad \text{implies} \quad y(t) \leq y(0)e^{\int_0^t A(\tau)d\tau} + \int_0^t B(s)e^{\int_s^t A(\tau)d\tau} ds, \quad (2.44)$$

where y , A , B are functions satisfying that $y(t) \geq 0$, $A(t)$, $B(t)$ are integrable over $[0, t_e]$. Defining

$$y(t) = \|S_t(t)\|^2, \quad A(t) = C(1 + \|S_x\|_{H^1(\Omega)}), \quad B(t) = C \left(1 + \|S_x\|_{H^1(\Omega)}^2 \right),$$

from the estimate $\|S_x\|_{H^1(Q_{t_e})} \leq C$ which is a consequence of (2.21) and the fact $|p|_{\kappa} \geq \kappa$, it follows that the above defined $A(t)$, $B(t)$ are integrable over $[0, t_e]$. Thus we can apply (2.44) to (2.43) and obtain

$$\|S_t(t)\|^2 \leq C,$$

whence

$$\|S_t(t)\|^2 + \int_0^t \int_{\Omega} |S_x|_{\kappa} |S_{xt}|^2 dx d\tau \leq C. \quad (2.45)$$

From which we obtain easily (2.35) since κ at this moment is a given number. Therefore we can use the equation to get easily (2.36). Thus the proof of this lemma is complete.

Remark 2.3. *Since we use the imbedding (2.42) which is valid only in one dimensional case, thus this lemma is only true for this one dimensional problem.*

Remark 2.4. *To derive (2.35) rigorously, we employ the technique of finite difference as, e.g., in [17]. We assume that there exists a unique classical solution (u, T, S) to problem (2.4) – (2.9) such that*

$$(u, T, S) \in C^{2,1}(\bar{Q}_{t_e}) \times C^{1,1}(\bar{Q}_{t_e}) \times C^{2+\alpha, 1+\alpha/2}(\bar{Q}_{t_e}), \quad S_{xt} \in L^2(Q_{t_e}).$$

Define $S_h(t, x) = (S(t+h, x) - S(t, x))/h$ for any $h > 0$. Then from (2.6) we obtain

$$S_{ht} = \frac{c\nu}{h} \left(\int_{S_x(t,x)}^{S_x(t+h,x)} |\xi|_{\kappa} d\xi \right)_x + \frac{c}{h} (T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) (|S_x|_{\kappa} - \kappa) \Big|_{(t,x)}^{(t+h,x)}, \quad (2.46)$$

for any $[t \in [0, t_e - \delta]]$, where δ is a fixed number such that $\delta \geq h$. Here and hereafter, we use the notations $f|_{(t,x)}^{(t+h,x)} = f(t+h, x) - f(t, x)$ and $f|_{(t,x)} = f(t, x)$. Multiplying (2.46) by S_h and integrating the resulting equation with respect to t, x over Q_t yield

$$\begin{aligned} & \|S_h(t)\|^2 + c\nu \int_0^{t_e-\delta} \int_{\Omega} \frac{1}{h} \int_{S_x(t,x)}^{S_x(t+h,x)} |\xi|_{\kappa} d\xi S_{hx} dx dt \\ &= \|S_h(0)\|^2 + c \int_0^{t_e-\delta} \int_{\Omega} (T_h \cdot \bar{\varepsilon} - \hat{\psi}''(S^*) S_h) (|S_x|_{\kappa} - \kappa) |_{(t+h,x)} S_h dx dt \\ &+ c \int_0^{t_e-\delta} \int_{\Omega} (T \cdot \bar{\varepsilon} - \hat{\psi}'(S)) |_{(t,x)} \frac{S_x(t+h, x) + S_x(t, x)}{|S_x(t+h, x)|_{\kappa} + |S_x(t, x)|_{\kappa}} S_{hx} S_h dx dt. \end{aligned} \quad (2.47)$$

Here S^* is a number between $S(t+h, x)$ and $S(t, x)$. Note that the second term on the left hand side of (2.47) is equal to

$$c\nu \int_0^{t_e-\delta} \int_{\Omega} |S_x(\eta, x)|_{\kappa} |S_{hx}|^2 dx dt \geq C \|S_{hx}\|_{L^2(Q_{t_e-\delta})}^2,$$

where $\eta \in [t, t+h]$ and we used $|p|_{\kappa} \geq \kappa$. By definition, we have

$$\frac{|S_x(t+h, x) + S_x(t, x)|}{|S_x(t+h, x)|_{\kappa} + |S_x(t, x)|_{\kappa}} \leq 1.$$

So the second and third terms on the right hand side of (2.47) are of lower orders and can be estimated in a similar way to (2.41). We thus arrive at

$$\|S_{hx}\|_{L^2(Q_{t_e-\delta})}^2 \leq C.$$

Further, we write

$$\begin{aligned} & \int_0^{t_e - \delta} \int_{\Omega} |S_x(\eta, x)|_{\kappa} |S_{hx}|^2 dx dt \\ &= \int_0^{t_e - \delta} \int_{\Omega} \left((|S_x(\eta, x)|_{\kappa} - |S_x(t, x)|_{\kappa}) + |S_x(t, x)|_{\kappa} \right) |S_{hx}|^2 dx dt. \end{aligned} \quad (2.48)$$

Invoking the Hölder continuity of S_x , applying the Fatou lemma for any fixed δ , we take the limit as $h \rightarrow 0$. Then letting $\delta \rightarrow 0$, we justify (2.35) and omit details.

We now turn back to prove Theorem 2.1.

Proof of Theorem 2.1. To complete the proof of the global existence of classical solution, we need to prove that $\|S_x\|_{C^{\alpha/2, \alpha}(\bar{Q}_{t_e})} \leq C$. To this end we make use of the estimates listed in Lemmas 2.2 – 2.5 and Lemma 2.7. To prove this, we invoke the following lemma see, e.g. [33]

Lemma 2.8 *Let $f(t, x)$ be a function on Q_{t_e} such that*

i) f is uniformly (with respect to x) Hölder continuous in t , with exponent $0 < \alpha \leq 1$, that is $|f(t, x) - f(s, x)| \leq C|t - s|^{\alpha}$,

and

ii) f_x is uniformly (with respect to t) Hölder continuous in x , with exponent $0 < \beta \leq 1$, that is $|f_x(t, x) - f_x(t, y)| \leq C'|y - x|^{\beta}$.

Then f_x is uniformly Hölder continuous in t with exponent $\gamma = \alpha\beta/(1 + \beta)$, such that

$$|f_x(t, x) - f_x(s, x)| \leq C''|t - s|^{\gamma}, \quad \forall x \in \bar{\Omega}, 0 \leq s \leq t \leq t_e.$$

where C'' is a constant which may depend on C, C' and α, β .

By applying this lemma we assert that there exists a constant $0 < \alpha < 1$ such that $\|S_x\|_{C^{\alpha/2, \alpha}} \leq C$. By the *a priori* estimate of the Schauder type for parabolic equations, we thus obtain that

$$\|S\|_{C^{1+\alpha/2, 2+\alpha}(\bar{Q}_{t_e})} \leq C,$$

which ensures us to apply the Leray-Schauder fixed point theorem, and the proof of global existence of classical solution is complete. Using the technique of difference quotient with respect to t to this classical solution see e.g. [33] we can prove (2.13). And the proof of Theorem 2.1 is thus complete.

3 Existence of weak solutions

3.1 Uniform a priori estimates

This subsection is devoted to derivation of some uniform *a priori* estimates, which are independent of $\kappa \in (0, 1]$, for the approximate solution to (2.4) – (2.9). However these estimates may depend on ν , this thus makes it difficult to discuss the sharp interface limit $\nu \rightarrow 0$. To investigate such a sharp interface limit, we need new techniques.

We now denote the approximate solution by $(u^{\kappa}, T^{\kappa}, S^{\kappa})$. Therefore we collect a priori estimates, which have been established in Section 2 and are independent of κ .

Lemma 3.1 *There hold for any $t \in [0, t_e]$ that*

$$\|S_x^\kappa(t)\|^2 + \int_0^t \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx d\tau \leq \bar{C}, \quad (3.1)$$

$$\int_0^t \int_\Omega \left((|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^{\frac{4}{3}} + |S_t^\kappa|^{\frac{4}{3}} \right) dx d\tau \leq \bar{C}, \quad (3.2)$$

$$\int_0^t \|S^\kappa\|_{H^1(\Omega)}^2 d\tau \leq \bar{C}. \quad (3.3)$$

Remark 3.1. *From (2.25) we see that the constant \bar{C} depends on ν .*

3.2 Limits

With the help of Lemma 2.6, applying the uniform a priori estimates established in Subsection 3.1, we shall investigate in this section the limits, as $\kappa \rightarrow 0$, of the approximate solutions and complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Firstly we apply again Lemma 2.6 to show that the sequence of the approximate solution S^κ has a subsequence which converges strongly. To this end, we choose

$$B_0 = H^1(\Omega), \quad B = C^\alpha(\bar{\Omega}), \quad B_1 = L^{\frac{4}{3}}(\Omega),$$

and

$$0 < \alpha < \frac{1}{2}, \quad p_1 = \frac{4}{3}.$$

It is easy to see that such defined B_0, B_1 are reflexive. Therefore we apply Lemma 2.6 and conclude that the sequence $\{S^\kappa\}_\kappa$ is a compact in $C([0, t_e]; C^\alpha(\bar{\Omega}))$. Thus we can select a subsequence of it, and denote it by $\{S^{\kappa_n}\}_n$, such that, as $n \rightarrow \infty$,

$$\kappa_n \rightarrow 0,$$

and

$$S^{\kappa_n} \rightarrow S, \text{ in } C([0, t_e]; C^\alpha(\bar{\Omega})), \quad (3.4)$$

from which we obtain that

$$\|S^{\kappa_n} - S\|_{C([0, t_e] \times \bar{\Omega})} \rightarrow 0. \quad (3.5)$$

On the other hand, by Lemma 2.5, we assert that there exists a subsequence of T^κ such that

$$T^{\kappa_n} \rightarrow T \text{ in } C^\alpha(\bar{Q}_{t_e}), \quad (3.6)$$

from this, (3.5) and (2.16), we obtain consequently that

$$(u^{\kappa_n}, T^{\kappa_n}) \rightarrow (u, T), \text{ uniformly in } C^\alpha(\bar{Q}_{t_e}), \quad (3.7)$$

as $n \rightarrow \infty$.

In what follows, we are going to prove that the limit function (u, T, S) is just a weak solution to problem (1.11) – (1.16) in the sense of Definition 1.1. It is not difficult to show that (1.11) and (1.12) are satisfied by the linearity of those two equations and the uniform convergence of u^κ, T^κ . The remaining part of this section is to prove that (1.13) is satisfied.

We shall make use of the theorem on the stability of viscosity solutions, see e.g. [23].

Theorem 3.2 (Stability of viscosity solutions) *Assume that F_n converges to F locally uniformly (as $n \rightarrow \infty$) in the domain of definition of F . Assume that v_n is a viscosity solution to*

$$(v_n)_t + F_n(t, x, v_n, \nabla v_n, \nabla^2 v_n) \leq 0 \text{ (resp. } \geq 0) \text{ in } Q_{t_e},$$

and that v_n converges to v locally uniformly in Q_{t_e} as $n \rightarrow \infty$. Then v is a viscosity solution to

$$v_t + F(t, x, v, \nabla v, \nabla^2 v) \leq 0 \text{ (resp. } \geq 0) \text{ in } Q_{t_e}.$$

To apply this theorem, we define $v_n = S^{\kappa_n}$ and

$$F_n(t, x, p, q, r) = H_{T^{\kappa_n}}(t, x, p, q, r) = c \left(T^{\kappa_n}(t, x) \cdot \bar{\varepsilon} - \hat{\psi}'(p) + \nu r \right) (|q|_{\kappa_n} - \kappa_n).$$

Invoking (3.7) and (3.5) we conclude that

- i) S^{κ_n} converges to S locally uniformly in any compact subset in Q_{t_e} .
- ii) $H_{T^{\kappa_n}}(t, x, p, q, r)$ converges to H_T locally uniformly in any compact subset in $(0, t_e) \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.
- iii) Since S^{κ_n} is a classical solution to equation (2.6) when T^{κ_n} is regarded temporarily fixed, S^{κ_n} is also a viscosity solution to (2.6).

Therefore, we can apply Theorem 4.1 and conclude that the limit S is a viscosity solution to $S_t = H_T(t, x, S_x, S_{xx})$. Hence, recalling the properties $S^*(t, x) = S_*(t, x) = S(t, x)$ and $(H_T)^*(t, x, p, q, r) = (H_T)_*(t, x, p, q, r) = H_T(t, x, p, q, r)$, we assert that (u, T, S) is a weak solution to problem (1.11) – (1.16) in the sense of Definition 1.1, and the proof of Theorem 1.2 is thus complete.

4 Appendix

Since our model is quite new, we briefly sketch, for the sake of readers' convenience, the physical background and the derivation of the diffusive interface model (1.1) – (1.6) from a sharp interface model. We also refer the reader to [3, 4, 5]. Our model differs from the Allen-Cahn model by a gradient term. The main reason is: In the Allen-Cahn model, the driving force for the motion of interface is the mean curvature, while the motion of interface considered in this article is driven by configurational forces, see e.g. [25, 35].

Material phases are characterized by the structure of the crystal lattice, in which the atoms are arranged. An interface between different material phases moves if the crystal lattice in front of the interface is transformed from one structure to the other. Often phase transformations are triggered by diffusion processes. A well-known model for diffusion dominated transformations is the Allen-Cahn equation when the order parameter is not conserved (or the Cahn-Hilliard equation if the order parameter is conserved). We derive our model (1.1) – (1.6) from a sharp interface model for diffusionless transformations, also called martensitic transformations, see e.g. [26, p. 162]. This sharp interface model is an initial-boundary value problem for the unknown functions u , T and for the unknown interface $\Gamma(t) \subset \Omega$ between two material phases, which is a free boundary. It consists of

(1.1) – (1.2) and the interface conditions

$$V(t, x)[S](t, x) = c \left(-\langle T \rangle(t, x) \cdot \bar{\varepsilon}[S](t, x) + [\hat{\psi}(S)](t, x) \right), \quad (4.1)$$

$$[u](t, x) = 0, \quad [T](t, x)n(t, x) = 0, \quad (4.2)$$

which must hold for $x \in \Gamma(t)$, and of a Dirichlet boundary condition for u the initial condition (1.6). We use the notation $[f] = f_+ - f_-$ and $\langle f \rangle = \frac{1}{2}(f_+ + f_-)$, where f_+ , f_- are the limit values of the function f on both sides of $\Gamma(t)$. Moreover, $V(t, x) \in \mathbb{R}^3$ denotes the normal speed of the interface $\Gamma(t)$, which is measured as positive in the direction for which $[S](t, x)$ is positive. Here c is a positive constant. Equation (4.1), a constitutive equation, determines the normal speed V of the phase interface as a function of the term $-\langle T \rangle \cdot \bar{\varepsilon}[S] + [\hat{\psi}(S)]$. Some computations show that this term is equal to the expression $n \cdot [E]n$ with the Eshelby tensor E (an energy-momentum tensor, see [19, p753-p767]) and the normal vector n to $\Gamma(t)$ (cf. [4]) and thus is a configurational force. We assume that V depends linearly on the configurational force, which is the most simple constitutive assumption. Thus, in this model the evolution of the phase interface is driven by the configurational force along the interface, an assumption appropriate for martensitic transformations.

Though configurational forces were introduced in the first half of the last century, it was clearly stated for the first time in [1] that (1.1), (1.2), (4.1), (4.2) form a closed initial-boundary value problem. Applications of this model can be found, for example, in [11, 36, 39], where equilibrium configurations for materials with phase transitions are determined, and in [30], where the evolution of phase interfaces in ferroelectric materials is modeled. In a sense, this free initial-boundary value problem from solid mechanics is comparable to the Stefan problem in fluid mechanics.

The initial-boundary value problem (1.1) – (1.6) can be considered to be a regularization of this sharp interface model, which could be used to prove existence of solutions of the sharp interface model, and it can also be considered to be a diffusive interface model for martensitic phase transitions, which is useful by itself and avoids some disadvantages of the model with sharp interfaces. We are interested in both aspects.

The derivation of (1.1) – (1.6) given in [2, 4] uses a rigorous method. To make the model plausible, we derive the model here in a different, short, but formal way. To this end we replace the phase interface $\Gamma(t)$, across which the order parameter jumps from 0 to 1, by finitely many interfaces parallel to the original interface, and consider a new order parameter, again denoted by S , with small jumps across these interfaces, such that the sum of the jumps is equal to 1. We assume that the new order parameter satisfies (4.1) and (4.2) along all interfaces. If we increase the number of interfaces and decrease the jump height, the new order parameter will converge to a continuous or even differentiable order parameter, for which the normal speed of the level manifolds is equal to the limit of the normal speed of the interfaces. For this limit speed we obtain from (4.1)

$$V(t, x) = c \lim_{[S] \rightarrow 0} (\langle T \rangle \cdot \bar{\varepsilon} + \hat{\psi}'(S^*)) = c(-T \cdot \bar{\varepsilon} + \hat{\psi}'(S)) = c\psi_S(\varepsilon(\nabla_x u), S). \quad (4.3)$$

The limit order parameter thus satisfies the Hamilton-Jacobi transport equation

$$S_t = -c\psi_S(\varepsilon(\nabla_x u), S)|\nabla_x S|, \quad (4.4)$$

since the level manifolds of solutions of equation (4.4) have this normal speed. The idea suggests itself to approximate the solution of the sharp interface model by smooth

solutions (u, T, S) of the system (1.1), (1.3), (4.4). Yet, examples in one space dimension show that in general the function S in such a smooth solution develops a jump after finite time. The reason for this is that the function $\hat{\psi}'$ appearing in ψ_S is not monotone, since $\hat{\psi}$ is a double well potential. After S has developed a jump, (4.4) can no longer be used to govern the evolution of S . To avoid this problem and to force solutions to stay smooth, (4.4) has been replaced by (1.3), which contains the regularizing term $\nu|\nabla_x S|\Delta_x S$ with the small positive parameter ν . This yields the model (1.1) – (1.6).

The choice of this special regularizing term follows from the second law of thermodynamics, which every model must satisfy. This law requires that there exist a free energy ψ and a flux q such that $\frac{\partial}{\partial t}\psi + \operatorname{div}_x q \leq b \cdot u_t$ holds; cf. [8]. If we choose a free-energy and a flux as (1.7) and (1.9), it follows by a short computation for solutions (u, T, S) of (1.1), (1.3) that

$$\frac{\partial}{\partial t}\psi - \operatorname{div}_x (T u_t + \nu S_t \nabla_x S) - b \cdot u_t = (\psi_S(\varepsilon, S) - \nu \Delta_x S) S_t.$$

Inserting (1.6) into this equation shows that the right-hand side is non-positive, whence the second law is fulfilled. However this would *not* be true by using, as in the theory of conservation laws, the standard regularization (i.e. adding an artificial viscosity term) $S_t = -c \psi_S(\varepsilon(\nabla_x u), S)|\nabla_x S| + \nu \Delta_x S$ of (4.4).

Acknowledgement.

The author would like to express his sincere thanks to Prof. H.-D. Alber for valuable discussions, and to the referee(s) for valuable opinions. This work has been partly supported by Grant MTM2008-03541 of the Ministerio de Educación y Ciencia of Spain, and by Project PI2010-04 of the Basque Government.

References

- [1] Abeyaratne, R. and Knowles, J. (1990) *On the driving traction acting on a surface of strain discontinuity in a continuum*, J. Mech. Phys. Solids, **38**, No. 3, 345 – 360.
- [2] Alber, H.-D. (2000) *Evolving microstructure and homogenization*, Continuum Mech. Thermodyn, **12**, 235 – 287. DOI 10.1007/s001610050137
- [3] Alber, H.-D. and Zhu, Peicheng (2006) *Solutions to a model with nonuniformly parabolic terms for phase evolution driven by configurational forces*, SIAM J. Appl. Math., **66** No. 2, 680 – 699. DOI 10.1137/050629951
- [4] Alber, H.-D. and Zhu, Peicheng (2007) *Evolution of phase boundaries by configurational forces*, Arch. Rational Mech. Anal., **185**, 235 – 286. DOI 10.1007/s00205-007-0054-8
- [5] Alber, H.-D. and Zhu, Peicheng (2008) *Solutions to a Model for Interface Motion by Interface Diffusion*, Proc. Royal Soc. Edinburgh, **138A**, 923 – 955.
- [6] Alber, H.-D. and Zhu, Peicheng (2009) *Interface motion by interface diffusion driven by bulk energy: justification of a diffusive interface model*. Continuum Mech Thermodyn. Online Aug. 10, 2010

- [7] Alber, H.-D. and Zhu, Peicheng (2010) *Solutions to a model with Neumann boundary conditions for phase transitions driven by configurational forces*. Submitted.
- [8] Alt, H. and Pawlow, I. (1996) *On the entropy principle of phase transition models with a conserved order parameter*, Adv. Math. Sci. Appl., **6**, 291 - 376.
- [9] Barles, G., Evans, L. and Souganidis, P. (1990) *Wavefront propagation for reaction-diffusion systems of PDE's*, Duke Univ. Math. J. **61**, 835 - 858.
- [10] Barles, G. and Perthame, B. (1987) *Discontinuous solutions of deterministic optimal stopping time problems*, RAIRO, Model Math. Anal. Nume., **21** 557 - 579.
- [11] Buratti, G., Huo, Y. and Müller, I. (2003) *Eshelby tensor as a tensor of free enthalpy*, J. Elasticity, **72**, 31 - 42.
- [12] Capuzzo Dolcetta, I. and Lions, P. (1997) *Viscosity solutions and Applications*, Springer Lecture Notes in Mathematics **1660** Berlin, Springer-Verlag.
- [13] Chen, Y., Giga, Y. and Goto, S. (1991) *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff Geom. **33**, 749 - 786.
- [14] Crandall, M., Ishii, H. and Lions, P. (1992) *User's guide to viscosity solutions of second order elliptic partial differential equations*, Bull. Amer. Math. Soc., **27** 1 - 67.
- [15] Crandall, M. and Lions, P. (1983) *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **277** 1 - 42.
- [16] Crandall, M. and Lions, P. (1986) *On existence and uniqueness of solutions of Hamilton-Jacobi equations*, Nonlinear Anal. T.M.A., **10** 353 - 370.
- [17] Dafermos, C. (1969) *The mixed initial boundary value problem for the equations of nonlinear one dimensional viscoelasticity*, J. Diff. eq., **6** 71 - 86.
- [18] Engler, H. and Lenhart, S. (1991) *Viscosity solutions for weakly coupled systems of Hamilton-Jacobi equations*, Proc. London Math. Soc., **63**(3)No. 1, 212 - 240.
- [19] Eshelby, J. (2006) *Collected works of J. D. Eshelby, The mechanics of defects and inhomogeneities*. Springer-Verlag.
- [20] Evans, L. and Gariepy, R. (1992) *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton.
- [21] Evans, L. and Sprunck, J. (1991) *Motion of level sets by mean curvature, I*, J. Diff. Geom., **33**, 635 - 681.
- [22] Giaquinta, M. (1993) *Introduction to regularity theory for nonlinear elliptic systems*, Birkhäuser, Basel, Boston, Berlin.
- [23] Giga, Y. (2006) *Surface evolution equations: A level set approach* Birkhäuser Verlag Basel, Boston, Berlin

- [24] Giga, Y., Goto, S. and Ishii, H. (1992) *Global existence of weak solutions for interface equations coupled with diffusion equations*, SIAM J. Math. Anal., **23**, No. 4, 821 – 835.
- [25] Gurtin, M. (2000) *Configurational forces as basic concepts of continuum physics*. Vol **137** Applied Math Sci., Springer-Verlag.
- [26] Hornbogen, E.; Warlimont, H. (2001) *Metallkunde*, 4. Auflage. Springer-Verlag.
- [27] Ishii, H. (1987) *Perron's method for Hamilton-Jacobi equations*, Duke Math. J. **55** 369 – 384.
- [28] Ishii, H. and Koike, S. (1991) *Viscosity solutions for monotone systems of second order elliptic PDEs*, Commun. PDEs, **16** 1095 – 1128.
- [29] Ishii, H. and Lions, P. (1990) *Viscosity solutions of fully nonlinear second order elliptic partial differential equations*, J. Diff. Equa., **83** 26 – 78.
- [30] James, R. (2002) *Configurational forces in magnetism with application to the dynamics of a smallscale ferromagnetic shape memory cantilever*, Contin. Mech. Thermodyn., **14**, 55 - 86.
- [31] Jensen, R. (1988) *The maximum principle for viscosity solutions of fully nonlinear second order elliptic partial differential equations*, Arch. Ration. Mech. Anal., **101** 1 – 27.
- [32] Kawashima, S. and Zhu, Peicheng (2010) *Traveling waves for models of phase transitions of solids driven by configurational forces*, to appear in Discr. Conti. Dyna. Systems.
- [33] Ladyzenskaya, O., Solonnikov, V. and Uralceva, N. (1968) *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs **23**, Amer. Math. Soc., Providence.
- [34] Lions, J. (1969) *Quelques methodes de resolution des problemes aux limites non lineaires*, Dunod Gauthier-Villars, Paris.
- [35] Maugin, G. (1993) *Material inhomogeneities in elasticity*, **3**, Appl Math and Math Comput. Chapman & Hall.
- [36] Müller, R. and D. Gross, D. (1998) *3D simulation of equilibrium morphologies of precipitates*, Computational Materials Sci., **11**, 35 - 44.
- [37] Roubíček, T. (1990) *A generalization of the Lions-Temam compact imbedding theorem*, Casopis Pest. Mat. **115**, 338 – 342.
- [38] Simon, J. (1987) *Compact sets in the space $L^p(0, T; B)$* , Annali di Matematica Pura ed Applicata, **146** 64 – 96.
- [39] Socrate, S. and Parks, D. (1993) *Numerical determination of the elastic driving force for directional coarsening in Ni-superalloys*, Acta Metall. Mater., **40**, 2185 - 2209.
- [40] Souganidis, P. (1985) *Existence of viscosity solutions of Hamilton-Jacobi equations*. J. Differential Equations, **56** 345 - 390.