

Nonlinear weakly curved rod by Γ -convergence

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Abstract We present a nonlinear model of weakly curved rod, namely the type of curved rod where the curvature is of the order of the diameter of the cross-section. We use the approach analogous to the one for rods and curved rods and start from the strain energy functional of three dimensional nonlinear elasticity and do not presuppose any constitutive behavior. To derive the model, by means of Γ -convergence, we need to propose how is the order of strain energy related to the thickness of the body h . We analyze the situation when the strain energy (divided by the order of volume) is of the order h^4 . That is the same approach as the one when Föppl-von Kármán model for plates and the analogous model for rods are obtained. The obtained model is analogous to Marguerre-von Kármán for shallow shells and its linearization is the linear shallow arch model which can be found in the literature.

Keywords weakly curved rod · Gamma convergence · shallow arch · asymptotic analysis

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1 Introduction

The study of thin structures is the subject of numerous works in the theory of elasticity. There is a vast literature on the subject of rods, plates and shells (see [5, 8, 9]).

The derivation and justification of the lower dimensional models, equilibrium and dynamic, of rods, curved rods, weakly curved rods, plates and shells in linearized elasticity, by using formal asymptotic expansion, is well established (see [8, 9] and the references therein). In all these approaches one starts from the equations

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of three-dimensional linearized elasticity and then via formal asymptotic expansion justify the lower dimensional models. One can also obtain the convergence results. In [3,4] the linear model of weakly curved rod (or as it is called shallow arch) is derived and the convergence result is obtained. We call weakly curved rods or shallow arches those characterized by the fact that the curvature of their centerline has the same order of magnitude as the diameter of the cross section, both being much smaller than their length.

Formal asymptotic expansion is also applied to derive non linear models of rods, plates and shells (see [8,9,22] and the references therein), starting from three-dimensional isotropic elasticity (usually Saint-Venant-Kirchoff material). Hierarchy of the models is obtained, depending on the order of the external loads related to the thickness of the body h (see also [11] for plates).

However, formal asymptotic expansion does not provide us a convergence result. The first convergence result, in deriving lower dimensional models from three-dimensional non linear elasticity, is obtained applying Γ -convergence, very powerful tool introduced by Degiorgi (see [6,10]). Using Γ -convergence, elastic string models, membrane plate and membrane shell models are obtained (see [1,16,17]). It is assumed that the external loads are of order h^0 . The obtained models are different from those ones obtained by the formal asymptotic expansion in the sense that additional relaxation of the energy functional is done.

Recently, hierarchy of models of rods, curved rods, plates and shells is obtained via Γ -convergence (see [12–15,19,24,25,30,31]). Influence of the boundary conditions and the order and the type of the external loads is largely discussed for plates (see [13,18]). Let us mention that Γ -convergence results provide us the convergence of the global minimizers of the total energy functional. Recently, compensated compactness arguments are used to obtain the convergence of the stationary points of the energy functional (see [23,28]).

Here we apply the tools developed for rods, plates and shells to obtain weakly curved rod model by Γ -convergence. It is assumed that we have free boundary conditions and that the strain energy (divided by the order of volume) is of the order h^4 , where h is the thickness of the rod. This corresponds to the situation when external transversal dead loads are of order h^3 (see Remark 8). The order h^4 of the strain energy gives Föppl-von Kármán model for plates, Marguerre-von Kármán model for shallow shells and the analogous model for rods (see [14,25,32]). The obtained model is non linear model of the highest order in the hierarchy of models and its linearization is shallow arch model, obtained in [3,4] for isotropic, homogenous case (see for comparison Remark 7 d)). Here we do not presuppose any constitutional behavior and thus work in a more general framework. The main result is stated in Theorem 5.

Throughout the paper \bar{A} or $\{A\}^-$ denotes the closure of the set. By a domain we call a bounded open set with Lipschitz boundary. \mathbf{I} denotes the identity matrix, by $\text{SO}(3)$ we denote the rotations in \mathbb{R}^3 , by $\text{so}(3)$ the set of antisymmetric matrices 3×3 and $\mathbb{R}_{\text{sym}}^{3 \times 3}$ denotes the set of symmetric matrices. By $\text{sym } \mathbf{A}$ we denote the symmetric part of the matrix, $\text{sym } \mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$. $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the vectors of the canonical base in \mathbb{R}^3 . By ∇_h we denote $\nabla_h = \nabla_{\mathbf{e}_1} + \frac{1}{h} \nabla_{\mathbf{e}_2, \mathbf{e}_3}$. $\|f\|_{C^1(\Omega)}$ stands for C^1 norm of the function $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ i.e. $\|f\|_{C^1(\Omega)} = \max_{x \in \Omega} |f| + \sum_{i=1}^n \max_{x \in \Omega} |\partial_i f|$. \rightarrow denotes the strong convergence and \rightharpoonup the weak convergence.

2 Setting up the problem

Let $\omega \subset \mathbb{R}^2$ be an open set having area equal to A and Lipschitz boundary. For all h such that $0 < h \leq 1$ and for given L we define

$$\omega^h = h\omega, \quad \Omega^h = (0, L) \times h\omega. \quad (2.1)$$

We shall leave out superscript when $h = 1$, i.e. $\Omega = \Omega^1$, $\omega = \omega^1$. Let us by $\mu(\omega)$ denote

$$\mu(\omega) = \int_{\omega} (x_2^2 + x_3^2) dx_2 dx_3. \quad (2.2)$$

Let us choose coordinate axis such that

$$\int_{\omega} x_2 dx_2 dx_3 = \int_{\omega} x_3 dx_2 dx_3 = \int_{\omega} x_2 x_3 dx_2 dx_3 = 0. \quad (2.3)$$

For every h we define the curve C^h of the form

$$C^h = \{\theta^h(x_1) = (x_1, \theta_2^h(x_1), \theta_3^h(x_1)) \in \mathbb{R}^3 : x_1 \in (0, L)\}. \quad (2.4)$$

where $\theta_k^h(x_1)$, for $k = 2, 3$, are given functions satisfying $\theta_k^h \in C^3(0, L)$. Let $(\mathbf{t}^h, \mathbf{n}^h, \mathbf{b}^h)$ be the Frenet trihedron associated with the curve C^h

$$\mathbf{t}^h = \frac{1}{\sqrt{1 + ((\theta_2^h)')^2 + ((\theta_3^h)')^2}} (1, (\theta_2^h)', (\theta_3^h)'), \quad (2.5)$$

$$\mathbf{n}^h = \frac{(\mathbf{t}^h)'}{\|(\mathbf{t}^h)'\|}, \quad (2.6)$$

$$\mathbf{b}^h = \mathbf{t}^h \times \mathbf{n}^h. \quad (2.7)$$

We suppose $\mathbf{n}^h \in C^1(0, L)$ which is satisfied if $(\theta_2^h)''$, $(\theta_3^h)''$ do not vanish at the same time (which is equivalent to the fact that the curvature of C^h is strictly positive for any $x_1 \in (0, L)$). The case where C^h has null curvature points can be treated in the same fashion, provided that we suppose that along these points we have the same degree of smoothness as before with \mathbf{t}^h , \mathbf{n}^h and \mathbf{b}^h appropriately chosen (see Remark 3). We define the map $\Theta^h : \bar{\Omega}^h \rightarrow \Theta^h(\bar{\Omega}^h) = \{\hat{\Omega}^h\}^- \subset \mathbb{R}^3$, where $\hat{\Omega}^h := \Theta^h(\bar{\Omega}^h)$, in the following manner:

$$\Theta^h(x^h) = (x_1, \theta_2^h(x_1), \theta_3^h(x_1)) + x_2^h \mathbf{n}^h(x_1) + x_3^h \mathbf{b}^h(x_1) \quad (2.8)$$

and we assume that Θ^h is a C^1 diffeomorphism which can be proved if h is small enough and θ_k^h , for $k = 2, 3$, are of the form considered here. Namely, we take $\theta_2^h = h\theta_2$, $\theta_3^h = h\theta_3$ where $\theta_k \in C^3(0, L)$. Let us suppose

$$((\theta_1'')^2(x_1) + ((\theta_2'')^2(x_1) \neq 0, \quad (2.9)$$

for all $x_1 \in (0, L)$. A generic point in $\bar{\Omega}^h$ or $\{\hat{\Omega}^h\}^-$ will be denoted by $x^h = (x_1, x_2^h, x_3^h)$.

Like in [24, 25, 12–15, 19] we start from three dimensional non linear elasticity functional of strain energy (see [7] for an introduction to non linear elasticity)

$$I^h(\mathbf{y}) := \frac{1}{h^2} \int_{\hat{\Omega}^h} W^h(x^h, \nabla \mathbf{y}) dx^h. \quad (2.10)$$

It is natural to divide the strain energy with h^2 , since the volume is vanishing with the order of h^2 . We are interested in finding Γ -limit (in some sense i.e. in characterizing the limits of minimizers) of the functionals $\frac{1}{h^4}I^h$. The reason why we divide with h^4 is that we want to obtain theory analogous to Föppl-von Kármán for plates and rods (see [13, 14, 25]) and Marguerre-von Kármán for shallow shells (see [32]). We do not look the total energy functional because the part with the strain energy contains the highest order derivatives (at least for the external dead loads) and thus makes the most difficult part of the analysis (see Remarks 8 and 10). We shall not impose Dirichlet boundary condition and assume that the body is free at the boundary. The consideration of the other boundary conditions is also possible. We rewrite the functional I^h on the domain Ω , i.e. we write

$$I^h(\mathbf{y}) := \int_{\Omega} W^h(\boldsymbol{\Theta}^h \circ P^h(x), (\nabla \mathbf{y}) \circ \boldsymbol{\Theta}^h \circ P^h) \det((\nabla \boldsymbol{\Theta}^h) \circ P^h(x)) dx, \quad (2.11)$$

where by $P^h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ we denote the mapping $P^h(x_1, x_2, x_3) = (x_1, hx_2, hx_3)$. $(\nabla \mathbf{y}) \circ \boldsymbol{\Theta}^h \circ P^h$ denotes $\nabla \mathbf{y}$ evaluated at the point $\boldsymbol{\Theta}^h(P^h(x))$. We assume that for each h it is valid

$$\det((\nabla \boldsymbol{\Theta}^h) \circ P^h(x)) W^h(\boldsymbol{\Theta}^h \circ P^h(x), \mathbf{F}) = W(x, \mathbf{F}), \forall x \in \Omega, \forall \mathbf{F} \in \mathbb{R}^{3 \times 3}, \quad (2.12)$$

where the stored energy function W is independent of h and satisfies the following assumptions (the same ones as in [25]):

- i) $W : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty]$ is a Carathéodory function; for some $\delta > 0$ the function $\mathbf{F} \mapsto W(x, \mathbf{F})$ is of class C^2 for $\text{dist}(\mathbf{F}, \text{SO}(3)) < \delta$ and for a.e. $x \in \Omega$;
- ii) the second derivative $\frac{\partial^2 W}{\partial \mathbf{F}^2}$ is a Carathéodory function on the set $\Omega \times \{\mathbf{F} \in \mathbb{R}^{3 \times 3} : \text{dist}(\mathbf{F}, \text{SO}(3)) < \delta\}$ and there exists a constant $\gamma > 0$ such that

$$\left| \frac{\partial^2 W}{\partial \mathbf{F}^2}(x, \mathbf{F})[\mathbf{G}, \mathbf{G}] \right| \leq \gamma |\mathbf{G}|^2 \text{ if } \text{dist}(\mathbf{F}, \text{SO}(3)) < \delta \text{ and } \mathbf{G} \in \mathbb{R}_{\text{sym}}^{3 \times 3};$$

- iii) W is frame-indifferent, i.e. $W(x, \mathbf{F}) = W(x, \mathbf{R}\mathbf{F})$ for a.e. $x \in \Omega$ and every $\mathbf{F} \in \mathbb{R}^{3 \times 3}, \mathbf{R} \in \text{SO}(3)$;
- iv) $W(x, \mathbf{F}) = 0$ if $\mathbf{F} \in \text{SO}(3)$; $W(x, \mathbf{F}) \geq C \text{dist}^2(\mathbf{F}, \text{SO}(3))$ for every $\mathbf{F} \in \mathbb{R}^{3 \times 3}$, where the constant $C > 0$ is independent of x .

Under these assumptions we first show the compactness result (Theorem 3) i.e. we take the sequence $y^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$ such that

$$\limsup_{h \rightarrow 0} \frac{1}{h^4} I^h < +\infty$$

and conclude how that fact affects the limit displacement. In Lemma 2 we prove the lower bound, in Theorem 4 we prove the upper bound and that enables us to identify the limit functional (Theorem 5). First we start with some basic properties of the mappings $\boldsymbol{\Theta}^h$ which are necessary for further analysis.

3 Properties of the mappings Θ^h

We introduce for $k = 2, 3$,

$$p_k(x_1) = \frac{\theta_k''(x_1)}{\sqrt{(\theta_2'')^2(x_1) + (\theta_3'')^2(x_1)}}. \quad (3.1)$$

Notice that

$$p_2^2 + p_3^2 = 1, \quad p_2 p_2' + p_3 p_3' = 0. \quad (3.2)$$

Let us denote $p = p_2 p_3' - p_2' p_3$.

Theorem 1 *Let the functions θ_k^h be such that*

$$\theta_k^h(x_1) = h\theta_k(x_1), \text{ for all } x_1 \in (0, L), k = 2, 3.$$

where $\theta_k \in C^3(0, L)$ is independent of h . Then there exists $h_0 = h_0(\theta) > 0$ such that the Jacobian matrix $\nabla \Theta^h(x^h)$, where the mappings Θ^h are defined with (2.8), is invertible for all $x^h \in \bar{\Omega}^h$ and all $h \leq h_0$. Also there exists $C > 0$ such that for $h \leq h_0$ we have

$$\det \nabla \Theta^h = 1 + h\delta^h(x^h), \quad (3.3)$$

and

$$\mathbf{t}^h(x_1) = \mathbf{e}_1 + h\theta_2'(x_1)\mathbf{e}_2 + h\theta_3'(x_1)\mathbf{e}_3 + h^2\mathbf{o}_1(x_1), \quad (3.4)$$

$$\begin{aligned} \mathbf{n}^h(x_1) &= p_2(x_1)\mathbf{e}_2 + p_3(x_1)\mathbf{e}_3 - h(\theta_2'p_2 + \theta_3'p_3)(x_1)\mathbf{e}_1 \\ &\quad + h^2\mathbf{o}_2(x_1), \end{aligned} \quad (3.5)$$

$$\begin{aligned} \mathbf{b}^h(x_1) &= -p_3(x_1)\mathbf{e}_2 + p_2(x_1)\mathbf{e}_3 + h(\theta_2'p_3 - \theta_3'p_2)(x_1)\mathbf{e}_1 \\ &\quad + h^2\mathbf{o}_3(x_1), \end{aligned} \quad (3.6)$$

$$\begin{aligned} \nabla \Theta^h(x^h) &= \mathbf{R}_e(x_1) + h\mathbf{C}(x_1) + x_2^h\mathbf{D}(x_1) + x_3^h\mathbf{E}(x_1) \\ &\quad + h^2\mathbf{O}_1^h(x^h), \end{aligned} \quad (3.7)$$

$$\begin{aligned} (\nabla \Theta^h(x^h))^{-1} &= \mathbf{R}_e^T(x_1) - h\mathbf{C}_1(x_1) - x_2^h\mathbf{D}_1(x_1) - x_3^h\mathbf{E}_1(x_1) \\ &\quad + h^2\mathbf{O}_2^h(x^h), \end{aligned} \quad (3.8)$$

$$\left\| (\nabla \Theta^h) - \mathbf{R}_e \right\|_{L^\infty(\Omega^h; \mathbb{R}^{3 \times 3})} < Ch, \quad (3.9)$$

$$\left\| (\nabla \Theta^h)^{-1} - \mathbf{R}_e^T \right\|_{L^\infty(\Omega^h; \mathbb{R}^{3 \times 3})} < Ch, \quad (3.10)$$

where

$$\mathbf{R}_e = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} p_2 & -p_3 \\ p_3 & p_2 \end{pmatrix}, \quad (3.11)$$

$$\mathbf{C} = \begin{pmatrix} 0 & -(\theta_2'p_2 + \theta_3'p_3) & -(\theta_3'p_2 - \theta_2'p_3) \\ \theta_2' & 0 & 0 \\ \theta_3' & 0 & 0 \end{pmatrix}, \quad (3.12)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ p_2' & 0 & 0 \\ p_3' & 0 & 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 0 & 0 & 0 \\ -p_3' & 0 & 0 \\ p_2' & 0 & 0 \end{pmatrix}, \quad (3.13)$$

$$\mathbf{C}_1 = \begin{pmatrix} 0 & -\theta'_2 & -\theta'_3 \\ \theta'_2 p_2 + \theta'_3 p_3 & 0 & 0 \\ \theta'_3 p_2 - \theta'_2 p_3 & 0 & 0 \end{pmatrix}, \quad (3.14)$$

$$\mathbf{D}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.15)$$

and $\delta^h : \bar{\Omega}^h \rightarrow \mathbb{R}$, $\mathbf{o}_i : (0, L) \rightarrow \mathbb{R}^3$, $i = 1, 2, 3$, $\mathbf{O}_k^h : \bar{\Omega}^h \rightarrow \mathbb{R}^{3 \times 3}$, $k = 1, 2$ are functions which satisfy

$$\sup_{0 < h \leq h_0} \max_{x^h \in \bar{\Omega}^h} |\delta^h(x^h)| \leq C_0,$$

$$\sup_{0 < h \leq h_0} \max_{x_1 \in (0, L)} \|\mathbf{o}_i^h(x_1)\| \leq C_0, \quad \sup_{0 < h \leq h_0} \max_{x_1 \in (0, L)} \|(\mathbf{o}_i^h)'\| \leq C_0$$

$$\sup_{0 < h \leq h_0} \max_{i, j} \max_{x^h \in \bar{\Omega}^h} \|\mathbf{O}_{k, ij}^h(x^h)\| \leq C_0, \quad k = 1, 2,$$

for some constant $C_0 > 0$.

Proof. It can be easily seen that

$$\mathbf{t}^h(x_1) = \mathbf{e}_1 + h\theta'_2(x_1)\mathbf{e}_2 + h\theta'_3(x_1)\mathbf{e}_3 - \frac{h^2}{2}((\theta'_2)^2 + (\theta'_3)^2)\mathbf{e}_1 + h^3\mathbf{o}_4^h(x^h), \quad (3.16)$$

where $\|\mathbf{o}_4\|_{C^2(0, L)} \leq C$. The relations (3.5) and (3.6) are the direct consequences of the relation (3.16). Let us by $\mathbf{u}^h : (0, L) \rightarrow \mathbb{R}^3$ denote the function

$$\mathbf{u}^h = (1, h\theta'_2, h\theta'_3)^T. \quad (3.17)$$

It is easy to see that

$$\nabla \boldsymbol{\Theta}^h(x^h) = (\mathbf{u}^h(x_1) + x_2^h(\mathbf{n}^h)'\|x_1) + x_3^h(\mathbf{b}^h)'\|x_1) \mid \mathbf{n}^h(x_1) \mid \mathbf{b}^h(x_1)). \quad (3.18)$$

The relations (3.3), (3.7), (3.9), (3.10) are the direct consequences of the relations (3.4)-(3.6) and (3.18). The relation (3.8) is the direct consequence of the fact that, for a regular matrix \mathbf{A} and arbitrary \mathbf{B} , which satisfies $\|\mathbf{A}^{-1}\mathbf{B}\| < 1$ ($\|\cdot\|$ is the operational norm), the matrix $\mathbf{A} + \mathbf{B}$ is invertible and

$$\|(\mathbf{A} + \mathbf{B})^{-1} - (\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})\| \leq \frac{\|\mathbf{A}^{-1}\mathbf{B}\|^2\|\mathbf{A}\|^{-1}}{1 - \|\mathbf{A}^{-1}\mathbf{B}\|}.$$

To end the proof observe that

$$\mathbf{C}_1 = \mathbf{R}_e^T \mathbf{C} \mathbf{R}_e^T, \quad \mathbf{D}_1 = \mathbf{R}_e^T \mathbf{D} \mathbf{R}_e^T, \quad \mathbf{E}_1 = \mathbf{R}_e^T \mathbf{E} \mathbf{R}_e^T.$$

Remark 1 By a careful computation it can be seen that \mathbf{o}_2^h , \mathbf{o}_3^h and \mathbf{o}_4^h (defined in the relation (3.16)) are dominantly in $\mathbf{e}_2, \mathbf{e}_3$ plane i.e. that we have for $i = 2, 3, 4$:

$$\mathbf{o}_i^h(x_1) = f_i^2(x_1)\mathbf{e}_2 + f_i^3(x_1)\mathbf{e}_3 + h\mathbf{r}_i^h(x_1), \quad (3.19)$$

where $f_i^2, f_i^3 \in C^1(0, L)$, $\sup_{0 < h \leq h_0} \|\mathbf{r}_i^h\|_{C^1(0, L)} \leq C$, for some $C > 0$.

Remark 2 By a further inspection it can be seen that

$$f_4^2 = -\frac{1}{2}\theta_2'((\theta_2')^2 + (\theta_3')^2), \quad f_4^3 = -\frac{1}{2}\theta_3'((\theta_2')^2 + (\theta_3')^2), \quad (3.20)$$

$$f_2^2 = p_2\left(f_2(\theta', \theta'') - \frac{1}{2}((\theta_2')^2 + (\theta_3')^2)\right) - \theta_2'(\theta_2'p_2 + \theta_3'p_3), \quad (3.21)$$

$$f_2^3 = p_3\left(f_2(\theta', \theta'') - \frac{1}{2}((\theta_2')^2 + (\theta_3')^2)\right) - \theta_3'(\theta_2'p_2 + \theta_3'p_3), \quad (3.22)$$

$$f_3^2 = p_3((\theta_2')^2 + (\theta_3')^2) - p_3f_2(\theta', \theta''), \quad (3.23)$$

$$f_3^3 = -p_2((\theta_2')^2 + (\theta_3')^2) + p_2f_2(\theta', \theta''), \quad (3.24)$$

where $f_2(\theta', \theta'') \in C^1(0, L)$ is the expression that includes θ', θ'' :

$$f_2(\theta', \theta'') = \frac{1}{2}\left((p_2 + p_3)((\theta_2')^2 + (\theta_3')^2) + 2(\theta_2' + \theta_3')(\theta_2'p_2 + \theta_3'p_3) - \sqrt{(\theta_2'')^2 + (\theta_3'')^2}(\theta_2'p_2 + \theta_3'p_3)^2\right).$$

Remark 3 It is not necessary to impose the condition (2.9). All we need is the existence of the expansions given by (3.4)-(3.6), where $p_2, p_3 \in C^1(0, L)$, including the statement of Remark 1.

Remark 4 Although Θ^h makes the small perturbation of the central line, $(x_1, 0, 0)$, for $x_1 \in [0, L]$, it is not true that $\nabla\Theta^h$ is close to the identity (like in the shallow shell model, see [32]). In fact, there are torsional effects of order 0 on every cross section. This is the main reason why is the change of coordinates introduced in the next chapter useful.

4 Γ -convergence

We shall need the following theorem which can be found in [12].

Theorem 2 (on geometric rigidity) *Let $U \subset \mathbb{R}^m$ be a bounded Lipschitz domain, $m \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $\mathbf{v} \in W^{1,2}(U; \mathbb{R}^m)$ there is an associated rotation $\mathbf{R} \in \text{SO}(m)$ such that*

$$\|\nabla\mathbf{v} - \mathbf{R}\|_{L^2(U)} \leq C(U)\|\text{dist}(\nabla\mathbf{v}, \text{SO}(m))\|_{L^2(U)}. \quad (4.1)$$

The constant $C(U)$ can be chosen uniformly for a family of domains which are Bilipschitz equivalent with controlled Lipschitz constants. The constant $C(U)$ is invariant under dilatations.

The following version of the Korn's inequality is needed.

Lemma 1 *Let $\omega \subset \mathbb{R}^2$ with Lipschitz boundary and $\mathbf{u} \in L^2(\omega; \mathbb{R}^2)$. Let us by $e_{ij}(\mathbf{u})$ denote $e_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i\mathbf{u} + \partial_j\mathbf{u})$. Let us suppose that for every $i, j = 1, 2$ we have that $e_{ij}(\mathbf{u}) \in L^2(\omega)$. Then we have that $\mathbf{u} \in W^{1,2}(\omega; \mathbb{R}^2)$. Also there exists constant $C(\omega)$, depending only on the domain ω , such that we have*

$$\|\mathbf{u}\|_{W^{1,2}(\omega; \mathbb{R}^2)} \leq C(\omega)\left(|\int_{\omega}\mathbf{u}dx_1dx_2| + |\int_{\omega}(x_1\mathbf{u}_2 - x_2\mathbf{u}_1)dx_1dx_2| + \sum_{i,j=1,2}\|e_{ij}(\mathbf{u})\|_{L^2(\omega)}\right). \quad (4.2)$$

Let us suppose that the domains ω_s are changing in the sense that they are equal to $\omega_s = \mathbf{A}_s \omega$, where $\mathbf{A}_s \in \mathbb{R}^{2 \times 2}$, and there exists a constant C such that $\|\mathbf{A}_s\|, \|\mathbf{A}_s^{-1}\| \leq C$. Then the constant in the inequality (4.2) can be chosen independently of s .

Proof. The first part of the lemma (the fixed domain) is a version of the Korn's inequality (see e.g. [29]). The last part we shall prove by a contradiction. Let us suppose the contrary that for each $n \in \mathbf{N}$ there exists s^n and $\mathbf{u}^n \in W^{1,2}(\omega_{s^n}; \mathbb{R}^2)$ such that we have

$$\begin{aligned} & \left| \int_{\omega_{s^n}} \mathbf{u}^n dx_1 dx_2 \right| + \left| \int_{\omega_{s^n}} (x_1 \mathbf{u}_2^n - x_2 \mathbf{u}_1^n) dx_1 dx_2 \right| \\ & + \sum_{i,j=1,2} \|e_{ij}^{s^n}(\mathbf{u}^n)\|_{L^2(\omega_{s^n})} \leq \frac{1}{n} \|\mathbf{u}^n\|_{W^{1,2}(\omega_{s^n}; \mathbb{R}^2)}, \end{aligned} \quad (4.3)$$

where we have by $e_{ij}^{s^n}(\cdot)$ denoted the symmetrized gradient on the domain ω_{s^n} . Without any loss of generality we can suppose that $\|\mathbf{u}^n\|_{W^{1,2}(\omega_{s^n}; \mathbb{R}^2)} = 1$. Let us take the subsequence of (s^n) (still denoted by (s^n)) such that $\mathbf{A}_{s^n} \rightarrow \mathbf{A}$ and $\mathbf{A}_{s^n}^{-1} \rightarrow \mathbf{A}^{-1}$ in $\mathbb{R}^{2 \times 2}$.

Let us look the sequence $\mathbf{u}_c^n = \mathbf{u}^n \circ \mathbf{A}_{s^n} \circ \mathbf{A}^{-1}$. It is clear that there exist $C_1, C_2 > 0$ such that

$$C_1 \leq \|\mathbf{u}_c^n\|_{W^{1,2}(\omega_\infty; \mathbb{R}^2)} \leq C_2, \quad (4.4)$$

where we have put $\omega_\infty := \mathbf{A}\omega$. Thus there exists $\mathbf{u} \in W^{1,2}(\omega_\infty; \mathbb{R}^2)$ such that $\mathbf{u}_c^n \rightharpoonup \mathbf{u}$ weakly in $W^{1,2}(\omega_\infty; \mathbb{R}^2)$. Specially, by the compactness of the embedding $L^2 \hookrightarrow W^{1,2}$ (see e.g. [2]), we also conclude the strong convergence $\mathbf{u}_c^n \rightarrow \mathbf{u}$ in $L^2(\omega_\infty; \mathbb{R}^2)$. Since it is valid $\mathbf{A}_{s^n} \mathbf{A}^{-1} \rightarrow \mathbf{I}$, it can be easily seen that, from the weak convergence, it follows $e_{ij}^{s^n}(\mathbf{u}^n) \circ (\mathbf{A}_{s^n} \circ \mathbf{A}^{-1}) \rightharpoonup e_{ij}^\infty(\mathbf{u})$, weakly in $L^2(\omega_\infty; \mathbb{R}^2)$, where we have by $e_{ij}^\infty(\cdot)$ denoted the symmetrized gradient on the domain ω_∞ . From the weak convergence we can conclude that

$$\|e_{ij}^\infty(\mathbf{u})\|_{L^2(\omega_\infty)} \leq \liminf_{n \rightarrow \infty} \|e_{ij}^{s^n}(\mathbf{u}^n) \circ (\mathbf{A}_{s^n} \circ \mathbf{A}^{-1})\|_{L^2(\omega_\infty)} = 0, \quad (4.5)$$

for every $i, j = 1, 2$. We can also from (4.3) conclude that

$$\int_{\omega_\infty} \mathbf{u} dx_1 dx_2 = 0, \quad \int_{\omega_\infty} (x_1 \mathbf{u}_2 - x_2 \mathbf{u}_1) dx_1 dx_2 = 0. \quad (4.6)$$

Applying the standard Korn's inequality on the domain ω_∞ , i.e.

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_c^n\|_{W^{1,2}(\omega_\infty; \mathbb{R}^2)} & \leq C(\omega_\infty) \left(\|\mathbf{u} - \mathbf{u}_c^n\|_{L^2(\omega_\infty; \mathbb{R}^2)} \right. \\ & \left. + \sum_{i,j=1,2} \|e_{ij}(\mathbf{u}) - e_{ij}(\mathbf{u}_c^n)\|_{L^2(\omega_\infty; \mathbb{R}^2)} \right), \end{aligned}$$

we conclude that $\mathbf{u}_c^n \rightarrow \mathbf{u}$ strongly in $W^{1,2}(\omega_\infty; \mathbb{R}^2)$. But then (4.4), (4.5), (4.6) make a contradiction with the version of the Korn's inequality (4.2) on the domain ω_∞ .

Remark 5 The same proof can be done under the assumption that $\omega_s = F_s(\omega)$, where F_s is the family of Bilipschitz mappings whose Bilipschitz constants we can control (i.e. the Lipschitz constants of F_s and F_s^{-1} are bounded by a universal constant), provided that the family F_s is strongly compact in $W^{1,\infty}(\omega; \mathbb{R}^2)$. It would require more analysis to conclude the same result only for Bilipschitz mappings whose Bilipschitz constants we can control.

Let us by $\mathbf{x}' : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denote the change of coordinates

$$(x'_1, x'_2, x'_3) = \mathbf{x}'(x_1, x_2, x_3) := \mathbf{R}_e(x_1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (4.7)$$

By Ω' we denote $\mathbf{x}'(\Omega)$ and $\omega'(x_1) \subset \mathbb{R}^2$ denotes $\mathbf{x}'(\{x_1\} \times \omega)$, for $x_1 \in [0, L]$. The generic point in Ω' is denoted with $x' = (x_1, x'_2, x'_3)$. Let us observe that by (2.2) and (2.3)

$$\int_{\omega'(x_1)} x'_2 dx'_2 dx'_3 = \int_{\omega'(x_1)} x'_3 dx'_2 dx'_3 = 0, \quad (4.8)$$

$$\int_{\omega'(x_1)} x'_2 x'_3 dx'_2 dx'_3 = p_2 p_3 \int_{\omega} (x_2^2 - x_3^2) dx_2 dx_3, \quad (4.9)$$

$$\mu(\omega) = \int_{\omega} (x_2^2 + x_3^2) dx_2 dx_3 = \int_{\omega'(x_1)} ((x'_2)^2 + (x'_3)^2) dx'_2 dx'_3, \quad (4.10)$$

for all $x_1 \in [0, L]$. By $(\partial_i \mathbf{y}_j) \circ \Theta^h \circ P^h$ we denote $\partial_i \mathbf{y}_j$ evaluated at the point $\Theta^h(P^h(x))$.

In the sequel we suppose $h_0 \geq 1$ (see Theorem 1). If this was not the case, what follows could be easily adapted. Using theorem 2 we can prove the following theorem

Theorem 3 *Let $\mathbf{y}^h \in W^{1,2}(\hat{\Omega}^h; \mathbb{R}^3)$ and let*

$$E^h = \frac{1}{h^2} \int_{\hat{\Omega}^h} \text{dist}^2(\nabla \mathbf{y}^h, \text{SO}(3)) dx.$$

Let us suppose that

$$\limsup_{h \rightarrow 0} \frac{E^h}{h^4} < +\infty. \quad (4.11)$$

Then there exist maps $\mathbf{R}^h : [0, L] \rightarrow \text{SO}(3)$ and $\tilde{\mathbf{R}}^h : [0, L] \rightarrow \mathbb{R}^{3 \times 3}$, with $|\tilde{\mathbf{R}}| \leq C$, $\tilde{\mathbf{R}} \in W^{1,2}([0, L], \mathbb{R}^{3 \times 3})$ and constants $\bar{\mathbf{R}}^h \in \text{SO}(3)$, $c^h \in \mathbb{R}^3$ such that the functions $\tilde{\mathbf{y}}^h := (\bar{\mathbf{R}}^h)^T \mathbf{y}^h - c^h$ satisfy

$$\|(\nabla \tilde{\mathbf{y}}^h) \circ \Theta^h \circ P^h - \mathbf{R}^h\|_{L^2(\Omega)} \leq Ch^2, \quad (4.12)$$

$$\|\mathbf{R}^h - \tilde{\mathbf{R}}^h\|_{L^2([0, L])} \leq Ch^2, \quad \|(\tilde{\mathbf{R}}^h)'\|_{L^2([0, L])} \leq Ch, \quad (4.13)$$

$$\|\mathbf{R}^h - \mathbf{I}\|_{L^\infty([0, L])} \leq Ch. \quad (4.14)$$

Moreover if we define

$$u^h = \frac{1}{A} \int_{\omega} \frac{\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h - x_1}{h^2} dx_2 dx_3, \quad (4.15)$$

$$v_k^h = \frac{1}{A} \int_{\omega} \frac{\tilde{\mathbf{y}}_k^h \circ \Theta^h \circ P^h - h\theta_k}{h} dx_2 dx_3, \quad \text{for } k = 2, 3, \quad (4.16)$$

$$w^h = \frac{1}{A\mu(\omega)} \int_{\omega} \frac{x'_2(\tilde{\mathbf{y}}_3 \circ \Theta^h \circ P^h) - x'_3(\tilde{\mathbf{y}}_2 \circ \Theta^h \circ P^h)}{h^2} dx_2 dx_3 \quad (4.17)$$

then, up to subsequences, the following properties are satisfied

- (a) $u^h \rightharpoonup u$ in $W^{1,2}(0, L)$;
 (b) $v_k^h \rightarrow v_k$ in $W^{1,2}(0, L)$, where $v_k \in W^{2,2}(0, L)$ for $k = 2, 3$.
 (c) $w^h \rightharpoonup w$ weakly in $W^{1,2}(0, L)$;
 (d) $\frac{(\nabla \tilde{\mathbf{y}}^h) \circ \Theta^h \circ P^h - \mathbf{I}}{h} \rightarrow \mathbf{A}$, in $L^2(\Omega)$, where $\mathbf{A} \in W^{1,2}(0, L)$ is given by

$$\mathbf{A} = \begin{pmatrix} 0 & -v'_2 & -v'_3 \\ v'_2 & 0 & -w \\ v'_3 & w & 0 \end{pmatrix}. \quad (4.18)$$

- (e) $\text{sym} \frac{\mathbf{R}^h - \mathbf{I}}{h^2} \rightarrow \frac{\mathbf{A}^2}{2}$ uniformly on $(0, L)$;
 (f) the sequence γ^h defined by

$$\begin{aligned} \gamma_1^h(x) &= \frac{1}{h} \left(\frac{(\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h)(x) - x_1}{h^2} - u^h(x_1) \right. \\ &\quad \left. + x'_2((v_2^h)' + \theta_2')(x_1) + x'_3((v_3^h)' + \theta_3')(x_1) \right), \\ \gamma_k^h(x) &= \frac{1}{h^2} \left(\frac{(\tilde{\mathbf{y}}_k^h \circ \Theta^h \circ P^h)(x) - h\theta_k - hx'_k}{h} \right. \\ &\quad \left. - v_k^h(x_1) - h(x'_k)^\perp \omega^h(x_1) \right), \text{ for } k = 2, 3, \end{aligned}$$

where $(x')^\perp := (0, -x'_3, x'_2)$, is weakly convergent in $L^2(\Omega)$ to a function γ belonging to the space \mathcal{C} , where

$$\begin{aligned} \mathcal{C} &= \{ \gamma \in L^2(\Omega; \mathbb{R}^3) : \int_\omega \gamma = 0, \partial_2 \gamma, \partial_3 \gamma \in L^2(\Omega; \mathbb{R}^3), \\ &\quad \int_\omega (x'_3 \gamma_2(x_1, \cdot) - x'_2 \gamma_3(x_1, \cdot)) dx_2 dx_3 = 0, \text{ for a.e. } x_1 \in (0, L) \}. \end{aligned} \quad (4.19)$$

Moreover $\partial_k \gamma^h \rightharpoonup \partial_k \gamma$ in $L^2(\Omega)$ for $k = 2, 3$,

Proof. We follow the proof of Theorem 2.2 in [25]. Applying Theorem 2 as in the compactness result of [24] (using the boundedness of $\nabla \Theta^h$ and $(\nabla \Theta^h)^{-1}$ we can find a sequence of piecewise constant maps $\mathbf{R}^h : [0, L] \rightarrow \text{SO}(3)$ such that

$$\int_\Omega \|(\nabla \mathbf{y}^h) \circ \Theta^h \circ P^h - \mathbf{R}^h\|^2 dx \leq Ch^4, \quad (4.20)$$

and

$$\int_{I'} \|\mathbf{R}^h(x_1 + \xi) - \mathbf{R}^h(x_1)\|^2 dx_1 \leq Ch^2(|\xi| + h)^2, \quad (4.21)$$

where I' is any open interval in $(0, L)$ and $\xi \in \mathbb{R}$ satisfies $|\xi| \leq \text{dist}(I', \{0, L\})$. Let $\eta \in C_0^\infty(0, 1)$ be such that $\eta \geq 0$ and $\int_0^1 \eta(s) ds = 1$. We set $\eta_h = \frac{1}{h} \eta(\frac{\cdot}{h})$ and we define

$$\tilde{\mathbf{R}}^h(x_1) := \int_{-h}^h \eta_h(s) \mathbf{R}^h(x_1 - s) ds,$$

where we have extended \mathbf{R}^h outside $[0, L]$ by taking $\mathbf{R}^h(x_1) = \mathbf{R}^h(0)$ for every $x_1 < 0$, $\mathbf{R}^h(x_1) = \mathbf{R}^h(L)$ for every $x_1 > L$. Clearly $\|\tilde{\mathbf{R}}^h\| \leq C$ for every h while properties (4.13) follow from properties (4.21). Moreover since by construction (see [24])

$$\|\mathbf{R}^h(x_1 + s) - \mathbf{R}^h(x_1)\|^2 \leq \frac{C}{h^3} \int_{\hat{\Omega}^h} \text{dist}^2(\nabla \mathbf{y}^h, \text{SO}(3)) \leq Ch^3,$$

for every $|s| \leq h$ we have by Jensen's inequality that

$$\|\tilde{\mathbf{R}}^h - \mathbf{R}^h\|_{L^\infty((0, L); \mathbb{R}^{3 \times 3})}^2 \leq Ch^3 \quad (4.22)$$

By the Sobolev-Poincaré inequality and the second inequality in (4.13), there exist constants $\mathbf{Q}^h \in \mathbb{R}^{3 \times 3}$ such that $\|\tilde{\mathbf{R}}^h - \mathbf{Q}^h\|_{L^\infty((0, L); \mathbb{R}^{3 \times 3})} \leq Ch$. Combining this inequality with (4.22), we have that $\|\mathbf{R}^h - \mathbf{Q}^h\|_{L^\infty((0, L); \mathbb{R}^{3 \times 3})} \leq Ch$. This implies that $\text{dist}(\mathbf{Q}^h, \text{SO}(3)) \leq Ch$; thus, we may assume that \mathbf{Q}^h belongs to $\text{SO}(3)$ and by modifying \mathbf{Q}^h by order h , if needed. Now choosing $\bar{\mathbf{R}}^h = \mathbf{Q}^h$ and replacing \mathbf{R}^h by $(\mathbf{Q}^h)^T \mathbf{R}^h$ and $\tilde{\mathbf{R}}^h$ by $(\mathbf{Q}^h)^T \tilde{\mathbf{R}}^h$, we obtain (4.14). By suitable choice of constants $\mathbf{c}^h \in \mathbb{R}^3$ we may assume that

$$\int_{\Omega} (\tilde{\mathbf{y}}_1^h \circ \boldsymbol{\theta}^h \circ P^h - x_1) = 0, \quad \int_{\Omega} (\tilde{\mathbf{y}}_k^h \circ \boldsymbol{\theta}^h \circ P^h - h\theta_k) = 0, \quad \text{for } k = 2, 3. \quad (4.23)$$

Let $\mathbf{A}^h = \frac{\tilde{\mathbf{R}}^h - \mathbf{I}}{h}$. By (4.14) there exists $\mathbf{A} \in L^\infty((0, L); \mathbb{R}^{3 \times 3})$ such that, up to subsequences,

$$\mathbf{A}^h \rightharpoonup \mathbf{A} \text{ weakly } * \text{ in } L^\infty((0, L); \mathbb{R}^{3 \times 3}). \quad (4.24)$$

On the other hand it follows from (4.13) and (4.14) that

$$\frac{\tilde{\mathbf{R}}^h - \mathbf{I}}{h} \rightharpoonup \mathbf{A} \text{ weakly in } W^{1,2}((0, L); \mathbb{R}^{3 \times 3}). \quad (4.25)$$

In particular, $\mathbf{A} \in W^{1,2}((0, L); \mathbb{R}^{3 \times 3})$ and $h^{-1}(\tilde{\mathbf{R}}^h - \mathbf{I})$ also converges uniformly. Using (4.22) we deduce that

$$\mathbf{A}^h \rightarrow \mathbf{A} \text{ uniformly.} \quad (4.26)$$

In view of (4.12) this clearly implies the convergence property in (d). Since $\mathbf{R}^h \in \text{SO}(3)$ we have

$$\mathbf{A}^h + (\mathbf{A}^h)^T = -h\mathbf{A}^h(\mathbf{A}^h)^T.$$

Hence, $\mathbf{A} + \mathbf{A}^T = 0$. Moreover, after division by $2h$ we obtain property (e) by (4.26). For adapting the proof to the proof of Theorem 2.2 in [25] it is essential to see

$$\begin{aligned} (\nabla \tilde{\mathbf{y}}^h) \circ \boldsymbol{\theta}^h \circ P^h &= (\nabla(\tilde{\mathbf{y}}^h \circ \boldsymbol{\theta}^h) \circ P^h)((\nabla \boldsymbol{\theta}^h)^{-1} \circ P^h) \\ &= \nabla_h(\tilde{\mathbf{y}}^h \circ \boldsymbol{\theta}^h \circ P^h)((\nabla \boldsymbol{\theta}^h)^{-1} \circ P^h). \end{aligned} \quad (4.27)$$

From (4.27) it follows

$$((\nabla \tilde{\mathbf{y}}^h) \circ \boldsymbol{\theta}^h \circ P^h)((\nabla \boldsymbol{\theta}^h) \circ P^h) = \nabla_h(\tilde{\mathbf{y}}^h \circ \boldsymbol{\theta}^h \circ P^h). \quad (4.28)$$

and

$$\begin{aligned} (\nabla \tilde{\mathbf{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{I} &= (\nabla_h(\tilde{\mathbf{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h))((\nabla \boldsymbol{\Theta}^h)^{-1} \circ P^h - \mathbf{R}_e^T) \\ &\quad + (\nabla_h(\tilde{\mathbf{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h))\mathbf{R}_e^T. \end{aligned} \quad (4.29)$$

Let us notice that from (2.8), (3.5), (3.6) we can conclude

$$\boldsymbol{\Theta}_k = h\theta_k + hx'_k + O_k(h^3) \text{ for } k = 2, 3, \quad (4.30)$$

where $\|O_k(h^3)\|_{C^1(\Omega)} \leq Ch^3$.

By multiplying (d) with $(\nabla \boldsymbol{\Theta}^h) \circ P^h = \nabla_h(\boldsymbol{\Theta}^h \circ P^h)$ and using (3.9), (4.28) we obtain

$$\frac{\nabla_h(\tilde{\mathbf{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)}{h} \rightarrow \mathbf{A}\mathbf{R}_e \text{ in } L^2(\Omega). \quad (4.31)$$

Property (b) immediately from (4.31) by using (3.7), (4.23) and (4.30). Moreover, $v'_k = \mathbf{A}_{k1}$ for $k = 2, 3$ so that $v_k \in W^{2,2}(0, L)$ since $\mathbf{A} \in W^{1,2}(0, L)$. By using (e), (3.10), (4.12) and (4.31) from (4.29) we conclude that

$$\left\| \frac{1}{h^2} \text{sym}(\nabla_h(\tilde{\mathbf{y}}^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}^h \circ P^h)\mathbf{R}_e^T) \right\|_{L^2((0, L); \mathbb{R}^{3 \times 3})} \leq C \quad (4.32)$$

The weak convergence of u^h follows from (3.7), (4.32) and the definition of \mathbf{R}_e . By using the convergence (4.31) and Poincare inequality on each cut $\{x_1\} \times \omega$ we can conclude

$$\begin{aligned} \frac{\tilde{\mathbf{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}_2^h \circ P^h}{h^2} - \frac{1}{h^2 A} \int_{\omega} (\tilde{\mathbf{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}_2^h \circ P^h) \\ \rightarrow (\mathbf{A}\mathbf{R}_e)_{22}x_2 + (\mathbf{A}\mathbf{R}_e)_{23}x_3 \text{ in } L^2(\Omega). \end{aligned} \quad (4.33)$$

By using (2.3), (4.7) and (4.30) we conclude from (4.33)

$$w_2^h := \frac{\frac{1}{h}\tilde{\mathbf{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h - x'_2}{h} - \frac{1}{h^2 A} \int_{\omega} \tilde{\mathbf{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h \rightarrow \mathbf{A}_{23}x'_3 \text{ in } L^2(\Omega). \quad (4.34)$$

Let us note that since the left hand side of (4.33) i.e. (4.34) is bounded in $W^{1,2}(\Omega)$ the convergence in (4.34) is in fact weak in $W^{1,2}(\Omega)$. The only nontrivial thing to prove is the boundedness of $\partial_1 w_2^h$ in $L^2(\Omega)$. By the chain rule we have for $i = 1, 2, 3$

$$\begin{aligned} \partial_1(\tilde{\mathbf{y}}_i^h \circ \boldsymbol{\Theta}^h \circ P^h) &= ((\partial_1 \tilde{\mathbf{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h)((\partial_1 \boldsymbol{\Theta}_1^h) \circ P^h) \\ &\quad + ((\partial_2 \tilde{\mathbf{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h)((\partial_1 \boldsymbol{\Theta}_2^h) \circ P^h) + ((\partial_3 \tilde{\mathbf{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h)((\partial_1 \boldsymbol{\Theta}_3^h) \circ P^h) \end{aligned} \quad (4.35)$$

and for $k = 2, 3$

$$\begin{aligned} \partial_k(\tilde{\mathbf{y}}_i^h \circ \boldsymbol{\Theta}^h \circ P^h) &= h \left[((\partial_1 \tilde{\mathbf{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h)((\partial_k \boldsymbol{\Theta}_1^h) \circ P^h) \right. \\ &\quad \left. + ((\partial_2 \tilde{\mathbf{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h)((\partial_k \boldsymbol{\Theta}_2^h) \circ P^h) + ((\partial_3 \tilde{\mathbf{y}}_i^h) \circ \boldsymbol{\Theta}^h \circ P^h)((\partial_k \boldsymbol{\Theta}_3^h) \circ P^h) \right] \end{aligned} \quad (4.36)$$

From (4.12), (4.34) and (4.35) we conclude that the boundedness of $\partial_1 w_2^h$ in $L^2(\Omega)$ is equivalent to the boundedness of

$$\begin{aligned} z_2^h &= \frac{\mathbf{R}_{21}^h \partial_1 \Theta_1 + \mathbf{R}_{22}^h \partial_1 \Theta_2 + \mathbf{R}_{23}^h \partial_1 \Theta_3 - h(p_2' x_2 - p_3' x_3)}{h^2} \\ &\quad - \frac{1}{h^2 A} (\mathbf{R}_{21}^h \int_{\omega} \partial_1 \Theta_1 + \mathbf{R}_{22}^h \int_{\omega} \partial_1 \Theta_2 + \mathbf{R}_{23}^h \int_{\omega} \partial_1 \Theta_3), \end{aligned} \quad (4.37)$$

in $L^2(\Omega)$. By using (2.3) and (3.18) we conclude

$$\begin{aligned} z_2^h &= \frac{\mathbf{R}_{21}^h (x_2(\mathbf{n}_1^h)' + x_3(\mathbf{b}_1^h)') + \mathbf{R}_{22}^h (x_2(\mathbf{n}_2^h)' + x_3(\mathbf{b}_2^h)')}{h} \\ &\quad + \frac{\mathbf{R}_{23}^h (x_2(\mathbf{n}_3^h)' + x_3(\mathbf{b}_3^h)') - (p_2' x_2 - p_3' x_3)}{h}. \end{aligned} \quad (4.38)$$

The boundedness of z_2^h in $L^2(\Omega)$ is the consequence of (3.5), (3.6) and (4.14). Now we have proved $w_2^h \rightharpoonup \mathbf{A}_{23} x_3'$ weakly in $W^{1,2}(\Omega)$.

Analogously we conclude

$$w_3^h := \frac{\frac{1}{h} \tilde{\mathbf{y}}_3^h \circ \Theta^h \circ P^h - x_3'}{h} - \frac{1}{h^2 A} \int_{\omega} \tilde{\mathbf{y}}_3^h \circ \Theta^h \circ P^h \rightharpoonup -\mathbf{A}_{23} x_2', \quad (4.39)$$

weakly in $W^{1,2}(\Omega)$. Now, since w^h can be written as

$$w^h(x_1) = \frac{1}{A\mu(\omega)} \int_{\omega} (x_2' w_3^h - x_3' w_2^h) dx_2 dx_3, \quad (4.40)$$

it is clear that w^h converges weakly to the function $w = -\mathbf{A}_{23} = \mathbf{A}_{32}$ in $W^{1,2}(0, L)$. Let us define for $\beta^h : \Omega' \rightarrow \mathbb{R}^3$, $\beta^h = \gamma \circ \mathbf{x}'^{-1}$. By the chain rule we have

$$\begin{aligned} \partial_1 \beta_i^h &= (\partial_1 \gamma_i^h) \circ (\mathbf{x}')^{-1} + (p_2' x_2' + p_3' x_3') (\partial_2 \gamma_i^h) \circ (\mathbf{x}')^{-1} \\ &\quad + (-p_3' x_2' + p_2' x_3') (\partial_3 \gamma_i^h) \circ (\mathbf{x}')^{-1}, \\ \partial_2 \beta_i^h &= p_2 (\partial_2 \gamma_i^h) \circ (\mathbf{x}')^{-1} - p_3 (\partial_3 \gamma_i^h) \circ (\mathbf{x}')^{-1}, \\ \partial_3 \beta_i^h &= p_3 (\partial_2 \gamma_i^h) \circ (\mathbf{x}')^{-1} + p_2 (\partial_3 \gamma_i^h) \circ (\mathbf{x}')^{-1}. \end{aligned} \quad (4.41)$$

By differentiating β_1 with respect to x_k' , with $k=2,3$, we have

$$\partial_2 \beta_1^h = \frac{1}{h^3} \partial_2 (\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) + \frac{1}{h} ((v_2^h)' + \theta_2'), \quad (4.42)$$

$$\partial_3 \beta_1^h = \frac{1}{h^3} \partial_3 (\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) + \frac{1}{h} ((v_3^h)' + \theta_3'). \quad (4.43)$$

Let us analyze only $\partial_2 \beta_1^h$. We have by (3.18), (4.36) and the chain rule

$$\begin{aligned} \partial_2 \beta_1^h &= \frac{((\partial_1 \tilde{\mathbf{y}}_1^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_2 \mathbf{n}_1^h - p_3 \mathbf{b}_1^h)}{h^2} \\ &\quad + \frac{((\partial_2 \tilde{\mathbf{y}}_1^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_2 \mathbf{n}_2^h - p_3 \mathbf{b}_2^h)}{h^2} \\ &\quad + \frac{((\partial_3 \tilde{\mathbf{y}}_1^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_2 \mathbf{n}_3^h - p_3 \mathbf{b}_3^h)}{h^2} \\ &\quad + \frac{1}{h} ((v_2^h)' + \theta_2'). \end{aligned} \quad (4.44)$$

By using (3.5), (3.6), (3.18), (4.12), (4.14), (4.35) and the definition of v_k^h we can conclude that for proving the boundedness of $\partial_2 \beta_1^h$ it is enough to prove the boundedness of $\delta_{1,2}^h$ in $L^2(\Omega)$ where

$$\delta_{1,2}^h := \frac{-h\mathbf{R}_{11}^h \theta_2' + \mathbf{R}_{12}^h}{h^2} + \frac{\mathbf{R}_{21}^h + h\theta_2'}{h^2} = \frac{1 - \mathbf{R}_{11}^h}{h} \theta_2' + \frac{\mathbf{R}_{12}^h + \mathbf{R}_{21}^h}{h^2}. \quad (4.45)$$

The boundedness of $\delta_{1,2}^h$ in $L^\infty(\Omega)$ is then the consequence of the property (e). In the same way we can prove the boundedness of $\partial_3 \beta_1^h$. Using the Poincaré inequality and the fact that $\int_{\omega'(x_1)} \beta_1^h dx_2, dx_3 = 0$, we deduce that there exists a constant $C > 0$ such that

$$\int_{\omega'(x_1)} (\beta_1^h(x))^2 dx_2 dx_3 \leq C \int_{\omega'(x_1)} [(\partial_2 \beta_1^h(x))^2 + (\partial_3 \beta_1^h(x))^2] dx_2 dx_3$$

for a.e. $x_1 \in (0, L)$ and for every h . Although the constant C depends on the domain, since all domains are translations and rotations of the domain ω , the constant C can be chosen uniformly. Integrating both sides with respect to x_1 , we obtain that the sequence (β_1^h) is bounded in $L^2(\Omega')$ so, up to subsequences $\beta_1^h \rightharpoonup \beta_1$ and $\partial_k \beta_1^h \rightharpoonup \partial_k \beta_1$ weakly in $L^2(\Omega')$, for $k = 2, 3$. From the relations (4.41) it can be concluded that $\gamma_1^h \rightharpoonup \gamma_1$ and $\partial_k \gamma_1^h \rightharpoonup \partial_k \gamma_1$ weakly in $L^2(\Omega)$, for $k = 2, 3$, where $\gamma = \beta \circ \mathbf{x}'$. For the sequences $(\beta_2^h), (\beta_3^h)$, we have by differentiation that for $j, k = 2, 3$

$$\partial_j \beta_k^h = \frac{1}{h^2} \left(\frac{1}{h} \partial_j (\tilde{\mathbf{y}}_k^h \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) - h \delta_{jk} - hw^h (1 - \delta_{jk}) (-1)^k \right). \quad (4.46)$$

By using the chain rule we see that for $k = 2, 3$,

$$\begin{aligned} \partial_2 (\tilde{\mathbf{y}}_k^h \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) &= h \left(((\partial_1 \tilde{\mathbf{y}}_k^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_2 \mathbf{n}_1^h - p_3 \mathbf{b}_1^h) \right. \\ &\quad + ((\partial_2 \tilde{\mathbf{y}}_k^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_2 \mathbf{n}_2^h - p_3 \mathbf{b}_2^h) \\ &\quad \left. + ((\partial_3 \tilde{\mathbf{y}}_k^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_2 \mathbf{n}_3^h - p_3 \mathbf{b}_3^h) \right), \\ \partial_3 (\tilde{\mathbf{y}}_k^h \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) &= h \left(((\partial_1 \tilde{\mathbf{y}}_k^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_3 \mathbf{n}_1^h + p_2 \mathbf{b}_1^h) \right. \\ &\quad + ((\partial_2 \tilde{\mathbf{y}}_k^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_3 \mathbf{n}_2^h + p_2 \mathbf{b}_2^h) \\ &\quad \left. + ((\partial_3 \tilde{\mathbf{y}}_k^h) \circ \Theta^h \circ P^h \circ (\mathbf{x}')^{-1}) (p_3 \mathbf{n}_3^h + p_2 \mathbf{b}_3^h) \right). \end{aligned}$$

Now we want to check that for $j, k = 2, 3$

$$e_{jk}(\beta^h) := \frac{1}{2} (\partial_j \beta_k^h + \partial_k \beta_j^h). \quad (4.47)$$

is bounded in $L^2(\Omega')$. In the similar way as for β_1 (relations (4.44) and (4.45)) we can using (3.5), (3.6), (4.12), (4.14) and the property (e) conclude that for every

$j, k = 2, 3$, $e_{jk}(\beta^h) \in L^2(\Omega')$. By using Korn's inequality (Lemma 1) we have that there exists $C > 0$ such that

$$\begin{aligned} & \|\beta_2^h\|_{W^{1,2}(\omega'(x_1))}^2 + \|\beta_3^h\|_{W^{1,2}(\omega'(x_1))}^2 \leq C \left(\left| \int_{\omega'(x_1)} \beta_2^h dx'_2 dx'_3 \right| \right. \\ & \quad \left. + \left| \int_{\omega'(x_1)} \beta_3^h dx'_2 dx'_3 \right| + \left| \int_{\omega'(x_1)} (x'_3 \beta_2^h - x'_2 \beta_3^h) dx'_2 dx'_3 \right| \right. \\ & \quad \left. + \sum_{j,k=1,2} \|e_{jk}(\beta^h)\|_{L^2(\omega'(x_1))} \right), \end{aligned} \quad (4.48)$$

for a.e. $x_1 \in (0, L)$. From the definition of v_k^h and w^h we see that the functions $(\beta_2^h(x_1, \cdot), \beta_3^h(x_1, \cdot))$ belong to the space

$$\begin{aligned} \mathcal{B}_{x_1} = \{ \beta = (\beta_2, \beta_3) \in W^{1,2}(\omega'(x_1); \mathbb{R}^2) : \int_{\omega'(x_1)} \beta dx'_2 dx'_3 = 0, \\ \int_{\omega'(x_1)} (x'_2 \beta_3 - x'_3 \beta_2) dx'_2 dx'_3 = 0 \} \end{aligned} \quad (4.49)$$

for every x_1 . By integrating (4.48) with respect to x_1 we conclude that β_2^h, β_3^h are bounded in $L^2(\Omega')$ as well as their derivatives with respect to x_2, x_3 . From this we can conclude the same fact about γ_2^h, γ_3^h . The fact that the weak limit belongs to the space \mathcal{C} can be concluded from the fact that for every h and a.e. x_1 $(\beta_2^h(x_1, \cdot), \beta_3^h(x_1, \cdot)) \in \mathcal{B}_{x_1}$. This finishes the proof of (f).

4.1 Lower bound

Lemma 2 *Let $\mathbf{y}^h, \tilde{\mathbf{y}}^h, E^h, \mathbf{R}^h, u^h, v^h, w^h, \gamma^h, \beta^h = \gamma^h \circ (\mathbf{x}')^{-1}, \gamma, \beta = \gamma \circ (\mathbf{x}')^{-1}, \mathbf{A}$ be as in Theorem 3 and let us suppose that the condition (4.11) is satisfied and that $\gamma^h \rightharpoonup \gamma, \partial_2 \gamma^h \rightharpoonup \partial_2 \gamma, \partial_3 \gamma^h \rightharpoonup \partial_3 \gamma$ weakly in $L^2(\Omega)$ i.e. $\beta^h \rightharpoonup \beta, \partial_2 \beta^h \rightharpoonup \partial_2 \beta, \partial_3 \beta^h \rightharpoonup \partial_3 \beta$ weakly in $L^2(\Omega')$. Let us define*

$$\begin{aligned} \eta_1^h(x) = \frac{1}{h} \left(\frac{(\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h)(x) - \Theta_1^h \circ P^h}{h^2} - u^h(x_1) \right. \\ \left. + x'_2 (v_2^h)'(x_1) + x'_3 (v_3^h)'(x_1) \right), \end{aligned} \quad (4.50)$$

$$\begin{aligned} \eta_k^h(x) = \frac{1}{h^2} \left(\frac{(\tilde{\mathbf{y}}_k^h \circ \Theta^h \circ P^h)(x) - \Theta_k^h \circ P^h}{h} \right. \\ \left. - v_k^h(x_1) - h(x'_k)^\perp \omega^h(x_1) \right), \text{ for } k = 2, 3, \end{aligned} \quad (4.51)$$

and $\kappa^h = \eta^h \circ (\mathbf{x}')^{-1}$. Then we have that $\eta^h \rightharpoonup \eta$ weakly in $L^2(\Omega)$ and $\partial_k \eta^h \rightharpoonup \partial_k \eta$ weakly in $L^2(\Omega)$ i.e. $\kappa^h \rightharpoonup \kappa, \partial_2 \kappa^h \rightharpoonup \partial_2 \kappa, \partial_3 \kappa^h \rightharpoonup \partial_3 \kappa$ weakly in $L^2(\Omega')$. Here

$$\eta_1 = \gamma_1, \quad (4.52)$$

$$\eta_2 = \gamma_2 + f_2^2 x_2 + f_2^3 x_3 = \gamma_2 + g_2^2 x'_2 + g_2^3 x'_3, \quad (4.53)$$

$$\eta_3 = \gamma_3 + f_3^2 x_2 + f_3^3 x_3 = \gamma_3 + g_3^2 x'_2 + g_3^3 x'_3, \quad (4.54)$$

$$\kappa = \eta \circ (\mathbf{x}')^{-1}, \quad (4.55)$$

f_k^j are defined in Remark 2 and g_k^j can be easily defined for the above identities to be valid i.e. for $k = 2, 3$, we define

$$g_k^2 = p_2 f_k^2 - p_3 f_k^3, \quad g_k^3 = p_3 f_k^2 + p_2 f_k^3. \quad (4.56)$$

The following strain convergence is valid

$$\mathbf{G}^h := \frac{(\mathbf{R}^h)^T((\nabla \hat{\mathbf{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h) - \mathbf{I}}{h^2} \rightharpoonup \mathbf{G} \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}). \quad (4.57)$$

and the symmetric part of \mathbf{G} denoted by $\tilde{\mathbf{G}}$, satisfies

$$\tilde{\mathbf{G}} = \text{sym}(\mathbf{J} - \frac{1}{2} \mathbf{A}^2 + \mathbf{K}), \quad (4.58)$$

where

$$\mathbf{J} = \begin{pmatrix} u' + v_2' \theta_2' + v_3' \theta_3' & 0 & 0 \\ w \theta_3' & v_2' \theta_2' & v_2' \theta_3' \\ -w \theta_2' & v_3' \theta_2' & v_3' \theta_3' \end{pmatrix} \quad (4.59)$$

$$\mathbf{K} = \begin{pmatrix} -x_2' v_2'' - x_3' v_3'' & & \\ -x_3' w' & \partial_2 \boldsymbol{\kappa} & \partial_3 \boldsymbol{\kappa} \\ x_2' w' & & \end{pmatrix}. \quad (4.60)$$

Moreover,

$$\begin{aligned} \liminf_{h \rightarrow 0} \frac{1}{h^6} \int_{\hat{\Omega}^h} W^h(x, \nabla \hat{\mathbf{y}}^h) dx &= \liminf_{h \rightarrow 0} \frac{1}{h^4} \int_{\Omega} W(x^h, (\nabla \hat{\mathbf{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h) dx^h \\ &\geq \frac{1}{2} \int_{\Omega} Q_3(x, \tilde{\mathbf{G}}(x)) dx, \end{aligned}$$

where Q_3 is twice the quadratic form of linearized elasticity, i.e.,

$$Q_3(x, \mathbf{F}) = \frac{\partial^2 W}{\partial \mathbf{F}^2}(\mathbf{I})[\mathbf{F}, \mathbf{F}]. \quad (4.61)$$

Proof. We follow the proof of Lemma 2.3 in [25]. Firstly, using Remark 1, it can be seen that

$$\begin{aligned} \boldsymbol{\eta}_1^h &= \boldsymbol{\gamma}_1^h + h \boldsymbol{o}_1, \\ \boldsymbol{\eta}_2^h &= \boldsymbol{\gamma}_2^h + f_2^2 x_2 + f_2^3 x_3 + h \boldsymbol{o}_2, \\ \boldsymbol{\eta}_3^h &= \boldsymbol{\gamma}_3^h + f_3^2 x_2 + f_3^3 x_3 + h \boldsymbol{o}_3, \end{aligned}$$

where $\|\boldsymbol{o}_i\|_{C^1(\Omega)} \leq C$, for some $C > 0$. The convergence of $\boldsymbol{\eta}^h$ is an easy consequence of the convergence of $\boldsymbol{\gamma}^h$. The estimate (4.12) implies that the L^2 norm of \mathbf{G}^h is bounded; therefore up to subsequences, there exists $\mathbf{G} \in L^2(\Omega; \mathbb{R}^{3 \times 3})$ such that (4.57) is satisfied. In order to identify the symmetric part of \mathbf{G} we decompose $\mathbf{R}^h \mathbf{G}^h$ as follows:

$$\mathbf{R}^h \mathbf{G}^h = \frac{(\nabla \tilde{\mathbf{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{I}}{h^2} - \frac{\mathbf{R}^h - \mathbf{I}}{h^2}, \quad (4.62)$$

so that

$$\mathbf{F}^h := \text{sym} \frac{(\nabla \tilde{\mathbf{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h - \mathbf{I}}{h^2} = \text{sym}(\mathbf{R}^h \mathbf{G}^h) + \text{sym} \frac{\mathbf{R}^h - \mathbf{I}}{h^2}. \quad (4.63)$$

The right hand side converges weakly to $\tilde{\mathbf{G}} + \frac{\mathbf{A}^2}{2}$ by (4.14), (4.57) and property (e) of the Theorem 3. Therefore the sequence \mathbf{F}^h has a weak limit in $L^2(0, L)$, satisfying $\mathbf{F} = \tilde{\mathbf{G}} + \frac{\mathbf{A}^2}{2}$. To conclude we only need to identify \mathbf{F} . Consider the functions

$$\phi_1^h := \frac{\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h - x_1}{h^2}. \quad (4.64)$$

From property (f) of Theorem 3 it follows that the functions $\phi_1^h - u^h + x_2'((v_2^h)' + \theta_2') + x_3'((v_3^h)' + \theta_3')$, which are equal to $h\gamma_1^h$ converge strongly to 0 in $L^2(\Omega)$. Thus by property (a) and (b) of Theorem 3 we conclude that

$$\phi_1^h \rightarrow u - x_2'(v_2' + \theta_2') - x_3'(v_3' + \theta_3') \text{ in } L^2(\Omega). \quad (4.65)$$

By using the chain rule, the property (d) of Theorem 3, (3.5), (3.6), (4.30) we can conclude that

$$\partial_1 \phi_1^h \rightarrow \mathbf{F}_{11} - x_2(\theta_2' p_2 + \theta_3' p_3)' + x_3(\theta_2' p_3 - \theta_3' p_2)' - v_2'(\partial_1 x_2' + \theta_2') - v_3'(\partial_1 x_3' + \theta_3'), \quad (4.66)$$

weakly in $L^2(\Omega)$. From (4.65) and (4.66) we conclude that

$$\begin{aligned} & u' - \partial_1 x_2'(v_2' + \theta_2') - x_2'(v_2'' + \theta_2'') - \partial_1 x_3'(v_3' + \theta_3') - x_3'(v_3'' + \theta_3'') \\ &= \mathbf{F}_{11} - x_2(\theta_2' p_2 + \theta_3' p_3)' + x_3(\theta_2' p_3 - \theta_3' p_2)' \\ & \quad - v_2'(\partial_1 x_2' + \theta_2') - v_3'(\partial_1 x_3' + \theta_3'). \end{aligned} \quad (4.67)$$

After some calculation we obtain

$$\mathbf{F}_{11} = u' + v_2' \theta_2' + v_3' \theta_3' - x_2' v_2'' - x_3' v_3''. \quad (4.68)$$

To identify \mathbf{F}_{12} we have to do some straight forward computations. By using the chain rule, (3.5), (3.6), Remark 1, property (d) of Theorem (3) we can conclude

$$\begin{aligned} & \frac{1}{h^2} \partial_1 (\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h) + \frac{1}{h^3} \left(p_2 \partial_2 (\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h) - p_3 \partial_3 (\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h) \right) = \\ & 2\mathbf{F}_{12}^h + \frac{\partial_1 x_2'}{h} - w(\theta_3' + \partial_1 x_3') + \mathbf{O}_1^h, \end{aligned} \quad (4.69)$$

where $\lim_{h \rightarrow 0} \|\mathbf{O}_1^h\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0$. On the other hand it can be easily seen that

$$\begin{aligned} \partial_2 \beta_1^h &= p_2 \partial_2 \gamma_1^h - p_3 \partial_3 \gamma_1^h \\ &= \frac{1}{h^3} \left(p_2 \partial_2 (\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h) - p_3 \partial_3 (\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h) \right) \\ & \quad + \frac{1}{h} ((v_2^h)' + \theta_2'). \end{aligned} \quad (4.70)$$

From (4.34), (4.69), (4.70) we conclude

$$\begin{aligned} 2\mathbf{F}_{12}^h &= \frac{1}{h^2} \partial_1 (\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h - h x_2') - \frac{1}{h} ((v_2^h)' + \theta_2') \\ & \quad + \partial_2 \beta_1^h + w \theta_3' + w \partial_1 x_3' - \mathbf{O}_1^h \\ &= \partial_1 w_2^h + \partial_2 \beta_1^h + w \theta_3' + w \partial_1 x_3' - \mathbf{O}_1^h. \end{aligned} \quad (4.71)$$

By using (4.34) we conclude that the right hand side of (4.71) converges in $W^{-1,2}(\Omega)$ to

$$\partial_1(-wx'_3) + \partial_2\beta_1 + w\theta'_3 + w\partial_1x'_3 = -x'_3w' + w\theta'_3 + \partial_2\kappa_1, \quad (4.72)$$

since $\beta_1 = \kappa_1$. On the other hand we know that the left hand side of (4.71) converges strongly in $L^2(\Omega)$ to $2\mathbf{F}_{12}$ and thus we can conclude

$$\mathbf{F}_{12} = \frac{1}{2}(-x'_3w' + w\theta'_3 + \partial_2\kappa_1). \quad (4.73)$$

In the same way one can prove

$$\mathbf{F}_{13} = \frac{1}{2}(x'_2w' - w\theta'_2 + \partial_3\kappa_1). \quad (4.74)$$

To identify \mathbf{F}_{22} let us observe that by the chain rule, (3.5), (3.6) and the property (d) of Theorem (3) we have

$$\begin{aligned} & \frac{1}{h^3} \left(p_2\partial_2(\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) - p_3\partial_3(\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right) = \\ & \mathbf{F}_{22}^h - v'_2\theta'_2 + \mathbf{O}_2^h, \end{aligned} \quad (4.75)$$

where $\lim_{h \rightarrow 0} \|\mathbf{O}_2^h\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0$. On the other hand we can conclude

$$\begin{aligned} \partial_2\kappa_2^h &= p_2\partial_2\eta_2^h - p_3\partial_3\eta_3^h \\ &= \frac{1}{h^3} \left(p_2\partial_2(\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) - p_3\partial_3(\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right). \end{aligned} \quad (4.76)$$

In the same way as before we conclude that

$$\mathbf{F}_{22} = v'_2\theta'_2 + \partial_2\kappa_2. \quad (4.77)$$

Analogously we can conclude

$$\mathbf{F}_{33} = v'_3\theta'_3 + \partial_3\kappa_3. \quad (4.78)$$

To identify $\mathbf{F}_{23} = \mathbf{F}_{32}$ we, by using the chain rule, (3.5), (3.6) and the property (d) of Theorem (3), can conclude:

$$\begin{aligned} & \frac{1}{h^3} \left(p_2\partial_2(\tilde{\mathbf{y}}_3^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) - p_3\partial_2(\tilde{\mathbf{y}}_3^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right) = \\ & \frac{1}{h^2} (\partial_2\tilde{\mathbf{y}}_3^h) \circ \Theta^h \circ P^h - v'_2\theta'_3 + \mathbf{O}_3^h, \end{aligned} \quad (4.79)$$

where $\lim_{h \rightarrow 0} \|\mathbf{O}_3^h\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0$. In the same way we conclude

$$\begin{aligned} & \frac{1}{h^3} \left(p_3\partial_2(\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) + p_2\partial_3(\tilde{\mathbf{y}}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right) = \\ & \frac{1}{h^2} (\partial_2\tilde{\mathbf{y}}_3^h) \circ \Theta^h \circ P^h - v'_3\theta'_2 + \mathbf{O}_4^h, \end{aligned} \quad (4.80)$$

where $\lim_{h \rightarrow 0} \|\mathbf{O}_4^h\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0$. It can be also concluded

$$p_2 \partial_2 \boldsymbol{\eta}_3^h - p_3 \partial_3 \boldsymbol{\eta}_3^h = \frac{1}{h^3} \left(p_2 \partial_2 (\tilde{\mathbf{y}}_3^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}_2 \circ P^h) - p_3 \partial_3 (\tilde{\mathbf{y}}_3^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}_2 \circ P^h) \right) + \frac{1}{h^2} w^h, \quad (4.81)$$

$$p_3 \partial_2 \boldsymbol{\eta}_2^h + p_2 \partial_3 \boldsymbol{\eta}_2^h = \frac{1}{h^3} \left(p_3 \partial_2 (\tilde{\mathbf{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}_2 \circ P^h) + p_2 \partial_3 (\tilde{\mathbf{y}}_2^h \circ \boldsymbol{\Theta}^h \circ P^h - \boldsymbol{\Theta}_2 \circ P^h) \right) - \frac{1}{h^2} w^h. \quad (4.82)$$

By summing the relations (4.79)-(4.82) and letting $h \rightarrow 0$ it can be concluded that

$$2\mathbf{F}_{23} = v_2' \theta_3' + v_3' \theta_2' + \partial_2 \boldsymbol{\kappa}_3 + \partial_3 \boldsymbol{\kappa}_2. \quad (4.83)$$

To prove the lower bound we can continue in the same way as in the proof of Lemma 2.3 in [25], by using the Taylor expansion, the cutting and Scorza-Dragnoni theorem.

4.2 Upper bound

Theorem 4 (optimality of lower bound) *Let $u, w \in W^{1,2}(0, L)$ and $v_k \in W^{2,2}(0, L)$ for $k = 2, 3$. Let γ be a function in \mathcal{C} where*

$$\mathcal{C} = \left\{ \gamma \in L^2(\Omega; \mathbb{R}^3) : \int_{\omega} \gamma = 0, \partial_2 \gamma, \partial_3 \gamma \in L^2(\Omega; \mathbb{R}^3), \int_{\omega} (x_3' \gamma_2(x_1, \cdot) - x_2' \gamma_3(x_1, \cdot)) dx_2 dx_3 = 0, \forall x_1 \in (0, L) \right\}. \quad (4.84)$$

Set

$$\tilde{\mathbf{G}} = \text{sym}(\mathbf{J} - \frac{1}{2} \mathbf{A}^2 + \mathbf{K}). \quad (4.85)$$

Here $\mathbf{A}, \mathbf{J}, \mathbf{K}$ are defined by the expressions (4.18), (4.59) and (4.60) and $\boldsymbol{\eta}, \boldsymbol{\kappa}$ are defined by the expressions (4.52)-(4.55).

Then there exists a sequence $(\hat{\mathbf{y}}^h) \subset W^{1,2}(\hat{\Omega}^h, \mathbb{R}^3)$ such that for u^h, v_k^h, w defined by the expressions (4.15)-(4.17) the properties (a)-(d) of Theorem 3 are valid. Also we have that the property (f) of Theorem 3 is valid (which is equivalent that for $\boldsymbol{\eta}^h$ defined by the expressions (4.50)-(4.51) it is valid $\boldsymbol{\eta}^h \rightharpoonup \boldsymbol{\eta}$ weakly in $L^2(\Omega)$ and $\partial_k \boldsymbol{\eta}^h \rightharpoonup \partial_k \boldsymbol{\eta}$ weakly in $L^2(\Omega)$). Also the following convergence is valid

$$\lim_{h \rightarrow 0} \frac{1}{h^6} \int_{\hat{\Omega}^h} W^h(x^h, \nabla \hat{\mathbf{y}}^h) dx^h = \frac{1}{2} \int_{\Omega} Q_3(x, \mathbf{G}(x)) dx \quad (4.86)$$

Proof. Let us first assume that $u, w, v_k, \boldsymbol{\eta}$ are smooth. Then we define for $(x_1, x_2^h, x_3^h) \in \bar{\Omega}^h$:

$$\begin{aligned} \hat{\mathbf{y}}^h(\boldsymbol{\Theta}^h(x_1, x_2^h, x_3^h)) &= \boldsymbol{\Theta}^h(x_1, x_2^h, x_3^h) + \begin{pmatrix} h^2 u(x_1) \\ h v_2(x_1) \\ h v_3(x_1) \end{pmatrix} \\ &+ h^2 \begin{pmatrix} -x_2(v_2' p_2 + v_3' p_3)(x_1) - x_3(v_3' p_2 - v_2' p_3)(x_1) \\ -x_2(p_3 w)(x_1) - x_3(p_2 w)(x_1) \\ x_2(p_2 w)(x_1) - x_3(p_3 w)(x_1) \end{pmatrix} \\ &+ h^3 \boldsymbol{\eta}(x_1, \frac{x_2^h}{h}, \frac{x_3^h}{h}), \end{aligned} \quad (4.87)$$

where $\boldsymbol{\eta} : \Omega \rightarrow \mathbb{R}^3$ is going to be chosen later. The convergence (a)-(d) and that $\boldsymbol{\eta}^h \rightharpoonup \boldsymbol{\eta}$ weakly in $L^2(\Omega)$ and $\partial_k \boldsymbol{\eta}^h \rightharpoonup \partial_k \boldsymbol{\eta}$ weakly in $L^2(\Omega)$ can easily be seen to be valid for this sequence. We also have

$$\begin{aligned} \nabla \hat{\boldsymbol{y}}^h \nabla \boldsymbol{\Theta}^h &= \nabla \boldsymbol{\Theta}^h + \begin{pmatrix} h^2 u' & -h(v_2' p_2 + v_3' p_3) & -h(v_3' p_2 - v_2' p_3) \\ h v_2' & -h p_3 w & -h p_2 w \\ h v_3' & h p_2 w & -h p_3 w \end{pmatrix} \\ &+ h^2 \begin{pmatrix} -x_2(v_2' p_2 + v_3' p_3)' + x_3(v_2' p_3 - v_3' p_2)' & & \\ -x_2(p_3 w)' - x_3(p_2 w)' & \left| \partial_2 \boldsymbol{\eta} \right| & \left| \partial_3 \boldsymbol{\eta} \right| \\ x_2(p_2 w)' - x_3(p_3 w)' & & \end{pmatrix} \\ &+ O(h^3). \end{aligned} \quad (4.88)$$

From (4.88), by using (3.8), we conclude

$$\begin{aligned} \nabla \hat{\boldsymbol{y}}^h &= \mathbf{I} + h \begin{pmatrix} 0 & -v_2' & -v_3' \\ v_2' & 0 & -w \\ v_3' & w & 0 \end{pmatrix} \\ &+ h^2 \begin{pmatrix} u' + v_2' \theta_2' + v_3' \theta_3' & 0 & 0 \\ w \theta_3' & v_2' \theta_2' & v_3' \theta_3' \\ -w \theta_2' & v_3' \theta_2' & v_3' \theta_3' \end{pmatrix} \\ &+ h^2 \begin{pmatrix} -x_2(v_2'' p_2 + v_3'' p_3) + x_3(v_2'' p_3 - v_3'' p_2) & & \\ -x_2(p_3 w') - x_3(p_2 w') & \left| \partial_2 \boldsymbol{\kappa} \right| & \left| \partial_3 \boldsymbol{\kappa} \right| \\ x_2(p_2 w') - x_3(p_3 w') & & \end{pmatrix} \\ &+ O(h^3). \end{aligned} \quad (4.89)$$

Using the identity $(\mathbf{I} + \mathbf{M})^T (\mathbf{I} + \mathbf{M}) = \mathbf{I} + 2 \operatorname{sym} \mathbf{M} + \mathbf{M}^T \mathbf{M}$ we obtain

$$(\nabla \hat{\boldsymbol{y}}^h)^T (\nabla \hat{\boldsymbol{y}}^h) = \mathbf{I} + 2h^2 \operatorname{sym} \mathbf{J} + 2h^2 \operatorname{sym} \mathbf{K} + h^2 \mathbf{A}^T \mathbf{A} + O(h^3),$$

where $\|O(h^3)\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})} \leq Ch^3$, for some $C > 0$.

Taking the square root we obtain

$$[(\nabla \hat{\boldsymbol{y}}^h)^T (\nabla \hat{\boldsymbol{y}}^h)]^{1/2} = \mathbf{I} + h^2 \tilde{\mathbf{G}} + O(h^3). \quad (4.90)$$

We have $\det(\nabla \hat{\boldsymbol{y}}^h) > 0$ for sufficiently small h . Hence by frame-indifference $W(x, (\nabla \hat{\boldsymbol{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h) = W(x, [(\nabla \hat{\boldsymbol{y}}^h)^T (\nabla \hat{\boldsymbol{y}}^h)]^{1/2} \circ \boldsymbol{\Theta}^h \circ P^h)$; thus by (4.90) and Taylor expansion we obtain:

$$\frac{1}{h^4} W(x, (\nabla \hat{\boldsymbol{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h) \rightarrow \frac{1}{2} Q_3(x, \tilde{\mathbf{G}}(x)) \text{ a.e}$$

and by the property ii) of W for h small enough

$$\frac{1}{h^4} W(x, (\nabla \hat{\boldsymbol{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h) \leq \frac{1}{2} C (\|\mathbf{J}\|^2 + \|\mathbf{K}\|^2 + \|\mathbf{A}\|^4) + Ch.$$

The equality (4.86) follows by the dominated convergence theorem. Namely, we have

$$\begin{aligned} \frac{1}{h^6} \int_{\Omega^h} W^h(x, \nabla \hat{\boldsymbol{y}}^h) dx &= \frac{1}{h^4} \int_{\Omega} W(x, (\nabla \hat{\boldsymbol{y}}^h) \circ \boldsymbol{\Theta}^h \circ P^h) dx \\ &\rightarrow \frac{1}{2} \int_{\Omega} Q_3(x, \tilde{\mathbf{G}}) dx. \end{aligned}$$

In the general case, it is enough to smoothly approximate u, w in the strong topology of $W^{1,2}$, v_k in the strong topology of $W^{2,2}$, and $\boldsymbol{\eta}, \partial_k \boldsymbol{\eta}$ in the strong topology of L^2 and to use the continuity of the right hand side of (4.86) with respect to these convergences.

Remark 6 Notice that

$$\mathbf{K} = \left(\mathbf{A}' \begin{pmatrix} 0 \\ x'_2 \\ x'_3 \end{pmatrix} \mid \partial_2 \boldsymbol{\kappa} \mid \partial_3 \boldsymbol{\kappa} \right) = \mathbf{L} + \left(\mathbf{A}' \begin{pmatrix} 0 \\ x'_2 \\ x'_3 \end{pmatrix} \mid \partial_2 \boldsymbol{\beta} \mid \partial_3 \boldsymbol{\beta} \right).$$

Here $\boldsymbol{\beta} = \boldsymbol{\gamma} \circ (\boldsymbol{x}')^{-1}$ and

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_2^2 & g_2^3 \\ 0 & g_3^2 & g_3^3 \end{pmatrix}. \quad (4.91)$$

From the fact that $\boldsymbol{\gamma} \in \mathcal{C}$ we can conclude $\boldsymbol{\beta} \in \mathcal{B}$, where

$$\begin{aligned} \mathcal{B} = \{ & \boldsymbol{\beta} \in L^2(\Omega'; \mathbb{R}^3) : \int_{\omega} \boldsymbol{\beta} = 0, \partial_2 \boldsymbol{\beta}, \partial_3 \boldsymbol{\beta} \in L^2(\Omega'; \mathbb{R}^3), \\ & \int_{\omega'(x_1)} (x'_3 \boldsymbol{\beta}_2(x_1, \cdot) - x'_2 \boldsymbol{\beta}_3(x_1, \cdot)) dx'_2 dx'_3 = 0, \text{ for a.e. } x_1 \in (0, L) \}. \end{aligned} \quad (4.92)$$

4.3 Identification of the Γ -limit

Let $Q : (0, L) \times \mathbb{R} \times \text{so}(3) \rightarrow [0, +\infty)$ be defined as

$$\begin{aligned} Q(x_1, t, \mathbf{F}) = \\ \min_{\boldsymbol{\alpha} \in W^{1,2}(\omega'(x_1); \mathbb{R}^3)} \int_{\omega'(x_1)} Q_3 \left(x, \left(\mathbf{F} \begin{pmatrix} 0 \\ x'_2 \\ x'_3 \end{pmatrix} + t \mathbf{e}_1 \mid \partial_2 \boldsymbol{\alpha} \mid \partial_3 \boldsymbol{\alpha} \right) \right) dx'_2 dx'_3, \end{aligned} \quad (4.93)$$

where Q_3 is the quadratic form defined in (4.61). For $u, w \in W^{1,2}(0, L)$ and $v_2, v_3 \in W^{2,2}(0, L)$ we introduce the functional

$$I^0(u, v_2, v_3, w) := \frac{1}{2} \int_0^L Q(x_1, u' + v'_2 \theta'_2 + v'_3 \theta'_3 + \frac{1}{2}((v'_2)^2 + (v'_3)^2), \partial_1 \mathbf{A}) dx_1, \quad (4.94)$$

where $\mathbf{A} \in W^{1,2}((0, L); \text{so}(3))$ is defined by (4.18). We shall state the result of Γ -convergence of the functionals $\frac{1}{h^2} I^h$ to I^0 . Before stating the theorem we analyze some properties of the limit density Q .

Remark 7 By using the remarks in the beginning of chapter 4 in [25] the following facts can be concluded:

- a) The functional $Q_3(x, \mathbf{G})$ is coercive on symmetric matrices i.e. there exists a constant $C > 0$, independent of x , such that $Q_3(x, \mathbf{G}) \geq C \|\text{sym } \mathbf{G}\|^2$, for every \mathbf{G} (this is the direct consequence of the assumption iv) on W). The minimum in (4.93) is attained. Since the functional $Q_3(x, \mathbf{G})$ depends only on the symmetric

part of \mathbf{G} , it is invariant under transformation $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} + c_1 + c_2(x')^\perp$ and hence the minimum can be computed on the subspace

$$V_{x_1} := \left\{ \boldsymbol{\alpha} \in W^{1,2}(\omega'(x_1), \mathbb{R}^3) : \int_{\omega'(x_1)} \boldsymbol{\alpha} = 0, \right. \\ \left. \int_{\omega'(x_1)} (x'_3 \boldsymbol{\alpha}_2 - x'_2 \boldsymbol{\alpha}_3) dx'_2 dx'_3 = 0 \right\}.$$

Strict convexity of $Q_3(x, \cdot)$ on symmetric matrices ensures that the minimizer is unique in V_{x_1} .

- b) Fix $x_1 \in (0, L)$, $t \in \mathbb{R}$ and $\mathbf{F} \in \text{so}(3)$. Let $\boldsymbol{\alpha}^{min} \in V$ be the unique minimizer of the problem (4.93). We set

$$g(x'_2, x'_3) = \mathbf{F} \begin{pmatrix} 0 \\ x'_2 \\ x'_3 \end{pmatrix} + t\mathbf{e}_1, \quad b_{ij}^{hk} = \frac{\partial^2 W}{\partial \mathbf{F}_{ih} \partial \mathbf{F}_{jk}}(x, \mathbf{I}),$$

and we call B^{hk} the matrix in $\mathbb{R}^{3 \times 3}$ whose elements are given by $(B^{hk})_{ij} = b_{ij}^{hk}$. Then $\boldsymbol{\alpha}^{min}$ satisfies the following Euler-Lagrange equation:

$$\int_{\omega'(x_1)} \sum_{h,k=2,3} (B^{hk} \partial_k \boldsymbol{\alpha}^{min}, \partial_h \varphi) dx'_2 dx'_3 = - \int_{\omega'(x_1)} \sum_{h=2,3} (B^{h1} g, \partial_h \varphi) dx'_2 dx'_3, \quad (4.95)$$

for every $\varphi \in W^{1,2}(\omega'(x_1); \mathbb{R}^{3 \times 3})$. From this equation it is clear that $\boldsymbol{\alpha}^{min}$ depends linearly on (t, \mathbf{F}) . Moreover Q is uniformly positive definite, i.e.

$$Q(x_1, t, \mathbf{F}) \geq C(t^2 + \|\mathbf{F}\|^2), \quad \forall t \in \mathbb{R}, \forall \mathbf{F} \in \text{so}(3), \quad (4.96)$$

and the constant C does not depend on x_1 .

- c) By mimicking the proof of Remark 4.3 in [25] it can be seen that there exists a constant C' (independent of x_1 , t and \mathbf{F}) such that

$$\|\partial_2 \boldsymbol{\alpha}^{min}\|_{L^2(\omega'(x_1); \mathbb{R}^{3 \times 3})} + \|\partial_3 \boldsymbol{\alpha}^{min}\|_{L^2(\omega'(x_1); \mathbb{R}^{3 \times 3})} \leq C' \|g\|_{L^2(\omega'(x_1); \mathbb{R}^{3 \times 3})}^2, \quad (4.97)$$

for a.e. $x_1 \in (0, L)$. To adapt the proof we only need to have that the constant in the Korn's inequality

$$\int_{\omega'(x_1)} \sum_{j,k=2,3} |\partial_k \boldsymbol{\alpha}_j^{min}|^2 dx'_2 dx'_3 \leq C_1 \int_{\omega'(x_1)} \sum_{j,k=2,3} |e_{jk}(\boldsymbol{\alpha}^{min})|^2 dx'_2 dx'_3 \quad (4.98)$$

can be chosen independently of x_1 . This is proved in Lemma 1.

- d) When Q_3 does not depend on x_2, x_3 we can find a more explicit representation for Q . More precisely Q can be decomposed into the sum of two quadratic forms

$$Q(x_1, t, \mathbf{F}) = Q_1(x_1, t) + Q_2(x_1, \mathbf{F}),$$

where

$$Q_1(x_1, t) := \min_{\mathbf{a}, \mathbf{b} \in \mathbb{R}^3} Q_3(x_1, (t\mathbf{e}_1 | \mathbf{a} | \mathbf{b})), \quad (4.99)$$

$$Q_2(x_1, 0, \mathbf{F}) := Q(x_1, 0, \mathbf{F}). \quad (4.100)$$

The relations (4.8) are only needed for this. If we assume the isotropic and homogenous case i.e.

$$Q_3(\mathbf{F}) = 2\mu \left| \frac{\mathbf{F} + \mathbf{F}^T}{2} \right|^2 + \lambda(\text{trace } \mathbf{F})^2,$$

then after some calculation (see Remark 3.5 in [24]) it can be shown that

$$\begin{aligned} Q_1(t) &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} t^2 \\ Q_2(x_1, \mathbf{F}) &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \left(\mathbf{F}_{12} \int_{\omega'(x_1)} (x_2')^2 dx_2' dx_3' \right. \\ &\quad \left. + 2\mathbf{F}_{12}\mathbf{F}_{13} \int_{\omega'(x_1)} x_2' x_3' dx_2' dx_3' + \mathbf{F}_{13} \int_{\omega'(x_1)} (x_3')^2 dx_2' dx_3' \right) \\ &\quad + \mu\tau \mathbf{F}_{23}, \end{aligned}$$

where the constant τ is so-called torsional rigidity, defined as

$$\tau(\omega'(x_1)) = \tau(\omega) = \int_{\omega} (x_2^2 + x_3^2 - x_2 \partial_3 \varphi + x_3 \partial_2 \varphi) dx_2 dx_3,$$

and φ is the torsion function i.e. the solution of the Neumann problem

$$\begin{cases} \Delta \varphi = 0 & \text{in } \omega \\ \partial_\nu \varphi = -(x_3, -x_2) \cdot \nu & \text{on } \partial\omega \end{cases}$$

The following theorem can be proved in the same way as Theorem 4.5 in [23] (we need Theorem 3, Lemma 2, Theorem 4, Remark 6 and Remark 7).

Theorem 5 *As $h \rightarrow 0$, the functionals $\frac{1}{h^4} I^h$ are Γ -convergent to the functional I^0 given in (4.94), in the following sense:*

- i) (compactness and liminf inequality) if $\limsup_{h \rightarrow 0} h^{-4} I^h < +\infty$ then there exists constants $\overline{\mathbf{R}}^h \in \text{SO}(3)$, $c^h \in \mathbb{R}^3$ such that (up to subsequences) $\overline{\mathbf{R}}^h \rightarrow \overline{\mathbf{R}}$ and the functions defined by

$$\begin{aligned} \tilde{\mathbf{y}}^h &:= (\overline{\mathbf{R}}^h)^T \mathbf{y}^h - c^h, \quad u^h = \frac{1}{A} \int_{\omega} \frac{\tilde{\mathbf{y}}_1^h \circ \Theta^h \circ P^h - x_1}{h^2} dx_2 dx_3 \\ v_k^h &= \frac{1}{A} \int_{\omega} \frac{\tilde{\mathbf{y}}_k^h \circ \Theta^h \circ P^h - h\theta_k}{h} dx_2 dx_3 \\ w^h &= \frac{1}{A\mu(\omega)} \int_{\omega} \frac{x_2'(\tilde{\mathbf{y}}_3 \circ \Theta^h \circ P^h) - x_3'(\tilde{\mathbf{y}}_2 \circ \Theta^h \circ P^h)}{h^2} dx_2 dx_3 \end{aligned}$$

satisfy

- (a) $(\nabla \tilde{\mathbf{y}}^h) \circ \Theta^h \circ P^h \rightarrow \mathbf{I}$ in $L^2(\Omega)$.
(b) there exist $u, w \in W^{1,2}(0, L)$ such that $u^h \rightharpoonup u$ and $w^h \rightharpoonup w$ weakly in $W^{1,2}(0, L)$.
(c) there exists $v_k \in W^{2,2}(0, L)$ such that $v_k^h \rightarrow v_k$ strongly in $W^{1,2}(0, L)$ for $k = 2, 3$.

Moreover we have

$$\liminf_{h \rightarrow 0} \frac{1}{h^4} I^h(\mathbf{y}^h) \geq I^0(u, v_2, v_3, w). \quad (4.101)$$

ii) (limsup inequality) for every $v, w \in W^{1,2}(0, L)$, $v_2, v_3 \in W^{2,2}(0, L)$ there exists $(\hat{\mathbf{y}}^h)$ such that (a)-(c) hold (with $\tilde{\mathbf{y}}^h$ replaced by $\hat{\mathbf{y}}^h$) and

$$\lim_{h \rightarrow 0} \frac{1}{h^4} I^h(\hat{\mathbf{y}}^h) = I^0(u, v_2, v_3, w) \quad (4.102)$$

Remark 8 Let $f_2, f_3 \in L^2(0, L)$. We introduce the functional

$$J^0 = I^0(u, v_2, v_3, w) - \int_0^L \sum_{k=2,3} f_k v_k, \quad (4.103)$$

for every $u \in W^{1,2}(0, L)$, $v_2, v_3 \in W^{2,2}(0, L)$, and $w \in W^{1,2}(0, L)$. The functional J^0 can be obtained as Γ -limit of the energies $\frac{1}{h^4} I^h$ by adding a term describing transversal body forces of order h^3 (see [13], see also [32]). For longitudinal body forces see [18]. The problem for longitudinal body forces arises because the longitudinal forces should be of order h^2 , the same order as for the model in [24]. One needs to impose certain stability condition to see which model describes the behavior of the body for the longitudinal forces of order h^2 .

Remark 9 The term $u' + v_2' \theta_2' + v_3' \theta_3' + \frac{1}{2}((v_2')^2 + (v_3')^2)$ in the strain measures the extension of the central line (which is of the second order). Namely, if we approximate the deformation of the weakly curved rod by:

$$\varphi_1(x_1, x_2, x_3) = x_1 + h^2 u + h^2 x_2'(v_2' + \theta_2') + h^2 x_3'(v_3' + \theta_3') \quad (4.104)$$

$$\varphi_k(x_1, x_2, x_3) = h \theta_k + h x_k' + h v_k + h^2 (x_k')^\perp w, \text{ for } k = 2, 3, \quad (4.105)$$

we see, that it is valid

$$\|\partial_1 \varphi(x_1, 0, 0)\|^2 - \|\partial_1 \Theta^h(x_1, 0, 0)\|^2 = h^2 \left(2u' + 2v_2' \theta_2' + 2v_3' \theta_3' + (v_2')^2 + (v_3')^2 \right).$$

Remark 10 The existence of the solution for the functional J^0 under the Dirichlet boundary condition for v_k at both ends of the rod can be proved directly. It is also enough that we impose v_2, v_2', v_3, v_3' at the one end. The existence can also be proved for the free boundary condition under the hypothesis that $\int_0^L \mathbf{f}_k dx_1 = 0$, $\int_0^L x_1 \mathbf{f}_k dx_1 = 0$ for $k = 2, 3$. It can be done in the same way as the proof of Lemma 5 in [32].

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