Control and stabilization of waves on 1-d networks

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Abstract

We present some recent results on control and stabilization of waves on $1-d$ networks.

The fine time-evolution of solutions of wave equations on networks and, consequently, their control theoretical properties, depend in a subtle manner on the topology of the network under consideration and also on the number theoretical properties of the lengths of the strings entering in it. Therefore, the overall picture is quite complex.

In this paper we summarize some of the existing results on the problem of controllability that, by classical duality arguments in control theory, can be reduced to that of observability of the adjoint uncontrolled system. The problem of observability refers to that of recovering the total energy of solutions by means of measurements made on some internal or external nodes of the network. They lead, by duality, to controllability results guaranteeing that L^2-controls located on those nodes may drive sufficiently smooth solutions to equilibrium at a final time. Most of our results in this context, obtained in collaboration with R. Dáger, refer to the problem of controlling the network from one single external node. It is, to some extent, the most complex situation since, obviously, increasing the number of controllers enhances the controllability properties of the system. Our methods of proof combine sidewise energy estimates (that in the particular case under consideration can be derived by simply applying the classical d'Alembert's formula), Fourier series representations, non-harmonic Fourier analysis, and number theoretical tools.

These control results belong to the class of the so-called open-loop control systems.

We then discuss the problem of closed-loop control or stabilization by feedback. We present a recent result, obtained in collaboration with J. Valein, showing that the observability results previously derived, regardless of the method of proof employed, can also be recast a posteriori in the context of stabilization, so to derive explicit decay rates (as $t \to \infty$) for the energy of smooth solutions. The decay rate depends in a very sensitive manner on the topology of the network and the number theoretical properties of the lengths of the strings entering in it.

In the end of the article we also present some challenging open problems.

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Contents

1 Introduction and main results

This article is devoted to the presentation of some results on wave propagation phenomena in multi-link or multi-body structures constituted by a planar network of linear vibrating strings and undergoing vertical displacements.

There exists a rich mathematical literature on multi-body mechanical systems constituted by coupled flexible or elastic elements as strings, beams, membranes or plates since their practical relevance is huge. In most cases they are systems of Partial Differential Equations (PDE) on networks or graphs. The interested reader is referred to the books [82] and [6] for an introduction to the theory of Partial Differential Equations on networks which is an active subject since the early 80's ([75], [88]). In [58] and [63] wide information may be found on modeling and control issues. We also refer to [61] for a systematic analysis of the application of domain decomposition techniques for networks. But elasticity and flexible structures are not the only motivation for dealing with wave equations on graphs or networks. This topic is also closely related to many other applications such as water supply and irrigation, in which case the relevant models are often the Saint Venant equations, a first order hyperbolic system (see [19], [20]).

The model we address in these notes is, to some extent, the simplest one in this context but, as we shall see, it is complex enough to present a rich variety of new qualitative properties. Indeed, the interaction between the different components of the multi-link structure generates new dynamics that can not be predicted by simply analyzing the dynamics of each component separately. Doing that requires taking into account various ingredients as the topology of the graph of the network, the lengths of the strings entering in it, the boundary conditions on the external nodes, the joint conditions, etc.

The goal of these notes is to present some by now well-known results that illustrate this complex dynamics, indicating the needed analytical tools and pointing towards some open problems and directions of research. We mainly focus on the control theoretical problems of observation, control and stabilization. These issues are intrinsically interesting but, in fact, constitute a way of analyzing and describing the fine propagation properties of waves in these media. We mainly focus on the case where controllers, observers and dampers are located in one single external node of the network. This is somehow the most degenerate situation, in which, control theoretical properties are harder to be fulfilled. The methods and ideas we develop for addressing this case can then easily be adapted to deal with other problems in which, in particular, several controllers are located in different nodes (internal or external ones) of the network.

We follow closely our previous book on the subject [37], devoted mainly to the problem of controllability and our more recent article on the stabilization [98], incorporating some new results and material.

As we mentioned before, we consider the scalar $1-d$ wave equation on a finite planar network of strings. Deformations are assumed to be perpendicular to the reference plane. The main advantage of considering this model, as compared to other more complex equations or systems along the graph, is that, while waves propagate within one of the strings, one can have a complete and explicit representation through the classical d'Alembert formula. This allows to easily follow the propagation of the energy along each individual string. But, the overall dynamics turns out to be rather complex, due to the interaction of the various strings at the joints. Indeed, when waves reach a node or junction point, part of the energy bounces back and part of it is transmitted to the other strings with the same common node.

This occurs whenever some wave reaches a node or the external boundary (in which, in the case of conservative boundary conditions, the whole energy bounces back).

Then, the overall picture necessarily depends on a number of ingredients:

- The topology of the graph;
- The lengths of the various strings constituting the graph;
- The boundary conditions imposed at the extremes of the graph;
- The joint conditions.

In these notes we consider the simplest model involving the so-called Kirchhoff type joint conditions. Other joint conditions can also be considered so that the model under consideration is well-posed. That is for instance the case when imposing dynamical point-mass equations on the joints. But, in that case, the dynamics is even more complex since the phenomena we address here have to be complemented with the possibility that waves have a different degree of regularity on the various strings involved in the network, a fact that was observed in [49] in the simplest case of two vibrating strings connected by a point mass and later extended to the multi-dimensional case in [55] and [66].

Thus, the results we present here are not exhaustive, by any means. However, most of the ideas and methods we develop here can be adapted and extended to more sophisticated and realistic wave models in networks.

As we mentioned above, one of the issues we address is that of *observability*. It concerns, roughly speaking, the issue of determining whether one can estimate the total energy of vibrations by partial measurements made, for instance, in one or several interior or external nodes of the network. [1] It is therefore intimately related to the way the energy of solutions is distributed along the various components of the multi-structure, as time evolves. This problem is relevant, not only because it is a way of analyzing deeply the nature of vibrations, but because it is also of immediate application in the context of inverse and control problems.

We also present the consequences of the observability properties in what concerns *controllability* issues. In this context, we are interested in driving the solutions to a given final state by means of the action of one or several controllers located in some of the internal nodes and/or the extremes of the network. [2] The problems of observability and controllability are dual one to each other and, therefore, the observability inequalities have immediate consequences in the controllability setting.

It is however important to underline that one of the difficulties related to dealing with networks and not the standard wave equation in an open domain of the Euclidean space or a smooth manifold is that, even if observability holds, the observed norm is weaker than the energy of the system, in analogy of the well-known behavior for the $1-d$ wave equation with point-wise interior observations. [3]

As we shall see, for instance, when the network is a tree, observing/controlling in all but one external vertices suffices to get full observation or control in the natural energy spaces (see [63]). This case is similar to that of the wave equation in a bounded domain with a

[1] As mentioned above, most of this article is devoted to the case in which the observation is only done on one external node of the network.

[2] Once more, we shall focus in the case in which one single controller acts on one of the external nodes of the network.

[3] We refer to [111] and [113] for relatively complete and updated surveys on the state of the art of the observability and controllability of PDE's.

control on a sufficient large subset of the boundary, fulfilling the so called *Geometric Control Condition (GCC)* by Bardos, Lebeau and Rauch [18]. But the problem becomes immediately much more complex when the control misses two external vertices. Then, diophantine approximation issues enter, as it happens for the internal point-wise control of the $1-d$ wave equation ([44]). The situation is even more complex when the graph contains closed circuits. Then there may exist eigen-vibrations of the network that remain concentrated and trapped in that circuit, without being propagated to the rest of the network. In those cases, obviously, it is impossible to achieve the observation and/or control property if the observer or controller is not located on the circuit where the solution is trapped. But whether a circuit may support a localized eigen-vibration depends also strongly on the number theoretical properties of the lengths of the strings composing the circuit. This is an issue that is not completely well understood.

Our main result for general networks asserts that the problem of observability or controllability for a sufficiently large time (twice the total length of the network) is equivalent to the property that all eigen-vibrations to be observable. The later is, obviously, a necessary condition for observability and controllability. Our result shows that it is also sufficient for observability/controllability to take place in spaces that can be described in Fourier series in terms of a summability condition of the Fourier coefficients with suitable weights. This is done using a corollary due to Haraux and Jaffard ([50]) of the celebrated Beurling-Malliavin's Theorem. However, characterizing the rate of decay of these weights for high frequencies (or, in other words, the spaces in which observability/controllability holds) in terms of the topological and geometrical properties of the graph is an open problem.

The overall picture is quite complex, and still not complete. We shall summarize the known results in this topic in Section 3.

In what concerns the problem of stabilization, recently, a black-box strategy has been developed in [98] allowing to automatically transfer the known observability/controllability results into stabilization ones. This provides a new way of getting stabilization results and complements the existing literature on the subject (some of the main references are collected in the bibliography at the end of the paper). Roughly speaking, whenever the wave process in the network is observable/controllable by some internal or exterior nodes, then the system can also be stabilized by feedback laws acting on the same nodes. But, of course, there is also a price to pay for the fact that the observation/control properties only hold in weaker spaces. In the context of stabilization, this amounts to get slow decay rates for smooth solutions (say, in the domain of the generator of the semigroup) and not exponential ones. The decay rate, roughly speaking, is polynomial when there is a loss of a finite number of derivatives in the observation/control process, but it may be even slower, say, logarithmic, when an infinite number of derivatives is lost in the observation/control process. Once again, the precise weak norm in which observability and/or controllability holds, depends on diophantine properties of the mutual lengths of the strings of the network.

The same issues arise for all other models like beams, Schrödinger or heat equations. The theory of observation and control of these models in open domains of $\mathbb{R}^n$ is by now quite well developed (we refer to the survey articles [109] , [111] and [113] for an updated account of the developments in this field). However, very little is known in the context of PDE's on networks. However, as pointed out in [37], one can transform the results obtained in the context of the wave equation in networks into results on the control of these systems in the same networks. In [37] this was proved to be true using the classical strategy by D. L. Russell [94] that was the first one to observe that the control to zero of the heat equation can be derived as a consequence of the exact controllability of the wave equation in domains of the Euclidean space. Recently, this issue has been further developed and clarified by L. Miller by the so-called transmutation method (see [79]), using the Kannai transform. We shall not develop this issue here but, for these models, as expected, due to the infinite speed of propagation, the observability inequalities hold in an arbitrarily small time ([37]). It is however important to underline that, so far, the direct analysis of the control/observation properties of the Schrödinger and heat equations on networks has not been addressed.

As we have already mentioned, this article collects the existing results on simple $1-d$ models on networks. Much remains to be done in this field. At the end of this article we include a list of open problems and possible subjects of future research.

For those who will address these topics for the first time, we refer to [77] for an introduction to some of the most elementary tools on the controllability of PDE's and to the survey articles [109], [111] and [113], for a description of the state of the art in this field.

This article is organized as follows. Section 2 is devoted to present the model under consideration: the wave equation on a $1-d$ network of strings. In Section 3 we make a brief presentation of known results on the observability and controllability of this model. In Section 4 we present known results on the problem of stabilization. In Section 5 we present and discuss some possible further developments of the methods and results in the paper and some open problems and future directions of research.

2 The wave equation on a network

Let us first recall some definitions and notations about $1-d$ networks used in the paper. We refer to [2, 81, 99, 37] for more details.

A $1-d$ network $\mathcal{N}$ is a connected set of $\mathbb{R}^n$, $n \geq 1$, defined by

$$\mathcal{N} = \bigcup_{j=1}^{M} e_j$$

where e_j is a curve that we identify with the interval $(0, l_j)$, $l_j > 0$, and such that for $k \neq j$, $\overline{e_j} \cap \overline{e_k}$ is either empty or a common end called a vertex or a node (here $\overline{e_j}$ stands for the closure of e_j).

For a function $u : \mathcal{N} \longrightarrow \mathbb{R}$, we set $u^j = u_{|e_j}$ the restriction of u to the edge e_j.

We denote by $\mathcal{E} = \{e_j \,;\, 1 \leq j \leq M\}$ the set of edges of $\mathcal{N}$, by $\mathcal{V}$ the set of external nodes of $\mathcal{N}$, and by N the number of these external nodes. For a fixed vertex v, let

$$\mathcal{E}_v \quad = \quad \{j \in \{1, ..., M\} \,;\, v \in \overline{e_j}\}$$

be the set of edges having v as vertex. If card $(\mathcal{E}_v) = 1$, v is an exterior node, while if card $(\mathcal{E}_v) \geq 2$, v is an interior one. We denote by $\mathcal{V}_{ext}$ the set of exterior nodes and by $\mathcal{V}_{int}$ the set of interior ones. For $v \in \mathcal{V}_{ext}$, the single element of $\mathcal{E}_v$ is denoted by j_v.

Now we consider a planar network of elastic strings that undergo small perpendicular vibrations. At rest, the network coincides with a planar graph G contained in that plane.

Let us suppose that the function $u^j = u^j(t,x) : \mathbb{R} \times [0, \ell_j] \to \mathbb{R}$ describes the transversal displacement in time t of the string that coincides at rest with the edge $\mathbf{e}_j$. Then, for every $t \in \mathbb{R}$, the functions u^j, $j = 1, ..., M$, define a function $\bar{u}(t)$ on G with components u^j $: \mathbb{R} \times [0, \ell_j] \to \mathbb{R}$ given by $u^j(t,x) = u^j(t, x_j(x))$.

As a model of the motion of the network, we assume that the displacements u^j satisfy the following non-homogeneous system

$$\begin{cases} u^j_{tt} - u^j_{xx} = 0 & \text{in } \mathbb{R} \times [0, \ell_j], \quad j = 1, ..., M, \\ u^{j_{\mathbf{v}_1}}(t, \mathbf{v}_1) = h(t) & t \in \mathbb{R}, \\ u^{j_{\mathbf{v}_i}}(t, \mathbf{v}_i) = 0 & t \in \mathbb{R}, \quad i = 2, ..., N, \\ u^j(t, \mathbf{v}) = u^k(t, \mathbf{v}) & t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{int}, \ j, k \in \mathcal{E}_{\mathbf{v}}, \\ \sum_{j \in I_{\mathbf{v}}} \partial_n u^i(t, \mathbf{v}) = 0 & t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{int}, \\ u^j(0, x) = u^j_0(x), \quad u^j_t(0, x) = u^j_1(x) & x \in [0, \ell_j], \quad j = 1, ..., M. \end{cases} \tag{1}$$

The first equation in this system represents the classical 1-d wave equation on the network. Within each of the M strings of the network the d'Alembert equation is fulfilled. The second and third equalities reflect the condition that over the exterior node $\mathbf{v}_1$ a control $h = h(t)$ acts to regulate its displacement, while the remaining $N - 1$ exterior nodes, are fixed. The fourth and fifth relations constitute the Kirchhoff joint conditions, expressing the continuity of the network and the balance of forces at the interior nodes. Finally, the last equation imposes the initial deformation and velocity of the strings (i.e., at time $t = 0$). The pair $(\bar{u}_0, \bar{u}_1)$ is called *initial state* of the network.

Here and in the sequel $\partial_n u^j(t, \mathbf{v})$ denotes the exterior normal derivative of u^j at the node $\mathbf{v}$.

Thus, (1) corresponds to a network with one controlled exterior node. Similar problems can be formulated when the controller acts on an interior node or when several controllers act simultaneously, either on interior or exterior nodes. We refer to [37] for a discussion of some of these problems.

For a proper functional analysis of this system, it is convenient to introduce the following Hilbert spaces:

$$V = \{\bar{u} \in \prod_{i=1}^{M} H^1(0, \ell_i) : u^i(\mathbf{v}) = u^j(\mathbf{v}) \text{ if } \mathbf{v} \in \mathcal{V}_{int} \text{ and } u^i(\mathbf{v}) = 0 \text{ if } \mathbf{v} \in \mathcal{V}_{ext}\},$$

$$H = \prod_{i=1}^{M} L^2(0, \ell_i),$$

endowed with the Hilbert structures

$$< \bar{u}, \bar{w} >_V := \sum_{i=1}^{M} < u^i, w^i >_{H^1(0,\ell_i)} = \sum_{i=1}^{M} \int_0^{\ell_i} u^i_x w^i_x dx,$$
$$< \bar{u}, \bar{w} >_H := \sum_{i=1}^{M} < u^i, w^i >_{L^2(0,\ell_i)} = \sum_{i=1}^{M} \int_0^{\ell_i} u^i w^i dx,$$

respectively. Besides, we will denote by $U = L^2(0, T)$, the space of controls. We also denote by W the product eergy space $W = V \times H$.

Since the imbedding $V \subset H$ is dense and compact, when H is identified with its dual H' by means of the Riesz-Fréchet isomorphism, we can define the operator $-\Delta_G : V \to V'$ by

$$\langle -\Delta_G \bar{u}, \bar{v} \rangle_{V' \times V} = \langle \bar{u}, \bar{v} \rangle_V.$$

The operator $-\Delta_G$ is an isometry from V to V'. The notation $-\Delta_G$ is justified by the fact that, for smooth functions $\bar{u} \in V$, the operator $-\Delta_G$ coincides with the Laplace operator.

The spectrum of the operator $-\Delta_G$ is formed by an increasing positive sequence $(\mu_n)_{n\in\mathbb{N}}$ of eigenvalues. The corresponding eigenfunctions $(\bar{\theta}_n)_{n\in\mathbb{N}}$ may be chosen to form an orthonormal basis of H.

The spaces V and H may be characterized as

$$V = \left\{ \bar{u} = \sum_{n\in\mathbb{N}} u_n\bar{\theta}_n : \quad ||\bar{u}||_V^2 := \sum_{n\in\mathbb{N}} \mu_n u_n^2 < \infty \right\},$$

$$H = \left\{ \bar{u} = \sum_{n\in\mathbb{N}} u_n\bar{\theta}_n : \quad ||\bar{u}||_H^2 := \sum_{n\in\mathbb{N}} u_n^2 < \infty \right\},$$

and the norms of V and H are equivalent to $||.||_V$ and $||.||_H$, respectively. The spaces V and H are Hilbert spaces with respect to the scalar products that generate the corresponding norms.

System (1) can be shown to be well-posed in an appropriate functional setting by means of the standard *transposition method* (see [72]).

To implement the method of transposition we need to consider the adjoint system:[4]

$$\begin{cases} \phi^j_{tt} - \phi^j_{xx} = 0 & \text{in } \mathbb{R}\times[0,\ell_j], \quad j=1,...,M, \\ \phi^{j_{\mathbf{v}_j}}(t,\mathbf{v}_j) = 0 & t\in\mathbb{R}, \quad j=1,...,N, \\ \phi^j(t,\mathbf{v}) = \phi^k(t,\mathbf{v}) & t\in\mathbb{R}, \quad \mathbf{v}\in\mathcal{V}_{int},\ j,k\in\mathcal{E}_{\mathbf{v}}, \\ \sum_{j\in I_{\mathbf{v}}} \partial_n\phi^j(t,\mathbf{v}) = 0 & t\in\mathbb{R}, \quad \mathbf{v}\in\mathcal{V}_{int}, \\ \phi^j(0,x) = \phi^j_0(x), \quad \phi^j_t(0,x) = \phi^j_1(x) & x\in[0,\ell_j], \quad j=1,...,M. \end{cases} \tag{2}$$

The solution of the adjoint system (2) with initial data

$$\bar{\phi}_0 = \sum_{n\in\mathbb{N}} \phi_{0,n}\bar{\theta}_n, \qquad \bar{\phi}_1 = \sum_{n\in\mathbb{N}} \phi_{1,n}\bar{\theta}_n, \tag{3}$$

can be written in Fourier series as follows:

$$\bar{\phi}(t,x) := \sum_{n\in\mathbb{N}} (\phi_{0,n}\cos\sqrt{\mu_n}t + \frac{\phi_{1,n}}{\sqrt{\mu_n}}\sin\sqrt{\mu_n}t)\bar{\theta}_n(x). \tag{4}$$

When $(\bar{\phi}_0,\bar{\phi}_1)\in V\times H$, by standard variational or semigroup methods it can be shown that the solution $\bar{\phi}$ satisfies

$$\bar{\phi}\in C([0,T];V)\bigcap C^1([0,T];H), \tag{5}$$

for all $T>0$.

For a classical smooth solution $\bar{u}$ of (1), the energy is defined as the sum of the energies of its components, that is,

$$\mathbf{E}_{\bar{u}}(t) := \sum_{j=1}^M \mathbf{E}_{u^j}(t) \quad \text{with} \quad \mathbf{E}_{u^j}(t) := \frac{1}{2}\int_0^{\ell_j} \left(\left|u^j_t(t,x)\right|^2 + \left|u^j_x(t,x)\right|^2 \right) dx.$$

This energy satisfies

$$\frac{d}{dt}\mathbf{E}_{\bar{u}}(t) = \sum_{j=1}^M u^j_t(t,\mathbf{v}_j)\partial_n u^j(t,\mathbf{v}_j). \tag{6}$$

[4]More rigorously, for the adjoint system, the initial data should be given at time $t=T$, but the system under consideration being time-reversible, we may consider equally that the initial data are given at $t=0$.

In particular, for the adjoint system (2), the energy is conserved for all t:

$$\mathbf{E}_{\bar{\phi}}(t) = \mathbf{E}_{\bar{\phi}}(0),$$

for every $t \in \mathbb{R}$. Besides, if the initial data are as in (3) then

$$\mathbf{E}_{\bar{\phi}} = \frac{1}{2}\sum_{n\in\mathbb{N}}(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2) = \frac{1}{2}(||\bar{\phi}_0||_V^2 + ||\bar{\phi}_1||_H^2). \tag{7}$$

For every $s \in \mathbb{R}$ we also consider the Hilbert spaces

$$V^s := \left\{ \bar{u} = \sum_{n\in\mathbb{N}} u_n \bar{\theta}_n : \quad \|\bar{u}\|_s^2 := \sum_{n\in\mathbb{N}} \mu_n^s |u_n|^2 < \infty \right\}, \tag{8}$$

$$h^s := \left\{ (u_n) : \quad \|(u_n)\|_s^2 := \sum_{n\in\mathbb{N}} \mu_n^s |u_n|^2 < \infty \right\}, \tag{9}$$

endowed with the norms $\|\cdot\|_s$, where (u_n) denotes a sequence of real numbers u_n. The canonical isomorphism $\sum_{n\in\mathbb{N}} u_n\bar{\theta}_n \to (u_n)$ is an isometry between V^s and h^s.

Let us observe that V^s is the domain of $(-\Delta_G)^{\frac{s}{2}}$ considered as an unbounded operator from H to H. Besides, $V = V^1$ and $H = V^0$.

Further, we introduce the Hilbert spaces

$$\mathcal{W}^s := V^s \times V^{s-1},$$

endowed with the natural product structures. We then have

$$\mathcal{W}^1 = V \times H, \qquad \mathcal{W}^0 = H \times V'.$$

For initial state $(\bar{\phi}_0, \bar{\phi}_1) \in \mathcal{W}^s$ the solution of the homogeneous problem ((2) may be defined by (5) and

$$\bar{\phi} \in C(\mathbb{R}; V^s) \bigcap C^1(\mathbb{R}; V^{s-1}).$$

Furthermore, the solutions of the adjoint system, for all $T > 0$ finite and every exterior node $\mathbf{v} \in \mathcal{V}_{ext}$ satisfy the following *hidden regularity* inequality

$$\int_0^T |\partial_n \phi^j(t, \mathbf{v})|^2 dt \le C\mathbf{E}_{\bar{\phi}}. \tag{10}$$

The inequality (10) may be proved using d'Alembert formula for the representation of the solutions of the wave equation in each string of the network, or multiplier techniques (see [**?**]).

If we multiply the first equation in (2) by u^j and integrate over $[0,t] \times [0, \ell_i]$ it holds, after integration by parts,

$$\int_0^t h\partial_n \phi^1(\tau, \mathbf{v}_1) d\tau = \sum_{j=1}^M \int_0^{\ell_j} \left(u^j(t,x)\phi_t^j(t,x) - u_t^j(t,x)\phi^j(t,x) \right) dx \,|_0^t.$$

We consider this identity as the definition of weak solution $\bar{u}$ of (1) in the sense of distributions. Given $h \in L^2(0,T)$, as a consequence of (10), this solution is well-defined, unique and, by (10), has the property

$$\bar{u} \in C([0,T] : H) \bigcap C^1([0,T]; V'), \tag{11}$$

together with the estimate

$$||\bar{u}||_{L^\infty(0,T:H)} + ||\bar{u}_t||_{L^\infty(0,T:V')} \leq C[||(\bar{u}_0, \bar{u}_1)||_{H\times V'} + ||h||_{L^2(0,T)}]. \tag{12}$$

The control problem in time T consists in determining for which initial states it is possible to choose the control $h \in L^2(0,T)$, such that the system reaches the equilibrium position at time T. Depending on how strict we are on requiring the state to reach equilibrium, several notions or degrees of controllability may be distinguished.

More precisely, given $T > 0$, we say that the initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$, is **exactly controllable** (or simply controllable) **in time** T, if there exists a function $h \in L^2(0,T)$, such that the solution of (1) with initial state $(\bar{u}_0, \bar{u}_1)$ satisfies

$$\bar{u}|_{t=T} = \bar{u}_t|_{t=T} = \bar{0}.$$

The system is said to be **approximately controllable in time** T when for every $\varepsilon > 0$ there exists a control h such that the corresponding solutions $\bar{u}^\varepsilon$ verifies

$$\|(\bar{u}^\varepsilon|_T, \bar{u}^\varepsilon_t|_T)\|_{H\times V'} < \varepsilon.$$

Here we shall mainly focus on the problem of controllability and present the existing results guaranteeing that the system is controllable within a class of initial data that one might identify.

Using the definition of solutions of the state equation by means of transposition the control property can be characterized in the following manner:

Proposition 2.1 *The initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T with control $h \in U$ if, and only if, for every $(\bar{\phi}_0, \bar{\phi}_1) \in V \times H$ the following equality holds*

$$-\langle \bar{u}_0, \bar{\phi}_1\rangle_H + \langle \bar{u}_1, \bar{\phi}_0\rangle_{V'\times V} = \int_0^T h(t)\partial_n \phi^{j_{\mathbf{v_1}}}(t, \mathbf{v_1})dt, \tag{13}$$

where $\bar{\phi}$ is the solution of system (2) with initial state $(\bar{\phi}_0, \bar{\phi}_1)$.

The relation (13) suggests a minimization algorithm for the construction of the control h. If we look for the control in the form $h = -\partial_n \bar{\psi}(\mathbf{v}_1, t)$, where $\bar{\psi}$ is a solution of the homogeneous system (2), then the equality (13) is the Euler equation $I'(\bar{\psi}_0, \bar{\psi}_1) = 0$ corresponding to the quadratic functional $I : V \times H \to \mathbb{R}$ defined by

$$I(\bar{\phi}_0, \bar{\phi}_1) = \frac{1}{2}\int_0^T |\partial_n \phi^i(t, \mathbf{v}_1)|^2 dt + \langle \bar{u}_0, \bar{\phi}_1\rangle - \langle \bar{u}_1, \bar{\phi}_0\rangle.$$

Therefore, if $(\bar{\psi}_0, \bar{\psi}_1)$ is a minimizer of I, the relation (13) will be verified. The functional I is continuous and convex. So, in order to guarantee the controllability of an initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ it is sufficient that I be coercive. This is the central idea of the Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions in [71].

The coercivity of the functional is equivalent to the following observability inequality:

$$||(\bar{\phi}_0, \bar{\phi}_1)||_*^2 \leq C\int_0^T |\partial_n \phi^i(t, \mathbf{v}_1)|^2 dt, \tag{14}$$

for all solutions of the adjoint system, where $||\cdot||_*$ stands for a norm to be identified.

Once the norm $||\cdot||_*$ has been identified and the observability inequality (14) proved, the controllability property can be guaranteed to hold for all initial data $(\bar{u}_0, \bar{u}_1)$ in $(\mathcal{W}^*)'$, the dual of $\mathcal{W}^*$, the Hilbert space obtained as the closure of $\mathcal{W}^1$ with respect to the norm $||\cdot||_*$.

The main issue is then the obtention of inequalities of the form (14), giving quantitative informations about the norm $||\cdot||_*$, so that the spaces in which observability and controllability hold, $\mathcal{W}^*$ and $(\mathcal{W}^*)'$, respectively, might be identified. These issues depend in a very sensitive manner on the topological and number theoretical properties of the network under consideration.

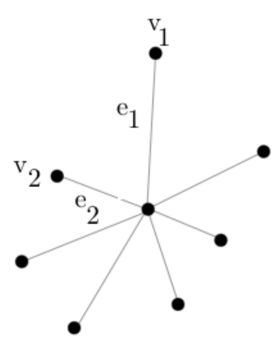
As we shall see later, the problem of stabilization can also be solved once the observability inequality (14) is well understood. Indeed, we shall present a black-box strategy recently developed in collaboration with J. Valein [98], allowing to get observability inequalities (and, consequently, decay properties) for the solutions of wave equations in networks with dissipative boundary conditions, as a consequence of their conservative counterparts.

3 Main results on observability and controllability

3.1 Summary of known results

The state of the art in what concerns the observability/controllability problem is more or less the one presented in [37], where the following three cases, in increasing complexity, were discussed. We summarize here the known main results.

- **The star**. In the star-like network a finite number of strings are connected on a single point by one of their extremes. This is a particular case of a tree-like network that we shall discuss below.



A star-shaped network.

If the observation/control acts on all but one external vertices of the star, one gets observability/controllability in the optimal energy spaces. In other words, we get (14) with $\mathcal{W}^* = \mathcal{W}^1$.

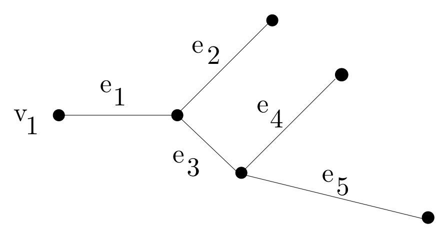
To the contrary, in the opposite case in which the observation/control is only applied in one external vertex, as we are doing here, then the space of observation and/or control can be described in Fourier series by means of suitable weights depending on the lengths of the strings entering in the star. These weights depend on the ratios of the lengths of the strings and, in particular, on their irrationality properties. In case when some of the non-controlled strings are mutually rational, some of these weights vanish and then the observability/controllability properties fail to hold. To the contrary, when they are all mutually irrational, all these weights are strictly positive but their lower envelop tends to zero for high frequencies so that there is always some loss in the spaces in which the observability/controllability problems are solvable. How important this loss is depends on the diophantine approximation properties of the quotients of the lengths. In particular, the weights may degenerate exponentially when some of the quotient of the lengths is a Liouville number. But, regardless of the diophantine properties and the nature of the spaces in which the observability/controllability properties hold, the time needed for observation turns out to be twice the sum of all the lengths of the strings of the networks.

It is interesting to analyze the relation of this result with the so-called Geometric Control Condition (GCC) introduced by Bardos, Lebeau and Rauch [18] in the context of the boundary observation and/or control of the wave equation in bounded domains of $\mathbb{R}^n$. The GCC requires that all the rays of Geometric Optics enter the observation region in a finite, uniform time, which turns out to be the minimal one for observation/control. In the case of the star shaped network this would correspond to the maximum of sum of the lengths of any pair of two strings.

But this time is insufficient for the control from only one end-point. As we mentioned, above, indeed, the time needed is twice the sum of all the lengths of all the strings of the star-shaped network. This control time is closer to the one one gets when one string is controlled at an interior point or two strings are controlled by a single control on a common vertex. In that case the minimal control time is $2(\ell_1+\ell_2)$ and not $2\max(\ell_1,\ell_2)$, ℓ_1,ℓ_2 being the lengths of the two strings. The wave equation is a second order problem and therefore, even in $1-d$, for a point-wise observation mechanism to be efficient we need to measure not only the position, but also the space derivative. This implies that a necessary condition for observation/control is that all waves pass twice through the observation point. This is guaranteed when the time of control is larger than $2(\ell_1+\ell_2)$. But, in fact, passing twice by the observation point is not sufficient either. The irrationality of the ratio ℓ_1/ℓ_2 is needed to guarantee that, when passing through the observation point the second time, the solution is not exactly at the configuration as in the first crossing, which, of course, would make the second observation to be insufficient too. Finally, even when ℓ_1/ℓ_2 is irrational, we cannot get a uniform bound of the energy of the solution but rather a weaker measurement in a weaker norm. The nature of this norm, which is represented in Fourier series by means of some weights depending on ℓ_1/ℓ_2, depends very strongly on the irrationality class to which the number ℓ_1/ℓ_2 belongs. In fact, even in the most favorable case, i.e., when ℓ_1/ℓ_2 is an algebraic number of degree two, one looses one derivative with respect to the expected energy norm.

We refer to [37] for an in depth discussion of the problem of simultaneous control of a finite number of strings and its connections with the problem of the control of star networks.

- **The tree**. The tree-like network is a generalization of the star-like one. As we said above, it is well known that, when all but one external nodes of the network are observed on a tree-like configuration, the whole energy of solutions may be observed (see [**?**]). This can be easily seen by sidewise energy estimates for the solutions of the wave equation. In this case the observation inequality holds in the sharp energy space in a time which is twice the length of the longest path joining the points of the network with some of the observed ends, which is much smaller than twice the total length of the network, which was the time needed for the observation from a single end in the case of stars mentioned above. This smaller observability time is the one that coincides with the one given by the GCC in the case of waves in domains of the Euclidean space.



A tree-shaped network.

In the opposite case in which the observation is made at one single extreme of the tree-like network, the observation time turns out to be, again, twice the sum of the lengths of the strings forming the network.

But for the observability inequality to be true in the case of the tree one needs a condition extending the one that, in the case of stars, requires the strings to have mutually irrational lengths. In [37] it was observed that this condition can be recast in spectral terms: two strings have mutually irrational lengths if and only if their Dirichlet spectra have empty intersection.

The latter condition turns out to be the appropriate one to be extended to general trees: the wave equation on a tree is observable from one end if and only if the spectra of all pairs of subtrees of the tree that match on an interior node are disjoint.

This allows showing, in particular, that, generically within the class of trees (i. e. for almost all tress), this property is satisfied and then, the wave process is observable/controllable from one single node. But the space in which the observability/controllability holds depends in a subtle manner on the distance between the various spectra of the corresponding subtrees and how it vanishes asymptotically at high frequencies.

Note however that the identification of the precise norm $||\cdot||_*$ in which the observability inequality (14) holds is a delicate issue.

- **General networks**. The characterization we have given of controllable stars and trees is hard to be extended to general graphs. Indeed, in the general case, we lack of a natural ordering on the graph to analyze the propagation of waves and, for instance, when the graph contains cycles, the condition of empty intersection of subgraphs is hard to extend. Actually, as we mentioned above, the presence of closed circuits may trap the waves thus making impossible the controllability/observability properties to hold from an external node.

Thus, in the analysis of general graphs, we proceed in a different way by applying a consequence of the celebrated Beurling-Malliavin's Theorem on the completeness of families of real exponentials obtained by Haraux and Jaffard in [50] when analyzing the control of plates. Using the min-max principle, one can show that the spectral density of a general graph is the same as that of a single string whose length, L, is the sum of the lengths of all the strings entering in the network. Then, when the time is greater than twice the total length, as a consequence of Beurling-Malliavin's Theorem, we deduce that there exist some Fourier weights so that the observation property holds in the corresponding weighted norm if and only if all the eigenfunctions of the network are observable.

So far we do not know of any necessary and sufficient condition guaranteeing that all the eigenfunctions are observable in the general case. However, this condition, in the particular case of stars and trees discussed above turns out to be sharp and equivalent to the ones we have identified in each particular case: (a) the condition that lengths of the strings are mutually irrational in the case of stars or (b) that the spectra of all pairs of subtrees with a common end-point to be mutually disjoint in the more general case of trees.

3.2 The weighted observability inequality

In the previous section we have described the main existing results on the observability of graphs distinguishing three different cases, in increasing complexity: the star, the tree and general graphs. In each case, under suitable assumptions, we obtain the observability inequality (14) for a suitable norm $||\cdot||_*$. This norm can be characterized in terms of the Fourier coefficients by suitable weights. This subsection is devoted to explain this fact, which plays a critical role in the control and stabilization results one can get out of this analysis, and that will be discussed in the next section.

Recall that if we suppose that $(\bar{\phi}^{(0)}, \bar{\phi}^{(1)}) \in \mathcal{W}^1$, then problem (2) admits a unique solution

$$\bar{\phi} \in C(\mathbb{R};\, V) \cap C^1(\mathbb{R};\, H).$$

The observability inequalities we have described can be rewritten in terms of the Fourier expansion (4) as follows:

$$\sum_{n\geq 1} c_n^2(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2) \leq C \int_0^T \left| \frac{\partial \phi_1}{\partial x}(\mathbf{v}_1,\, t) \right|^2 dt. \tag{15}$$

This holds in the situations described above, under the corresponding assumptions on the network, for T large enough (twice the sum of the lengths of all the strings enetring in the network, $T > 2L$) and for a suitable observability constant $C > 0$ and weights $\{c_n\}_{n\geq 1}$. The norm $||\cdot||_*$ arising in the observability inequality is therefore as follows:

$$||(\bar{\phi}_0, \bar{\phi}_1)||_* = \left[\sum_{n\geq 1} c_n^2(\mu_n \phi_{0,n}^2 + \phi_{1,n}^2) \right]^{1/2}. \tag{16}$$

Obviously, the nature of this norm depends on how fast the weights $\{c_n\}_{n\geq 1}$ tend to zero as $n \to \infty$.

Recall however that, in each case, extra assumptions are needed to ensure that the weights c_n^2 are strictly positive for every $n \in \mathbb{N}^*$.

One of the most interesting open problems in this context is to give sharp sufficient conditions on the network so that these weights have a given asymptotic lower bound as $n \to \infty$. At this respect, the case of the star network is the simplest one: it then suffices to impose diophantine conditions on the quotients of the lengths of the strings entering in the network to get those lower bounds.

4 Stabilization

4.1 Problem formulation

So far we have considered an open-loop control problem. In this section we discuss the closed-loop counterpart in which the goal is to find suitable feedback mechanisms ensuring the decay as $t \to \infty$ of solutions.

Recall that, in the control and observation problems above we have distinguished one vertex $\mathbf{v}_1$ among all the exterior ones $\mathcal{V}_{ext}$: the one in which the control or the observation is being applied. The rest of the nodes in which the homogeneous Dirichlet boundary condition holds for the control problem is denoted by $\mathcal{V}_{\mathcal{D}}$. In this way, we distinguish the conservative exterior nodes, $\mathcal{V}_{\mathcal{D}}$, in which we impose Dirichlet homogeneous boundary conditions, and the one in which the damping term is effective, $\mathbf{v}_1$. To simplify the notation, we will assume that $\mathbf{v}_1$ is located at the end 0 of the edge e_1.

The system under consideration then reads as follows:

$$\begin{cases} \frac{\partial^2 y^j}{\partial t^2} - \frac{\partial^2 y^j}{\partial x^2} = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1, ..., M\}, \\ y^j(\mathbf{v},\, t) = y^l(\mathbf{v},\, t) & \forall j,\, l \in \mathcal{E}_{\mathbf{v}},\, \mathbf{v} \in \mathcal{V}_{int}, t > 0, \\ \sum_{j\in\mathcal{E}_{\mathbf{v}}} \frac{\partial y^j}{\partial n_j}(\mathbf{v},\, t) = 0 & \forall \mathbf{v} \in \mathcal{V}_{int}, t > 0, \\ y^{j_{\mathbf{v}}}(\mathbf{v},\, t) = 0 & \forall \mathbf{v} \in \mathcal{V}_{\mathcal{D}}, t > 0, \\ \frac{\partial y^1}{\partial x}(0,\, t) = \frac{\partial y^1}{\partial t}(0,\, t) & \forall t > 0, \\ \bar{y}(0) = \bar{y}_0,\, \frac{\partial \bar{y}}{\partial t}(0) = \bar{y}_1, & \end{cases} \tag{17}$$

where $\partial y^j/\partial n_j(\mathbf{v},\, .)$ stands for the outward normal (space) derivative of y^j at the vertex $\mathbf{v}$. Similarly the normal derivative at the vertex $\mathbf{v}_1 = 0$ where the dissipative boundary condition is imposed is denoted by $-\partial y^1(0,\, t)/\partial x$, y^1 being the deformation of the first edge with extreme $\mathbf{v}_1 = 0$. The deformation of the network at that point is given by $y^1(0, t)$. As usual, we denote by $\overline{y}$ the vector $\overline{y} = (y^j)_{j=1,\ldots,M}$.

The above system has been considered in a number of articles where the decay rate of solutions has been investigated in some specific examples and, recently, an unified treatment has been given in [98]. We briefly present here the main ideas and results.

In order to study system (17) we need a proper functional setting which is slightly different to the one considered until now because of the damped boundary condition on one of the nodes. To be more precise, the space V above has to be replaced by:

$$V_{\mathcal{D}} = \{\bar{y} \in \prod_{j=1}^{M} H^1(0, \ell_j) : y^j(\mathbf{v}) = y^k(\mathbf{v}) \text{ if } \mathbf{v} \in \mathcal{V}_{int}, \forall j,\, k \in \mathcal{E}_{\mathbf{v}} \text{ and } y^{j_{\mathbf{v}}}(\mathbf{v}) = 0 \text{ if } \mathbf{v} \in \mathcal{V}_{\mathcal{D}}\}.$$

The only difference between the space V above and the new one $V_{\mathcal{D}}$ is that, in the later, we do not impose the homogeneous Dirichlet boundary condition on $\mathbf{v}_1 = 0$.

It is easy to see by semigroup methods that this dissipative system is well posed in in the Hilbert space

$$\mathcal{W}_{\mathcal{D}} := V_{\mathcal{D}} \times H,$$

equipped with the canonical norm.

Then, for an initial datum in $\mathcal{W}_{\mathcal{D}} := V_{\mathcal{D}} \times H$, there exists a unique solution such that

$$\bar{y} \in C([0,\infty); V_{\mathcal{D}}) \bigcap C^1([0,\infty); H). \tag{18}$$

Moreover, the solutions remain in $D(\mathcal{A}_{\mathcal{D}})$, the domain of the operator $\mathcal{A}_{\mathcal{D}}$, for all $t > 0$ whenever the initial data belong to $D(\mathcal{A}_{\mathcal{D}})$:

$$D(\mathcal{A}_{\mathcal{D}}) := \{(y,\, z) \in (V_{\mathcal{D}} \cap \prod_{j=1}^{M} H^2(0,\, l_j)) \times V_{\mathcal{D}} \,:\, \frac{\partial y^1}{\partial x}(0) = z^1(0)\,;\, \sum_{j\in\mathcal{E}_v} \frac{\partial y^j}{\partial n_j}(\mathbf{v}) = 0,\, \forall \mathbf{v} \in \mathcal{V}_{int}\}.$$

For this dissipative system the energy satisfies the energy dissipation law

$$\frac{d}{dt}\mathbf{E}_{\bar{y}}(t) = -\left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 \le 0, \tag{19}$$

and therefore it is decreasing.

Integrating the expression (19) between 0 and T, we obtain

$$\int_0^T \left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 dt = \mathbf{E}_{\bar{y}}(0) - \mathbf{E}_{\bar{y}}(T) \le \mathbf{E}_{\bar{y}}(0).$$

This estimate implies that $\frac{\partial y^1}{\partial t}(0,\, \cdot)$ belongs to $L^2(0,\, T)$ for finite energy solutions.

The main goal of this section is to show how the results of previous sections on observability/controllability can be used to derive energy decay rates as $t \to \infty$ for smooth solutions in $D(\mathcal{A}_{\mathcal{D}})$. Obviously, the better the observability/controllability results, faster decay rates will be obtained.

Note, however, that in the context of observability/controllability we have considered only Dirichlet boundary conditions while in here we are imposing a dissipative boundary condition on one node. Thus, we need to reduce the problem of getting decay results for the damped systems into the one of observability inequalities for the conservative one with Dirichlet boundary conditions on all the exterior nodes. To do this we proceed in two steps:

- We first reduce it to the case of conservative Dirichlet-Neumann boundary conditions,
- to later reduce it to the case of purely Dirichlet conditions.

As we shall see, overall, this reduction argument allows obtaining an observability inequality for $\bar{y}$ out of the known ones for the solutions of the Dirichlet problem. The obtained observability inequality reads

$$\mathbf{E}^{-}_{\bar{y}}(0) \le C \int_0^T \left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 dt, \tag{20}$$

for an energy $\mathbf{E}^{-}_{\bar{y}}(0)$ that we shall make precise below but that, definitely, will be weaker than the energy norm.

To obtain explicit decay rates out of this weak observability inequality we use an interpolation inequality which is a variant of the one from Bégout and Soria [21] and which is a generalization of Hölder's inequality. For this to be done we need to assume more regularity of the initial data.

To be more precise we shall consider initial data $(\bar{y}_0, \bar{y}_1) \in X_s := [D(\mathcal{A}_{\mathcal{D}}), \mathcal{W}_{\mathcal{D}}]_{1-s}$ for $0 < s < 1/2$ and deduce an interpolation inequality of the form

$$1 \leq \Phi_s \left(\frac{\mathbf{E}^-_{\bar{y}}(0)}{C\mathbf{E}_{\bar{y}}(0)} \right) \frac{\|(\bar{y}_0, \bar{y}_1)\|^2_{X_s}}{C'\mathbf{E}_{\bar{y}}(0)}, \tag{21}$$

where Φ_s is an increasing function which depends on s and on the weak energy $\mathbf{E}^-_{\bar{y}}$ under consideration.

The previous interpolation inequality implies

$$\mathbf{E}^-_{\bar{y}}(0) \geq C\mathbf{E}_{\bar{y}}(0)\Phi_s^{-1} \left(\frac{\mathbf{E}_{\bar{y}}(0)}{C' \|((\bar{y}_0, \bar{y}_1)\|^2_{X_s}} \right).$$

With (19) and (20), we obtain

$$\mathbf{E}_{\bar{y}}(0) - \mathbf{E}_{\bar{y}}(T) \geq C\mathbf{E}_{\bar{y}}(0)\Phi_s^{-1} \left(\frac{\mathbf{E}_{\bar{y}}(0)}{C' \|(\bar{y}_0, \bar{y}_1)\|^2_{X_s}} \right),$$

which implies, by the semigroup property (see Ammari and Tucsnak [10])

$$\forall t > 0,\, \mathbf{E}_{\bar{y}}(t) \leq C\Phi_s \left(\frac{1}{t+1} \right) \|(\bar{y}_0, \bar{y}_1)\|^2_{X_s}. \tag{22}$$

Obviously, the decay rate in (22) depends on the behaviour of the function Φ_s near 0. Thus, in order to determine the explicit decay rate, we need to have a sharp description of the function Φ_s, which depends on s and on the energies $\mathbf{E}_{\bar{y}}$ and $\mathbf{E}^-_{\bar{y}}$ and thus on the nature of the weak energy $\mathbf{E}^-_{\bar{y}}$ in an essential way and this depends on the topology of the network and the number theoretical properties of the lengths of the strings entering in it.

This approach allows getting in a systematic way decay rates for the energy of smooth solutions of the damped system as a consequence of the observability properties of the undamped one.

The key ingredients of the proof that remain to be developed are the following:

- To get the weak observability inequality (20) out of the previous results on the observability of the Dirichlet problem;
- To derive the interpolation inequality (21) with a precise estimate on the behavior of Φ_s.

4.2 Observability for the damped system

This subsection is devoted to explain how the weak observability inequality (20) can be proved as a consequence of the results of previous sections on the Dirichlet problem on the same network.

Let us explain how the observability results of the purely Dirichlet case discussed in the previous sections can be applied directly to get an inequality of the form (20) for the solutions of (17).

For that, we decompose $\bar{y}$, the solution of (17), as the sum of $\bar{\phi}$, solution of (2), and a reminder term $\bar{\epsilon}$:

$$\bar{y} = \bar{\phi} + \bar{\epsilon}.$$

Recall that $\bar{\phi}$ is a solution of (2) with appropriate initial data $(\bar{y}_0 - y_0^1(0)\bar{\gamma},\, \bar{y}^1)$, where $\bar{\gamma}$ is a given smooth function such that $\gamma^1(0) = 1$ and vanishing on all other external nodes.

Applying (15) to the solution $\bar{\phi}$ of (2), we obtain the following weighted observability estimate (note that $y^1(0,0) = y_0^1(0)$)

$$||(\bar{y}_0 - y_0^1(0)\bar{\gamma},\, \bar{y}_1)||_*^2 \le C_T \int_0^T \left(\frac{\partial \phi^1}{\partial x}(0,\, t)\right)^2 dt, \tag{23}$$

where the weak norm $||\cdot||_*$ is defined with weights $(c_n^2)_n$ that tend to zero as $n \to \infty$, depending on the network, as described in the previous sections. It is however important to underline that (23) holds under the same assumptions on the network needed for observability to hold for the Dirichlet problem (2) and provided $T > 2L$.

The reminder term ϵ is the solution of the following non-homogeneous Dirichlet problem:

$$\begin{cases} \frac{\partial^2 \epsilon^j}{\partial t^2} - \frac{\partial^2 \epsilon^j}{\partial x^2} = 0 & \forall x \in (0,\, l_j),\, t > 0,\, \forall j \in \{1, ..., M\}, \\ \epsilon^j(v,\, t) = \epsilon^l(\mathbf{v},\, t) & \forall j,\, l \in \mathcal{E}_{\mathbf{v}},\, \mathbf{v} \in \mathcal{V}_{int}, t > 0, \\ \sum_{j \in \mathcal{E}_v} \frac{\partial \epsilon^j}{\partial n_j}(\mathbf{v},\, t) = 0 & \forall \mathbf{v} \in \mathcal{V}_{int}, t > 0, \\ \epsilon_{j_{\mathbf{v}}(\mathbf{v},t)=0} & \forall \mathbf{v} \in \mathcal{V}_{\mathcal{D}}, t > 0, \\ \epsilon(0,\, t) = y(0,\, t) & t > 0, \\ \bar{\epsilon}(0) = y_0^1(0)\bar{\gamma},\, \frac{\partial \bar{\epsilon}}{\partial t}(0) = \bar{0}. & \end{cases} \tag{24}$$

Note that ϵ satisfies a non-homogeneous Dirichlet boundary condition at $x = 0$. Actually it coincides with the initial value of the solution y^1 of (17) at that point. We know that the solution $\bar{y}$ of the dissipative problem, because of the energy dissipation law, is such that $\partial y^1(0,\, \cdot)/\partial t \in L^2(0,\, T)$, so that the non-homogeneous Dirichlet boundary condition belongs to $H^1(0,\, T)$.

Proceeding in this manner, the following result was proved in [98]:

Theorem 4.1 **([98])**. *Assume that the network is such that the weighted observability inequality (23) is satisfied for $T > 2L$ for the conservative system (2) with Dirichlet boundary conditions at all the exterior nodes. Define the weak energy $\mathbf{E}^-_{\bar{y}}(0)$ by*

$$\mathbf{E}^-_{\bar{y}}(0) := \frac{1}{2}\left[||(\bar{y}_0 - y_0^1(0)\bar{\gamma},\, \bar{y}_1)||_*^2 + y_0^1(0)^2\right]. \tag{25}$$

Then for all $T > 2L$, there exists $C_T > 0$ such that all solution $\bar{y}$ of (17) satisfies the weak observability inequality (20).

Note that $\left(\mathbf{E}^-_{\bar{y}}(0)\right)^{\frac{1}{2}}$ as above defines a norm in the space of initial data $(\bar{y}_0, \bar{y}_1) \in \mathcal{W}_{\mathcal{D}}$. Indeed, when $\mathbf{E}^-_{\bar{y}}(0)$ vanishes, $y_0^1(0) = 0$. Thus $(\bar{y}_0, \bar{y}_1) \in \mathcal{W}$ and then $\mathbf{E}^-_{\bar{y}}(0) = \mathbf{E}_{\bar{y}}(0)$, and, by assumption, $(\mathbf{E}_{\bar{y}}(0))^{\frac{1}{2}}$ defines a norm in $\mathcal{W}$.

Let us now present a sketch of the proof of Theorem 4.1.

Note that, standard results on the hidden regularity of the wave equation guarantee that, for all $T > 0$ there exists $C_T > 0$ such that the solutions $\bar{y}$ of (17) and $\bar{\epsilon}$ of (24) satisfy the following estimate

$$\int_0^T \left(\frac{\partial \epsilon^1}{\partial x}(0,\, t)\right)^2 dt \le C_T \int_0^T \left[\left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 + \left(y^1(0,\, t)\right)^2\right] dt. \tag{26}$$

Despite of the fact that we are here working with the wave equation on a network, this result is of local nature and therefore it is sufficient to apply the standard multiplier techniques of the scalar wave equation in the string with vertex at $\mathbf{v_1} = 0$. With a little extra work the right hand side term of this inequality can be slightly weakened to yield

$$\int_0^T \left(\frac{\partial \epsilon^1}{\partial x}(0,\, t)\right)^2 dt \le C_T \left(\int_0^T \left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 dt + \left(y_0^1(0)\right)^2\right). \tag{27}$$

Combining (23) and (27) and the fact that

$$\int_0^T \left(\frac{\partial \phi^1}{\partial x}(0,\, t)\right)^2 dt \le 2\int_0^T \left(\frac{\partial y^1}{\partial x}(0,\, t)\right)^2 dt + 2\int_0^T \left(\frac{\partial \epsilon^1}{\partial x}(0,\, t)\right)^2 dt$$

and that, due to the choice of the dissipative boundary condition,

$$\int_0^T \left(\frac{\partial y^1}{\partial x}(0,\, t)\right)^2 dt = \int_0^T \left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 dt$$

we have

$$\mathbf{E}^-_{\bar{y}}(0) \le C\left[\int_0^T \left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 dt + \left|y_0^1(0)\right|^2\right]. \tag{28}$$

In fact, we can remove the last term in the right hand side of (28). To do this, it is sufficient to show that

$$\left|y_0^1(0)\right|^2 \le C_T \int_0^T \left(\frac{\partial y^1}{\partial t}(0,\, t)\right)^2 dt$$

for a positive constant C_T depending on T.

This can be done by a classical compactness-uniqueness argument using the fact that the perturbation is of rank one (and therefore compact with respect to any norm) and the fact that whenever $\partial y_1(0,\, t)/\partial t$ and $\partial y_1(0,\, t)/\partial x$ vanish for $t \in (0, T)$ during a sufficiently long time interval ($T > 2L$), then, necessarily, $y_0(0) = 0$.

In this way, we conclude that the wanted inequality (20) is true.

4.3 The interpolation inequality

In this subsection we recall the main ingredients of the proof of the interpolation inequality (21). Its proof uses a discrete interpolation inequality, similar to that in [21], introduced in [98] and a description of the various energies and norms entering in the estimates we have obtained so far in terms of Fourier series.

The discrete interpolation inequality reads as follows:

Let $m \in [0,\, 1)$, $0 < s < 1/2$ and assume that

$$\omega : (m,\, \infty) \to (0,\, \omega(m)) \text{ is convex and decreasing with } \omega(\infty) = 0, \tag{29}$$

$$\Phi_s : (0,\, \omega(m)) \to (0,\, \infty) \text{ is concave and increasing with } \Phi_s(0) = 0, \tag{30}$$

$$\forall t \in [1,\, \infty),\, 1 \le \Phi_s(\omega(t))t^{2s}, \tag{31}$$

$$\text{The function } t \mapsto \frac{1}{t}\Phi_s^{-1}(t) \text{ is nondecreasing on } (0,\, 1). \tag{32}$$

Under the conditions (30)-(31), we have the following result which is a generalized Hölder's inequality, a variant of Theorem 2.1 given in [21]:

LEMMA 4.1 *Let* (ω, Φ_s) *be as above satisfying (29)-(31). Then for any* $f = (f_n)_{n\in\mathbb{N}^*} \in l^1(\mathbb{N}^*)$*,* $f \neq 0$*, we have*

$$1 \leq \Phi_s\left(\frac{\sum_{n\geq 1}|f_n|\,\omega(n)}{\sum_{n\geq 1}|f_n|}\right)\frac{\sum_{n\geq 1}|f_n|\,n^{2s}}{\sum_{n\geq 1}|f_n|}, \tag{33}$$

as soon as $(f_n\omega(n))_n \in l^1(\mathbb{N}^*)$ *and* $(f_n n^{2s})_n \in l^1(\mathbb{N}^*)$*.*

We now give some examples of pairs (ω, Φ_s) satisfying (29)-(32):

- 1. If
$$\omega(t) = \frac{c}{t^p},$$
for some $p \geq 1$, we can take Φ_s of the form
$$\Phi_s(t) = \left(\frac{t}{c}\right)^{\frac{2s}{p}}.$$
We can easily prove that (ω, Φ_s) satisfy (29)-(30) with $m = 0$ and (31)-(32).

- 2. If
$$\omega(t) = Ce^{-At}$$
where $A > 2(2s+1)$ and $C > 0$, we can take Φ_s of the form
$$\Phi_s(t) = \left(\frac{A}{\ln\left(\frac{C}{t}\right)}\right)^{2s}.$$
We can easily prove that $t \mapsto \frac{1}{t}\Phi_s^{-1}(t)$ is nondecreasing on $(0, 1)$ and that the pair (ω, Φ_s) satisfies (31) on $[1, \infty)$. Thus (ω, Φ_s) satisfy (29)-(32) with $m = 1/2$.

When applying this argument, $\sum_{n\geq 1}|f_n|\,\omega(n)$ will play the role of the weak energy $\mathbf{E}_{\bar{y}}^-$, $\sum_{n\geq 1}|f_n|$ the role of the standard energy $\mathbf{E}_{\bar{y}}$ and $\sum_{n\geq 1}|f_n|\,n^{2s}$ that of the norm in X_s. But for this to be done, these energies and norms have to written in a suitable discrete manner.

We explain how this can be done distinguishing each of the terms:

- **The X_s-norm.** At this level, the fact that $0 < s < 1/2$ plays a key role. The following Lemma was proved in [98]:

  LEMMA 4.2 **([98])** *Assume that* $(\bar{y}_0, \bar{y}_1)$ *belongs to* X_s*, where* $0 < s < 1/2$*, and* $(\bar{\phi}_0, \bar{\phi}_1) = (\bar{y}_0 - y_0^1(0)\bar{\gamma}, \bar{y}_1)$*, where* $\bar{\gamma}$ *is a given smooth function such that* $\gamma^1(0) = 1$ *and vanishing on all other external nodes. Then there exists a positive constant* C *such that*
  $$\left\|(\bar{\phi}_0, \bar{\phi}_1)\right\|_{D((-\Delta_G)^s)}^2 + \left|y_0^1(0)\right|^2 \leq C\,\|(\bar{y}_0, \bar{y}_1)\|_{X_s}^2,$$
  where $D((-\Delta_G)^s)$ *is the domain of the operator* $(-\Delta_G)^s$*, which is the* s*-th power of the Laplacian on the graph,* $-\Delta_G$*, with Dirichlet boundary conditions at all exterior nodes.*

This means that it is sufficient to prove the interpolation inequality (21) with the norm in X_s replaced by $\left[\left\|\left(\bar{\phi}_0,\,\bar{\phi}_1\right)\right\|^2_{D((-\Delta_G)^s)}+\left|y^1_0(0)\right|^2\right]^{1/2}$.

On the other hand, the norm $\left\|\left(\bar{\phi}_0,\,\bar{\phi}_1\right)\right\|^2_{D((-\Delta_G)^s)}$ can be written easily in terms of the Fourier coefficients of $\left(\bar{\phi}_0,\,\bar{\phi}_1\right)$ in the basis of eigenfunctions of $-\Delta_G$:

$$\left\|\left(\bar{\phi}_0,\,\bar{\phi}_1\right)\right\|^2_{D((-\Delta_G)^s)}=\sum_{n\geq 1}\left[\mu_n^{1+s}|\phi_{0,n}|^2+\mu_n^s|\phi_{1,n}|^2\right],$$

where $(\phi_{0,n},\phi_{1,n})$ are the Fourier coefficients of the data $\left(\bar{\phi}_0,\,\bar{\phi}_1\right)$.

- **The weak energy $\mathbf{E}^-_{\bar{y}}$.** According to the results of the previous sections and, in particular, (15), the observed weak energy can be rewritten as

$$\mathbf{E}^-_{\bar{y}}(0)=\sum_{n\geq 1}c_n^2(\mu_n\phi^2_{0,n}+\phi^2_{1,n})+\left|y^1_0(0)\right|^2. \tag{34}$$

- **The energy $\mathbf{E}_{\bar{y}}$.** Similarly, the energy $\mathbf{E}_{\bar{y}}$ is equivalent to the discrete norm:

$$\mathbf{E}_{\bar{y}}\sim\left\|\left(\bar{\phi}_0,\,\bar{\phi}_1\right)\right\|^2_{\mathcal{W}}+\left|y^1_0(0)\right|^2=\sum_{n\geq 1}\left[\mu_n|\phi_{0,n}|^2+|\phi_{1,n}|^2\right]+\left|y^1_0(0)\right|^2.$$

Once this is done, the interpolation inequality (21) is a consequence of the abstract discrete interpolation result (33).

In the next subsection we state the main stabilization result that this analysis yields.

4.4 The main result

Before moving further we observe that, as proved in [98], for $0<s<1/2$,

$$X_s=\left(V_{\mathcal{D}}\cap\prod_j H^{1+s}(0,l_j)\right)\times\prod_j H^s(0,l_j).$$

Thus, the space X_s of smooth initial data can be identified in classical Sobolev terms.

We assume that the network is such that the weighted observability inequality (15) holds. In the previous sections we have given sufficient conditions on the network for that to hold with positive weights $c_n>0$.

The main stabilization result is as follows:

THEOREM 4.2 *Assume that the weighted observability inequality (15) holds for every solution of (2) with $\liminf_{n\to\infty}c_n=0$ and $c_n\neq 0$ for all $n\in\mathbb{N}^*$. Let ω be defined by a lower envelope of the sequence of weights (c_n^2) satisfying (29). Assume that the initial data $(\bar{y}_0,\,\bar{y}_1)$ belong to X_s where $0<s<1/2$. Let Φ_s be a function such that the pair $(\omega,\,\Phi_s)$ satisfies (29)-(32). Then there exists a constant $C>0$ such that the corresponding solution $\bar{y}$ of (17) verifies*

$$\forall t\geq 0,\,\mathbf{E}_{\bar{y}}(t)\leq C\Phi_s\left(\frac{1}{t+1}\right)\|(\bar{y}_0,\,\bar{y}_1)\|^2_{X_s}. \tag{35}$$

We see that the decay rate of the energy directly depends on the behavior of the interpolation function Φ_s near 0 and thus of ω and of the weights c_n^2 as $n \to \infty$.

Using (35) and making a particular and explicit choice of the concave function Φ_s, we obtain a more explicit decay rate. To be more precise, we set

$$\forall t > 0,\, \varphi(t) = \frac{\omega(t)}{t^2}.$$

Then there exists a constant $C > 0$ such that for any initial data $(\bar{y}_0,\, \bar{y}_1) \in X_s$ $(0 < s < 1/2)$, the corresponding solution $\bar{y}$ of (17) verifies

$$\forall t \geq 0,\, \mathbf{E}_{\bar{y}}(t) \leq \frac{C}{\left(\varphi^{-1}\left(\frac{1}{t+1}\right)\right)^{2s}} \left\|(\bar{y}_0,\, \bar{y}_1)\right\|_{X_s}^2 . \tag{36}$$

We refer to [37] and [98] for explicit examples of networks in which explicit estimates on the rate of vanishing on the weights (c_n^2) and, accordingly, of the decay rate of the energy for the dissipative system are given.

5 Further comments and open problems

As we have mentioned thoughout the article, there are many interesting questions (most of them are difficult) to be investigated in connection with the topics we have addressed here and some other closely related ones, in connection with PDE in networks. We mention here some of them. Of course the list is non exhaustive. We refer to [57], for instance, for a recent survey on this area.

- **Lower bounds on the weights.** As we have seen, the weights entering the observability inequalities, and, more precisely, their decay at high frequencies, play a key role when identifying the control/observation spaces and also the decay rates on the dissipative framework. It would be very interesting to analyze how the degeneracy of these weights at high frequencies depends on the properties of the network under consideration.

- **Wave equations with potentials.** We have considered here the pure wave model. What happens when the equations are perturbed by lower order terms? In the context of the equation in domains of the Euclidean space, it is well known that these lower order perturbations do not matter in the sense that they add compact perturbations that can be get rid-of by a compactness-uniqueness argument. But the situation is different in networks because the best expected observability results are weak and require the loss of at least one derivative. This derivative is precisely the one that the zero order potentials allow gaining, but it is not enough to ensure the compactness of the perturbations. Note moreover that this happens in very special situations where the diophantine theory can be applied. But, in general, the loss of derivatives can be arbitrary. Thus, the problem of whether the observability/controllability/stabilizability properties we have proved here are preserved when one adds arbitrary bounded potentials on the various strings of the network is open.

- **Wave equations with variable coefficients.** The same problem above can be formulated for wave equations on $1-d$ networks with variable and sufficiently smooth coefficients (say BV-ones). Note that, even for the $1-d$ wave equation on an interval the BV-regularity is the minimal one required for the observability property to holds ([28]).

- **Semilinear wave equations.** Similar issues arise for semilinear wave equations. In the case of domains of the Euclidean space, sharp estimates on the cost of controlling wave equations with potentials, together with fixed point techniques, allow proving the controllability of semilinear wave equations, under suitable growth conditions on the nonlinearity at infinity. This is an open issue in the context of networks, the first difficulty being, as mentioned above, that of dealing with wave equations with potentials.

- **Transmutation.** As we have mentioned above, most of the analysis of control problems on networks has been developed for the wave equation. Then, the obtained result, using the method of transmutation based on Kannai's transform, leads to null control results for the heat equation. This can also be done establishing a continuity result on the property of null controllability between the wave and the heat equation through the damped wave equation (see [74], [90]).

  But, in the case of the heat equation in bounded domains of the Euclidean space, the corresponding observability inequalities are often obtained applying Carleman inequalities directly to the heat model. This is still to be done in the context of the heat equation on graphs. Note however that the evidence that the expected results need to depend on the topological and number theoretical properties of the network makes this method very hard to be applied in this context. In any case the issue of applying Carleman inequalities to obtain directly observability inequalities for PDE in networks is widely open.

- **Multipliers.** The results in this paper were obtained using a fine analysis of the propagation properties of waves along the network. However, in the context of the wave equation in the Euclidean space, relevant results can be obtained much more easily by using the method of multipliers (see [73]). It would be interesting to explore if the observability results for waves on networks (other than the one guaranteeing the observability of the energy of a tree-like network when measurements are done on all but one external node) can be obtained by the method of multipliers.

- **Thermoelasticity.** In the context of PDE in domains of the Euclidean space one can combine the theory of the wave and heat equations to obtain results on the controllability of several relevant systems, including the system of thermoelasticity (see [65]).

- **Hyperbolic-Parabolic systems.** Recently, motivated by problems of fluid-structure interaction, there has been work done for models coupling a wave and a heat equation along an interface. The coupling turns out to be quite weak so that the corresponding system does not even decay uniformly exponentially ([104]). Similar issues could be considered on networks where, in principle, one could choose arbitrarily the location of the heat and wave equations. In the context of the control of those systems in $1-d$ (a wave equation and a heat equation coupled through a point-wise interface) it is well known that the controllability properties depend on the location of the controller. In particular, the system is much more easily controllable when the control is on the external boundary of the wave domain than in the one of the heat domain (see [110]). Using the methods in [110], which combine sidewise energy estimates with known controllability results on the heat equation, and the results on the heat equation that one can derive from the results on the wave equation we have presented here by transmutation, one could prove controllability results on general networks provided: a) All the wave components are located on external segments so that the system under consideration is the heat equation on a graph surrounded by external vibrating controlled strings; b) the resulting heat-like configuration is controllable. But all the other situations are still to be investigated.

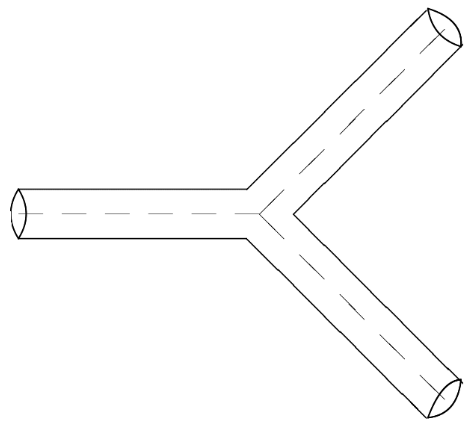
- **Other joint conditions.** All the results presented here refer to the Laplacian on networks defined through the so-called Kirchhoff conditions. But the systems under consideration have a physical meaning and are well-posed for other joint conditions. In particular we could assume that the external and/or internal nodes contain point masses. Very likely similar results will hold in that case but, even in the case of two strings connected by a point-mass it is well-known that the control theoretical properties change dramatically because of the presence of the mass. In particular it is well known that, in those cases, the observability/controllability spaces are asymmetric to both sides of the point-mass (see [49]). Similar asymmetry properties may be expected in the case of networks with point masses on the joints.

- **Switching control.** Recently, a theory of switching controls has been developed for PDE with various actuators or controllers. This is particularly suited for networks endowed with different controllers, located in various nodes (internal or external ones). It would be interesting to analyze systematically the possibilities of controlling networks (in particular for the heat equation in which the time-analiticity of solutions can be guaranteed) by means of switching controls (see [105]). The same can be said in the context of stabilization, in which the various feedback controllers are requested to be activated in a switching manner. At this respect, the work [48] is worth mentioning. There the authors consider a star-like network composed by M strings endowed with M feedback controllers on the exterior nodes, each of which can be deactivated by a time-dependent switching law. They provide conditions on the switching laws guaranteeing that the network can be stabilized asymptotically to rest.

- **Infinite networks of finite length.** It would be interesting to investigate the possible extension of the results of this paper to newtorks involving an infinite number of strings, but of finite total length.

- **Optimal placement of controllers.** We have discussed here the problem of observation, control or stabilization from a given external vertex. But it would be of interest to discuss the problem of the choice of the optimal placement of the controller. This is a widely open subject. We refer to [7] for some of the few existing results in context of the string equation on a segment.

- **Graph-like thin manifolds.** In these notes we have considered the control and stabilization of the wave equation on $1-d$ networks. We have also discussed similar issues for other models as the heat or Schrödinger equations. It is very natural to analyze the same issues in thin $2-d$ domains obtained by simply adding a thickness of size ε to the network on the perpendicular direction to each string.



A graph-like thin manifold or a $3-d$ branching-domain

The control of PDE's in thin cylinders is reasonably understood. In the case of the wave equation, due to the existence of trapped rays in the perpendicular directions, the wave and plate process can not be controlled from the lateral boundary and the filtering of the high frequency trapped rays is needed to get uniform controllability results ([45]). To the contrary, in the case of the heat equation, the intrinsic strongly dissipative effect damps out the high frequency components that the added dimension generates, and the limit of null controls in thin domains is a null control in the limit cross section (see [108] and [39]).

It would be natural and interesting to analyze similar questions in the context of "thick networks" when the thickness tends to zero. The subject will however be more complex than in domains of the Euclidean space since, as the results concerning $1-d$ networks show (see [37]), the results one has to expect when passing to the limit, necessarily, will depend on the number theoretical properties of the lengths of the edges of the network.

We refer to [43] for recent results on the behavior of the spectrum of the Laplacian under this singular perturbation and to [46] for a recent survey on the subject.

- **Numerics.** In recent years the problem of numerical approximation of control problems, especially for waves, has been the object of intensive research (see [112]). But very little is known in the context of networks. The abstract results in [42] can be applied in this context and we can obtain controllability results for time-discrete wave equations on networks, provided the high frequency components are appropriately filtered out. But the analysis of space-discretizations is a widely open subject.

- **Strichartz inequalities.** There are other interesting features of PDE on domains of the Euclidean space that are badly understood in the context of networks. That is for instance the case of the dispersive or Strichartz estimates for the Schrödinger equation. This issue is still to be investigated in a systematic manner in the context of networks. We refer to [52] for the first results in this direction in the case of some particular tree like infinite networks.

  Note also that these dispersive estimates play a key role when analyzing the solvability of the corresponding nonlinear problems and in their numerical approximation ([52]).

- **Inverse problems.** Inverse problems for waves on networks are intimately related to the control problems we have considered in this paper. The issue consists roughly on determining the topological and geometric properties of the network through measurements done on the exterior vertices. We refer to the recent paper [12] for the analysis of tree-like networks through the so-called boundary-control-approach developed in [22] and to the references in [57]. These kind of problems are widely open in the case of networks containing circuits.

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