

A result concerning the global approximate controllability of the Navier Stokes system in dimension

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Abstract

In this paper we deal with the three-dimensional Navier-Stokes system, posed in a cube. In this context, we prove a result concerning its global approximate controllability by means of boundary controls which act in some part of the boundary.

Résumé

Dans ce papier nous travaillons avec le système de Navier-Stokes sur un cube. Dans ce cadre, on démontre un résultat global concernant la contrôlabilité approchée en utilisant des contrôles frontières sur une partie du bord.

Keywords: Navier-Stokes system, global approximate controllability

1. Introduction

Let $T > 0$. We consider the three-dimensional Navier-Stokes system posed in the unit cube:

$$\begin{cases} (\partial_t u - \Delta u + (u, \nabla)u + \nabla p)(t, x) = f(t, x) & (t, x) \in Q = (0, T) \times \Omega, \\ \nabla \cdot u = 0 & (t, x) \in (0, T) \times \Omega, \\ u(t, 0, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times (0, 1)^2, \\ u(0, x) = u_0(x) & x \in \Omega. \end{cases} \quad (1)$$

Here Ω is the open set given by:

$$\Omega = \{x = (x_1, x_2, x_3) : x_1, x_2, x_3 \in (0, 1)\},$$

whose boundary is denoted by $\partial\Omega$, $f \in L^2(0, T; L^2(\Omega))$ is a given source term and $u_0 \in H(\Omega)$ where for any open subset Ω in \mathbb{R}^3 we define

$$H(\Omega) = \{w \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) : \nabla \cdot w = 0 \text{ in } \Omega, w \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad (2)$$

where ν is the outward unit normal vector on $\partial\Omega$.

Besides the space $H(\Omega)$ we introduce the space

$$V_0(\Omega) = \{w \in H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) : \nabla \cdot w = 0 \text{ in } \Omega\}, \quad (3)$$

and for a cylindrical domain $\Omega = (0, 1) \times D$ where D is an open subset of \mathbb{R}^2 we define

$$V(\Omega) = \{w \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) : \nabla \cdot w = 0 \text{ in } \Omega, w = 0 \text{ on } \{0, 1\} \times D\}. \quad (4)$$

Here $H^s(\Omega)$ is the standard Sobolev space of distributions integrable in $L^2(\Omega)$ together with their derivatives up to the order s , $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in norm of the space $H^s(\Omega)$. Below, in order to simplify notations we also denote by $H^1(\Omega)$ the vector space $H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ and the analogous for $H^{-1}(\Omega)$.

Our main goal in this paper is to prove a result related to global approximate null controllability for system (1). Precisely, we shall look for a boundary control (which acts on $\partial\Omega \setminus \{0\} \times (0, 1)^2$) such that, for a sequence f_ϵ of approximations of the right hand side f , at least one corresponding solution u_ϵ is close to zero at time $t = T$ in some norm.

It is a well-known fact that we cannot expect exact controllability for the Navier-Stokes equations with an arbitrary target function, in particular because of the dissipative and non reversible properties of the system. On the other hand, we do not know the answer for the approximate controllability of this system as long as the control is acting in a interior domain or on part of the boundary.

An important point in the study of the controllability of Navier-Stokes systems has been the so-called *return method*, introduced by J.-M. Coron in [2] to study the stabilization of some control systems and then used in [3] to prove global exact controllability for the Euler equations in dimension 2 (see also [12]) and then in [4] to prove global approximate controllability result for the Navier-Stokes system with Navier-slip boundary conditions (or when the control is acting on the whole boundary; this result was generalized later in [5] for the case of the Navier-Stokes system on a manifold without boundary). For the analysis of the similar three-dimensional situation for the Euler equations, see [10]. The global approximate controllability and the global controllability to trajectories for the Boussinesq system in the torus are proved in [7], the global exact controllability of the Camassa-Holm equation in the circle is established in [11], while the global null controllability for the 2-D Burgers equation is proved in [15].

In the paper [8], the *local exact controllability to the trajectories* for the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + (u, \nabla u) + \nabla p = v1_\omega & \text{in } Q, \\ \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0(\cdot) & \text{in } \Omega \end{cases} \quad (5)$$

is established. More precisely, it is proven that we can reach (in finite time T) any point on any trajectory of the same operator. That is to say, for (\bar{u}, \bar{p}) solution of the uncontrolled Navier-Stokes system

$$\begin{cases} \bar{u}_t - \Delta \bar{u} + (\bar{u}, \nabla \bar{u}) + \nabla \bar{p} = 0 & \text{in } Q, \\ \nabla \cdot \bar{u} = 0 & \text{in } Q, \\ \bar{u} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \bar{u}(0, \cdot) = \bar{u}_0 & \text{in } \Omega, \end{cases} \quad (6)$$

one have to find a control v such that at least one solution of (5) satisfies

$$u(T, \cdot) = \bar{u}(T, \cdot) \text{ in } \Omega.$$

The precise regularity assumptions that one has to impose to \bar{u} can be found in [8]. For a previous result in the same line, see [14].

At present, we do not know any global result concerning exact controllability for (5). In this work, we give a result related to *global approximate controllability* for the Navier-Stokes equations. We separate the case of weak solutions and the case of strong solutions. We have

Theorem 1. *Let $u_0 \in H(\Omega)$ and $f \in L^2((0, T) \times \Omega)$. Then, there exists a sequence of functions $\{f_\epsilon\}_{\epsilon>0}$ with $f_\epsilon \in L^2(\Omega \times (0, T))$ such that*

$$f_\epsilon \rightarrow f \quad \text{in } L^{p_0}(0, T; H^{-1}(\Omega)) \quad p_0 \in (1, 4/3)$$

and there exists at least one solution $u_\epsilon \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ to the controllability problem

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon + (u_\epsilon, \nabla)u_\epsilon + \nabla p_\epsilon = f_\epsilon & \text{in } Q, \\ \nabla \cdot u_\epsilon = 0 & \text{in } Q, \\ u_\epsilon(t, 0, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times (0, 1)^2, \\ u_\epsilon(0, x) = u_0(x), \quad u_\epsilon(T, x) = 0 & x \in \Omega. \end{cases}$$

Theorem 2. *Let $u_0 \in V_0(\Omega)$ and $f \in L^2((0, T) \times \Omega)$. Then, there exists a sequence of functions $\{f_\epsilon\}_{\epsilon>0}$ with $f_\epsilon \in L^2((0, T) \times \Omega)$ such that*

$$f_\epsilon \rightarrow f \quad \text{in } L^{p_0}(0, T; H^{-1}(\Omega)) \quad p_0 \in (1, 4/3)$$

and there exists a strong solution $u_\epsilon \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ to the controllability problem

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon + (u_\epsilon, \nabla)u_\epsilon + \nabla p_\epsilon = f_\epsilon & \text{in } Q, \\ \nabla \cdot u_\epsilon = 0 & \text{in } Q, \\ u_\epsilon(t, 0, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times (0, 1)^2, \\ u_\epsilon(0, x) = u_0(x), \quad u_\epsilon(T, x) = 0 & x \in \Omega. \end{cases}$$

Notice that this is not a classical result of global approximate controllability as we need to introduce a sequence of approximate right hand sides. The way we find the function u_ϵ satisfying Theorem 1 and Theorem 2 is constructive. The control appears on the part of the boundary $\partial\Omega \setminus \{0\} \times (0, 1)^2$ and in the approximate right hand side f_ϵ . Remark that in the case

of Theorem 2, given the constructed control the corresponding solution is unique. Let us explain the general ideas of our construction for the case of weak solutions. We divide our time interval $(0, T)$ in four subintervals:

- In the first one $(0, T_1)$ no control is needed, so we let the Navier-Stokes system evolve from our initial condition u_0 and u_ϵ equals the solution of (1) with zero Dirichlet boundary conditions.

- In the second time interval we explicitly give our solution u_ϵ . In this stage, we ‘disturb’ our solution a little bit from the state $u(T_1, \cdot)$, driving it to some compactly supported state u_{1, ϵ_1} .

- In the next step, we construct our solution u_ϵ in a much more intrinsic way. Indeed, we search for u_ϵ as the sum of three functions: a solution of a transport equation (which will be denoted y), a very particular solution of the Navier-Stokes system constructed from a solution of a controlled heat equation (which will be denoted U , multiplied by a large parameter) and the solution of a (linear) Stokes system (which will be denoted W).

- Finally, on the last time interval, we will reduce the question to drive u_ϵ to zero at time $t = T$ into a null controllability problem for a linear heat equation.

- For the case of strong solutions, we will show that there exists $\hat{T} > 0$ with $K\hat{T} = T$ such that the same construction can be performed on each (small) interval $((k-1)\hat{T}, k\hat{T})$, $k = 1, \dots, K$.

The rest of this paper is organized as follows. In the next section, we will construct the functions U and y in the first paragraph while the function W will be constructed in a second paragraph, together with some estimates. Finally, in the last section, we will provide the proofs of Theorem 1 and Theorem 2.

To obtain a global null controllability result would require to be able to take $f_\epsilon = f$. We cannot obtain this here as our method is based on the transport of the initial data by N^2U where U is a particular solution of Navier-Stokes equations and N is a large parameter, and this transport has no effect in the neighborhood of $x_1 = 0$ because of the boundary conditions on U .

Generalizations of our result to coupled systems like Boussinesq system are not considered here as we do not know particular solutions of these systems which would be analogous to our function U .

Our result is given in the 3 dimensional context but apart from uniqueness, the 2 dimensional case does not allow better results.

2. Construction of some intermediate functions

In this first section, we will construct some specific solution U of the Navier-Stokes system and we will explain how we look for the solution u_ϵ . Let $z = z(t, x_1)$ be a solution to the following problem

$$\begin{cases} \partial_t z - \partial_{x_1 x_1}^2 z = c(t) & (t, x_1) \in (0, T) \times (0, 1), \\ z(t, 0) = 0, \quad z(t, 1) = w(t) & t \in (0, T), \\ z(0, x_1) = 0 & x_1 \in (0, 1). \end{cases} \quad (7)$$

Here, $c \in C^2([0, T])$ is a positive function and w is a nonnegative function satisfying

$$w(t) \in C^\infty[0, T], \quad w(0) = 0, \quad w'(0) = c(0), \quad w''(0) = c'(0). \quad (8)$$

We will specify the choice of the function $z(t, x_1)$ (and therefore of c and w) later on but observe that from the above conditions and thanks to the maximum principle, there exists a constant $C > 0$ such that

$$|z(t, x_1)| \leq Ct \quad (t, x_1) \in (0, T) \times (0, 1). \quad (9)$$

Thanks to the compatibility condition (8) the function $z(t, x_1)$ has the following regularity (see e.g.[16]):

$$z \in L^2(0, T, H^2(0, 1)), \quad \partial_t z \in L^2((0, T) \times (0, 1)).$$

On the other hand, we have

$$\begin{cases} \partial_t(\partial_t z) - \partial_{x_1 x_1}^2(\partial_t z) = c'(t) & (t, x_1) \in (0, T) \times (0, 1), \\ \partial_t z(t, 0) = 0, \quad \partial_t z(t, 1) = w'(t) & t \in (0, T), \\ \partial_t z(0, x_1) = c(0) & x_1 \in (0, 1). \end{cases}$$

This gives

$$\partial_t z \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)). \quad (10)$$

Consequently,

$$\partial_{x_1 x_1}^2 z \text{ in } C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)) \quad (11)$$

and, in particular,

$$\partial_{x_1} z \in L^\infty((0, T) \times (0, 1)). \quad (12)$$

Next we note that the function $\tilde{z}(t, x_1) = x_1^2 z(t, x_1)$ solves the problem

$$\begin{cases} \partial_t \tilde{z} - \partial_{x_1}^2 \tilde{z} = x_1^2 c(t) - 4x_1 \partial_{x_1} z - 2z & (t, x_1) \in (0, T) \times (0, 1), \\ \tilde{z}(t, 0) = 0, \quad \tilde{z}(t, 1) = w(t) & t \in (0, T), \\ \tilde{z}(0, x_1) = 0 & x_1 \in (0, 1). \end{cases}$$

Hence by (10), (11) (see e.g. [16]) we have

$$\tilde{z} \in \{y \in L^2(0, T; H^4(0, 1)), \partial_t y \in L^2(0, T; H^2(0, 1))\}.$$

Using this fact, we obtain more regularity on the function $x_1^3 z$, more precisely

$$x_1^3 z \in \{y \in L^2(0, T; H^6(0, 1)), \partial_t y \in L^2(0, T; H^3(0, 1))\}.$$

Hence

$$z \in C^2([0, T] \times [\delta, 1]) \quad \forall \delta > 0. \quad (13)$$

Let us define the open set \mathcal{G} containing Ω by

$$\mathcal{G} = \{x = (x_1, x_2, x_3) \in (0, 1) \times \mathbf{R}^2\}. \quad (14)$$

Then, using the function z we construct the functions $U(t, x) = (0, z(t, x_1), z(t, x_1))$ and $q = (x_2 + x_3)c(t)$ for $(t, x) \in (0, T) \times \mathcal{G}$. These functions solve the boundary value problem

$$\begin{cases} \partial_t U - \Delta U + (U, \nabla)U = \nabla q & \text{in } (0, T) \times \mathcal{G}, \\ \nabla \cdot U = 0 & \text{in } (0, T) \times \mathcal{G}, \\ U(t, 0, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ U(0, x) = 0 & x \in \mathcal{G}. \end{cases} \quad (15)$$

Note that since $(U, \nabla)U = 0$, for any $N \in \mathbf{R}$ the couple (NU, Nq) also fulfills system (15).

Now, after having let the system evolve freely in some interval of time, and an additional regularization step which will be described later on, we look for a function u solution of (1) in the form

$$u(t, x) = \tilde{y}(t, x) + N^2 U(t, x) \quad (t, x) \in (T_1, T) \times \Omega.$$

Consequently, we search a function \tilde{y} which fulfills the following system:

$$\begin{cases} \partial_t \tilde{y} - \Delta \tilde{y} + F_N(\tilde{y}, U) + \nabla \tilde{p} = f & (t, x) \in (T_1, T) \times \mathcal{G}, \\ \nabla \cdot \tilde{y} = 0 & (t, x) \in (T_1, T) \times \mathcal{G}, \\ \tilde{y}(t, 0, x_2, x_3) = 0 & (t, x_2, x_3) \in (T_1, T) \times \mathbf{R}^2, \\ \tilde{y}(T_1, x) = v_0(x) & x \in \mathcal{G}, \end{cases} \quad (16)$$

where

$$F_N(\tilde{y}, U) = N^2(U, \nabla)\tilde{y} + N^2(\tilde{y}, \nabla)U + (\tilde{y}, \nabla)\tilde{y}.$$

Note that

$$(U, \nabla)\tilde{y} + (\tilde{y}, \nabla)U = (z(\partial_{x_2}\tilde{y}_1 + \partial_{x_3}\tilde{y}_1), z(\partial_{x_2}\tilde{y}_2 + \partial_{x_3}\tilde{y}_2) + \tilde{y}_1\partial_{x_1}z, z(\partial_{x_2}\tilde{y}_3 + \partial_{x_3}\tilde{y}_3) + \tilde{y}_1\partial_{x_1}z).$$

As we explained in the introduction, we will look for \tilde{y} solution of (16) in the form

$$\tilde{y}(t, x) = y(t, x) - W(t, x) \quad (t, x) \in (T_1, T) \times \mathcal{G},$$

where y is a particular solution of the transport equation associated to (16) and W is solution of some linear Stokes system. In the next two paragraphs, we construct y and W with explicit estimates of their norms in terms of N as N is large.

2.1. Null controllability for a transport equation

In this paragraph for an arbitrary initial condition $v_0 \in C_0^3(\Omega) \cap V_0(\Omega)$ extended by zero on \mathcal{G} we solve the following null controllability problem for the transport equation:

$$\begin{cases} \partial_t y + N^2(U, \nabla)y + N^2(y, \nabla)U = 0 & (t, x) \in Q_{2/N}, \\ y(t, 0, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, 2/N) \times \mathbf{R}^2, \quad y(t, x) \rightarrow 0 \text{ as } |x| \rightarrow +\infty, \\ y(0, x) = v_0(x) & x \in \mathcal{G}, \quad y(t, x) = 0 \quad t \in [1/N, 2/N], \quad x \in \Omega. \end{cases} \quad (17)$$

Here, we have denoted

$$Q_{2/N} = (0, 2/N) \times \mathcal{G} = (0, 2/N) \times (0, 1) \times \mathbf{R}^2. \quad (18)$$

Observe that U depends on z (solution of (7)) and therefore on c and w . We will also need particular estimates for y , keeping an explicit dependence with respect to N when N is large enough. The precise result is provided in the following lemma.

Lemma 3. *Let $v_0 \in C_0^3(\Omega) \cap V_0(\Omega)$, $\text{supp } v_0 \subset K$ where K is an open set such that $\bar{K} \subset \Omega$. Then, there exist $N_0(K)$ such that for $N \geq N_0(K)$ there exists a solution y to problem (17) and a positive constant $C(K)$ independent of N , such that*

$$\|y\|_{C^0([0, \frac{2}{N}]; C^2(\bar{\mathcal{G}}))} + \frac{1}{N} \|\partial_t y\|_{C^0([0, \frac{2}{N}]; C^2(\bar{\mathcal{G}}))} \leq C(K) \|v_0\|_{C^3(\bar{\Omega})} \quad (19)$$

and

$$y(t, x) = 0 \quad t \in [1/N, 2/N], \quad x \in \Omega. \quad (20)$$

Proof. Let $\gamma > 0$ be small enough such that

$$\text{supp } v_0 \subset [\gamma, 1 - \gamma] \times [0, 1] \times [0, 1]. \quad (21)$$

Let us consider the function

$$Z(t, x_1) = \int_0^t z(s, x_1) ds \quad (t, x_1) \in (0, T) \times (0, 1).$$

Recall that the function z fulfills system (7). We can write

$$z(t, x_1) = tc(0) + \theta(t, x_1) \quad (t, x_1) \in (0, T) \times (\gamma/2, 1), \quad (22)$$

where, from (8), (13), $\theta \in C^2([0, T] \times [\gamma/2, 1])$ and satisfies

$$\theta(0, x_1) = 0 \text{ and } \partial_t \theta(0, x_1) = 0.$$

Thanks to Taylor's formula we then have

$$\|\theta(\tau, \cdot)\|_{C^1([\gamma/2, 1])} = o(\tau) \quad \text{as } \tau \rightarrow 0^+.$$

Let us denote $v_0 = (v_{0,1}, v_{0,2}, v_{0,3})$. Then, from the fact that $\partial_t Z = z$, we readily obtain that the function

$$y_1(t, x) = v_{0,1}(x_1, -N^2 Z(t, x_1) + x_2, -N^2 Z(t, x_1) + x_3) \quad (t, x) \in (0, T) \times \mathcal{G} \quad (23)$$

is a solution to the equation

$$\partial_t y_1 + N^2 z \partial_{x_2} y_1 + N^2 z \partial_{x_3} y_1 = 0 \quad (t, x) \in (0, T) \times \mathcal{G}.$$

Of course by (21) and (23), $y_1(t, 0, x_2, x_3) = v_{0,1}(0, x_2, x_3) = 0$ for every $(t, x_2, x_3) \in (0, T) \times \mathbf{R}^2$ and $y_1(0, x) = v_{0,1}(x)$.

Concerning the estimates, we clearly have that

$$\|y_1\|_{C^0(\overline{Q}_{2/N})} + \|\partial_{x_2}y_1\|_{C^0(\overline{Q}_{2/N})} + \|\partial_{x_3}y_1\|_{C^0(\overline{Q}_{2/N})} \leq C\|v_0\|_{C^1(\overline{\Omega})}.$$

Moreover, (22) implies that

$$N^2|\partial_{x_1}Z(t, x_1)| + N^2|Z(t, x_1)| + N|\partial_t Z(t, x_1)| \leq C \quad \forall (t, x_1) \in (0, 2/N) \times (\frac{\gamma}{2}, 1)$$

with $C > 0$ independent of N . Then, using (21), we obtain

$$\|\partial_{x_1}y_1\|_{C^0(\overline{Q}_{2/N})} + \frac{1}{N}\|\partial_t y_1\|_{C^0(\overline{Q}_{2/N})} \leq C\|v_0\|_{C^1(\overline{\Omega})}.$$

If we differentiate the function Z twice and three times, by (13), for any positive $\delta' \in (\gamma/2, 1)$ we have

$$N^2|\partial_{x_1x_1}^2 Z(t, x_1)| + N|\partial_t \partial_{x_1x_1}^2 Z(t, x_1)| \leq C(\delta') \quad \forall (t, x_1) \in (0, 2/N) \times (\delta', 1)$$

with $C > 0$ independent of N and so (19) is established for the first component of the function y .

Next we consider the equations

$$\partial_t y_2 + N^2 z(\partial_{x_2} y_2 + \partial_{x_3} y_2) = -N^2 \partial_{x_1} z y_1 \quad (t, x) \in (0, T) \times \mathcal{G} \quad (24)$$

and

$$\partial_t y_3 + N^2 z(\partial_{x_2} y_3 + \partial_{x_3} y_3) = -N^2 \partial_{x_1} z y_1 \quad (t, x) \in (0, T) \times \mathcal{G}, \quad (25)$$

where y_1 is given by (23). These equations, together with the boundary and initial conditions in (17), determine our functions y_2 and y_3 . Let us consider the case of y_2 , the other one being similar.

If we write

$$x_2(s) = x_2 - N^2 Z(s, x_1), \quad x_3(s) = x_3 - N^2 Z(s, x_1),$$

we can write

$$y_2(t, x_1, x_2, x_3) = v_{0,2}(x_1, x_2(t), x_3(t)) - N^2 \int_0^t \partial_{x_1} z(s, x_1) y_1(s, x_1, x_2(s), x_3(s)) ds.$$

First of all it is clear that

$$y_2(t, x_1, x_2, x_3) = 0, \quad t \in [0, 2/N], \quad x_1 \in [0, \frac{\gamma}{2}].$$

Now as $t \in [0, 2/N]$, using the previous estimates on y_1 and z we immediately obtain

$$\|y_2\|_{C^0(Q_{2/N})} \leq C\|v_0\|_{C^0(\bar{\Omega})}.$$

Taking the derivative of (24) with respect to x_2 and x_3 and using again the estimates on y_1 we obtain

$$\|\partial_{x_2}y_2\|_{C^0(Q_{2/N})} \leq C\|v_0\|_{C^1(\bar{\Omega})}, \quad \|\partial_{x_3}y_2\|_{C^0(Q_{2/N})} \leq C\|v_0\|_{C^1(\bar{\Omega})},$$

and then

$$\frac{1}{N}\|\partial_t y_2\|_{C^0(Q_{2/N})} \leq C\|v_0\|_{C^1(\bar{\Omega})}.$$

Taking now the derivative of (24) with respect to x_1 and using the above estimate and the estimates on $\partial_{x_1x_1}z$ we obtain

$$\|\partial_{x_1}y_2\|_{C^0(Q_{2/N})} \leq C\|v_0\|_{C^1(\bar{\Omega})}.$$

We can repeat this argument for the second derivatives to obtain (19) for y_2 and then for y_3 .

Let us now make a choice of N_0 in order to have

$$y_1(t, x) = 0 \quad t \in [1/N, 2/N], \quad x \in \Omega. \quad (26)$$

Recall that $c(t)$ in (8) is a positive function. Without loss of generality we may assume that

$$c(0) > 8.$$

Indeed, since $c(0)$ is positive in system (17) one can make the change of $U \rightarrow \hat{N}^2U$ and $N \rightarrow N/\hat{N}$. For \hat{N} sufficiently large the above inequality holds true.

Thanks to (22) we might always assume that for all N greater or equal to the some sufficiently large N_0 , we have

$$-N^2Z(t, x_1) < -4, \quad t \in [1/N, 2/N], \quad x_1 \in (\gamma/2, 1). \quad (27)$$

Consequently, for any $t \in [1/N, 2/N]$ and $(x_2, x_3) \in (0, 1)^2$, we have that

$$-N^2Z(t, x_1)+x_2 < -3 \quad \text{and} \quad -N^2Z(t, x_1)+x_3 < -3 \quad \text{in} \quad (\frac{\gamma}{2}, 1) \times (0, 1) \times (0, 1)$$

and so (26) readily follows for all x in the set $(\frac{\gamma}{2}, 1) \times (0, 1) \times (0, 1)$. For all x in the set $(0, \frac{\gamma}{2}) \times (0, 1) \times (0, 1)$ the equality (26) follows from (21) and (23).

Let us finally check that

$$y_2(t, x) = y_3(t, x) = 0 \quad t \in [1/N, 2/N], \quad x \in \Omega.$$

Let us prove it, for instance, for y_2 . We may rewrite the equation satisfied by y_2 into the form

$$\partial_t y_2 + N^2 z(\partial_{x_2} y_2 + \partial_{x_3} y_2) + N^2 \partial_{x_1} z v_{0,1}(x_1, -N^2 Z(t, x_1) + x_2, -N^2 Z(t, x_1) + x_3) = 0.$$

Take $(x_2, x_3) \in (0, 1)^2$ and $t \in [1/N, 2/N]$. Consider the curve

$$(\tilde{x}_2(s), \tilde{x}_3(s)) = (N^2 Z(s, x_1) + \alpha, N^2 Z(s, x_1) + \beta),$$

with α and β constants such that $(\tilde{x}_2(t), \tilde{x}_3(t)) = (x_2, x_3)$. Note that for the function $y_2(s, x_1, \tilde{x}_2(s), \tilde{x}_3(s))$, we have

$$\frac{dy_2}{ds} = \partial_s y_2 + \partial_{x_2} y_2 \frac{d\tilde{x}_2}{ds} + \partial_{x_3} y_2 \frac{d\tilde{x}_3}{ds} = \partial_s y_2 + N^2 z(\partial_{x_2} y_2 + \partial_{x_3} y_2) = -N^2 \partial_{x_1} z(x_1) v_{0,1}(x_1, \alpha, \beta). \quad (28)$$

If $x_1 \in [0, \frac{\gamma}{2}]$ by (21) then $\frac{dy_2}{ds} = 0$. From the initial value of y , we have that $y_2|_{[1/N, 2/N] \times [0, \frac{\gamma}{2}] \times (0, 1)^2} \equiv 0$.

On the other hand, if $(x_1, x_2, x_3) \in (\frac{\gamma}{2}, 1) \times (0, 1)^2$, since by (27)

$$N^2 Z(t, x_1) > 4,$$

we have

$$\alpha < -3 \text{ and } \beta < -3.$$

Then from (28) and (21) the function y_2 for a fixed x_1 is a constant on the curve $(\tilde{x}_2(s), \tilde{x}_3(s))$. But $(\tilde{x}_2(s), \tilde{x}_3(s))|_{s=0} = (\alpha, \beta)$. Since $y_2(0, x_1, \tilde{x}_2(0), \tilde{x}_3(0)) = y_2(0, x_1, \alpha, \beta) = v_{0,2}(x_1, \alpha, \beta) = 0$ we have

$$y_2(t, x_1, x_2, x_3) = 0 \quad x = (x_1, x_2, x_3) \in (\frac{\gamma}{2}, 1) \times (0, 1)^2.$$

This finishes the proof of the lemma 3. ■

2.2. Construction of W

Consider the following Stokes problem:

$$\left\{ \begin{array}{ll} \partial_t W - \Delta W + \nabla r = 0 & (t, x) \in Q_{2/N}, \\ W(t, 0, x_2, x_3) = W(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, 2/N) \times \mathbf{R}^2, \\ W(t, x_1, x_2, x_3) \rightarrow 0 & |x_2| + |x_3| \rightarrow +\infty, \\ \nabla \cdot W = \nabla \cdot y & (t, x) \in Q_{2/N}, \\ W(0, x) = 0 & x \in \mathcal{G}, \end{array} \right. \quad (29)$$

where y is the function provided by lemma 3. We will be looking for a solution to the problem (29) such that $W - W_* \in L^2(0, 2/N; V_0(\mathcal{G}))$ and $\partial_t W \in L^2(0, 2/N; V_0'(\mathcal{G}))$ where $W_* \in L^2(0, 2/N; H^2(\mathcal{G}))$, $\partial_t W_* \in L^2(Q_{2/N})$. We note that since $\nabla \cdot y(0, \cdot) = \nabla \cdot v_0 \equiv 0$ and, because of the boundary values of y , $\int_{\mathcal{G}} \nabla \cdot y(t, x) dx = 0$ for all positive t the compatibility conditions hold true. Therefore such a solution exists. Indeed, thanks to the compatibility conditions we can choose a function $W_* \in L^2(0, 2/N; H^2(\mathcal{G}))$, $\partial_t W_* \in L^2(Q_{2/N})$ such that $\nabla \cdot W_* = \nabla \cdot y$ on $Q_{2/N}$ and $W_*(t, 0, x_2, x_3) = W_*(t, 1, x_2, x_3) = 0$ for any (t, x_2, x_3) in $(0, 2/N) \times \mathbf{R}^2$. Observe that $W_*(0, \cdot)$ belongs to the space $V(\mathcal{G})$. Then one can construct a solution to (29) in the form $W = W_* + W_{**}$ and $r = r_{**}$ where the pair $(W_{**}, r_{**}) \in L^2(0, 2/N; H^2(\mathcal{G})) \times L^2(Q_{2/N})$ is the solution to the boundary value problem

$$\left\{ \begin{array}{ll} \partial_t W_{**} - \Delta W_{**} + \nabla r_{**} = -\partial_t W_* + \Delta W_* & (t, x) \in Q_{2/N}, \\ W_{**}(t, 0, x_2, x_3) = W_{**}(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, 2/N) \times \mathbf{R}^2, \\ W_{**}(t, x_1, x_2, x_3) \rightarrow 0 & |x_2| + |x_3| \rightarrow +\infty, \\ \nabla \cdot W_{**} = 0 & (t, x) \in Q_{2/N}, \\ W_{**}(0, x) = -W_*(0, x) & x \in \mathcal{G}. \end{array} \right. \quad (30)$$

The uniqueness of a velocity field W in the problem (29) follows from standard energy estimates. The pressure is unique modulo a some constant valued function $h(t)$. On the other hand, since we are looking for the pressure in $L^2(Q_{2/N})$, as $Q_{2/N}$ is unbounded, we can take this constant function to be zero.

We have

Proposition 1. *Let W be the solution to problem (29). Then, the following a priori estimates hold true:*

- For any $p \in (1, \infty)$

$$\|W\|_{L^p(Q_{2/N})} \leq C(p)/N^{1/p} \|v_0\|_{C^3(\bar{\Omega})}. \quad (31)$$

- There exists a positive constant $C > 0$ independent of N such that

$$\|W\|_{C^0([0, 2/N]; L^2(\mathcal{G}))} + \|\partial_{x_2} W\|_{C^0([0, 2/N]; L^2(\mathcal{G}))} + \|\partial_{x_3} W\|_{C^0([0, 2/N]; L^2(\mathcal{G}))} \leq \frac{C}{N^{1/4}}. \quad (32)$$

Before providing the proof of Proposition 1, we state a technical result concerning the regularity of the pressure term when the velocity vector field is said to be a *weak solution*. More precisely, let us consider $w \in L^2(0, T; V(\mathcal{G})) \cap C^0([0, T]; H(\mathcal{G}))$ (together with some pressure h) the weak solution of the Stokes system:

$$\begin{cases} w_t - \Delta w + \nabla h = f, & \nabla \cdot w = 0 & (t, x) \in (0, T) \times \mathcal{G}, \\ w(t, 0, x_2, x_3) = w(t, 1, x_2, x_3) = 0 & & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ w(0, x) = w_0 & & x \in \mathcal{G}, \end{cases} \quad (33)$$

where f is a given source term. Here, $T > 0$ is a positive number and we recall that \mathcal{G} was defined in the introduction, just before (15).

The following lemma provides the regularity result for the pressure when a fluid flow described by the Stokes system (see also [6] for the case of a bounded domain).

Lemma 4. *Let $w_0 \in H(\mathcal{G})$, $f \in L^2(0, T; H^{-1}(\mathcal{G}))$. Then, the pressure term h in (33) satisfies*

$$h \in H^{-1/4}(0, T; L^2(\mathcal{G}))$$

and there exists a positive constant $C = C(\mathcal{G})$ independent of T such that

$$\|h\|_{H^{-1/4}(0, T; L^2(\mathcal{G}))} \leq C(\|w_0\|_{L^2(\mathcal{G})} + \|f\|_{L^2(0, T; H^{-1}(\mathcal{G}))}). \quad (34)$$

We give the proof of this lemma in appendix.

Proof of proposition 1: First, recall that the definition of the set $Q_{2/N}$ was given in (18). Let us then consider the backward Stokes system

$$\begin{cases} -\partial_t \mathcal{W} - \Delta \mathcal{W} + \nabla \tilde{q} = g & (t, x) \in Q_{2/N}, \\ \mathcal{W}(t, 0, x_2, x_3) = \mathcal{W}(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, 2/N) \times \mathbf{R}^2, \\ \nabla \cdot \mathcal{W} = 0 & (t, x) \in Q_{2/N}, \\ \mathcal{W}(2/N, x) = \mathcal{W}_0(x) & x \in \mathcal{G}. \end{cases} \quad (35)$$

Let $p \in (1, \infty)$. For any $g \in L^p(Q_{2/N})$ and any $\mathcal{W}_0 \in W^{1,p}(\mathcal{G}) \cap H(\mathcal{G})$ there exists a unique solution $(\mathcal{W}, \tilde{q}) \in W_p^{1,2}(Q_{2/N}) \times L^p(0, \frac{2}{N}; W^{1,p}(\mathcal{G}))$ of system (35) which satisfies the estimate

$$\|\mathcal{W}\|_{W_p^{1,2}(Q_{2/N})} + \|\nabla \tilde{q}\|_{L^p(Q_{2/N})} \leq C(\|g\|_{L^p(Q_{2/N})} + \|\mathcal{W}_0\|_{W^{1,p}(\mathcal{G})}), \quad (36)$$

for a positive constant C independent of N . Here, we have denoted

$$W_p^{1,2}(Q_{2/N}) = \{w \in L^p(0, 2/N; W^{2,p}(\mathcal{G})) : \partial_t w \in L^p(Q_{2/N})\}.$$

For a proof of (36), see for instance [9] in the case of a bounded domain and [1] for the case of an infinite layer.

We set in (35) $\mathcal{W}_0 \equiv 0$ and $g = W|W|^{p-2}$. Then, inequality (36) for $p/(p-1)$ instead of p tells that

$$\|\mathcal{W}\|_{W_p^{1,2}(Q_{2/N})} + \|\nabla \tilde{q}\|_{L^{p/(p-1)}(Q_{2/N})} \leq C \|W\|_{L^p(Q_{2/N})}^{p-1}. \quad (37)$$

Multiplying (35) (with $g = W|W|^{p-2}$ and $\mathcal{W}_0 \equiv 0$) by W , using that W solves (29) and integrating by parts, we obtain

$$\|W\|_{L^p(Q_{2/N})}^p = -(\nabla \cdot y, \tilde{q} - \bar{q}(t))_{L^2(Q_{2/N})} = -(\nabla \cdot y, \tilde{q} - \bar{q}(t))_{L^2((0,2/N) \times [0,1] \times [-C_*, C_*]^2)} \quad (38)$$

where C_* is some positive constant independent of N and

$$\bar{q}(t) = \frac{1}{4C_*^2} \int_{[0,1] \times [-C_*, C_*]^2} \tilde{q}(t, x) dx.$$

Using (37) we obtain from (38):

$$\|W\|_{L^p(Q_{2/N})}^p \leq C \|\nabla \cdot y\|_{L^p(Q_{2/N})} \|\tilde{q} - \bar{q}(t)\|_{L^{p/p-1}((0,2/N) \times [0,1] \times [-C_*, C_*]^2)} \leq C \|\nabla \cdot y\|_{L^p(Q_{2/N})} \|W\|_{L^p(Q_{2/N})}^{p-1}$$

Hence

$$\|W\|_{L^p(Q_{2/N})} \leq (C/N^{1/p}) \|\nabla \cdot y\|_{L^\infty(Q_{2/N})}.$$

Thanks to (19), this inequality implies (31).

In order to prove estimate (32), we will use the regularity of pressure result stated in lemma 4.

Let $t \in (0, 2/N)$ and let us consider system (35) in the time interval $(0, t)$ with $\mathcal{W}_0 = W(t, \cdot)$ and $g = 0$. Then, using again that W solves (29), we obtain

$$\|W(t, \cdot)\|_{L^2(\mathcal{G})}^2 = (\nabla \cdot y, \tilde{q})_{L^2((0,t) \times \mathcal{G})}.$$

From this identity, we have

$$\|W\|_{L^\infty(0,2/N; L^2(\mathcal{G}))}^2 \leq C \|\nabla \cdot y\|_{H^{1/4}(0,2/N; L^2(\mathcal{G}))} \|\tilde{q}\|_{H^{-1/4}(0,2/N; L^2(\mathcal{G}))}.$$

We now use here (34) for the pressure term $\|\tilde{q}\|_{H^{-1/4}(0,2/N;L^2(\mathcal{G}))}$ and the estimates we established in lemma 3 for the function y (see (19)), and we deduce that

$$\begin{aligned} \|W\|_{L^\infty(0,2/N;L^2(\mathcal{G}))}^2 &\leq C\|\nabla \cdot y\|_{L^2((0,2/N)\times\mathcal{G})}^{3/4}\|\nabla \cdot y\|_{H^1(0,2/N;L^2(\mathcal{G}))}^{1/4}\|W(2/N, \cdot)\|_{L^2(\mathcal{G})} \\ &\leq C\frac{1}{N^{3/8}}N^{1/8}\|W(2/N, \cdot)\|_{L^2(\mathcal{G})}, \end{aligned}$$

so for the first term in the left hand side of (32) the estimate is established. In order to finish the proof of (32), observe that the role of $\partial_{x_2}W$ and $\partial_{x_3}W$ are analogous, so let us restrict ourselves to prove estimate (32) just for $\partial_{x_2}W$. First, we realize that $W_{x_2} := \partial_{x_2}W$ solves the following problem:

$$\begin{cases} \partial_t W_{x_2} - \Delta W_{x_2} + \nabla r_{x_2} = 0 & (t, x) \in Q_{2/N}, \\ W_{x_2}(t, 0, x_2, x_3) = W_{x_2}(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, 2/N) \times \mathbf{R}^2, \\ \nabla \cdot W_{x_2} = \nabla \cdot y_{x_2} & (t, x) \in (0, 2/N) \times \mathcal{G}, \\ W_{x_2}(0, x) = 0 & x \in \mathcal{G}. \end{cases}$$

Then, using the same argument as before, one can prove that

$$\|W_{x_2}\|_{L^\infty(0,2/N;L^2(\mathcal{G}))} \leq C\|\nabla \cdot y_{x_2}\|_{L^2(Q_{2/N})}^{3/4}\|\nabla \cdot y_{x_2}\|_{H^1(0,2/N;L^2(\mathcal{G}))}^{1/4},$$

and the conclusion follows again from (19). ■

3. Proofs of Theorem 1 and Theorem 2

3.1. Proof of Theorem 1

First we claim that instead of the statement of Theorem 1 it suffices to prove a weaker result. Namely, to construct a sequence of functions $\{f_\epsilon\}_\epsilon$, with $f_\epsilon \in L^2(0, T; L^2(\Omega))$, satisfying

$$f_\epsilon \rightarrow f \quad \text{in } L^{p_0}(0, T; V_0'(\Omega)) \quad \forall p_0 \in (1, 4/3) \quad (39)$$

and such that there exists a solution of the Navier-Stokes system

$$\begin{cases} (\partial_t u_\epsilon - \Delta u_\epsilon + (u_\epsilon, \nabla)u_\epsilon + \nabla p_\epsilon)(t, x) = f_\epsilon(t, x) & (t, x) \in (0, T) \times \Omega, \\ u_\epsilon(t, 0, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times (0, 1)^2, \\ (\nabla \cdot u_\epsilon)(t, x) = 0 & (t, x) \in (0, T) \times \Omega, \\ u_\epsilon(0, x) = u_0(x) & x \in \Omega, \end{cases} \quad (40)$$

which satisfies

$$u_\epsilon(T, x) = 0 \quad x \in \Omega. \quad (41)$$

Indeed, suppose that such a sequence is constructed. We claim that there exists a sequence $\{q_\epsilon\} \subset L^{p_0}((0, T); L^2(\Omega))$ such that

$$f_\epsilon + \nabla q_\epsilon \rightarrow f \quad \text{in } L^{p_0}(0, T; H^{-1}(\Omega)) \quad \forall p_0 \in (1, 4/3). \quad (42)$$

Then obviously the sequence $\{(u_\epsilon, p_\epsilon + q_\epsilon)\}$ is one we are looking for in the statement of Theorem 1.

In order to show this, for any t from $[0, T]$ we consider the stationary Stokes problem

$$\Delta z + \nabla r = f(t, \cdot) \quad \text{in } \Omega, \quad \nabla \cdot z = 0, \quad z|_{\partial\Omega} = 0$$

and

$$\Delta z_\epsilon + \nabla r_\epsilon = f_\epsilon(t, \cdot) \quad \text{in } \Omega, \quad \nabla \cdot z_\epsilon = 0, \quad z_\epsilon|_{\partial\Omega} = 0.$$

Then

$$\|z - z_\epsilon\|_{L^{p_0}(0, T; V_0(\Omega))} \rightarrow 0 \quad \text{as } \epsilon \rightarrow +0. \quad (43)$$

Therefore we have

$$\|f - f_\epsilon - \nabla r + \nabla r_\epsilon\|_{L^{p_0}(0, T; H^{-1}(\Omega))} = \|\Delta z - \Delta z_\epsilon\|_{L^{p_0}(0, T; H^{-1}(\Omega))} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (44)$$

Setting $q_\epsilon = r - r_\epsilon$ we have (42).

Now we prove (39)-(41).

Let $\epsilon > 0$ be chosen. We are going to construct (u_ϵ, p_ϵ) and $f_\epsilon \in L^2(0, T; L^2(\Omega))$ satisfying (39)-(41) with $\|f - f_\epsilon\|_{L^2(0, T; V_0'(\Omega))} \leq \epsilon$.

First of all there exists $\delta_0 > 0$ such that

$$\|f\|_{L^2(T-\delta_0, T; V_0'(\Omega))} \leq \frac{\epsilon}{5}.$$

Denote

$$\mathcal{L}u = \partial_t u - \Delta u + (u, \nabla)u.$$

We know that there exists at least one weak solution (u, p) of the problem

$$\begin{aligned} \mathcal{L}u + \nabla p &= f \quad \text{in } (0, T) \times \Omega, \\ \nabla \cdot u &= 0 \quad \text{in } (0, T) \times \Omega, \\ u &= 0 \quad \text{on } (0, T) \times \partial\Omega, \\ u(0, x) &= u_0(x) \quad \text{in } \Omega, \end{aligned}$$

with $u \in L^2(0, T; V_0(\Omega)) \cap L^\infty(0, T; H(\Omega))$.

Let $\tilde{T}_1 \in (T - \delta_0, T)$ be such that $\tilde{u}_1 = u(\tilde{T}_1) \in V_0(\Omega)$. We know (see for example [17]) that for a small time interval $(\tilde{T}_1, \tilde{T}_1 + \eta)$, with $\eta < \delta$, there exists a unique strong solution u to the above Navier Stokes problem such that $u(\tilde{T}_1) = \tilde{u}_1$. Therefore, there exists $T_1 \in (\tilde{T}_1, \tilde{T}_1 + \eta)$ such that $u(T_1) \in V_0(\Omega) \cap H^2(\Omega)$.

• First step. On the interval $(0, T_1)$ we do not exert any control and we take

$$u_\epsilon = u, \quad p_\epsilon = p, \quad f_\epsilon = f.$$

We will write

$$u_1 = u(T_1) \in V_0(\Omega) \cap H^2(\Omega).$$

• Second step. Let $T_2 \in (T_1, T)$ which will be defined precisely later on, close to T_1 . We consider a sequence of functions $u_{1, \epsilon_1} \in V_0(\Omega) \cap C_0^\infty(\Omega)$ such that

$$u_{1, \epsilon_1} \rightarrow u_1 \quad \text{in } V_0(\Omega) \quad \text{as } \epsilon_1 \rightarrow 0^+, \quad \|u_{1, \epsilon_1}\|_{V_0(\Omega)} \leq 2\|u_1\|_{V_0(\Omega)}. \quad (45)$$

On the time interval (T_1, T_2) we define

$$p_\epsilon \equiv 0, \quad (46)$$

$$u_\epsilon(t) = \frac{(t - T_1)}{(T_2 - T_1)} u_{1, \epsilon_1} + \frac{(T_2 - t)}{(T_2 - T_1)} u_1, \quad (47)$$

$$f_\epsilon = \mathcal{L}u_\epsilon. \quad (48)$$

Then, we have

$$u_\epsilon(T_1) = u_1, \quad u_\epsilon(T_2) = u_{1, \epsilon_1}, \quad \nabla \cdot u_\epsilon = 0, \quad f_\epsilon \in L^2(0, T; L^2(\Omega)),$$

and

$$\|f_\epsilon\|_{L^2(T_1, T_2; V_0'(\Omega))} \leq \|\partial_t u_\epsilon\|_{L^2(T_1, T_2; V_0'(\Omega))} + \|u_\epsilon\|_{L^2(T_1, T_2; V_0(\Omega))} + \|(u_\epsilon, \nabla)u_\epsilon\|_{L^2(T_1, T_2; V_0'(\Omega))}. \quad (49)$$

For the last two terms in inequality (49), using the continuous embedding $H_0^1(\Omega) \subset L^4(\Omega)$ we have

$$\|u_\epsilon\|_{L^2(T_1, T_2; V_0(\Omega))} + \|(u_\epsilon, \nabla)u_\epsilon\|_{L^2(T_1, T_2; V_0'(\Omega))} \leq C\sqrt{(T_2 - T_1)}(\|u_1\|_{H_0^1(\Omega)} + \|u_1\|_{H_0^1(\Omega)}^2 + 1). \quad (50)$$

Therefore, we can choose $T_2 - T_1$ small enough in order to guarantee that this term is smaller than $\frac{\epsilon}{5}$.

For the first term in the inequality (49) we have

$$\|\partial_t u_\epsilon\|_{L^2(T_1, T_2; V'_0(\Omega))} \leq \frac{1}{\sqrt{(T_2 - T_1)}} \|u_{1, \epsilon_1} - u_1\|_{L^2(\Omega)}, \quad (51)$$

and therefore, once T_2 is chosen, we can choose ϵ_1 small enough to bound this term by $\frac{\epsilon}{5}$.

From (50) and (51), choosing $T_2 - T_1$ small enough and then ϵ_1 small enough, we have proved that

$$\|f_\epsilon\|_{L^2(T_1, T_2; V'_0(\Omega))} \leq \frac{2\epsilon}{5},$$

and we have, at the end of this step

$$u_2 = u_\epsilon(T_2) = u_{1, \epsilon_1} \in V_0(\Omega) \cap C_0^\infty(\Omega). \quad (52)$$

• Third step. On the segment $[T_2, T_2 + 2/N]$, for N large enough, we look for the solution u_ϵ in the form

$$u_\epsilon(t, x) = N^2 \tilde{U}(t, x) + \mathbf{y}(t, x) - \tilde{W}(t, x), \quad p_\epsilon(t, x) = \tilde{r}(t, x), \quad (t, x) \in [T_2, T_2 + 2/N] \times \Omega.$$

Here,

$$\tilde{U}(t, x) = U(t - T_2, x), \quad \mathbf{y}(t, x) = y(t - T_2, x),$$

where U is the solution to problem (15), y is solution to the problem (17) with initial condition $v_0 = u_2$ (which obviously satisfies the hypothesis of lemma 3) and

$$\tilde{W}(t, x) = \theta(t - T_2)W(t - T_2, x), \quad \tilde{r}(t, x) = \theta(t - T_2)r(t - T_2, x),$$

where (W, r) is solution of (29) and $\theta = \theta(t) \in C^2[0, 2/N]$ satisfies

$$\theta(t) = 1 \quad t \in [0, 1/N] \quad \text{and} \quad \theta(t) = 0 \quad \text{in a neighborhood of } 2/N. \quad (53)$$

Due to the definition and properties of y (see Lemma 3), the definitions of U , W and θ , we have, for N large enough,

$$u_\epsilon(T_2) = u_2, \quad u_\epsilon(T_2 + 2/N) = N^2 U(2/N), \quad \nabla \cdot u_\epsilon = 0,$$

and

$$u_\epsilon(t, 0, x_2, x_3) = 0 \quad (t, x_2, x_3) \in (T_2, T_2 + 2/N) \times (0, 1)^2.$$

Let us now compute the Navier-Stokes operator acting on u_ϵ defined this way. We have (recalling that $(U, \nabla)U = 0$)

$$\begin{aligned} \mathcal{L}u_\epsilon(t, x) + \nabla \tilde{r}(t, x) &= (-\Delta \mathbf{y} - N^2(\tilde{U}, \nabla)\tilde{W} - N^2(\tilde{W}, \nabla)\tilde{U} + ((\mathbf{y} - \tilde{W}), \nabla)(\mathbf{y} - \tilde{W}))(t, x) \\ &\quad - ((\partial_t \theta)W)(t - T_2, x) + N^2 \nabla q(t - T_2, x) + \nabla \tilde{r}(t, x), \quad (t, x) \in (T_2, T_2 + 2/N) \times \Omega. \end{aligned}$$

On the one hand, from (19) in Lemma 3, we obtain

$$\|\Delta \mathbf{y}\|_{L^2(T_2, T_2 + 2/N; V'_0(\Omega))} \leq \frac{C}{N^{1/2}} \sup_{t \in (T_2, T_2 + 2/N)} \|\mathbf{y}(t, \cdot)\|_{H^1(\Omega)} \leq \frac{C}{N^{1/2}} \|u_2\|_{C^3(\bar{\Omega})}.$$

Concerning the transport term $N^2(\tilde{W}, \nabla)\tilde{U}$, we realize that

$$\|N^2(\tilde{W}, \nabla)\tilde{U}\|_{V'_0(\Omega)} = \sup_{b \in V_0(\Omega), \|b\|_{V_0(\Omega)}=1} \int_{\Omega} N^2(\tilde{W}, \nabla)\tilde{U} b \, dx.$$

Let us consider two functions $e_1, e_2 \in C^1([0, 1])$ such that

$$e_1(s) = 1 \, \forall s \in [0, \gamma/2], \quad e_1(s) = 0 \, \forall s \in [\gamma, 1], \quad (e_1 + e_2)(s) = 1 \, \forall s \in [0, 1]$$

(recall that γ was defined in (21)). Then, an integration by parts gives

$$\int_{\Omega} N^2(\tilde{W}, \nabla)\tilde{U} b e_1 \, dx = N^2 \left(- \int_{\Omega} (\nabla \cdot \mathbf{y}) \tilde{U} b e_1 \, dx - \int_{\Omega} \tilde{W} \cdot \nabla(b e_1) \tilde{U} \, dx \right). \quad (54)$$

The first term vanishes since $\nabla \cdot \mathbf{y} = 0$ when $x_1 \in (0, \gamma)$ (see (21), (23), (24) and (25)) and the second one can be estimated by

$$\|N \tilde{U}\|_{L^\infty(\Omega)} \|N \tilde{W}\|_{L^2(\Omega)}.$$

On the other hand, we directly get

$$\int_{\Omega} N^2(\tilde{W}, \nabla)\tilde{U} b e_2 \, dx \leq \|N \nabla \tilde{U}\|_{L^\infty((\gamma/2, 1) \times \mathbb{R}^2)} \|N \tilde{W}\|_{L^2(\Omega)}. \quad (55)$$

We observe that thanks to (9) we have that

$$\|N \tilde{U}\|_{L^\infty(T_2, T_2 + 2/N; L^\infty(\Omega))} \leq C$$

and thanks to (22), we have that

$$\|N \nabla \tilde{U}\|_{L^\infty(T_2, T_2 + 2/N; L^\infty((\gamma/2, 1) \times \mathbb{R}^2))} \leq C.$$

Consequently, from (54) and (55) we get that

$$\|N^2(\tilde{W}, \nabla)\tilde{U}\|_{V'_0(\Omega)} \leq C\|N\tilde{W}\|_{L^2(\Omega)}$$

and so, using (32), we obtain

$$\|N^2(\tilde{W}, \nabla)\tilde{U}\|_{L^{p_0}(T_2, T_2+2/N; V'_0(\Omega))} \leq C N N^{-1/4-1/p_0}.$$

Thanks to our choice of p_0 , the right hand side of this inequality goes to zero as $N \rightarrow +\infty$.

Analogous computations can be made for the term $N^2(\tilde{U} \cdot \nabla)\tilde{W}$.

Next, using again (32), we get

$$\|(\partial_t \theta)W\|_{L^{p_0}(0, 2/N; L^2(\Omega))} \leq CN\|W\|_{L^{p_0}(0, 2/N; L^2(\Omega))} \leq CN^{1-1/p_0}\|W\|_{L^\infty(0, 2/N; L^2(\Omega))} \leq CN^{3/4-1/p_0}.$$

Finally, from (19) and (31), we obtain

$$\|((\mathbf{y}-\tilde{W}), \nabla)(\mathbf{y}-\tilde{W})\|_{L^2(T_2, T_2+2/N; V'_0(\Omega))} \leq C\|\mathbf{y}-\tilde{W}\|_{L^4(T_2, T_2+2/N; L^4(\Omega))}^2 \rightarrow 0 \text{ as } N \rightarrow +\infty.$$

Therefore, by choosing N large enough, with the above construction of (u_ϵ, p_ϵ) we have

$$\|f_\epsilon\|_{L^{p_0}(T_2, T_2+2/N; V'_0(\Omega))} = \|\mathcal{L}u_\epsilon + \nabla p_\epsilon\|_{L^{p_0}(T_2, T_2+2/N; V'_0(\Omega))} = \|\mathcal{L}u_\epsilon\|_{L^{p_0}(T_2, T_2+2/N; V'_0(\Omega))} \leq \frac{\epsilon}{5}.$$

At the end of this step we have

$$u_\epsilon(T_2 + 2/N) = N^2U(2/N).$$

• Fourth step. Finally, on the interval $[T_2 + \frac{2}{N}, T]$, we may reduce the question to drive u_ϵ to zero at time $t = T$ into a null controllability problem for a linear heat equation.

Indeed, in the interval $[T_2 + \frac{2}{N}, T]$, we take $f_\epsilon \equiv 0$ and we try to find a boundary control which drives the associated solution of (40) which starts at time $t = T_2 + 2/N$ from the initial condition $N^2U(2/N, x)$ to zero at time $t = T$.

In a first sight, u_ϵ is solution of the Navier-Stokes system but the fact that its initial condition possesses the structure

$$N^2U(2/N, x) = (0, N^2z(2/N, x_1), N^2z(2/N, x_1)) \quad x \in \Omega$$

will lead us to a heat equation.

Actually, from well-known controllability results for the linear heat equation (see, for instance, [13]), for any $\bar{z}_0 \in L^2(0, 1)$, there exists a boundary control $\rho = \rho(t) \in L^2(0, T - T_2 - 2/N)$ such that the solution of

$$\begin{cases} \partial_t \bar{z} - \partial_{x_1 x_1}^2 \bar{z} = 0 & (t, x_1) \in (0, T - T_2 - 2/N) \times (0, 1), \\ \bar{z}(t, 0) = 0, \quad \bar{z}(t, 1) = \rho(t) & t \in (0, T - T_2 - 2/N), \\ \bar{z}(0, x_1) = \bar{z}_0(x_1) & x_1 \in (0, 1) \end{cases}$$

satisfies

$$\bar{z}(T - T_2 - 2/N, x_1) = 0 \quad x_1 \in (0, 1).$$

Then, it suffices to take

$$\begin{cases} u_\epsilon(t, x) = (0, \bar{z}(t - T_2 - 2/N, x_1), \bar{z}(t - T_2 - 2/N, x_1)) \\ (t, x) \in (T_2 + 2/N, T) \times \Omega, \end{cases}$$

where \bar{z} is the solution of the previous null controllability problem with initial condition

$$\bar{z}_0(x_1) = N^2 z(2/N, x_1) \quad x_1 \in (0, 1).$$

We then obtain

$$u_\epsilon(T, \cdot) = 0,$$

and we have

$$\|f - f_\epsilon\|_{L^{p_0}(0, T; V_0'(\Omega))} \leq \epsilon.$$

The proof of Theorem 1 is now complete. ■

3.2. Proof of Theorem 2

In this case we now take $u_0 \in V_0(\Omega)$. Let us define $T(u_0, f)$ to be the maximal time of existence for a strong solution to the Navier-Stokes problem

$$\mathcal{L}u + \nabla p = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \nabla \cdot u = 0, \quad u(0, \cdot) = u_0.$$

We know (see e.g. [17]) that $T(u_0, f) > 0$.

In the same way, if f is extended by zero for $t > T$, let $T^*(\tau)$ be the maximal time of existence for a strong solution to the Navier-Stokes system

$$\mathcal{L}u + \nabla p = f(t + \tau, \cdot) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \nabla \cdot u = 0, \quad u(0, \cdot) = 0.$$

We claim that there exists $\delta_* > 0$ such that for every $\tau \in [0, T]$, $T^*(\tau) \geq \delta_*$. Indeed, it is known (see e.g. [17]) that the initial value problem for the incompressible Navier-Stokes system with zero initial velocity and right hand side $\tilde{f} \in L^2(0, T; L^2(\Omega))$ admits a strong solution provided that $\|\tilde{f}\|_{L^2(0, T; L^2(\Omega))} \leq C(\Omega)$, with $C(\Omega)$ small enough. Observe that there exists $\delta_* > 0$ such that for every $\tau \in [0, T]$, $\|f(\cdot + \tau, \cdot)\|_{L^2(0, \delta_*; L^2(\Omega))} \leq C(\Omega)$. So, after extension of $f(\cdot + \tau, \cdot)$ by zero for $t > \delta_*$ applying the above result we obtain that $T^*(\tau) > \delta_*$.

Let now \hat{T} be such that $0 < \hat{T} < \min\{T(u_0, f), \inf_{\tau \in [0, T]} T^*(\tau)\}$ and $K\hat{T} = T$ for some integer K .

On the interval $(0, \hat{T})$ we proceed as for the proof of Theorem 1 with ϵ replaced by $\frac{\epsilon}{K}$ with the additional information that we here have a strong solution u_ϵ . We end up at \hat{T} with $u_\epsilon(\hat{T}) = 0$ and $\|f_\epsilon - f\|_{L^{p_0}(0, \hat{T}; V_0'(\Omega))} \leq \frac{\epsilon}{K}$.

Next we iterate this procedure on each interval $[(k-1)\hat{T}, k\hat{T}]$, with $k = 2, \dots, K$, noticing that at each step we start with initial condition $u_\epsilon((k-1)\hat{T}) = 0$. We therefore obtain f_ϵ and u_ϵ which is a strong solution on the interval $(0, T)$ such that

$$\|f_\epsilon - f\|_{L^{p_0}(0, T; V_0'(\Omega))} \leq \epsilon \quad \text{and} \quad u_\epsilon(T) = 0.$$

The proof of Theorem 2 is then complete. ■

Appendix: Regularity and energy estimate of the pressure

In this paragraph, we will prove the result stated in lemma 4 concerning the regularity of the pressure associated to energy solutions of the Stokes system.

Let us first recall the system we are dealing with:

$$\begin{cases} w_t - \Delta w + \nabla h = f & (t, x) \in (0, T) \times \mathcal{G}, \\ \nabla \cdot w = 0 & (t, x) \in (0, T) \times \mathcal{G}, \\ w(t, 0, x_2, x_3) = w(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ w(0, x) = w_0(x) & x \in \mathcal{G}. \end{cases} \quad (56)$$

Since $f \in L^2(0, T; H^{-1}(\mathcal{G}))$, there exists $\{f_i\}_{i=0}^3 \subset L^2((0, T) \times \mathcal{G})$ such that

$$f = f_0 + \partial_{x_1} f_1 + \partial_{x_2} f_2 + \partial_{x_3} f_3.$$

Then, instead of (34), we will prove its analog in this situation:

$$\|h\|_{H^{-1/4}(0, T; L^2(\mathcal{G}))} \leq C(\|w_0\|_{L^2(\mathcal{G})} + \sum_{i=0}^3 \|f_i\|_{L^2((0, T) \times \mathcal{G})}),$$

for a positive constant C independent of T .

Proof of lemma 4: First, we will prove that we can extend the solution to problem (56) for negative times $t < 0$.

Proposition 2. *Let $u_1 \in H(\mathcal{G})$. Then there exists $f_i \in L^2((0, T) \times \mathcal{G})$ ($i = 0, 1, 2, 3$) such that*

$$\begin{cases} \tilde{w}_t - \Delta \tilde{w} + \nabla h = f_0 + \sum_{i=1}^3 \partial_{x_i} f_i & (t, x) \in (0, T) \times \mathcal{G}, \\ \nabla \cdot \tilde{w} = 0 & (t, x) \in (0, T) \times \mathcal{G}, \\ \tilde{w}(t, 0, x_2, x_3) = w(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ \tilde{w}(0, x) = 0, \quad \tilde{w}(T, x) = u_1(x) & x \in \mathcal{G}. \end{cases}$$

and the following estimate holds true

$$\|\tilde{w}\|_{L^2(0, T; V(\Omega))} + \sum_{j=1}^3 \|f_j\|_{L^2((0, T) \times \mathcal{G})} + \|u\|_{L^2((0, T) \times \mathcal{G})} \leq C\|u_1\|_{L^2(\mathcal{G})}. \quad (57)$$

Proof. In order to prove the statement of the Proposition, we consider the extremal problem

$$J(\tilde{w}, \vec{f}, u) = \frac{1}{2} \sum_{j=1}^3 \|f_j\|_{L^2((0,T) \times \mathcal{G})}^2 + \frac{1}{2} \|u\|_{L^2((0,T) \times \mathcal{G})}^2 \rightarrow \inf, \quad (58)$$

where

$$\begin{cases} \tilde{w}_t - \Delta \tilde{w} + \nabla h = \sum_{i=1}^3 \partial_{x_i} f_i + u & (t, x) \in (0, T) \times \mathcal{G}, \\ \nabla \cdot \tilde{w} = 0 & (t, x) \in (0, T) \times \mathcal{G}, \\ \tilde{w}(t, 0, x_2, x_3) = \tilde{w}(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ \tilde{w}(0, x) = 0, \quad \tilde{w}(T, x) = u_1(x) & x \in \mathcal{G}. \end{cases}$$

Let us first assume that $u_1 \in C^\infty(\overline{\mathcal{G}}) \cap V(\Omega)$. Then, there obviously exists at least one admissible element for this extremal problem (58). Moreover, thanks to the a priori estimates for the Stokes system, we also know that there exists a solution $((w^*, h^*), \vec{f}^*, (u^*, p^*))$ to (58). By classical arguments, one can prove that the optimality system has the form

$$\begin{cases} \partial_t w^* - \Delta w^* = -\nabla h^* + \sum_{i=1}^3 \partial_{x_i} f_i^* + u^* & (t, x) \in (0, T) \times \mathcal{G}, \\ -\partial_t u^* - \Delta u^* = -\nabla p^* & (t, x) \in (0, T) \times \mathcal{G}, \\ \partial_{x_i} u^* = -f_i^* \quad \forall i \in \{1, 2, 3\} & (t, x) \in (0, T) \times \mathcal{G}, \\ \nabla \cdot w^* = 0, \quad \nabla \cdot u^* = 0 & (t, x) \in (0, T) \times \mathcal{G}, \\ w^*(t, 0, x_2, x_3) = w^*(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ u^*(t, 0, x_2, x_3) = u^*(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ w^*(0, x) = 0, \quad w^*(T, x) = u_1(x) & x \in \mathcal{G}. \end{cases}$$

Multiplying the equation of u^* by $t u^*$ in $L^2((0, T) \times \mathcal{G})$, we have the a priori estimate

$$\|u^*(T, \cdot)\|_{L^2(\mathcal{G})} \leq C(\|u^*\|_{L^2((0,T) \times \mathcal{G})} + \sum_{j=1}^3 \|f_j\|_{L^2((0,T) \times \mathcal{G})}). \quad (59)$$

Now, multiplying the equation of w^* by u^* , we obtain

$$(u_1, u^*(T, \cdot))_{L^2(\mathcal{G})} = 2J(w^*, \vec{f}^*, u^*). \quad (60)$$

Combining (59) and (60), we obtain the estimate

$$\|\vec{f}^*\|_{L^2((0,T)\times\mathcal{G})} + \|u^*\|_{L^2((0,T)\times\mathcal{G})} \leq C\|u_1\|_{L^2(\mathcal{G})}.$$

Using this estimate, we immediately can get rid of the assumption $u_1 \in V(\Omega) \cap C^\infty(\bar{\mathcal{G}})$. The proof of the technical result is completed. \square

We set $T = 1$ in Proposition 2 and denote \tilde{w}, \tilde{f} the solution given by this proposition with $u_1 = w_0$. We extend the solution of problem (56) for negative t by formula $w(t, \cdot) = \tilde{w}(t+1, \cdot)$. Similarly we set $f(t, \cdot) = \tilde{f}(t+1, \cdot)$. Then, in order to prove the statement of lemma 4 it suffices to establish the following inequality:

$$\|h\|_{H^{-1/4}(\mathbf{R}; L^2(\mathcal{G}))} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R}\times\mathcal{G})}, \quad (61)$$

where (w, h) is the solution of

$$\begin{cases} w_t - \Delta w + \nabla h = f & (t, x) \in \mathbf{R} \times \mathcal{G}, \\ \nabla \cdot w = 0 & (t, x) \in \mathbf{R} \times \mathcal{G}, \\ w(t, 0, x_2, x_3) = w(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in \mathbf{R}^3, \\ w(t, x) = 0 \quad \forall t \leq -1 & x \in \mathcal{G}. \end{cases} \quad (62)$$

Let us consider some sequences $\{f_{i,\epsilon}\}_{i=0}^3 \subset C_0^\infty(\mathbf{R} \times \mathcal{G})^3$ such that

$$\|f_{i,\epsilon} - f_i\|_{L^2(\mathbf{R}\times\mathcal{G})} \rightarrow 0 \quad \text{as } \epsilon \searrow 0^+, \text{ for } i = 0, 1, 2, 3.$$

Denote by (w_ϵ, h_ϵ) the solution to problem (62) with right hand side

$$f_{0,\epsilon} + \partial_{x_1} f_{1,\epsilon} + \partial_{x_2} f_{2,\epsilon} + \partial_{x_3} f_{3,\epsilon}.$$

Suppose for a moment that estimate (61) is already established for h_ϵ with a constant C which is independent of ϵ . Then, by the uniqueness of solutions of the Stokes system, the statement of our theorem holds true. Therefore, from this point of the proof on, we assume that

$$f_i \in C_0^\infty(\mathbf{R} \times \mathcal{G}) \quad \text{for } i = 0, 1, 2, 3. \quad (63)$$

Then, it is well-known that one can define a unique weak solution $(w, h) \in (L^2(\mathbf{R}; V(\Omega)) \cap L^\infty(\mathbf{R}; L^2(\mathcal{G}))) \times H^{-1/2}(\mathbf{R}; L^2(\mathcal{G}))$ of (62) and that it satisfies

$$\|w\|_{L^2(\mathbf{R}; H^1(\mathcal{G})) \cap L^\infty(\mathbf{R}; L^2(\mathcal{G}))} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}. \quad (64)$$

Taking the curl operator in the equation of (56), we deduce that $\varphi = \nabla \times w$ satisfies the equation

$$\varphi_t - \Delta \varphi = G = \nabla \times f \quad \text{in } \mathbf{R} \times \mathcal{G}. \quad (65)$$

Step 1. Estimates on normal derivatives of φ .

The next step is to estimate some normal derivatives of φ on $\partial \mathcal{G} = \{0, 1\} \times \mathbf{R}^2$. Since the task is identical on both sides ($x_1 = 0$ or $x_1 = 1$), we will do that, for instance, for $x_1 = 0$.

Let $\rho = \rho(x_1) \in C^\infty([0, 1])$ be a function such that it is zero in $(1 - \delta_1, 1)$ (with $\delta_1 > 0$ small) and $\rho(0) = 1$. Denote $\tilde{\varphi} = \rho \varphi$.

Now, we take the Fourier transform of (65) with respect to t , x_2 and x_3 . We introduce the following notation for the Fourier transform of $\tilde{\varphi}$:

$$\mathcal{F}(\tilde{\varphi}) = \hat{\tilde{\varphi}} =: \psi = (\psi_1, \psi_2, \psi_3) \quad (\tau, x_1, \xi_2, \xi_3) \in \mathbf{R} \times (0, 1) \times \mathbf{R}^2.$$

• *Estimates on ψ_2 .* Let us decompose the heat operator $L = -\partial_{x_1 x_1}^2 + i\tau + \xi_2^2 + \xi_3^2 = -L^+ L^- = -L^- L^+$, with

$$L^+ = \partial_{x_1} + \alpha(\tau, \xi_2, \xi_3)$$

and

$$L^- = \partial_{x_1} - \alpha(\tau, \xi_2, \xi_3),$$

where $\alpha(\tau, \xi_2, \xi_3)$ is a root of $i\tau + \xi_2^2 + \xi_3^2$ with positive real part.

Again from (65), we have

$$\begin{cases} -L^- L^+ \psi_2 = \mathbf{G}_2 + [\rho, \partial_{x_1 x_1}^2] \hat{\varphi}_2 & (\tau, x_1, \xi_2, \xi_3) \in \mathbf{R} \times \mathcal{G}, \\ \psi_2(\tau, 1, \xi_2, \xi_3) = \partial_{x_1} \psi_2(\tau, 1, \xi_2, \xi_3) = 0 & (\tau, \xi_2, \xi_3) \in \mathbf{R}^3, \end{cases}$$

where $\mathbf{G}_2 = \sum_{j=0}^2 (1 + \xi_1^2 + \xi_2^2)^{\frac{2-j}{2}} \partial_{x_1}^j G_{2j+1}$, with $G_{21}, G_{22}, G_{23} \in L^2(\mathbf{R} \times \mathcal{G})$, $\text{supp } G_{2i} \subset \mathbf{R} \times (\delta_1, 1 - \delta_1) \times \mathbf{R}^2$ and

$$\|G_{21}\|_{L^2(\mathbf{R} \times \mathcal{G})} + \|G_{22}\|_{L^2(\mathbf{R} \times \mathcal{G})} + \|G_{23}\|_{L^2(\mathbf{R} \times \mathcal{G})} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}. \quad (66)$$

Next, we will provide an estimate for the function $L^+\psi_2$ at $x_1 = 0$. Observe that the function $\Psi = L^-\psi_2$ satisfies

$$\begin{cases} -L^-\Psi = \mathbf{G}_2 + [\rho, \partial_{x_1 x_1}^2] \hat{\varphi}_2 & (\tau, x_1, \xi_2, \xi_3) \in \mathbf{R} \times \mathcal{G}, \\ \Psi(\tau, 1, \xi_2, \xi_3) = 0 & (\tau, \xi_2, \xi_3) \in \mathbf{R}^3. \end{cases} \quad (67)$$

Thus, for any $q_0 \in L^2(\mathbf{R}^3)$, let q be the solution of the following adjoint system:

$$\begin{cases} -(L^-)^* q = 0 & (\tau, x_1, \xi_2, \xi_3) \in \mathbf{R} \times \mathcal{G}, \\ q(\tau, 0, \xi_2, \xi_3) = q_0(\tau, \xi_2, \xi_3) & (\tau, \xi_2, \xi_3) \in \mathbf{R}^3, \end{cases} \quad (68)$$

where $(L^-)^* = -\partial_{x_1} - \overline{\alpha(\tau, \xi_2, \xi_3)}$ is the formal adjoint operator of L^- .

Let us denote

$$\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle = |\tau| + \xi_2^2 + \xi_3^2.$$

For solution of the Cauchy problem (68) the following a priori estimate holds true

$$\int_0^1 \langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{1/2} |q|^2 dx_1 \leq C |q_0(\tau, \xi_2, \xi_3)|^2 \quad \forall (\tau, \xi_2, \xi_3) \in \mathbf{R}^3.$$

Then, we also have the estimates

$$\int_0^1 (\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle |q|^2 + |\partial_{x_1} q|^2) dx_1 \leq C \langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{1/2} |q_0(\tau, \xi_2, \xi_3)|^2 \quad \forall (\tau, \xi_2, \xi_3) \in \mathbf{R}^3$$

and

$$\int_0^1 (|\partial_{x_1 x_1} q|^2 + \langle\langle \tau, \xi_2, \xi_3 \rangle\rangle |\partial_{x_1} q|^2) dx_1 \leq C \langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/2} |q_0(\tau, \xi_2, \xi_3)|^2 \quad \forall (\tau, \xi_2, \xi_3) \in \mathbf{R}^3.$$

Integrating with respect to τ , ξ_2 and ξ_3 , we deduce that

$$\begin{aligned} & \|\partial_{x_1 x_1}^2 q\|_{L^2(\mathbf{R} \times \mathcal{G})} + \|(\tau + \xi_2^2 + \xi_3^2) \partial_{x_1} q\|_{L^2(\mathbf{R} \times \mathcal{G})} \\ & \leq C \|\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/4} q_0\|_{L^2(\mathbf{R} \times \mathcal{G})}. \end{aligned} \quad (69)$$

Taking the scalar product of the equation of (67) and the function q we obtain

$$\int_{\mathbf{R} \times \mathcal{G}} (\mathbf{G}_2 + [\rho, \partial_{x_1 x_1}^2] \hat{\varphi}_2) \bar{q} d\tau dx_1 d\xi_2 d\xi_3 = - \int_{\mathbf{R}^3} \Psi(\tau, 0, \xi_2, \xi_3) \bar{q}_0 d\tau d\xi_2 d\xi_3.$$

Now, if we set

$$q_0(\tau, \xi_2, \xi_3) = \frac{\Psi(\tau, 0, \xi_2, \xi_3)}{\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/2}}$$

and we use estimates (64), (66) and (69), we obtain

$$\int_{\mathbf{R}^3} \frac{|L^+ \psi_2(\tau, 0, \xi_2, \xi_3)|^2}{\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/2}} d\tau d\xi_2 d\xi_3 \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}^2. \quad (70)$$

- *Estimates on ψ_3 .* Analogously as for ψ_2 , we can prove:

$$\int_{\mathbf{R}^3} \frac{|L^+ \psi_3(\tau, 0, \xi_2, \xi_3)|^2}{\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/2}} d\tau d\xi_2 d\xi_3 \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}^2. \quad (71)$$

From estimates (70) and (71), we deduce that

$$\begin{aligned} & \int_{\mathbf{R}^3} \frac{|(\partial_{x_1 x_1}^2 \hat{w}_3 + \alpha(\tau, \xi_2, \xi_3) \partial_{x_1} \hat{w}_3)(\tau, 0, \xi_2, \xi_3)|^2}{\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/2}} d\tau d\xi_2 d\xi_3 \\ & + \int_{\mathbf{R}^3} \frac{|(\partial_{x_1 x_1}^2 \hat{w}_2 + \alpha(\tau, \xi_2, \xi_3) \partial_{x_1} \hat{w}_2)(\tau, 0, \xi_2, \xi_3)|^2}{\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/2}} d\tau d\xi_2 d\xi_3 \\ & \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}^2. \end{aligned} \quad (72)$$

Step 2. Estimates on \hat{h} and conclusion.

Observe that from the Stokes equation in (56), we obtain

$$\partial_{x_1} \hat{h}(\tau, 0, \xi_2, \xi_3) = -(i\xi_2 \partial_{x_1} \hat{w}_2 + i\xi_3 \partial_{x_1} \hat{w}_3)(\tau, 0, \xi_2, \xi_3) \quad (\tau, \xi_2, \xi_3) \in \mathbf{R}^3.$$

and

$$i\xi_k \hat{h}(\tau, 0, \xi_2, \xi_3) = \partial_{x_1 x_1}^2 \hat{w}_k(\tau, 0, \xi_2, \xi_3) \quad (\tau, \xi_2, \xi_3) \in \mathbf{R}^3, \quad k = 2, 3.$$

Then, from (72), we have

$$\left(\partial_{x_1} \hat{h} + \frac{\xi_2^2 + \xi_3^2}{\alpha(\tau, \xi_2, \xi_3)} \hat{h} \right)(\tau, 0, \xi_2, \xi_3) = \sum_{k=2}^3 \frac{\langle\langle \tau, \xi_2, \xi_3 \rangle\rangle^{3/4}}{\alpha(\tau, \xi_2, \xi_3)} \sqrt{\xi_2^2 + \xi_3^2} g_k(\tau, \xi_2, \xi_3), \quad (73)$$

where $g_k \in L^2(\mathbf{R}^3)$, $k = 2, 3$ and

$$\sum_{k=2}^3 \|g_k\|_{L^2(\mathbf{R}^3)} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}. \quad (74)$$

From (56), we obtain

$$\partial_{x_1 x_1}^2 \hat{h} - (\xi_2^2 + \xi_3^2) \hat{h} = \mathbf{F} \quad (\tau, x_1, \xi_2, \xi_3) \in \mathbf{R} \times \mathcal{G}, \quad (75)$$

where $\mathbf{F} = \sum_{j=0}^2 (1 + \xi_2^2 + \xi_3^2)^{\frac{2-j}{2}} \partial_{x_1}^j F_{j+1}$, with $F_1, F_2, F_3 \in L^2(\mathbf{R} \times \mathcal{G})$ satisfying $\text{supp } F_i \subset \mathbf{R} \times [\delta_2, 1 - \delta_2] \times \mathbf{R}^2$ ($i = 1, 2, 3$) for some $\delta_2 > 0$ (see (63)) and

$$\|F_1\|_{L^2(\mathbf{R} \times \mathcal{G})} + \|F_2\|_{L^2(\mathbf{R} \times \mathcal{G})} + \|F_3\|_{L^2(\mathbf{R} \times \mathcal{G})} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}. \quad (76)$$

Let us now introduce the functions

$$\tilde{w}(t, x) = (w_1(t, 1 - x_1, x_2, x_3), -w_2(t, 1 - x_1, x_2, x_3), -w_3(t, 1 - x_1, x_2, x_3)),$$

$$\tilde{h}(t, x) = -h(t, 1 - x_1, x_2, x_3),$$

and

$$\tilde{w}_0(x) = (w_{0,1}(1 - x_1, x_2, x_3), -w_{0,2}(1 - x_1, x_2, x_3), -w_{0,3}(1 - x_1, x_2, x_3)).$$

Then

$$\begin{cases} \tilde{w}_t - \Delta \tilde{w} + \nabla \tilde{h} = \tilde{f} & (t, x) \in (0, T) \times \mathcal{G}, \\ \nabla \cdot \tilde{w} = 0 & (t, x) \in (0, T) \times \mathcal{G}, \\ \tilde{w}(t, 0, x_2, x_3) = \tilde{w}(t, 1, x_2, x_3) = 0 & (t, x_2, x_3) \in (0, T) \times \mathbf{R}^2, \\ \tilde{w}(0, x) = \tilde{w}_0(x) & x \in \mathcal{G}, \end{cases} \quad (77)$$

where $\tilde{f} \in L^2(0, T; H^{-1}(\mathcal{G}))$. Moreover we have

$$\|\tilde{f}\|_{L^2(0, T; H^{-1}(\mathcal{G}))} \leq C \|f\|_{L^2(0, T; H^{-1}(\mathcal{G}))}.$$

It is also readily seen that (73)-(74) also holds for \hat{h} . Consequently, we have

$$\begin{aligned} & (\partial_{x_1} \hat{h} - \frac{\xi_2^2 + \xi_3^2}{\alpha(\tau, \xi_2, \xi_3)} \hat{h})(\tau, 1, \xi_2, \xi_3) \\ &= \sum_{k=2}^3 \frac{\langle \langle \tau, \xi_2, \xi_3 \rangle \rangle^{3/4}}{\alpha(\tau, \xi_2, \xi_3)} \sqrt{\xi_2^2 + \xi_3^2} \tilde{g}_k(\tau, \xi_2, \xi_3), \end{aligned} \quad (78)$$

where $\tilde{g}_k \in L^2(\mathbf{R}^3)$, $k = 2, 3$ and

$$\sum_{k=2}^3 \|\tilde{g}_k\|_{L^2(\mathbf{R}^3)} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}.$$

taking the divergence of the first equation of system (77) we have

$$\Delta \tilde{h} = \nabla \cdot \tilde{f} \quad \text{in } \Omega, \quad t \in (0, T). \quad (79)$$

Case A Assume that $|\tau| > \max\{1, \frac{1}{\epsilon}(\xi_2^2 + \xi_3^2)\}$, for some $\epsilon > 0$ small.

Let \hat{h}_1 be the solution to the following boundary value problem

$$\begin{cases} \partial_{x_1 x_1}^2 \hat{h}_1 - (\xi_2^2 + \xi_3^2) \hat{h}_1 = \mathbf{F} & (\tau, x_1, \xi_2, \xi_3) \in \mathbf{R} \times \mathcal{G}, \\ \hat{h}_1|_{x_1=0} = \hat{h}_1|_{x_1=1} = 0 & (\tau, \xi_2, \xi_3) \in \mathbf{R}^3. \end{cases} \quad (80)$$

Here \mathbf{F} is the Fourier transform of $\nabla \cdot \tilde{f}$ respect to t, x_2, x_3 . Then, it is not difficult to check that \hat{h}_1 satisfies

$$\sum_{j=0}^1 \| |(\xi_2, \xi_3)|^{-\frac{3}{2}} \partial_{x_1} \hat{h}_1|_{x_1=j} \|_{L^2(\mathbf{R}^3)} + \|\hat{h}_1\|_{L^2(\mathbf{R} \times \mathcal{G})} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}. \quad (81)$$

Next we represent the function \hat{h} in the form $\hat{h} = \hat{h}_1 + (1 + |\tau|)^{\frac{1}{4}} \hat{h}_2$. From (79) and (80), we deduce that the function \hat{h}_2 solves the following boundary value problem:

$$\partial_{x_1 x_1}^2 \hat{h}_2 - (\xi_2^2 + \xi_3^2) \hat{h}_2 = 0 \quad (\tau, x_1, \xi_2, \xi_3) \in \mathbf{R} \times \mathcal{G}, \quad (82)$$

together with

$$(\partial_{x_1} \hat{h}_2 + b \hat{h}_2)(\tau, 0, \xi_2, \xi_3) = r_1, \quad (\partial_{x_1} \hat{h}_2 - b \hat{h}_2)(\tau, 1, \xi_2, \xi_3) = r_2 \quad (83)$$

with $b(\tau, \xi_2, \xi_3) = \frac{\xi_2^2 + \xi_3^2}{\alpha(\tau, \xi_2, \xi_3)}$, where r_i ($i = 1, 2$) satisfies

$$\sum_{i=1}^2 \| |(\xi_2, \xi_3)|^{-\frac{3}{2}} r_i \|_{L^2(\mathbf{R}^3)} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})}. \quad (84)$$

On the other hand, the (unique) solution to the problem (82)-(83) is given by

$$\hat{h}_2(\tau, x_1, \xi_2, \xi_3) = C_1 e^{i(\xi_2, \xi_3)x_1} + C_2 e^{-i(\xi_2, \xi_3)x_1},$$

where

$$C_2 = C_1 \left(\frac{|(\xi_2, \xi_3)| + b}{|(\xi_2, \xi_3)| - b} \right) - \frac{r_1}{|(\xi_2, \xi_3)| - b}$$

and

$$C_1 = \frac{r_2 + r_1 \left(\frac{|(\xi_2, \xi_3)| + b}{|(\xi_2, \xi_3)| - b} \right) e^{-i(\xi_2, \xi_3)x_1}}{(|(\xi_2, \xi_3)| - b) e^{i(\xi_2, \xi_3)x_1} + \frac{(b + |(\xi_2, \xi_3)|)^2}{|(\xi_2, \xi_3)| - b} e^{-i(\xi_2, \xi_3)x_1}}.$$

Direct computations provide the estimate

$$\|\hat{h}_2\|_{L^2(\mathbf{R} \times \mathcal{G})} \leq C \sum_{i=0}^1 \| |(\xi_2, \xi_3)|^{-\frac{3}{2}} r_i \|_{L^2(\mathbf{R}^3)}. \quad (85)$$

From estimates (85), (84) and (81), we readily deduce the desired inequality (61).

Case B. Assume that $|\tau| \leq (\xi_2^2 + \xi_3^2)/\epsilon$.

Taking the Fourier transform of equation (56) respect to (t, x_2, x_3) we have

$$(\partial_{x_1} \hat{h}, i\xi_2 \hat{h}, i\xi_3 \hat{h}) = -i\tau \hat{w} + (\partial_{x_1}^2 - \xi_2^2 - \xi_3^2) \hat{w} + \hat{f} \quad (86)$$

Then

$$\|(\partial_{x_1} \hat{h}_1, i\xi_2 \hat{h}_2, i\xi_3 \hat{h}_3)\|_{X'} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R} \times \mathcal{G})} \quad (87)$$

Here X is the Banach space with the norm $\|g\|_X = \left(\int_{[0,1] \times \mathbf{R}^2} ((\partial_{x_1} g)^2 + (1 + \xi_1^2 + \xi_3^2) g^2) dx_1 d\xi_2 d\xi_3 \right)^{\frac{1}{2}}$. Consider the following boundary value problem

$$\partial_{x_1} v_1 + i\xi_2 v_2 + i\xi_3 v_3 = p \quad \text{in } [0, 1] \times \mathbf{R}^2, \quad v_1|_{x_1=0} = v_1|_{x_1=1} = 0. \quad (88)$$

For any $p \in L^2([0, 1] \times \mathbf{R}^2)$ there exists a solution to problem (88) $v \in X$ and independent constant C such that

$$\|v\|_X \leq C \|p\|_{L^2([0,1] \times \mathbf{R}^2)}, \quad (89)$$

Setting in (88) $p = \hat{h}$ and taking the scalar product of v with (86) and using (89) and (87) we obtain

$$\|\hat{h}\|_{L^2([0,1]\times\mathbf{R}^2)} \leq C \sum_{i=0}^3 \|f_i\|_{L^2(\mathbf{R}\times\mathcal{G})}. \quad (90)$$

From (85), (90) and (81) we obtain (48). The proof is complete. ■

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