ON THE POSITIVITY OF SOLUTIONS OF SYSTEMS OF STOCHASTIC PDES

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On the occasion of the 75^{th} birthday of Prof. Dr. Dr.h.c. Wolfgang L. Wendland

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Abstract. We study the positivity of solutions of a class of semi-linear parabolic systems of stochastic partial differential equations by considering random approximations. For the family of random approximations we derive explicit necessary and sufficient conditions such that the solutions preserve positivity. These conditions imply the positivity of the solutions of the stochastic system for both Itô's and Stratonovich's interpretation of stochastic differential equations.

1. Introduction

We study the positivity of solutions of *systems* of semi-linear parabolic equations under stochastic perturbations. The systems are of the form

$$du^l(x,t) = \big(-\sum_{i=1}^{m} A_i^l(x,D)u^i(x,t) + f^l(x,t,u(x,t))\big)dt + \sum_{i=1}^{\infty} q_j g_j^l(x,t,u(x,t))dW_t^j, \tag{1}$$

$l = 1,\dots,m$, where $x \in O$, $O \subset \mathbb{R}^n$ is a bounded domain, and $t > 0$. The function $u = (u^1,\dots,u^m)$ is a vector-valued, A_i^l are linear elliptic operators of second order, and the non-linearity $f = (f^1,\dots,f^m)$ takes values in $\mathbb{R}^m$. Moreover, we assume that $\{W_t^j,\, t \geq 0\}_{j\in\mathbb{N}}$ is a family of independent standard scalar Wiener processes on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, and dW_t^j denotes the corresponding Itô differential. The non-negative parameters q_j are normalization factors and the functions g_j^l are real-valued, $l = 1,\dots,m, j \in \mathbb{N}$. The boundary conditions are given by the operators $(B^1,\dots,B^m)$,

$$B^l(x,D)u^l(x,t) = 0 \qquad x \in \partial O,\ t > 0,\ l = 1,\dots,m,$$

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and the solution satisfies the initial conditions

$$u^l(0,x) = u_0^l(x) \qquad x \in \bar{O},\ l = 1, \dots, m.$$

We interpret the stochastic system in the sense of Itô, denote by (A, f, g) stochastic systems of the form (1), and the corresponding unperturbed deterministic system by $(A, f, 0)$. Our aim is to derive explicit conditions on the coefficient functions of the operators A_i^l and the functions f and g to ensure that the solutions of System (1) preserve positivity. The explicit characterization is important as it allows to verify mathematical models arising in various applications, where the solutions describe positive quantities (see [5]). In this case, that is, if solutions emanating from non-negative initial data remain non-negative as long as they exist, we say that the system satisfies the *positivity property.* To study the positivity of solutions of the stochastic system we construct a family of random PDEs such that its solutions converge in expectation to the solution of the stochastic system. We are in particular interested in characterizing the class of stochastic perturbations g such that the family of random approximations satisfies the positivity property, which then implies the positivity of solutions of the stochastic system. Moreover, we prove that the positivity is preserved for both, Itôs and Stratonovich's interpretation of stochastic differential equations (see [7]).

Applications that fall into the class of stochastic models (1) are for instance predator-prey systems under stochastic perturbations. We give an example that was discussed in [2] (Section 5), further applications can be found in [3] (Section 6). The deterministic model is formulated as reaction-diffusion system for the predator u and the prey v in a bounded spatial domain $O \subset \mathbb{R}^3$,

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + D(u,v) \begin{pmatrix} u \\ v \end{pmatrix},$$

where the matrix $D(u,v)$ is of the form

$$D(u,v) = \begin{pmatrix} \beta_1\left(\left|\frac{v}{u}\right|\right) & c\beta_2\left(\left|\frac{v}{u}\right|\right) \\ 0 & \left[\gamma - \beta_2\left(\left|\frac{v}{u}\right|\right)\right] \end{pmatrix},$$

and the solutions satisfy homogeneous Neumann boundary conditions. The constants c and γ are positive, and the functions $\beta_1, \beta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ non-negative. It can be verified by our deterministic positivity criterion (see Theorem 3 in Section 2.2) that the system preserves positivity. The model includes a certain uncertainty since it is impossible to determine the exact model parameters γ, β_1 and β_2 (see [2]). One possibility to take this into account is to add noise, which leads to the following stochastic model

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \left[D(u,v) + dW_t \,\mathrm{Id}\right] \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2}$$

where Id denotes the identity matrix. Our main result (Theorem 5 in Section 3.2) implies that the positivity of solutions is also preserved by the stochastic system (2), and this is valid independent of the choice of Itô's or Stratonovich's interpretation of stochastic differential equations.

The outline of the paper is as follows: In the introductory sections we present our problem, explain our strategy to study the positivity property of stochastic systems and formulate our main result. We specify the class of deterministic systems in Section 2 and formulate necessary and sufficient conditions for the positivity of solutions of semi-linear parabolic PDEs. The positivity property of stochastic systems is analysed in Section 3. Our proof is based on the deterministic result and essentially uses an approximation theorem for stochastic systems. This random approximation theorem was obtained by Chueshov-Vuillermot in [3] and is recalled in Subsection 3.1. Finally, in Subsection 3.2 we formulate and prove our positivity criterion for stochastic systems.

1.1. **Deterministic case.** Random equations can be interpreted pathwise and allow to apply deterministic methods. The random approximations lead to a family of non-autonomous parabolic systems. For autonomous deterministic systems of quasi-linear and semi-linear PDEs *necessary and sufficient* conditions for the positivity of solutions were obtained in [5]. Generalizing the results for non-autonomous deterministic systems of the form $(A, f, 0)$ we deduce the following criterion (see Theorem 3 in Section 2).

Theorem 1. *Under appropriate conditions on the coefficient functions of the operators* A *and* B *the deterministic system* $(A, f, 0)$ *satisfies the positivity property if and only if the differential operators are diagonal, and the components of the interaction function satisfy*

$$f^l(x,t,u^1,\dots,u^{l-1},0,u^{l+1},\dots,u^m) \geq 0 \qquad \text{for } x \in O, t \geq 0,\ u^k \geq 0,\ k,l = 1,\dots,m.$$

This theorem yields explicit conditions on the coefficients of the differential operator A and the interaction function f, which are easy to verify, and allows to validate mathematical models (see [8], Section 4.3, p.76, and [5]). As a consequence, it suffices to consider stochastic systems with diagonal differential operators

$$du^l(x,t) = \Big(-A^l(x,D)u^l(x,t) + f^l(x,t,u(x,t))\Big)\,dt + \sum_{i=1}^{\infty} q_j g_j^l(x,t,u(x,t))dW_t^j, \tag{3}$$

$l = 1,\dots,m$, where $x \in O$ and $t > 0$. In the sequel we denote by (f, g) the system of SPDEs (3) and the corresponding unperturbed deterministic system by $(f, 0)$.

1.2. **Stochastic case.** To analyse the stochastic problem (f, g) with diagonal differential operator A we use a Wong-Zakaï type approximation theorem, which was obtained by Chueshov-Vuillermot in [3], and yields a family of random approximations $(f_{\epsilon,\omega}, 0)$ for the stochastic system. The solutions of the random approximations do not converge to the solution of the original system, but to the solution of a modified stochastic system. Hence, we first construct in Section 3.2 an auxiliary stochastic system (F, g) such that the solutions of the corresponding random approximations $(F_{\epsilon,\omega}, 0)$ converge to the solution of our original problem (f, g). We apply the deterministic result to derive explicit necessary and sufficient conditions for the positivity property of the random systems $(F_{\epsilon,\omega}, 0)$. Since these conditions are preserved by the random approximations and are invariant under the transformation relating the original and the modified system, our main result yields an *explicit characterization* of the stochastic perturbations g and interaction functions f to ensure that the stochastic system (f, g) satisfies the positivity property. Furthermore, the conditions for the positivity property of the random PDEs are invariant under the transformation relating the equations obtained through Itô's and Stratonovich's interpretation. Our main result, stated in the following theorem, is therefore independent of the choice of interpretation:

Theorem 2. *Let (f, g) be a system of stochastic PDEs of the form (3), which is interpreted in the sense of Itô or Stratonovich. Then, the associated family of random approximations $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property if and only if the interaction term satisfies*

$$f^l(x, t, u^1, \dots, u^{l-1}, 0, u^{l+1}, \dots, u^m) \geq 0 \qquad x \in O, t \geq 0, \quad \text{for } u^k \geq 0,$$

and the stochastic perturbation g fulfils

$$g^l_j(x, t, u^1, \dots, u^{l-1}, 0, u^{l+1}, \dots, u^m) = 0 \qquad x \in O, t \geq 0, \quad \text{for } u^k \geq 0,$$

for all $j \in \mathbb{N}$ and $k, l = 1, \dots, m$. In this case, the stochastic system (f, g) satisfies the positivity property.

Remark 1. *Up to the author's knowledge for systems of stochastic PDEs only sufficient conditions for the positivity of solutions are known. Initially, we were hoping to obtain a stronger result. Namely, that the stochastic system (f, g) satisfies the positivity property if and only if the stochastic perturbation g and the interaction function f satisfy the conditions in Theorem 2. P. Kotelenez proved this equivalence in* [6] *for scalar parabolic equations. His proof is not based on random approximations. While the sufficiency of the conditions is formulated in our theorem, showing the necessity is more involved since we cannot deduce the non-negativity of solutions of the random approximations from the non-negativity of the solutions of the stochastic system. Hence, we cannot directly apply the necessary conditions known in the deterministic case.*

2. The Deterministic Case - Necessary and Sufficient Conditions for Positivity

2.1. **Semi-linear Systems of Parabolic PDEs.** Necessary and sufficient conditions for autonomous systems of semi-linear and quasi-linear reaction-diffusion-convection-equations were studied in [5]. We cannot directly apply these results in our case since the Wong-Zakaï approximations lead to a family of parabolic systems with time-dependent interaction functions. In this section we formulate a generalization of one of the theorems in [5] allowing for non-autonomous interaction functions and arbitrary linear elliptic differential operators of second order. For its proof we refer to [4], which uses the same methods and ideas as applied in the mentioned article.

To be more precise, we consider the following class of systems of semi-linear parabolic equations

$$\partial_t u^l(x,t) = -\sum_{i=1}^{m} A_i^l(x,D)u^i(x,t) + f^l(x,t,u(x,t)) \qquad x \in O, t > 0, \tag{4}$$

where $u = (u^1, \dots, u^m)$ is a vector-valued function, and $O \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain with smooth boundary ∂O.

Assumptions on the operator A

The linear second order differential operators $A_i^l(x, D)$ are defined as

$$A_i^l(x,D) = -\sum_{k,j=1}^{n} a_{kj}^{il}(x)\partial_{x_k}\partial_{x_j} + \sum_{k=1}^{n} a_k^{il}(x)\partial_{x_k} \qquad \text{for } x \in O,\ i,l = 1,\dots,m.$$

Compared to the setting in [3] we omit the zero-order terms in the operator A as for our problem it seems more natural to absorb these terms in the interaction function f. We assume that the coefficient functions satisfy $a_{kj}^{il} = a_{jk}^{il}$, and the operators are uniformly elliptic,

$$\sum_{k,j=1}^{n} a_{kj}^{il}(x)\zeta_k\zeta_j \geq \mu|\zeta|^2 \qquad \text{for all } x \in O, \zeta \in \mathbb{R}^n,\ i,l = 1,\dots,m.$$

Moreover, all coefficient functions of the operator A are continuously differentiable and bounded in the domain O.

Assumptions on the boundary operators B

The boundary values of the solution are determined by the operators

$$B^l(x,D) = b_0^l(x) + \delta^l \sum_{k=1}^{n} b_k^l(x)\partial_{x_k} \qquad l = 1,\dots,m,$$

where $\delta^l \in \{0,1\}$. The functions b_k^l, b_0^l are smooth on the boundary ∂O and satisfy $b_0^l \geq 0$. Moreover, we assume $b_0^l \equiv 1$ for $\delta^l = 0$, and $b^l = (b_1^l, \dots, b_m^l)$ is an outward pointing, nowhere tangent vector-field on the boundary ∂O.

Assumptions on the non-linear interaction term f

For the interaction function we assume that the partial derivatives $\partial_u f^l$ exist and are continuous, $l = 1, \ldots, m$. Moreover, we assume that for $x \in O$ and $t > 0$ the functions $f^l = f^l(x,t,u)$ and $\partial_u f^l = \partial_u f^l(x,t,u)$ are bounded for bounded values of u.

2.2. **A Positivity Criterion.** To formulate our criterion for the positivity of solutions we define the positive cone in $L^2(O;\mathbb{R}^m)$.

Definition 1. *The* ***positive cone*** *is the set of all componentwise non-negative functions in* $L^2(O;\mathbb{R}^m)$*,*

$$K^+ := \{u \in L^2(O;\mathbb{R}^m) \mid u^i \geq 0 \ a.e. \text{ in } O, i = 1, \ldots, m\}.$$

Furthermore, we say that System (4) satisfies the ***positivity property*** *if every solution* $u(\,\cdot\,,\cdot\,;u_0) : O \times [0,T] \to \mathbb{R}^m$ *originating from non-negative initial data* $u_0 \in K^+$ *remains non-negative (as long as it exists); that is,* $u(\,\cdot\,,t;u_0) \in K^+$ *for* $t \in [0,T]$*. Thereby,* $[0,T]$ *denotes the maximal existence interval of the solution.*

Our concern is not to study the existence of solutions but their qualitative behaviour. Hence, in the sequel we assume that for any initial data $u_0 \in K^+$ there exists a unique solution, and for $t > 0$ the solution satisfies L^∞-estimates. The following theorem provides a criterion for the positivity property of System (4) and generalizes the previous result for semi-linear systems in [5].

Theorem 3. *Let the operators A and B be defined as in the beginning of this section and the above conditions on the coefficient functions of the operators and interaction functions be satisfied. Moreover, we assume the initial data $u_0 \in K^+$ is smooth and fulfils the compatibility conditions. Then, System (4) satisfies the positivity property if and only if the matrices* $\big(a^{il}_{kj}\big)_{1\leq i,l\leq m}$ *and* $\big(a^{il}_{k}\big)_{1\leq i,l\leq m}$ *are diagonal for all* $1 \leq j,k \leq n$*, and the components of the reaction term satisfy*

$$f^i(x,t,u^1,\ldots,\underbrace{0}_{i},\ldots,u^m) \geq 0, \qquad \text{for } x \in O, t > 0, u^k \geq 0,\ i,k = 1\ldots m. \tag{5}$$

This theorem can be proved by extending the method applied in [5], for a detailed proof we refer to the forthcoming article [4].

Consequently, it suffices to consider stochastic perturbations of systems of semi-linear PDEs of the form (3). A Wong-Zakaï approximation theorem for such systems was obtained in [3].

3. The Stochastic Case - Necessary and Sufficient Conditions for the Positivity of Random Approximations

We use a Wong-Zakaï-type approximation theorem and associate to a given stochastic system (f, g) a suitable family of random approximations. The result of the previous section yields necessary and sufficient conditions for the positivity property of the family of random PDEs. Since the solutions of the random approximations converge in expectation to the solution of the original problem, theses conditions ensure the positivity property of the stochastic system.

3.1. **Wong-Zakaï Approximation and Random Systems of PDEs.** E. Wong and M. Zakaï ([10],[11]) studied the relation between ordinary and stochastic differential equations and introduced a smooth approximation of the Brownian motion to approximate stochastic integrals by ordinary integrals. Doing so, they obtain an approximation of the stochastic differential equation by a family of random differential equations. However, when the smoothing parameter tends to zero the random solutions do not converge to the solution of the original stochastic differential equation, but a modified one. The appearing correction term is called *Wong-Zakaï correction term.* The Wong-Zakaï approximation theorem has been generalized in various directions. In this section, we briefly recall the main result by Chueshov-Vuillermot in [3] about a Wong-Zakaï-type approximation theorem for a class of stochastic systems of semi-linear parabolic PDEs, which is, in particular, applicable for the systems we consider.

Assumptions on the stochastic perturbations

We assume $\{W_t^j,\, t \geq 0\}_{j\in\mathbb{N}}$ is a family of mutually independent standard scalar Wiener processes on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, and dW_t^j denotes the corresponding Itô differential. The non-negative parameters q_j are normalization factors. Moreover, the functions $g_j^l : O \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are smooth and assumed to be bounded for bounded values of the solution, where $j \in \mathbb{N}, l = 1, \ldots, m$.

3.1.1. *Smooth Predictable Approximation of the Wiener Process.* A general notion of a smooth predictable approximation of the Wiener process is defined by Chueshov and Vuillermot in [3] (Definition 4.1, p.1440). In the following, we will take their main example as a definition (Proposition 4.2, p.1441).

Let $\{W_t, t \geq 0\}$ be a standard scalar Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t, t \geq 0\}$. The ***smooth predictable approximation*** of $\{W_t, t \geq 0\}$ is the family of random processes $\{W_\epsilon(t), t \geq 0\}_{\epsilon>0}$ defined by

$$W_\epsilon(t) = \int_0^\infty \phi_\epsilon(t - \tau) W_\tau d\tau,$$

where $\phi_\epsilon(t) = \epsilon^{-1}\phi(t/\epsilon)$, and $\phi(t)$ is a function with the properties

$$\phi \in C^1(\mathbb{R}),\ \text{supp}\phi \subset [0,1],\ \int_0^1 \phi(t)dt = 1.$$

We will need the following result ([3], p.1442), which states that the derivative of the smooth predictable approximation W_ϵ, denoted by $\dot{W}_\epsilon$, can be written as a stochastic integral of the form

$$\dot{W}_\epsilon(t) \quad = \quad \int_{t-\epsilon}^{t} \phi_\epsilon(t-\tau)dW_\tau,\ t \geq \epsilon. \tag{6}$$

As a consequence, $\dot{W}_\epsilon$ is Gaussian, which will be fundamental in our proof.

3.1.2. *Predictable Smoothing of Itô's Problem and Random Systems.* Using the previously defined family of smooth predictable approximations $\{W_\epsilon^j(t), t \geq 0\}_{\epsilon>0, j\in\mathbb{N}}$ of the Wiener processes $\{W_t^j, t \geq 0\}_{j\in\mathbb{N}}$ the predictable smoothing of Itô's problem (3) is the family of random equations

$$du^l(x,t) = \big(-A^l(x,t,D)u^l(x,t) + f^l(x,t,u(x,t))\big)dt + \big(\sum_{j=1}^{\infty} q_j g_j(x,t,u(x,t))\dot{W}_\epsilon^j(t)\big)dt, \tag{7}$$

where $l = 1, \ldots, m$. Using our notation, we are led to the following definition:

Definition 2. *The* ***smooth random approximation of the stochastic system*** (f,g) *of PDEs with respect to the smooth predictable approximation* $\{W_\epsilon(t), t \geq 0\}_{\epsilon>0}$ *is the family of random PDEs* $(f_{\epsilon,\omega}, 0)$*, where*

$$f_{\epsilon,\omega}^l(x,t,u(x,t)) = f^l(x,t,u(x,t)) + \sum_{j=1}^{\infty} q_j g_j^l(x,t,u(x,t))\dot{W}_\epsilon^j(t) \qquad \epsilon > 0.$$

3.1.3. *A Wong-Zakaï Approximation Theorem.* Following Chueshov and Vuillermot ([3], p.1436) we consider mild solutions of the stochastic system of PDEs (f,g):
In the following definition the family $\{U(t),\ t \geq 0\}$ denotes the linear semigroup generated by the operator $A = (A^1, \ldots, A^m)$ in $L^2(O;\mathbb{R}^m)$ with domain

$$W_B^{2,2}(O;\mathbb{R}^m) := \{u \in W^{2,2}(O;\mathbb{R}^m)\ :\ Bu = 0\}.$$

Here, B indicates the boundary operator and

$$W^{k,2}(O) := \{u \in L^2(O)\ :\ D^\alpha u \in L^2(O) \text{ for all } |\alpha| \leq k\}.$$

Definition 3. *A random function* $u(x,t,\omega) = (u^1(x,t,\omega), \ldots, u^m(x,t,\omega))$ *is called a* ***mild solution*** *of the stochastic problem* (f,g) *in the space* $V = W_B^{1,2}(O;\mathbb{R}^m)$ *on the interval* $[0,T]$*, if* $u(t) = u(x,t,\omega) \in C(0,T;L^2(\Omega\times O))$ *is a predictable process such that*

$$\int_0^T E \parallel u(t) \parallel_V^2 dt < \infty,$$

and satisfies the integral equation

$$u(t) = U(t)u_0 + \int_0^t U(t-\tau)f(\tau, u(\tau))d\tau + \sum_{j=1}^{\infty} q_j \int_0^t U(t-\tau)g_j(\tau, u(\tau))dW^j(\tau, \omega), \tag{8}$$

where we assume that all integrals in (8) exist.

For further details we refer to [3] and [1].

Definition 4. *Let (f, g) be a stochastic system of PDEs and u be its mild solution. We say that the mild solutions u_ϵ of a family of random PDEs $(F_{\epsilon,\omega}, 0)$* ***converge*** *to the mild solution of the stochastic system (f, g) if*

$$\lim_{\epsilon \to 0} \int_0^T E \parallel u(t) - u_\epsilon(t) \parallel^2_{W^{1,2}(O;\mathbb{R}^m)} dt = 0.$$

Thereby, the function u_ϵ is a mild solution of the family of random PDEs $(F_{\epsilon,\omega}, 0)$ if it satisfies the integral equation

$$u_\epsilon(t) = U(t)u_0 + \int_0^t U(t-\tau)F_{\epsilon,\omega}(\tau, u_\epsilon(\tau))d\tau.$$

The main result of Chueshov and Vuillermot in [3] is the following (Theorem 4.3, p.1443):

Theorem 4. *Assume that the stated assumptions on the operators A and B and the functions f and g are satisfied. Moreover, let $\sum_{j=1}^{\infty} q_j < \infty$, the initial data $u_0 \in C_B^2(O; \mathbb{R}^m)$ be $\mathcal{F}_0$-measurable and $E\|u_0\|^r_{C^2(O)} < \infty$ for some $r > 8$. We assume the associated system of random PDEs $(f_{\epsilon,\omega}, 0)$ has a mild solution u_ϵ belonging to the class $C(0, T; L^r(\Omega, X_{\alpha,p}))$ for all $0 \leq \alpha < 1$ and $p > 1$, and for this solution there exists a constant C independent of ϵ such that*

$$\sup_{t\in[0,T]} E \parallel u_\epsilon \parallel^r_{L^p(O)} \leq C \qquad \text{for all } p > 1.$$

Then, the mild solutions u_ϵ converge to a solution u_{cor} of the corrected stochastic system of PDEs (f_{cor}, g) when ϵ tends to zero, where

$$f^l_{\text{cor}} = f^l + \frac{1}{2} \sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g_j^i \frac{\partial g_j^l}{\partial u^i} \qquad \text{for } l = 1, \ldots m.$$

The spaces $X_{\alpha,,p}$ in Theorem 4 denote the fractional power spaces associated to the operator A. For further details we refer to [3].

3.2. **A Positivity Criterion for Systems of Stochastic PDEs.** Our aim is to study the positivity property of the stochastic system (f, g) of the form (3). Hence, in the sequel we assume that a unique solution of the stochastic initial value problem exists, and the solutions of the random approximations converge to the solution of the modified stochastic system (cf. Theorem 4). Sufficient conditions for the existence and uniqueness of solutions under even more general assumptions can be found in the article [3] (Section 3). Since the solutions of the random approximations do not converge to the solution of the original system we construct an auxiliary stochastic system as follows:

- Let (F, g) be a stochastic system of the form (3). The corresponding family of random approximations $(F_{\epsilon,\omega}, 0)$, $\epsilon > 0$, $\omega \in \Omega$ is explicit, depends on the definition of the smooth approximation W_ϵ of the Wiener process $\{W(t), t \geq 0\}$, and is given by
$$F^l_{\epsilon,\omega} = F^l + \sum_{j=1}^{\infty} q_j g^l_j \dot{W}^j_\epsilon \qquad l = 1, \ldots, m.$$
- Theorem 4 states that the solutions of the random systems converge in expectation to the solution of the modified stochastic system (F_{cor}, g), where
$$F^l_{cor} = F^l + \frac{1}{2} \sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g^i_j \frac{\partial g^l_j}{\partial u^i} \qquad l = 1, \ldots, m.$$
- To analyse a given stochastic system (f, g) of the form (3) we therefore construct an auxiliary system (F, g) (see Equation (9) below) such that the solutions of the associated system of random PDEs $(F_{\epsilon,\omega}, 0)$ converge to the solutions of our original system (f, g).
- We then use the deterministic positivity criterion to derive necessary and sufficient conditions for the positivity property of the family of random approximations $(F_{\epsilon,\omega}, 0)$. Finally, we show that this property is preserved by the transformation relating the original system and the modified system and by passing to the limit when ϵ goes to zero.

Let (f, g) be a given system of SPDEs. If we define the auxiliary stochastic system (F, g) by

$$F^l = f^l - \frac{1}{2} \sum_{j=1}^{\infty} q_j^2 \left(g^1_j \frac{\partial g^l_j}{\partial u^1} + \cdots + g^m_j \frac{\partial g^l_j}{u^m} \right) \qquad l = 1, \ldots, m, \tag{9}$$

then, the solutions of the family of random PDEs $(F_{\epsilon,\omega}, 0)$ converge to the solution of the original stochastic system (f, g).
Theorem 3 states that the deterministic system $(f, 0)$ satisfies the positivity property if and only if the the interaction term satisfies Condition (5) in Theorem 3. This motivates the following definition.

Definition 5. *We say that the function*

$$f : O \times \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m, \;\; f(x, t, u) = (f^1(x, t, u), \ldots, f^m(x, t, u)),$$

*satisfies the **positivity condition** if it satisfies Property (5).*

The next lemma will be essential for the proof of our main result.

Lemma 1. *Let (f, g) be a given stochastic system of PDEs. We assume that the functions g^l_j are twice continuously differentiable with respect to u^k and satisfy*

$$g^l_j(x, t, u^1, \ldots, \underbrace{0}_{l}, \ldots, u^m) = 0 \qquad x \in O, t > 0,\ u^k \geq 0, \tag{10}$$

for all $j \in \mathbb{N}$ and $k, l = 1, \ldots, m$. Then, the following statements are equivalent:

(a) *The function f satisfies the positivity condition.*
(b) *The modified function F satisfies the positivity condition.*
(c) *The associated random functions $F_{\epsilon,\omega}$ satisfy the positivity condition for all $\epsilon > 0$ and $\omega \in \Omega$.*

Proof. The proof is a simple computation. Let $j \in \mathbb{N}$ and $1 \leq l \leq m$. Since g^l_j is continuously differentiable with respect to u^l and satisfies $g^l_j(x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m) = 0$, we can represent it in the form $g^l_j(x, t, u) = u^l G^l_j(x, t, u)$ with a continuously differentiable function G^l_j. We obtain for the sum appearing in the Wong-Zakaï correction term

$$\sum_{i=1}^{m} g^i_j \frac{\partial g^l_j}{\partial u^i} = \sum_{i=1}^{m} g^i_j \frac{\partial (u^l G^l_j)}{\partial u^i} = \sum_{i \neq l} g^i_j u^l \frac{\partial G^l_j}{\partial u^i} + g^l_j \frac{\partial (u^l G^l_j)}{\partial u^l},$$

which leads to an associated function F of the form

$$F^l = f^l - \frac{1}{2} \sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g^i_j \frac{\partial g^l_j}{\partial u^i} = f^l - \frac{1}{2} \sum_{j=1}^{\infty} q_j^2 \left(\sum_{i \neq l} g^i_j u^l \frac{\partial G^l_j}{\partial u^i} + g^l_j \frac{\partial (u^l G^l_j)}{\partial u^l} \right).$$

Due to Assumption (10) we note that the modified function F satisfies the positivity condition if and only if f satisfies the positivity condition since all correction terms vanish if $u^l = 0$. Finally, the associated system of random PDEs $(F_{\epsilon,\omega}, 0)$ is given by

$$F^l_{\epsilon,\omega} = F^l + \sum_{j=1}^{\infty} q_j g^l_j \dot{W}^j_\epsilon.$$

Condition (10) therefore implies

$$F^l_{\epsilon,\omega}(x, t, u^1, \ldots, \underbrace{0}_{l}, \ldots, u^m) = F^l(x, t, u^1, \ldots, \underbrace{0}_{l}, \ldots, u^m) = f^l(x, t, u^1, \ldots, \underbrace{0}_{l}, \ldots, u^m),$$

which concludes the proof of the lemma. □

Applying Lemma 1 we derive necessary and sufficient conditions for the positivity property of the random approximations.

Theorem 5. *Let (f, g) be a system of stochastic PDEs and $(F_{\epsilon,\omega}, 0)$ be the associated family of random approximations. We assume that the functions g^l_j are twice continuously differentiable* *with respect to u^k, for all $j \in \mathbb{N}$ and $k, l = 1, \ldots, m$. Then, the family of random approximations $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property for all ω and (sufficiently small) $\epsilon > 0$ if and only if f satisfies the positivity condition and the stochastic perturbation g fulfils Property (10). In this case, the stochastic system (f, g) satisfies the positivity property.*

Proof. ***Sufficiency:*** By assumption, the function f satisfies the positivity condition. Since the stochastic perturbation fulfils Property (10), Lemma 1 implies the positivity condition for random functions $F_{\epsilon,\omega}$, $\omega \in \Omega, \epsilon > 0$. We now apply the deterministic positivity criterion (Theorem 3) to conclude the non-negativity of the solutions of the random approximations. Finally, the Wong-Zakaï approximation theorem states that the solutions of the random approximations $(F_{\epsilon,\omega}, 0)$ converge in expectation to the solution of the stochastic system (f, g), which implies that the stochastic system (f, g) satisfies the positivity property.

Necessity: We assume the family of random PDEs $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property. By Theorem 3 this is equivalent to the positivity condition for the random functions $F^l_{\epsilon,\omega}$; that is,

$$F^l_{\epsilon,\omega}(x,t,\tilde{u}) \quad = \quad F^l(x,t,\tilde{u}) + \sum_{j=1}^{\infty} q_j g^l_j(x,t,\tilde{u}) \dot{W}^j_\epsilon(t) \ \geq 0 \qquad x \in O, t > 0, \tag{11}$$

where $\tilde{u} := (u^1, \ldots, \underbrace{0}_{l}, \ldots, u^m)$, $u^k \geq 0$, $k, l = 1, \ldots, m$. The derivative of the smooth approximation $W_\epsilon(t)$ of the Wiener process can be represented as the stochastic integral (6) and takes arbitrary values. Assuming that the function $g^l_j(x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m)$ is not identically zero, then for sufficiently small $\epsilon > 0$ we always find an $\omega \in \Omega$ such that the inequality (11) is violated. This proves the necessity of the condition on the stochastic perturbation. If Property (10) holds, the positivity condition for the family of random approximations is equivalent to the positivity condition for the function f by Lemma 1. □

The same result is valid if we apply Stratonovich's interpretation of stochastic differential equations. In other words, the positivity property of solutions of the stochastic system is independent of the choice of interpretation, which was stated in Theorem 2 in the introduction.

Corollary 1. *Let (f, g) be a system of stochastic (Itô) PDEs. We assume the hypothesis of Theorem 5 are satisfied and the family of random approximations $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property. Then, the stochastic system $(f, g)_{Strat}$ obtained when we use Stratonovich's interpretation of the stochastic differential equations satisfies the positivity property.*

Proof. The Wong-Zakaï correction term coincides with the transformation relating Ito's and Stratonovich's interpretation of the stochastic system (see [9], Section 6.1). That is, the solutions of the random approximations $(f_{\epsilon,\omega}, 0)$ converge to the solution of the given stochastic system, when interpreted in the sense of Stratonovich. Hence, the corollary is an immediate consequence of Theorem 5 and Lemma 1. □

The intuitive interpretation of the condition on the stochastic perturbation is the following: In the critical case, when one component of the solution approaches zero, the stochastic perturbation needs to vanish. Otherwise, the positivity of the solution cannot be guaranteed.

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