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Parabolic H-measures

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Abstract

Classical H-measures introduced by Tartar (1990) and independently by Gérard (1991) are not well suited for the study of parabolic equations. Recently, several parabolic variants have been proposed, together with a number of applications. We introduce a new parabolic variant (and call it the parabolic H-measure), which is suitable for these known applications. Moreover, for this variant we prove the localisation and propagation principle, establishing a basis for more demanding applications of parabolic H-measures, similarly as it was the case with classical H-measures. In particular, the propagation principle enables us to write down a transport equation satisfied by the parabolic H-measure associated to a sequence of solutions of a Schrödinger type equation. Some applications to specific equations are presented, illustrating the possible use of this new tool. A comparison to similar results for classical H-measures has been made as well.

Keywords: parabolic H-measures, localisation principle, propagation principle

Mathematics subject classification: 35B27, 35K05, 35Q40, 35S05, 46F25, 46G10, 47G30

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1. Introduction

H-measures

In various situations concerning partial differential equations one often encounters weakly converging sequences (in order to fix the ideas, let us take $u_n \rightharpoonup u$ in $\mathrm{L}^2(\mathbf{R}^d)$), which do not converge strongly (numerous such examples can be found in [17]). For such a sequence it is natural to consider a L^1 bounded sequence $|u_n - u|^2$, which in general does not converge weakly in L^1, but only vaguely (after eventual extraction of a subsequence) in the space of bounded Radon measures ($\mathcal{M}_b = \mathrm{C}_0'$, the *convergence vague* being just the weak $*$ convergence) to a defect measure ν. This simple object was the starting point of Pierre-Louis Lions' study of variational elliptic problems [29, 30].

Essentially, there are two distinctive types of non-compact sequences (for these two examples we fix $\varphi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ such that $\|\varphi\|_{\mathrm{L}^2(\mathbf{R}^d)} = 1$, and note that in both cases below one has $u_n \rightharpoonup 0$):

a) *concentration*: $u_n(\mathbf{x}) := n^{d/2}\varphi(n(\mathbf{x} - \mathbf{x}_0))$, where $\nu = \delta_{\mathbf{x}_0}$, and

b) *oscillation*: $u_n(\mathbf{x}) := \varphi(\mathbf{x})e^{-2\pi i n \mathbf{x}\cdot\boldsymbol{\xi}}$, where ν is actually equal to $|\varphi|^2$ (i.e. to the measure having density $|\varphi|^2$ with respect to the Lebesgue measure).

In the seventies of the last century, Luc Tartar [39, 42] proposed a new mathematical approach for solving nonlinear partial differential equations of continuum mechanics, in particular the method of compactness by compensation (the term *compacité par compensation* was coined by Jacques-Louis Lions, and it is usually translated—imprecisely—as *compensated compactness*). More precisely, in continuum mechanics one distinguishes between two types of laws: the general balance laws, which are expressed as linear partial differential equations, and are amenable to treatment by the method of compactness by compensation, and pointwise nonlinear constitutive relations, which are treated by Young measures. This approach led to a number of successful applications, like those initiated by Ronald DiPerna in conservation laws [15] or by John Ball and Richard James in materials science [13] (see also [33]).

However, in a number of situations Young measures turned out to be inadequate [41]. One can get an idea of such an inadequacy by considering the above examples of oscillating sequences. The direction $\boldsymbol{\xi}$ of the oscillation can be important in some problems, but neither Young measures nor defect measures can capture that information.

This was one of the deficiencies motivating Tartar to introduce a new mathematical tool, H-measures [40] (essentially the same objects were introduced independently, under the name of *microlocal defect measures*, by Patrick Gérard [22] practically at the same time). These new objects indicate where in the physical space, and at which frequency in the Fourier space, are the obstacles to strong L^2 convergence. The corresponding defect measure can be obtained simply by integrating the H-measure with respect to the Fourier space variable $\boldsymbol{\xi}$.

Consider a domain $\Omega \subseteq \mathbf{R}^d$; an H-measure is a Radon measure on the product $\Omega \times S^{d-1}$ (in general, we can take a manifold Ω, and the corresponding cospherical bundle). In order to apply the Fourier transform, functions defined on the entire $\mathbf{R}^d$ should be considered and this can be achieved by extending the functions by zero outside of the domain. Such an extension preserves the weak convergence in L^2. After such adjustment, the following theorem can be stated [40, 22] (the reader might also choose to consult [18, 44], which are written more recently):

Theorem 1 (existence of H-measures) *If* (u_n) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$*, such that* $\mathsf{u}_n \xrightharpoonup{\mathrm{L}^2} \mathsf{0}$ *(weakly), then there exists a subsequence* $(\mathsf{u}_{n'})$ *and a complex* $r \times r$ *matrix Radon* *measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times S^{d-1}$ *such that for all* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(S^{d-1})$*:*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}_{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}_{n'}\Big) \psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times S^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi})\ .$$

∎

The above matrix Radon measure $\boldsymbol{\mu}$ is hermitian and it is called *H-measure*. We shall often abuse the notation and terminology, assuming that we have already passed to a subsequence determining an H-measure.

As an immediate consequence we have that any H-measure associated to a strongly convergent sequence is necessarily zero.

Notation. Throughout this paper $\otimes$ stands for the vector tensor product on $\mathbf{C}^d$, defined by $(\mathsf{a} \otimes \mathsf{b})\mathsf{v} = (\mathsf{v} \cdot \mathsf{b})\mathsf{a}$ (in components $(\mathsf{a} \otimes \mathsf{b})_{ij} = a_i \bar{b}_j$), while $\boxtimes$ stands for the tensor product of functions in different variables. By $\cdot$ we denote a (complex) scalar product for vectors and matrices (linear operators, elements of $\mathrm{M}_{d \times d}$) on $\mathbf{C}^d$, in the latter case defined as $\mathbf{A} \cdot \mathbf{B} = \mathsf{tr}(\mathbf{A}\mathbf{B}^*)$. On the other side, $\langle \cdot, \cdot \rangle$ denotes a sesquilinear dual product, which we take to be antilinear in the first variable, and linear in the second. When vector or matrix functions appear as both arguments in a dual product, we interpret it as $\langle \mathsf{u}, \mathsf{v} \rangle = \int \mathsf{v} \cdot \mathsf{u} = \langle\, \mathsf{v} \mid \mathsf{u} \,\rangle$ and $\langle \mathbf{A}, \mathbf{B} \rangle = \int \mathbf{B} \cdot \mathbf{A} = \int \mathsf{tr}(\mathbf{B}\mathbf{A}^*)$. This choice allows us to use the complex scalar product, while preserving the known formulæ for H-measures.

Variables in $\mathbf{R} \times \mathbf{R}^d$ are denoted by $(t, \mathbf{x}) = (t, x^1, \ldots, x^d) = (x^0, x^1, \ldots, x^d)$, whichever is more convenient, and $\partial_k = \frac{\partial}{\partial x^k}$. Similarly for the *dual* variable $(\tau, \boldsymbol{\xi}) = (\tau, \xi_1, \ldots, \xi_d) = (\xi_0, \xi_1, \ldots, \xi_d)$, where the derivatives are denoted by $\partial^l = \frac{\partial}{\partial \xi_l}$. We conveniently write $\nabla_{t,\mathbf{x}} = (\partial_0, \nabla_{\mathbf{x}})$, and use ∇ for $\nabla_{\mathbf{x}}$. When we deem it convenient, we explicitly write the variables of differentiation, like $\nabla^{\boldsymbol{\xi}}$ and $\nabla_{\mathbf{x}}$, or for Schwartz' notation $\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}$ (where $\boldsymbol{\alpha} \in \mathbf{N}^d$ is a multiindex). Summation with repeated indices, one upper and one lower, is assumed.

The Fourier transform is defined as $\hat{\mathsf{u}}(\boldsymbol{\xi}) := \mathcal{F}\mathsf{u}(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \mathsf{u}(\mathbf{x})\, d\mathbf{x}$. As $\widehat{\partial_j f} = 2\pi i \xi_j \hat{f}$, occasionally it will be more convenient to use the reduced derivative operator $D_j := \frac{1}{2\pi i}\partial_j$ (similarly we define $D^j := \frac{1}{2\pi i}\partial^j$) in order to have a simple relation $\widehat{D_j f} = \xi_j \hat{f}$, which we shall need for symbols of (pseudo)differential operators.

We denote the Lebesgue measure by λ and integration over $\mathbf{R}^d$ with respect to the Lebesgue measure by $d\mathbf{x} = d\lambda(\mathbf{x})$; for integration over a surface we simply write dA. Closed and open intervals in $\mathbf{R}$ we denote by $[a, b]$ and $\langle a, b \rangle$, and analogously for semiclosed $[a, b\rangle$ and $\langle a, b]$. ∎

A result similar to Theorem 1 is valid for weakly convergent sequences (u_n) in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{C}^r)$ as well; however, in that case $\boldsymbol{\mu}$ is not necessary a finite Radon measure, but only a distribution of order 0 (in the Bourbaki terminology, a *Radon measure*), which we shall denote by $\boldsymbol{\mu} \in \mathcal{M}(\mathbf{R}^d; \mathrm{M}_{d \times d}) = (\mathrm{C}_c(\mathbf{R}^d; \mathrm{M}_{d \times d}))'$.

H-measures, when applied to multiphase composite materials, appear as two point correlation functions which play a crucial rôle in estimating their effective properties. They were applied by Robert Kohn to multiple-wells problem (quadratic wells in linearised elasticity) [26], where the solution is equivalent to finding all possible H-measures of phase mixtures, being the relaxation of multiple-wells energies (the practical interest being in modelling coherent phase transitions).

One successful application of H-measures is also in extending the compactness by compensation theory from constant coefficient differential relations to variable coefficients [40]. Assume that $\mathsf{u}_n \rightharpoonup \mathsf{0}$ weakly in $\mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$ and satisfy the following differential relations in divergence form (summation over $k \in 1..d$ is implicitly assumed)

$$\partial_k(\mathbf{A}^k \mathsf{u}_n) = \mathsf{f}_n\ ,$$

where each $\mathbf{A}^k$ is a continuous $r \times r$ matrix function, while $\mathsf{f}_n \longrightarrow 0$ strongly in $\mathrm{H}^{-1}_{\rm loc}(\mathbf{R}^d; \mathbf{C}^r)$. Denoting (after passing to a subsequence if needed) the H-measure associated to (u_n) by $\boldsymbol{\mu}$, we have, with the principal symbol (to be precise, up to a multiplicative constant $2\pi i$) in the differential relation being $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$, the following *localisation identity* on $\mathbf{R}^d \times S^{d-1}$:

$$\mathbf{P}\boldsymbol{\mu} = \mathbf{0} \,.$$

This result implies that the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\{(\mathbf{x}, \boldsymbol{\xi}) \in \mathbf{R}^d \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\}$ of points where $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})$ is a singular matrix.

A more complicated result is the *propagation principle* [40], which in a more general setting of symmetric systems takes the form [1]:

Theorem 2 (propagation principle for symmetric systems) *Let the hermitian $r \times r$ matrix functions $\mathbf{A}^k$ be of class $\mathrm{C}^1_0(\Omega)$ and let $\mathbf{B}$ be of class $\mathrm{C}_0(\Omega; \mathrm{M}_{r\times r})$. If for every n the pair $(\mathsf{u}_n, \mathsf{f}_n)$ satisfies the system*

$$\mathbf{A}^k \partial_k \mathsf{u}_n + \mathbf{B}\mathsf{u}_n = \mathsf{f}_n \,,$$

and both sequences (u_n) and (f_n) converge to zero weakly in $\mathrm{L}^2(\Omega;\, \mathbf{C}^r)$, then any H-measure

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix}$$

associated to (a subsequence of) the sequence $(\mathsf{u}_n, \mathsf{f}_n)$ satisfies, in the sense of distributions on $\Omega \times S^{d-1}$, the following first order partial differential equation:

$$\partial_l(\partial^l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) - \partial^l_T(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11})\xi^l + (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2\mathsf{Re\,tr}\boldsymbol{\mu}_{12} \,,$$

where $\partial^l_T := \partial^l - \xi^l \xi_k \partial^k$ is the (l-th component of) tangential gradient on the unit sphere, while $\mathbf{S}$ is the hermitian part of matrix $\mathbf{B}$.

∎

The microlocal energy density for the wave equation, with smooth non-oscillating coefficients and oscillating initial data was successfully computed by Gilles Francfort and François Murat in [19], by considering the initial-value problem for H-measures (see also [23]). Well posedness of such evolution equations for H-measures associated to sequences of solutions of linear transport equations was studied by Sergey Sazhenkov [37].

While Tartar defined H-measures for sequences of functions taking values in finite dimensional spaces, Gérard [22], with a goal of an application to *averaging lemmata*, considered also the functions taking values in a separable Hilbert space. Another such extension, to H-measures parametrised by a continuous parameter, was proposed by Evgeniĭ Panov [34] (see also [27]), with applications in the theory of conservation laws. Some other variants were introduced in [10] and [32].

Related, but certainly different objects are *semiclassical measures* first introduced by Gérard [21], and later renamed by Pierre-Luis Lions and Thierry Paul [31] as *Wigner measures*. Their relation to H-measures was a source of some controversy and conflicting opinions, to which we do not want to contribute, but refer the interested reader to [31, 41, 44] instead.

The above mentioned variants are not the ones we are going to study in this paper. Our goal is to replace the 1:1 ratio between t and $\mathbf{x}$ variables in the definition with 1:2, which will be suited to parabolic problems, as well as to some other situations where such ratio occurs. For such a ratio, we would like to present a theory parallel to what Tartar did in [40], with respective applications. Clearly, based on this, additional variants along the above lines might be introduced. Also, the ratio 1:2 might be replaced by 1:3, 2:3, or some other, but we have not yet worked out any interesting applications in such cases.

This paper is a part of a long-term research programme started in [1] (a parallel recent development towards the same ultimate goal can be found in [4, 5]), where it was clear that for symmetric systems that change their type an improved version of H-measures was needed. Our hope is that better understanding of parabolic H-measures provided by this paper would, besides presenting a number of immediate applications, bring us closer to a new tool, applicable in more general situations.

Parabolic variants

The study of parabolic H-measures has been motivated by a similar problem to the one which motivated Tartar to introduce the original ones; namely the simple model based on the Navier-Stokes equation [40], where in the stationary case the correction term in the homogenised problem could be expressed using the corresponding H-measure. However, for the time-dependent case, where the equation is parabolic, this was not possible. This prompted Tartar to consider parabolic scaling, in some discussions with Konstantina Trivisa and Chun Liu in 1996/97. This was partially written down in [42, pp. 67–69], while quite recently the correction term was expressed via a variant parabolic H-measure in [9] , where it was also proved that this correction is symmetric.

As already stated, H-measures are defined as Radon measures on the product $\Omega \times S^{d-1}$; we often refer to $\Omega \subseteq \mathbf{R}^d$ as the *physical space*, while $\boldsymbol{\xi} \in S^{d-1}$ is the dual variable to $\mathbf{x} \in \mathbf{R}^d$. Symbols are functions on the phase space, in variables $\mathbf{x} \in \Omega$ and $\boldsymbol{\xi} \in \mathbf{R}^d_* := \mathbf{R}^d \setminus \{0\}$, satisfying certain properties; in the simplest case they are products $p(\mathbf{x}, \boldsymbol{\xi}) = a(\boldsymbol{\xi})b(\mathbf{x})$. In particular, it is assumed that a is defined on S^{d-1}, and then extended to $\mathbf{R}^d_*$ by homogeneity $a(\boldsymbol{\xi}) = a\left(\boldsymbol{\xi}/|\boldsymbol{\xi}|\right)$. This introduces the projection $\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$, of $\mathbf{R}^d \setminus \{0\}$ onto S^{d-1}. If we denote the origin by O, and for a given point S, by S_0 its projection to S^{d-1}, while by S' the intersection of the ray OS with the circle around O passing through another point T (see Figure 1), we have equal ratios:

$$d(S_0, T_0) : 1 = d(S, T') : d(O, S) = d(S', T) : d(O, T) ,$$

which easily leads to an important inequality

$$d(S_0, T_0) \leqslant 2 \frac{d(S, T)}{d(O, S) + d(O, T)} .$$

On the other hand, when dealing with the non-stationary Stokes equation, or the heat equation $u_t - u_{xx} = 0$, we are led to consider some variants [7]. The symbol of the heat operator is $i\tau + \xi^2$, and the natural scaling is no longer along the rays through the origin, but along the parabolas $\tau = c\xi^2$ in the dual space (Figure 2).

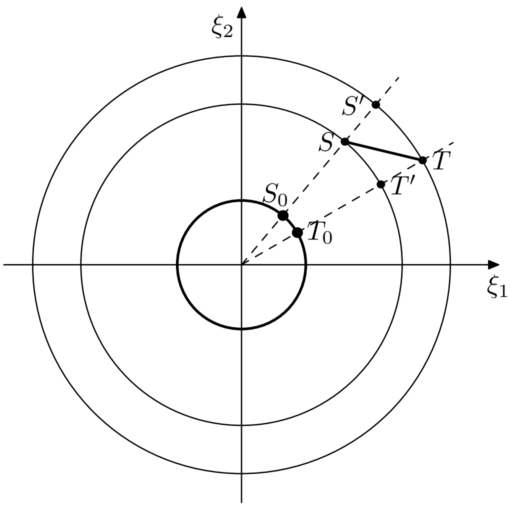

Figure 1. Projection on the unit sphere.

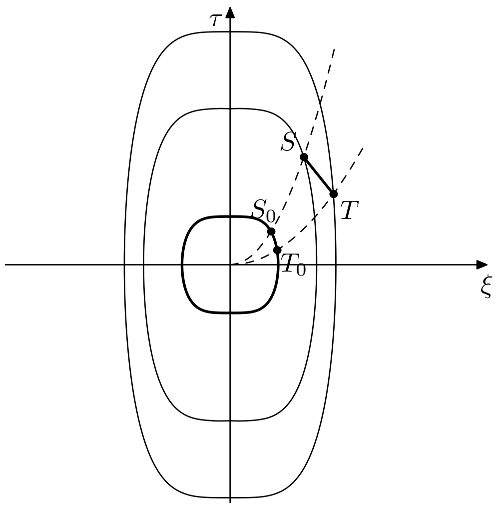

Figure 2. Projection on a parabolic surface of order four.

Function a should be constant along these parabolas, and we can choose a set of representative points of each parabola. One choice could be the points of intersection with unit sphere S^{d-1}; however, the coordinate expression for the projection, which we are going to use quite often, is not convenient in this case.

A smooth compact hypersurface we chose in [8] was implicitly given by:

$$\Pi^{d-1} \ \ldots \ \sigma^4(\tau, \boldsymbol{\xi}) := (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4 = 1 \, .$$

For any given point $(\tau, \boldsymbol{\xi}) \in \mathbf{R}^d_*$, its projection to Π^{d-1} is given by (as $\sigma^4 > 0$ on $\mathbf{R}^d_*$, by choosing positive determination of roots, the projection is uniquely defined)

$$\pi(\tau, \boldsymbol{\xi}) = \left(\frac{\tau}{\sigma^2(\tau, \boldsymbol{\xi})}, \frac{\boldsymbol{\xi}}{\sigma(\tau, \boldsymbol{\xi})} \right) \, .$$

In particular, from this formula it is clear that we indeed have a projection on Π^{d-1}.

On the other hand, in this case we were able to prove [8] an inequality of the form

$$d(S_0, T_0) \leqslant C \frac{d(S, T)}{\sigma(S) + \sigma(T)} \, ,$$

which is valid only for S and T far away from the origin.

In general, one can define a variant H-measure as long as one has a smooth compact hypersurface Σ in $\mathbf{R}^d$, and a smooth projection $\pi : \mathbf{R}^d_* \longrightarrow \Sigma$ (some details can be found in [44, Ch. 28]). However, for some additional results the choice of Σ and π, and the corresponding inequalities similar to those mentioned for particular cases above, are important.

Parabolic variants presented above have proved to be a successful tool when applied to homogenisation ([9, 11]), velocity averaging [28, 35]), and some other types of problems. However, in order to set up a complete theory of parabolic H-measures, parallel to the one developed by Tartar for the original H-measures, we had to introduce a new variant. The crucial ingredient was the correct choice of the domain (i.e. the smooth hypersurface in the dual space), making the partial integration in the dual space feasible. This allowed us to prove the propagation principle, resulting in a transport equation satisfied by the parabolic H-measure associated to a sequence of solutions of Schrödinger type equations. At the same time, we were able to preserve the localisation principle (for which an earlier variant was tailored for), and the referred applications remain feasible with this new variant as well. Even though we used the localisation principle earlier [6], this is the first published proof of the result for any parabolic variant (it requires fractional derivatives, which are nonlocal operators, a dissimilar situation compared to classical H-measures).

In the next section we introduce the new parabolic variant, describing the properties of the new scaling; in particular, by proving the integration by parts formula. We also state the existence result for the new variant, some of its immediate properties and two standard examples. A number of proofs has been omitted, as they are quite similar either to the proofs for classical H-measures, or already published proofs for earlier parabolic variants.

In the third section we prove the localisation principle, and the propagation principle in the last section. For this we had to introduce anisotropic Sobolev spaces and fractional derivatives.

Some applications to classical partial differential equations are presented, immediately following the introduction of general properties of parabolic H-measures, illustrating the possible uses of this new tool. Throughout the paper we tried to keep the regularity assumptions on symbols minimal, having in mind that in applications they are the coefficients in differential equations. At the same time we sketched the possibility of proving similar results in the smooth case by using classical results on pseudodifferential operators [25].

We conclude the paper with a comparison of the propagation results for classical and parabolic H-measures.

2. New parabolic variant

New scaling

In the study of parabolic equations it is convenient to single out one variable t (the *time*), so we shall systematically use $(t, \mathbf{x}) \in \mathbf{R}^{1+d}$ in the physical space, and analogously $(\tau, \boldsymbol{\xi}) \in \mathbf{R}^{1+d}$ in the dual space.

Let us denote by P^d a smooth hypersurface (in fact, a rotational ellipsoid) in $\mathbf{R}^{1+d}$ defined by $\tau^2 + \frac{|\boldsymbol{\xi}|^2}{2} = 1$, and by p a *parabolic projection* (this particular projection we denote by p; while discussing other projections we use π instead) of the set $\mathbf{R}^{1+d}_* := \mathbf{R}^{1+d} \setminus \{0\}$ onto P^d, defined by the relation

$$p(\tau, \boldsymbol{\xi}) := \left(\frac{\tau}{|\boldsymbol{\xi}/2|^2 + \sqrt{|\boldsymbol{\xi}/2|^4 + \tau^2}}, \frac{\boldsymbol{\xi}}{\sqrt{|\boldsymbol{\xi}/2|^2 + \sqrt{|\boldsymbol{\xi}/2|^4 + \tau^2}}} \right).$$

This is a parabolic projection indeed, as $p(\lambda^2\tau, \lambda\boldsymbol{\xi}) = p(\tau, \boldsymbol{\xi})$, so the whole branch of parabola $\{(\lambda^2\tau, \lambda\boldsymbol{\xi}) : \lambda \in \mathbf{R}^+\}$ (such a branch we call the *coordinate parabola*) is projected to the same point on P^d.

In the sequel we shall denote the introduced projection of a point $T = (\tau, \boldsymbol{\xi}) \in \mathbf{R}^{1+d}_*$ by

$$T_0 = (\tau_0, \boldsymbol{\xi}_0) := p(T) = p(\tau, \boldsymbol{\xi}) = \left(\frac{\tau}{\rho^2(\tau, \boldsymbol{\xi})}, \frac{\boldsymbol{\xi}}{\rho(\tau, \boldsymbol{\xi})} \right),$$

where ρ stands for the non-negative function defined by $\rho^2(\tau, \boldsymbol{\xi}) := |\boldsymbol{\xi}/2|^2 + \sqrt{|\boldsymbol{\xi}/2|^4 + \tau^2}$. Both T and T_0 lie on the paraboloid $\sigma = a\boldsymbol{\eta}^2$, $a = \tau/|\boldsymbol{\xi}|^2$ (on the line $\boldsymbol{\eta} = \mathsf{0}$ for $\boldsymbol{\xi} = \mathsf{0}$, and on the plane $\sigma = 0$ for $\tau = 0$). The projection p is constant along any meridian of the above paraboloid (on a coordinate parabola) through point T (Figure 3), while function ρ takes constant value $\lambda \in \mathbf{R}^+$ on each ellipsoid $\tau^2 + |\lambda\boldsymbol{\xi}|^2/2 = \lambda^4$. Note that the value $\lambda = 1$ corresponds to P^d, and that P^d can alternatively be characterised as the set of all points satisfying $\rho(\tau, \boldsymbol{\xi}) = 1$.

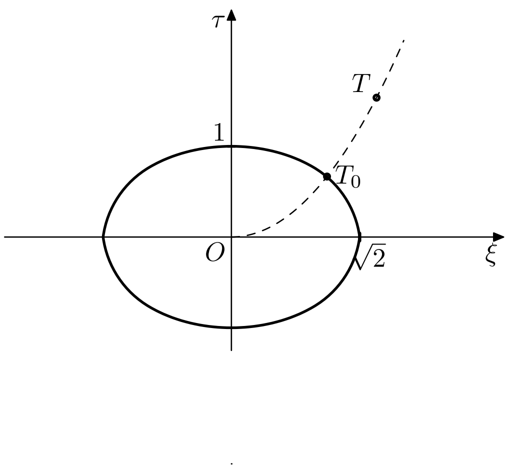

Figure 3. The new parabolic projection on P^d.

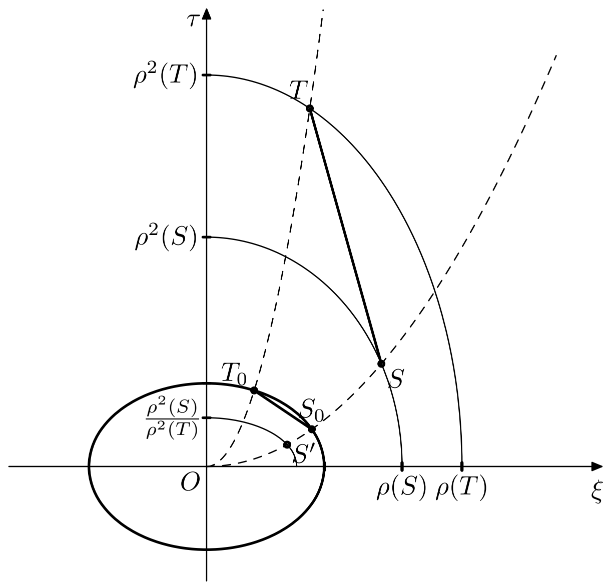

Figure 4. The variables appearing in the inequality.

By means of ρ we define a mapping associating to a pair of points $T = (\tau, \boldsymbol{\xi})$ and $S = (\sigma, \boldsymbol{\eta})$ the value

$$|TS|_p := \rho(T - S) = \rho(\tau - \sigma, \boldsymbol{\xi} - \boldsymbol{\eta}).$$

It can easily be checked that the introduced function is a *parabolic distance (metric)* on $\mathbf{R}^{1+d}$ (in fact, it is uniformly equivalent to the classical parabolic metric [20, 3.2]

$$d_p(S,T) := \sqrt{|\tau-\sigma| + |\boldsymbol{\xi}-\boldsymbol{\eta}|^2}\ ;$$

the Euclidean norm and metric being written without a subscript, while for the parabolic quantities we write the subscript p).

The crucial property of the introduced scaling is contained in the following lemma.

Lemma 1 *For points T, $S \in \mathbf{R}^{1+d}_*$ the following inequality holds*

$$|T_0S_0| \leqslant 2\Big(2+\sqrt{2\sqrt{2}}\Big)\frac{|TS|_p}{|OT|_p+|OS|_p},$$

where $O = (0, \mathsf{0})$ is the origin (see Figure 4).

Dem. Let $T = (\tau, \boldsymbol{\xi})$ and $S = (\sigma, \boldsymbol{\eta})$; without loosing generality we assume that $\rho(T) \geqslant \rho(S) > 0$.

By denoting $1/r = \rho(T) = |OT|_p$ and $1/s = \rho(S) = |OS|_p$, we can write the parabolic projections of T and S as $T_0 = (r^2\tau, r\boldsymbol{\xi})$ and $S_0 = (s^2\sigma, s\boldsymbol{\eta})$. Let $S' = (r^2\sigma, r\boldsymbol{\eta})$ be the intersection of the coordinate parabola through S (and S_0) with the coordinate ellipsoid $\rho = r/s < 1$, which is contained inside P^d. The triangle inequality for Euclidean distance gives us

$$|T_0S_0| \leqslant |T_0S'| + |S'S_0|\ .$$

For the first term on the right hand side, let us denote $t := |\tau_0 - \sigma'|$, $x := |\boldsymbol{\xi}_0 - \boldsymbol{\eta}'|$, and notice that both points are contained within P^d, where $t \leqslant 2$, so that $t^2 \leqslant 2t$. Therefore we have

$$\begin{aligned}|T_0S'|^2 = t^2 + x^2 &\leqslant 2\sqrt{2}\Big(\frac{t}{\sqrt{2}} + \frac{1+\sqrt{2}}{\sqrt{2}}\frac{x^2}{2^2}\Big) = 2\sqrt{2}\Big(\Big(\frac{x}{2}\Big)^2 + \frac{(x/2)^2+t}{\sqrt{2}}\Big)\\ &\leqslant 2\sqrt{2}\Big(\Big(\frac{x}{2}\Big)^2 + \sqrt{\Big(\frac{x}{2}\Big)^4 + t^2}\Big) = 2\sqrt{2}\rho^2\Big(r^2(\tau-\sigma), r(\boldsymbol{\xi}-\boldsymbol{\eta})\Big) = 2\sqrt{2}\,r^2|TS|_p^2\ ,\end{aligned}$$

which gives us the bound $|T_0S'| \leqslant \sqrt{2\sqrt{2}}\,r|TS|_p$.

It remains to estimate the second term:

$$|S'S_0| = \sqrt{(s^2-r^2)^2\sigma^2 + (s-r)^2|\boldsymbol{\eta}|^2} = (s-r)\sqrt{(s+r)^2\sigma^2 + |\boldsymbol{\eta}|^2}\ .$$

However, $(s+r)^2 \leqslant 4s^2 = 4/\rho^2(S) \leqslant 4/|\sigma|$, so $(s+r)^2\sigma^2 + |\boldsymbol{\eta}|^2 \leqslant 4|\sigma| + |\boldsymbol{\eta}|^2 \leqslant 4\rho^2(S)$.

On the other hand, as $\rho(\cdot - \cdot)$ is a metric, $s - r = rs(\rho(T) - \rho(S)) \leqslant rs\rho(T - S)$, which implies $|S'S_0| \leqslant 2r\rho(T - S) = 2r|TS|_p$.

Finally, taking into account the assumption $r < s$ we get the required symmetric formula

$$|T_0S_0| \leqslant 2\Big(2+\sqrt{2\sqrt{2}}\Big)\frac{|TS|_p}{|OT|_p+|OS|_p}\ .$$

Q.E.D.

Parabolic scaling and integration by parts

In comparison to other choices of hypersurfaces Π^d discussed in the Introduction, the advantage of this one is in the fact that the curves along which the projections are taken (the *coordinate parabolas*) intersect the ellipsoid P^d in the normal direction, as it was in the classical case, where the rays from origin normally intersected the unit sphere. Indeed, the normal to P^d has the form $(2\tau, \boldsymbol{\xi})^\top$, which is equal to the velocity $(2\lambda\tau, \boldsymbol{\xi})^\top$ (for $\lambda = 1$) of the above parabola parametrised by λ, as described above.

Each coordinate parabola is uniquely determined by the point of its intersection with ellipsoid P^d, while a point on the chosen coordinate parabola is determined by parameter $\lambda \in \mathbf{R}^+$. Based on this observation, and following the analogy with spherical coordinate system, we can introduce a curvilinear coordinate system on $\mathbf{R}^{1+d}$ (the only singularity being the origin O). However, this coordinate system is not orthogonal, except on the ellipsoid P^d, which will suffice for our needs to carry out integration by parts (being necessary for applications of the propagation principle; for the classical case see the last equation in Theorem 2).

Let us calculate the mean curvature H of the rotational ellipsoid (with semiaxes equal to a and b) in $\mathbf{R}^{1+d}$, which will be needed later.

The ellipsoid can be described as a level surface:

$$F(\xi_0, \xi_1, \ldots, \xi_d) = \frac{\xi_0^2}{a^2} + \frac{\xi_1^2}{b^2} + \cdots + \frac{\xi_d^2}{b^2} = 1 \,.$$

First we calculate ($|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^{1+d}$):

$$\begin{aligned} \nabla F(\xi_0, \boldsymbol{\xi}) &= 2\Big(\frac{\xi_0}{a^2}, \frac{\xi_1}{b^2}, \ldots, \frac{\xi_d}{b^2}\Big) \,, \\ |\nabla F|(\xi_0, \boldsymbol{\xi}) &= 2\sqrt{\frac{\xi_0^2}{a^4} + \frac{\xi_1^2}{b^4} + \cdots + \frac{\xi_d^2}{b^4}} \,, \end{aligned}$$

while the Hessian is of the form

$$\nabla\nabla F(\xi_0, \boldsymbol{\xi}) = \begin{bmatrix} \frac{2}{a^2} & 0 & \cdots & 0 \\ 0 & \frac{2}{b^2} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{2}{b^2} \end{bmatrix} .$$

Without loss of generality, we can consider the curvature at the point $\mathbf{p} = (\xi_0, \xi_1, 0, \ldots, 0)$, as the curvatures will stay the same along the whole parallel on the rotational ellipsoid.

For calculating the mean curvature we can use a well known formula [45, Exercise 12.16]

$$H(\mathbf{p}) = \frac{1}{d}\sum_{i=1}^{d} k(\mathsf{v}_i) \,,$$

where vectors $\mathsf{v}_1, \ldots, \mathsf{v}_d$ form an orthonormal basis for the tangential space at the given point $\mathbf{p}$, while $k(\mathsf{v})$ denotes the *normal curvature* in the direction v.

It can easily be seen that (the second expression does not change along a parallel)

$$\alpha := \frac{1}{\sqrt{\frac{\xi_0^2}{a^4} + \frac{\xi_1^2}{b^4}}} = \frac{1}{\sqrt{\frac{\xi_0^2}{a^4} + \frac{1}{b^2}(1 - \frac{\xi_0^2}{a^2})}} \,,$$

so we actually have the following orthonormal basis

$$\mathsf{v}_1 = (\alpha\xi_1/b^2, -\alpha\xi_0/a^2, 0, \ldots, 0),\ \mathsf{v}_2 = (0, 0, 1, 0, \ldots, 0), \ldots, \mathsf{v}_d = (0, \ldots, 0, 1)$$

for the tangent space at $\mathbf{p}$.

Now we can apply another well known formula [45, Exercise 12.1] (here $\mathcal{S}_{\mathbf{p}}$ denotes the *second fundamental form* to the ellipsoid at $\mathbf{p}$)

$$k(\mathsf{v}) = \mathcal{S}_{\mathbf{p}}(\mathsf{v}) = -\frac{1}{|\nabla F(\mathbf{p})|}\partial^i\partial^j F(\mathbf{p}) v_i v_j \,,$$

which gives us:

$$k(\mathsf{v}_1) = \frac{-\alpha^3}{a^2b^2}$$
$$k(\mathsf{v}_2) = -\alpha/b^2$$
$$\dots$$
$$k(\mathsf{v}_d) = -\alpha/b^2\ .$$

The mean value of these normal curvatures is the sought mean curvature $H(\mathbf{p})$.

One can easily check that the directions have been chosen in such a way that the above numbers are, in fact, the principal curvatures, their product being the Gauss-Kronecker curvature $K(\mathbf{p})$ [45, p. 91].

In particular, we have for P^d that $a = 1$ and $b = \sqrt{2}$, so the mean curvature is

$$H(\mathbf{p}) = -\frac{\alpha}{2d}(\alpha^2 + d - 1)\ ,$$

with $\alpha^2 = \frac{2}{\tau^2+1}$ (of course, $\tau = \xi_0$, as specified in Notation at the beginning of the first section).

Lemma 2 *(***integration by parts formula***) For p and q, a scalar and a vector function of class $\mathrm{C}^1(\mathbf{R}^{1+d})$, the following formula for integration on a C^2 level surface P in $\mathbf{R}^{1+d}$ holds*

$$\int_P (\nabla_T p \cdot q + p(\operatorname{div}\mathsf{q} - \nabla\mathsf{q}\cdot(\mathsf{n}\otimes\mathsf{n})))\, dA = -d\int_P Hp\,\mathsf{q}\cdot\mathsf{n}\, dA\,.$$

Here n stands for the outwardly directed unit normal vector to P, H for the mean curvature of surface P, while $\nabla_T := \nabla - \mathsf{n}\nabla_{\mathsf{n}}$ is the tangential gradient on P ($\nabla_{\mathsf{n}} f := \nabla f\cdot\mathsf{n}$ denotes the normal component of the gradient).

Dem. We start from the formula [24, Lemma 16.1]

$$\int_P (\nabla g - (\nabla g\cdot\mathsf{n})\mathsf{n})dA = -d\int_P gH\mathsf{n}dA,$$

where g is a C^1 scalar function.

Taking the i-th component of the formula above, and inserting $g = f_i$, where $\mathsf{f} = (f_0, \dots, f_d)^\top$ is a vector function, after summation with respect to i we get

$$\int_P \partial_T^i f_i\, dA = \int_P (\partial^i f_i - (\nabla f_i\cdot\mathsf{n})n^i)dA = -d\int_P Hf_i n^i dA\ ,$$

which can be written in an intrinsic form as:

$$\int_P (\operatorname{div}\mathsf{f} - \nabla\mathsf{f}\cdot(\mathsf{n}\otimes\mathsf{n}))dA = -d\int_P H\mathsf{f}\cdot\mathsf{n}\, dA\ .$$

Now we can take $\mathsf{f} = p\mathsf{q}$ to obtain

$$\int_P \Big(\nabla p\cdot\bar{\mathsf{q}} + p\operatorname{div}\mathsf{q} - (p\nabla\mathsf{q} + \nabla p\otimes\bar{\mathsf{q}})\cdot(\mathsf{n}\otimes\mathsf{n})\Big)dA = -d\int_P Hp\,\mathsf{q}\cdot\mathsf{n}dA,$$

i.e.

$$\int_P \Big((\nabla p - (\nabla p\cdot\mathsf{n})\mathsf{n})\cdot q + p(\operatorname{div}\mathsf{q} - \nabla\mathsf{q}\cdot(\mathsf{n}\otimes\mathsf{n}))\Big)dA = -d\int_P Hp\,\mathsf{q}\cdot\mathsf{n}dA. \tag{1}$$

Q.E.D.

Applying the lemma to the rotational ellipsoid P^d, where the mean curvature is $H = -\frac{\alpha}{2d}(\alpha^2 + d - 1)$, we get the following corollary.

Corollary 1 *For $p \in \mathrm{C}^1(\mathbf{R}^{1+d})$ parabolicly homogeneous, and $\mathsf{q} \in \mathrm{C}^1(\mathbf{R}^{1+d}; \mathbf{R}^{1+d})$, one has*

$$\int_{\mathrm{P}^d} (\nabla p \cdot q + p(\mathsf{div}\, \mathsf{q} - \nabla \mathsf{q} \cdot (\mathsf{n} \otimes \mathsf{n})))\, dA = \int_{\mathrm{P}^d} \frac{\alpha}{2}(\alpha^2 + d - 1) p \mathsf{q} \cdot \mathsf{n}\, dA\,.$$

Dem. As p is parabolicly homogeneous, it is constant along meridians of paraboloids $\tau = a|\boldsymbol{\xi}|^2$, while these parabolas are normal to the surface P^d at the intersection points, so the term $\nabla_{\mathsf{n}} p$ in (1) vanishes. **Q.E.D.**

Existence

When defining a new variant of H-measures, we consider two types of elementary operators on $\mathrm{L}^2(\mathbf{R}^{1+d})$. For a function $\psi \in \mathrm{C}(\mathrm{P}^d)$ we define an operator P_ψ on $\mathrm{L}^2(\mathbf{R}^{1+d})$ by $P_\psi u := ((\psi \circ p)\hat{u})^\vee$, i.e.

$$(P_\psi u)(t, \mathbf{x}) = \int_{\mathbf{R}^{1+d}} e^{2\pi i (t\tau + \mathbf{x}\cdot\boldsymbol{\xi})} \psi\Big(\frac{\tau}{\rho^2(\tau, \boldsymbol{\xi})}, \frac{\boldsymbol{\xi}}{\rho(\tau, \boldsymbol{\xi})}\Big) \hat{u}(\tau, \boldsymbol{\xi})\, d\tau d\boldsymbol{\xi}\,. \tag{2}$$

Clearly, P_ψ is a bounded operator, called *(the Fourier) multiplier*, with norm equal to $\|\psi\|_{\mathrm{L}^\infty}$.

For $\phi \in \mathrm{C}_0(\mathbf{R}^{1+d})$ a *(Sobolev) multiplication operator* M_ϕ on $\mathrm{L}^2(\mathbf{R}^{1+d})$ is defined,

$$M_\phi u := \phi u, \tag{3}$$

with norm $\|\phi\|_{\mathrm{L}^\infty}$. It might be of interest to note that both $\psi \mapsto P_\psi$ and $\phi \mapsto M_\phi$ are linear isometries.

By approximating ψ uniformly by Lipschitz functions, we can prove the First commutation lemma, essentially following the same steps as in [40, 7]. Of course, the crucial rôle in the proof is played by Lemma 1, as it is played by inequality $\Big|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} - \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}\Big| \leqslant 2\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|}{|\boldsymbol{\xi}|+|\boldsymbol{\eta}|}$ in [40], or Lemma 2 in [7] (Lemma 2 in the cited paper!).

Lemma 3 (first commutation lemma) *If $\psi \in \mathrm{C}(\mathrm{P}^d)$ and $\phi \in \mathrm{C}_0(\mathbf{R}^{1+d})$, the commutator $K := [P_\psi, M_\phi]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^{1+d})$.* ∎

The main theorem on existence of H-measures with parabolic scaling now follows as in [40, 7]. We believe that the reader will neither have any difficulties in repeating that proof, nor with the proof of next theorem. Let us just mention in passing that everything remains true also in the case where $\mathbf{R}^{1+d}$ is replaced by any open set $\Omega \subseteq \mathbf{R}^{1+d}$, as we can easily extend L^2 functions by zero to $\mathbf{R}^{1+d} \setminus \Omega$, preserving the weak convergence to zero. The resulting parabolic H-measure will not be supported in the interior of $\mathbf{R}^{1+d} \setminus \Omega$, due to Corollary 3 below.

Theorem 3 (existence of parabolic H-measures) *If (u_n) is a sequence in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$, such that $\mathsf{u}_n \xrightharpoonup{\mathrm{L}^2} 0$ (weakly), then there exists a subsequence $(\mathsf{u}_{n'})$ and a complex $r \times r$ matrix Radon measure $\boldsymbol{\mu}$ on $\mathbf{R}^{1+d} \times \mathrm{P}^d$ such that for all $\phi_1, \phi_2 \in \mathrm{C}_0(\mathbf{R}^{1+d})$ and $\psi \in \mathrm{C}(\mathrm{P}^d)$:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^{1+d}} \mathcal{F}(\phi_1 \mathsf{u}_{n'}) \otimes \mathcal{F}(\phi_2 \mathsf{u}_{n'})(\psi \circ p)\, d\boldsymbol{\xi} &= \Big\langle \boldsymbol{\mu}, (\phi_1 \bar{\phi}_2) \boxtimes \psi \Big\rangle \\ &= \int_{\mathbf{R}^{1+d} \times \mathrm{P}^d} \phi_1(\mathbf{x}) \bar{\phi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi})\,.\end{aligned}$$

∎

Measure $\boldsymbol{\mu}$ from the above theorem we call *the parabolic H-measure* associated to (a sub)sequence (of) (u_n). A sequence (u_n) is called *pure* if the associated parabolic H-measure is unique for any chosen subsequence. Of course, to a sequence converging strongly, it corresponds the parabolic H-measure zero. The opposite is true, but only in $\mathrm{L}^2_{\mathrm{loc}}$.

Indeed, a simple example is furnished by taking a nontrivial $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^{1+d})$, and considering the sequence $u_n(t,\mathbf{x}) := v((t,\mathbf{x}) - n\mathsf{e})$, e being a unit vector, which weakly tends to 0. The supports of u_n tend to infinity, but the convergence is not strong, while the parabolic H-measure is zero.

If P_ψ and M_ϕ are operators defined by (2) and (3) respectively, by the means of First commutation lemma the limit from the last theorem can be rephrased as:

$$\lim_n \langle\, P_\psi M_\phi \mathsf{u}_n \mid \mathsf{u}_n \,\rangle_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \lim_n \Big\langle\, (\psi\circ p)\, \hat{\mathsf{u}}_n \;\Big|\; \widehat{(\overline{\phi}\mathsf{u}_n)} \,\Big\rangle_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \langle \boldsymbol{\mu}, \phi\boxtimes\psi\rangle \;, \tag{4}$$

where p is the parabolic projection defined above, i.e. $\psi\circ p$ is *parabolicly homogeneous* extension of function ψ to $\mathbf{R}^{1+d}_*$.

If we restrict test functions ϕ to those with compact support (i.e. $\phi \in \mathrm{C}_c(\mathbf{R}^{1+d})$), the above theorem is valid for weakly convergent sequences (u_n) in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^r)$ as well (however, in that case $\boldsymbol{\mu}$ is not necessary a finite Radon measure, but a distribution of order 0). In the sequel, we shall use both forms, whichever will be better suited to the intended application.

Immediate consequences of Theorem 3 are the following corollaries; their simple proofs are almost identical to analogous statements in [8, Corollaries 4.1.1–3].

Corollary 2 *Parabolic H-measure $\boldsymbol{\mu}$ is hermitian and non-negative:*

$$\boldsymbol{\mu} = \boldsymbol{\mu}^* \qquad \text{and} \qquad (\forall\, \boldsymbol{\phi}\in \mathrm{C}_0(\mathbf{R}^{1+d};\mathbf{C}^r)) \quad \langle \boldsymbol{\mu}, \boldsymbol{\phi}\otimes\boldsymbol{\phi}\rangle \geqslant 0 \;,$$

where $\langle \boldsymbol{\mu}, \boldsymbol{\phi}\otimes\boldsymbol{\phi}\rangle$ is considered as a Radon measure on P^d. In particular, $\boldsymbol{\mu}$ is supported on the support of its trace $\mathsf{tr}\boldsymbol{\mu}$. ∎

For parabolic H-measures we have simple localisation as an immediate consequence of the definition.

Corollary 3 *Let the sequence (u_n) define a parabolic H-measure $\boldsymbol{\mu}$. If all the components $\mathsf{u}_n\cdot\mathsf{e}_i$ have their supports in closed sets $K_i \subseteq \mathbf{R}^{1+d}$ respectively, then the support of the component $\boldsymbol{\mu}\mathsf{e}_i\cdot\mathsf{e}_j$ is contained in $(K_i\cap K_j)\times \mathrm{P}^d$.* ∎

To an L^2 sequence (u_n) we can associate a defect measure corresponding to (u_n^2). Its connection to parabolic H-measures is given by the following corollary.

Corollary 4 *If $\mathsf{u}_n\otimes\mathsf{u}_n$ converges weakly$*$ in $(\mathrm{C}_0(\mathbf{R}^{1+d};\mathrm{M}_{r\times r}))'$ to a measure $\boldsymbol{\nu}$, then for every $\phi\in \mathrm{C}_0(\mathbf{R}^{1+d})$:*

$$\langle \boldsymbol{\nu}, \phi\rangle = \langle \boldsymbol{\mu}, \phi\boxtimes 1\rangle \;,$$

where $\boldsymbol{\mu}$ is any parabolic H-measure determined by a subsequence of (u_n). ∎

Next lemma gives us a relation between parabolic H-measures associated to conjugated (the complex conjugation being performed on each component of vector u_n) sequences (v. [9, Lemma 1]).

Lemma 4 *Let (u_n) be a pure sequence in $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$, and $\boldsymbol{\mu}$ the corresponding parabolic H-measure. Then the sequence $(\overline{\mathsf{u}}_n)$ is pure with associated parabolic H-measure $\boldsymbol{\nu}$, such that $\boldsymbol{\nu}(t,\mathbf{x},\tau,\boldsymbol{\xi}) = \boldsymbol{\mu}^\top(t,\mathbf{x},-\tau,-\boldsymbol{\xi})$.*

In particular, a parabolic H-measure μ associated to a real scalar sequence is antipodally symmetric, i. e. $\mu(t,\mathbf{x},\tau,\boldsymbol{\xi}) = \mu(t,\mathbf{x},-\tau,-\boldsymbol{\xi})$. ∎

First examples

As for the original H-measures, we are particularly interested in the examples of sequences converging weakly, but not strongly, i.e. to oscillating and concentrating sequences, as typical of such behaviour.

Example 1. (oscillation) Let $v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d})$ be a periodic function with (for simplicity) unit period in each of its variables. Thus it can be written as a Fourier expansion

$$v(t,\mathbf{x}) = \sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}}\, e^{2\pi i(\omega t + \mathsf{k}\cdot\mathbf{x})}\,,$$

where $\hat{v}_{\omega,\mathsf{k}}$ denote Fourier coefficients of function v. Furthermore, we assume it to have zero mean value, i.e. $\hat{v}_{0,0} = 0$.

For $\alpha, \beta \in \mathbf{R}^+$, let us define a sequence of periodic functions with periods approaching zero:

$$u_n(t,\mathbf{x}) := v(n^\alpha t, n^\beta \mathbf{x}) = \sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}}\, e^{2\pi i(n^\alpha \omega t + n^\beta \mathsf{k}\cdot\mathbf{x})}\,.$$

Then (u_n) is pure, and its parabolic H-measure (for details see [8]) is

$$\mu(t,\mathbf{x},\tau,\boldsymbol{\xi}) = \lambda(t,\mathbf{x}) \begin{cases} \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \omega\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}) + \sum_{\mathsf{k}\in\mathbf{Z}^d} |\hat{v}_{0,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}), & \alpha > 2\beta \\ \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \mathsf{k}\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}) + \sum_{\omega\in\mathbf{Z}} |\hat{v}_{\omega,0}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}), & \alpha < 2\beta \\ \displaystyle\sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{\left(\frac{\omega}{\rho^2(\omega,\mathsf{k})},\frac{\mathsf{k}}{\rho(\omega,\mathsf{k})}\right)}(\tau,\boldsymbol{\xi}), & \alpha = 2\beta. \end{cases}$$

∎

Example 2. (concentration) For given function $v \in \mathrm{L}^2(\mathbf{R}^{1+d})$ and $\alpha, \beta \in \mathbf{R}^+$ we consider a sequence of functions

$$u_n(t,\mathbf{x}) := n^{\alpha+\beta d} v(n^{2\alpha} t, n^{2\beta}\mathbf{x}).$$

The above sequence is pure, with the associated parabolic H-measure (the details, mutatis mutandis, are to be found in [8])

$$\langle \boldsymbol{\mu}, \phi \boxtimes \psi \rangle = \phi(0,0) \begin{cases} \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma,\boldsymbol{\eta})|^2 \psi\Big(\frac{\sigma}{|\sigma|},0\Big) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}^d} |\hat{v}(0,\boldsymbol{\eta})|^2 \psi\Big(0,\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}\Big)\, d\boldsymbol{\eta}, & \alpha > 2\beta \\ \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma,\boldsymbol{\eta})|^2 \psi\Big(0,\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}\Big) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}} |\hat{v}(\sigma,0)|^2 \psi\Big(\frac{\sigma}{|\sigma|},0\Big)\, d\sigma, & \alpha < 2\beta \\ \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma,\boldsymbol{\eta})|^2 \psi\left(\frac{\sigma}{\rho^2(\sigma,\boldsymbol{\eta})},\frac{\boldsymbol{\eta}}{\rho(\sigma,\boldsymbol{\eta})}\right) d\sigma d\boldsymbol{\eta}, & \alpha = 2\beta. \end{cases}$$

∎

It might be of interest to note that any non-negative Radon measure on $\Omega \times \mathrm{P}^d$ of total mass A^2 can be described as a parabolic H-measure of some sequence $u_n \rightharpoonup 0$ with $\mathrm{L}^2(\Omega)$ norm not greater than $A + \varepsilon$, as it was true for classical H-measures. The proof resembles the one in [40, Corollary 2.3], and is left to the reader.

3. Localisation principle

Anisotropic Sobolev spaces

In preparation to state the localisation principle, we are first going to define certain function spaces, and describe their essential properties. While we first introduced these spaces as a necessary framework for our immediate goal, we only much later learned that such spaces had previously been studied as a particular generalisation of Sobolev spaces.

In particular, the spaces we are interested in consist of temperate distributions u such that $k\hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d})$, for some weight function k, which are described in [25, 10.1] and denoted by $B_{2,k}$ there. While we are going to recall the main properties of these spaces in our notation, we refer the reader to look up the details and, in particular, the proofs in Hörmander's book.

Analogously to isotropic Sobolev spaces H^s, for $s \in \mathbf{R}$, we can define special anisotropic function spaces

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{S}' : k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d}) \right\},$$

where we define:

$$k_p(\tau, \boldsymbol{\xi}) := \sqrt[4]{1 + (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}\,.$$

Here, index p denotes the fact that k_p is adjusted to parabolic problems; we shall also use $k(\boldsymbol{\xi}) := \sqrt{1 + (2\pi|\boldsymbol{\xi}|)^2}$, which figures in the definition of isotropic spaces (in fact, $\mathrm{H}^s = B_{2,k^s}$, while $\mathrm{H}^{\frac{s}{2},s} = B_{2,k_p^s}$, using Hörmander's notation). With the scalar product

$$\langle\, u \mid v \,\rangle_{\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})} := \langle\, k_p^s\, \hat{u} \mid k_p^s\, \hat{v} \,\rangle_{\mathrm{L}^2(\mathbf{R}^{1+d})},$$

$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ is a Hilbert space. Also, for $s \in \mathbf{R}^+$, space $\mathrm{H}^{-\frac{s}{2}}(\mathbf{R}^{1+d})$ is continuously embedded into $\mathrm{H}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d})$. Indeed, for $u \in \mathrm{H}^{-\frac{s}{2}}(\mathbf{R}^{1+d})$ the integral $\int_{\mathbf{R}^{1+d}} |\hat{u}|^2 k_p^{-2s} d\tau d\boldsymbol{\xi}$ is finite, and the statement follows from the inequality

$$k_p^{-s} = \left(\frac{1}{\sqrt[4]{1 + (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}} \right)^s \leqslant 2 \left(\frac{1}{\sqrt{1 + (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^2}} \right)^{\frac{s}{2}} = 2k^{-s/2}\,.$$

Remark. It might be of interest to note that $k_p^4 = 1 + \sigma^4$, where σ was used to define the compact hypersurface Π^d, which was used in the definition of variant parabolic H-measure in [8], as described in the Introduction. Actually, it was this convenience in calculations that prompted us to consider such variant. However, in order to get the propagation principle, we had to replace σ by ρ. ∎

Lemma 5 *We have $\mathcal{S} \hookrightarrow \mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) \hookrightarrow \mathcal{S}'$, the embeddings being dense and continuous. Furthermore, $\mathrm{C}_c^\infty(\mathbf{R}^{1+d})$ is dense in $\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$.* ∎

Lemma 6 *If we define $\mathrm{H}^{0,r}(\mathbf{R}^{1+d})$ as the set of those $u \in \mathcal{S}'$ such that $(\sqrt[4]{1 + (2\pi|\boldsymbol{\xi}|)^4})^r \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d})$, then for any compact set $K \subseteq \mathbf{R}^{1+d}$ the embedding*

$$\mathrm{H}^{0,-r}(\mathbf{R}^{1+d}) \cap \mathcal{E}'(K) \hookrightarrow \mathrm{H}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d})$$

is compact, provided $r < s$. ∎

As the space $\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ is *semi-local* (v. [25, loc. cit.]), the smallest *local space* containing it is

$$\mathrm{H}^{\frac{s}{2},s}_{\mathrm{loc}}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{D}'(\mathbf{R}^{1+d}) : (\forall\, \varphi \in \mathrm{C}_c^\infty(\mathbf{R}^{1+d})) \quad \varphi u \in \mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) \right\}.$$

Naturally, it is endowed with the weakest topology in which every such mapping $u \mapsto \varphi u$ is continuous.

The principle

Besides a simple localisation result expressed in Corollary 3, as for classical H-measures, we are able to prove a more powerful localisation principle, which is quite useful in various applications (as it was in the classical case). Let us just mention the small amplitude homogenisation [11], where the precise argument relies on this principle. Also, when we shall later study the propagation principle, the localisation principle will be used first to reduce the form of the parabolic H-measure. Finally, compactness by compensation turns out to be a special case of the localisation principle.

First we need to define the fractional derivative: $\sqrt{\partial_t}$ denotes a pseudodifferential operator with a polyhomogeneous symbol $\sqrt{2\pi i\tau}$, i.e.

$$\sqrt{\partial_t} u = \overline{\mathcal{F}}\left(\sqrt{2\pi i\tau}\,\hat{u}(\tau)\right).$$

Here we assume that one branch of the square root has been selected.

The introduced operator is well defined on the union of Sobolev spaces $\mathrm{H}^{-\infty}(\mathbf{R}^{1+d}) = \bigcup_s \mathrm{H}^s(\mathbf{R}^{1+d})$.

Theorem 4 (localisation principle) *Let (u_n) be a sequence of functions uniformly compactly supported in t and converging weakly to zero in $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{C}^r)$, and let for $s \in \mathbf{N}$*

$$\sqrt{\partial_t}^{\,s}(\mathbf{A}^0\mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) \longrightarrow 0 \quad \textit{strongly in} \quad \mathrm{H}^{-\frac{s}{2},-s}_{\mathrm{loc}}(\mathbf{R}^{1+d}), \tag{5}$$

where $\mathbf{A}^0, \mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}_b(\mathbf{R}^{1+d};\mathrm{M}_{l\times r}(\mathbf{C}))$ (i.e. in $\mathrm{C}(\mathbf{R}^{1+d};\mathrm{M}_{l\times r}(\mathbf{C})) \cap \mathrm{L}^\infty(\mathbf{R}^{1+d};\mathrm{M}_{l\times r}(\mathbf{C}))$), for some $l \in \mathbf{N}$, while $\boldsymbol{\alpha} \in \mathbf{N}_0^d$.

Then for the associated parabolic H-measure $\boldsymbol{\mu}$ we have

$$\left((\sqrt{2\pi i\tau})^s\mathbf{A}^0 + \sum_{|\boldsymbol{\alpha}|=s}(2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}}\mathbf{A}^{\boldsymbol{\alpha}}\right)\boldsymbol{\mu}^\top = \mathbf{0}.$$

Dem. Let us first show that an analogue to relation (5) holds for any localised sequence $(\phi\mathsf{u}_n)$, where $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^{1+d})$. First, we take $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ (a function independent of t), for which we have

$$\begin{aligned}\sqrt{\partial_t}^{\,s}(\mathbf{A}^0\phi\mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\phi\mathsf{u}_n) &= \phi\Big(\sqrt{\partial_t}^{\,s}(\mathbf{A}^0\mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n)\Big)\\ &\quad + \sum_{|\boldsymbol{\alpha}|=s}\sum_{|\boldsymbol{\beta}|=1}^{s}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\partial_{\mathbf{x}}^{\boldsymbol{\beta}}\phi\,\partial_{\mathbf{x}}^{(\boldsymbol{\alpha}-\boldsymbol{\beta})}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n).\end{aligned}$$

By Lemma 6, the terms under the last summation converge strongly to zero in $\mathrm{H}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d})$. The remaining terms converge in the same space by the assumption (5) of the theorem, which proves the claim.

For an arbitrary $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^{1+d})$, we can take a time independent function $\varphi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ such that $\phi = \phi\varphi$. Taking into account that $\rho^4(\tau, \boldsymbol{\xi}) \leqslant (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4$, we get

$$
\begin{aligned}
&\Big\| \sqrt{\partial_t}^s (\mathbf{A}^0 \phi \mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \phi \mathsf{u}_n) \Big\|_{\mathrm{H}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d})} \\
&\quad \leqslant \Big\| \Big(\frac{\sqrt{2\pi i\tau}}{\sqrt[4]{(2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}} \Big)^s \widehat{\mathbf{A}^0 \phi \mathsf{u}_n} + \sum_{|\boldsymbol{\alpha}|=s} \frac{(2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}}}{\sqrt[4]{(2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}^{\,s}} \widehat{\mathbf{A}^{\boldsymbol{\alpha}} \phi \mathsf{u}_n} \Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \\
&\quad \leqslant \Big\| P_s(\phi \mathbf{A}^0 \varphi \mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} P_{\boldsymbol{\alpha}}(\phi \mathbf{A}^{\boldsymbol{\alpha}} \varphi \mathsf{u}_n) \Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \\
&\quad \leqslant \Big\| \phi \Big(P_s(\mathbf{A}^0 \varphi \mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} P_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \varphi \mathsf{u}_n) \Big) \Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \\
&\qquad\quad + \Big\| [P_s, M_\phi](\mathbf{A}^0 \varphi \mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} [P_{\boldsymbol{\alpha}}, M_\phi](\mathbf{A}^{\boldsymbol{\alpha}} \varphi \mathsf{u}_n) \Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} ,
\end{aligned}
$$

where P_s and $P_{\boldsymbol{\alpha}}$ denote Fourier multiplier operators associated to symbols $p_s = \left(\frac{\sqrt{2\pi i\tau}}{\rho(\tau,\boldsymbol{\xi})}\right)^s$ and $p_{\boldsymbol{\alpha}} = \frac{(2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}}}{\rho(\tau,\boldsymbol{\xi})^s}$, respectively. By the First commutation lemma the commutators $[P_s, M_\phi]$ and $[P_{\boldsymbol{\alpha}}, M_\phi]$ are compact operators on $\mathrm{L}^2(\mathbf{R}^{1+d})$, thus implying the convergence of the last term above to 0. According to the first part of the proof, the uniformly compactly supported sequence (φu_n) satisfies the analogous relation to (5), and by Lemma 7 below, the sequence of functions $P_s(\mathbf{A}^0\varphi\mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} P_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\varphi\mathsf{u}_n)$ converges strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d})$. Thus we have shown that $\sqrt{\partial_t}^s(\mathbf{A}^0\phi\mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\phi\mathsf{u}_n) \to 0$ in $\mathrm{H}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d})$, as well as

$$
\rho^{-s}(\tau, \boldsymbol{\xi}) \Big(\sqrt{2\pi i\tau}^s \widehat{\mathbf{A}^0 \phi \mathsf{u}_n} + \sum_{|\boldsymbol{\alpha}|=s} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}} \widehat{\mathbf{A}^{\boldsymbol{\alpha}} \phi \mathsf{u}_n} \Big) \longrightarrow \mathsf{0} \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^{1+d}).
$$

After multiplying the n-th term of the above sequence by $\widehat{\phi \mathsf{u}_n}$ and $\psi \circ p$, for $\psi \in \mathrm{C}(\mathrm{P}^d)$, we have that

$$
\begin{aligned}
0 &= \lim_n \int_{\mathbf{R}^{1+d}} (\psi \circ p) \left(\left(\frac{\sqrt{2\pi i\tau}}{\rho(\tau, \boldsymbol{\xi})} \right)^s \widehat{\mathbf{A}^0 \phi \mathsf{u}_n} + \sum_{|\boldsymbol{\alpha}|=s} \frac{(2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}}}{\rho(\tau, \boldsymbol{\xi})^s} \widehat{\mathbf{A}^{\boldsymbol{\alpha}} \phi \mathsf{u}_n} \right) \otimes \left(\widehat{\phi \mathsf{u}_n} \right) d\tau d\boldsymbol{\xi} \\
&= \Big\langle \frac{1}{\rho(\tau, \boldsymbol{\xi})^s} \overline{(\sqrt{2\pi i\tau})^s \mathbf{A}^0 + \sum_{|\boldsymbol{\alpha}|=s} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}} \boldsymbol{\mu} \, , \, |\phi|^2 \boxtimes \psi \Big\rangle.
\end{aligned}
$$

As $\rho(\tau, \boldsymbol{\xi}) = 1$ on the support of parabolic H-measure $\boldsymbol{\mu}$, the claim follows. **Q.E.D.**

Remark. The supports of u_n have to be contained in a fixed compact, as $\sqrt{\partial_t}$ is not local, and we lack an appropriate variant of the Leibniz product rule for fractional derivatives. Of course, for an even integer s the proof is much simpler.

This assumption can be substituted by the requirement that coefficients $\mathbf{A}^{\boldsymbol{\alpha}}$ have compact support in t.

∎

Remark. The principle can easily been generalised to sequences with mixed time and space partial derivatives. Thus the strong convergence

$$
\sum_{\beta + |\boldsymbol{\alpha}|=s} \sqrt{\partial_t}^{\beta} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} (\mathbf{A}_{\beta,\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{H}^{-\frac{s}{2},-s}_{\mathrm{loc}}(\mathbf{R}^{1+d}) \, ,
$$

implies

$$
\Big(\sum_{\beta + |\boldsymbol{\alpha}|=s} (\sqrt{2\pi i\tau})^{\beta} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}} \mathbf{A}_{\beta,\boldsymbol{\alpha}} \Big) \boldsymbol{\mu}^{\top} = \mathbf{0}.
$$

∎

Lemma 7 *Let* f *be a measurable vector function on* $\mathbf{R}^d$*, and* h *a continuous scalar function of the form* $h(\mathbf{x}) = \sum_{i=1}^{d} |2\pi x^i|^{\alpha_i}, \alpha_i \in \mathbf{N}$*. Suppose* (u_n) *is a sequence of vector functions with supports contained in a fixed compact set such that* $\mathsf{u}_n \rightharpoonup \mathsf{0}$ *in* $\mathrm{L}^2(\mathbf{R}^d; \mathbf{R}^r)$*, and*

$$\frac{\mathsf{f}}{(1+h)^\beta} \cdot \hat{\mathsf{u}}_n \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$$

for some constant $\beta \in \mathbf{R}^+$*. If* $h^{-\beta}\mathsf{f} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^r)$*, then also*

$$\frac{\mathsf{f}}{h^\beta} \cdot \hat{\mathsf{u}}_n \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d).$$

∎

The proof can essentially be found in [9, Lemma 3].

Application to the heat equation

A natural next step is to apply the localisation principle to oscillating sequences of solutions of some well-studied parabolic equations. We start with the heat equation on $\mathbf{R}^+ \times \mathbf{R}^d$:

$$\begin{cases} \partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u(0,\cdot) = u_0, \end{cases} \tag{6}$$

where $\mathbf{A} \in \mathrm{L}^\infty([0,T] \times \mathbf{R}^d; \mathrm{M}_{d\times d}(\mathbf{R}))$ is a positive definite matrix field, i.e. $\mathbf{A}(t,\mathbf{x})\mathsf{v}\cdot\mathsf{v} \geqslant \alpha|\mathsf{v}|^2$ (a.e. $(t,\mathbf{x}) \in [0,T] \times \mathbf{R}^d$), for some $\alpha > 0$. The existence and regularity of a solution to (6) is derived from the theory presented in [14, Ch. XVIII, §3], for which one needs the following result.

Lemma 8 *Let* X, Y *be (complex) Hilbert spaces, such that* $X \overset{cd}{\hookrightarrow} Y$ *(continuously and densely embedded), and let* $a, b \in \overline{\mathbf{R}}$*. Then*

$$W(a,b;X,Y) := \Big\{u : u \in \mathrm{L}^2([a,b];X), \partial_t u \in \mathrm{L}^2([a,b];Y)\Big\}$$

is a Hilbert space with the norm defined by $\|u\|_W^2 = \|u\|^2_{\mathrm{L}^2([a,b];X)} + \|\partial_t u\|^2_{\mathrm{L}^2([a,b];Y)}$*. Furthermore, the space* $\mathrm{C}_c^\infty([a,b];X)$ *(restrictions of functions from* $\mathrm{C}_c^\infty(\mathbf{R};X)$ *to* $[a,b]\cap\mathbf{R}$*) is dense in* $W(a,b;X,Y)$*.*

In particular, if $X = V$*,* $Y = V'$*, and* H *is a Hilbert space such that* $V \hookrightarrow H \equiv H' \hookrightarrow V'$ *(we identify* H *with its dual* H'*; the embeddings are continuous and dense), i.e.* V, H *and* V' *form a Gelfand triplet, then the following embedding holds*

$$W(a,b;V) := W(a,b;V,V') \hookrightarrow \mathrm{C}([a,b];H)\ .$$

∎

In the case $-a = b = \infty$ we shall usually write $W(\mathbf{R};X,Y)$ instead of $W(-\infty,\infty;X,Y)$.

The existence result for (6) is now given by the following theorem.

Theorem 5 *Let* $u_0 \in \mathrm{L}^2(\mathbf{R}^d)$ *and* $f \in \mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^d))$*. Then under the above assumptions on coefficients* $\mathbf{A}$ *there exists a unique solution* u *of (6),* $u \in W(0,T;\mathrm{H}^1(\mathbf{R}^d)) \hookrightarrow \mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^d))$*. Furthermore, the following bound holds (*C *being a positive constant, independent of* u_0 *and* f*)*

$$\|u\|_{W(0,T;\mathrm{H}^1(\mathbf{R}^d))} \leq C\left(\|u_0\|_{\mathrm{L}^2(\mathbf{R}^d)} + \|f\|_{\mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^d))}\right).$$

∎

The energy in this problem is

$$E(t) := \int_{\mathbf{R}^d} u^2(t,\mathbf{x})d\mathbf{x}\,,$$

with the time derivative

$$\begin{aligned}\frac{d}{dt}E(t) &= 2\,{}_{\mathrm{H}^1(\mathbf{R}^d)}\Big\langle\, u(t,\cdot), \partial_t u(t,\cdot)\,\Big\rangle_{\mathrm{H}^{-1}(\mathbf{R}^d)}\\ &= 2\,{}_{\mathrm{H}^1(\mathbf{R}^d)}\Big\langle\, u(t,\cdot), \mathsf{div}(\mathbf{A}(t,\cdot)\nabla u(t,\cdot))\,\Big\rangle_{\mathrm{H}^{-1}(\mathbf{R}^d)} + 2\,{}_{\mathrm{H}^1(\mathbf{R}^d)}\Big\langle\, u(t,\cdot), f(t,\cdot)\,\Big\rangle_{\mathrm{H}^{-1}(\mathbf{R}^d)}\\ &= -2\int_{\mathbf{R}^d}\mathbf{A}(t,\mathbf{x})\nabla u(t,\mathbf{x})\cdot\nabla u(t,\mathbf{x})d\mathbf{x} + 2\,{}_{\mathrm{H}^1(\mathbf{R}^d)}\Big\langle\, u(t,\cdot), f(t,\cdot)\,\Big\rangle_{\mathrm{H}^{-1}(\mathbf{R}^d)}\,,\end{aligned}$$

which holds for a.e. $t\in[0,T]$. The application of Gauss's theorem is justified, as the functions with compact support are dense in $\mathrm{L}^2(\mathbf{R}^d)$.

In the sequel we consider a sequence of initial-value problems

$$\left\{\begin{aligned}\partial_t u_n - \mathsf{div}\,(\mathbf{A}\nabla u_n) &= \mathsf{div}\,\mathsf{f}_n\\ u_n(0,\cdot) &= \gamma_n\,,\end{aligned}\right. \tag{7}$$

where $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^+;\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d))$, $\gamma_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, and we additionally assume $\mathbf{A}\in \mathrm{C}_b(\mathbf{R}^+\times\mathbf{R}^d;\mathrm{M}_{d\times d}(\mathbf{R}))$.

The energy dissipation is described by the quadratic term with ∇u_n converging weakly to zero in the space $\mathrm{L}^2([0,T]\times\mathbf{R}^d;\mathbf{R}^d)$, together with the linear term (involving also f_n).

Our goal is to express it via a parabolic H-measure associated to the sequences (u_n) and (f_n) (more precisely, we shall use some derivatives of u_n which will define the parabolic H-measure, see below), and then to relate it to the parabolic H-measure determined by the data, γ_n and f_n.

For the application of Theorem 4, besides the above given bounds on the solutions, we shall need a bound on sequence $(\sqrt{\partial_t}u_n)$. This is derived by using results on fractional derivatives [28, I.4].

Theorem 6 *Let X, Y be Hilbert spaces, and $u\in W(\mathbf{R},X,Y)$. Then for $r\in[0,1]$*

$$|\tau|^r\hat{u}\in\mathrm{L}^2(\mathbf{R};[X,Y]_r),$$

where $[X,Y]_r$ stands for the classical intermediate space of X and Y with index r, and

$$\Big\||\tau|^r\hat{u}\Big\|_{\mathrm{L}^2(\mathbf{R};[X,Y]_r)} \leqslant \|u\|_{W(\mathbf{R};X,Y)}.$$

In particular, for separable $X=V$ and $Y=V'$ it follows that $|\tau|^{1/2}\hat{u}\in\mathrm{L}^2(\mathbf{R};H)$, where H is a Hilbert space such that V,H,V' form a Gelfand triplet.

∎

In order to apply the last theorem, as well as the localisation principle, we need to localise (in time) sequences of functions u_n and f_n. Multiplication of (7_1) by $\theta\in\mathrm{C}^\infty_c(\mathbf{R}^+)$ gives

$$\partial_t(\theta u_n) - \mathsf{div}\,(\mathbf{A}\nabla\theta u_n) = \mathsf{div}\,(\theta\mathsf{f}_n) + q_n,$$

where, according to Theorem 5, $q_n=(\partial_t\theta)u_n$ is bounded in $\mathrm{L}^2(\mathbf{R}^{1+d})$ and converges strongly in $\mathrm{H}^{-\frac{1}{2},-1}_{\mathrm{loc}}(\mathbf{R}^{1+d})$. Thus multiplication by θ does not affect the application of the localisation principle. For that reason, in the sequel we shall not distinguish the solutions u_n from the localised functions θu_n.

Furthermore, for the sequence of (localised) functions u_n we have that $u_n \longrightarrow 0$ weakly in $W(\mathbf{R};\mathrm{H}^1(\mathbf{R}^d))$, and the last theorem together with the Plancherel formula gives that $\sqrt{\partial_t}\,u_n = \big(\sqrt{2\pi i\tau}\,\widehat{u_n}\big)^\vee$ is bounded in $\mathrm{L}^2(\mathbf{R}^{1+d})$.

Thus we can define a sequence of functions $\tilde{\mathsf{v}}_n = (v_n^0, \mathsf{v}_n, \mathsf{f}_n)^\top := (\sqrt{\partial_t} u_n, \nabla u_n, \mathsf{f}_n)^\top$ converging weakly to zero in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{R}^{1+2d})$. The associated parabolic H-measure has the block matrix form

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \mu_0 & \boldsymbol{\mu}_{01} & \boldsymbol{\mu}_{02} \\ \boldsymbol{\mu}_{10} & \boldsymbol{\mu} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{20} & \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_f \end{bmatrix},$$

where μ_0 and $\boldsymbol{\mu}$ stand for measures associated to $v_n^0 = \sqrt{\partial_t} u_n$ and $\mathsf{v}_n = \nabla u_n$, respectively, while $\boldsymbol{\mu}_f$ denotes the parabolic H-measure associated to f_n.

With the notation just introduced, we rewrite (7_1) as

$$\sqrt{\partial_t} v_n^0 - \mathsf{div}\,(\mathbf{A}\mathsf{v}_n) = \mathsf{div}\,\mathsf{f}_n,$$

and apply the localisation principle, thus obtaining

$$\begin{aligned} \mu_0 \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu}_{01} \cdot \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_{02} \cdot \boldsymbol{\xi} &= 0 \\ \boldsymbol{\mu}_{10} \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu} \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_{12}\, \boldsymbol{\xi} &= \mathsf{0} \\ \boldsymbol{\mu}_{20} \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu}_{21}\, \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_f \boldsymbol{\xi} &= \mathsf{0}. \end{aligned} \tag{8}$$

More information on the above measures can be obtained by using relations between components of $\tilde{\mathsf{v}}_n$.

After applying the Localisation principle to Schwarz relations $\partial_j v_n^k = \partial_k v_n^j$, for $j, k \in 1..d$, we get

$$\xi_j \tilde{\mu}^{mk} = \xi_k \tilde{\mu}^{mj}, \quad j, k \in 1..d, \quad m \in 0..2d\,. \tag{9}$$

Taking $m = j$, the summation with respect to $j \in 1..d$ gives

$$(\mathsf{tr}\boldsymbol{\mu})\, \boldsymbol{\xi} = \boldsymbol{\mu}\, \boldsymbol{\xi}\,.$$

Using (9) and the hermitian property of $\boldsymbol{\mu}$ one obtains $\xi_k \mu^{mj} = \xi_m \mu^{jk}$. Finally, multiplication by ξ_m and summation with respect to $m \in 1..d$ yields

$$|\boldsymbol{\xi}|^2 \boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\ \mathsf{tr}\boldsymbol{\mu}. \tag{10}$$

As parabolic H-measures are defined (in variable $\boldsymbol{\xi}$) on hypersurface P^d, measure $\boldsymbol{\mu}$ is real and determined by the scalar measure $\mathsf{tr}\boldsymbol{\mu}$ everywhere except on the South/North pole of the ellipsoid P^d (where $\boldsymbol{\xi} = \mathsf{0}$).

On the other hand, taking $m \in (d+1)..2d$ in (9) and applying a similar procedure as above we get

$$\boldsymbol{\mu}_{12} \boldsymbol{\xi} = (\mathsf{tr}\boldsymbol{\mu}_{12}) \boldsymbol{\xi}. \tag{11}$$

Similarly, the relation (of course, $[\cdot,\cdot]$ denotes the commutator of two operators)

$$\sqrt{\partial_t} v_n^k - \partial_k v_n^0 = [\sqrt{\partial_t}, \partial_k] u_n = 0\,,$$

together with the Localisation principle gives

$$\sqrt{2\pi i \tau} \tilde{\mu}^{mk} = 2\pi i \xi_k \tilde{\mu}^{m0}, \quad k \in 1..d,\ m \in 0..2d\,. \tag{12}$$

In particular, by choosing first $m = 0$ and then $m \in 1..d$, it respectively follows

$$\begin{aligned} \sqrt{2\pi i \tau} \boldsymbol{\mu}_{01}^\top &= 2\pi i \mu_0 \boldsymbol{\xi}\,, \\ \sqrt{2\pi i \tau} \boldsymbol{\mu} &= 2\pi i \boldsymbol{\mu}_{10} \otimes \boldsymbol{\xi}\,, \end{aligned} \tag{13}$$

implying that parabolic H-measure $\boldsymbol{\mu}$ is supported outside of the poles of ellipsoid P^d (where, let it be repeated, $\boldsymbol{\mu}$ is determined by a real measure $\mathsf{tr}\boldsymbol{\mu}$). As $\overline{\boldsymbol{\mu}}_{10} = \boldsymbol{\mu}_{01}^\top$, the last two expressions give a relation between μ_0 and $\boldsymbol{\mu}$:

$$|2\pi\tau|\boldsymbol{\mu} = 4\pi^2\mu_0\,\boldsymbol{\xi}\otimes\boldsymbol{\xi}. \tag{14}$$

On the other hand, after scalar multiplication of (13_2) by $\boldsymbol{\xi}$ we get

$$\boldsymbol{\mu}_{10} = \frac{\sqrt{2\pi i\tau}\boldsymbol{\mu}\boldsymbol{\xi}}{2\pi i|\boldsymbol{\xi}|^2}, \quad \boldsymbol{\xi}\neq \mathsf{0}. \tag{15}$$

In the same way as we have obtained the last relation, by taking $m \in (d+1)..2d$ in (12) one can get

$$\boldsymbol{\mu}_{20} = \frac{\sqrt{2\pi i\tau}\boldsymbol{\mu}_{21}\boldsymbol{\xi}}{2\pi i|\boldsymbol{\xi}|^2}, \quad \boldsymbol{\xi}\neq \mathsf{0}. \tag{16}$$

By inserting (10) and (15) in (7_2), we get

$$\frac{1}{|\boldsymbol{\xi}|^2}\left(\tau - 2\pi i\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi}\right)\mathsf{tr}\boldsymbol{\mu}\,\boldsymbol{\xi} = 2\pi i\boldsymbol{\mu}_{12}\boldsymbol{\xi}, \quad \boldsymbol{\xi}\neq\mathsf{0}.$$

Taking the scalar product with $\boldsymbol{\xi}$, after comparing to (10), we obtain

$$\left(\tau - 2\pi i\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi}\right)\mathsf{tr}\boldsymbol{\mu} = 2\pi i|\boldsymbol{\xi}|^2\mathsf{tr}\boldsymbol{\mu}_{12}.$$

Analogously, by inserting (16) in (7_3), the scalar multiplication by $\boldsymbol{\xi}$, conjugation and relation (10) give

$$\left(\tau + 2\pi i\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi}\right)\mathsf{tr}\boldsymbol{\mu}_{12} = -2\pi i\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}.$$

By combining the last two expressions we obtain a relation between the unknown measure $\boldsymbol{\mu}$ and the parabolic H-measure associated to f_n:

$$\mathsf{tr}\boldsymbol{\mu} = \frac{(2\pi\boldsymbol{\xi})^2}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi},$$

or, after taking into account (9),

$$\boldsymbol{\mu} = \frac{(2\pi)^2}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}(\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}.$$

Making use of relation (14) between μ_0 and $\boldsymbol{\mu}$, we finally obtain

$$\mu_0 = \frac{|2\pi\tau|}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}.$$

Thus we have explicitly described unknown macroscopic energy terms by using only given data (the sequence f_n) for (7). In particular, for $\mathbf{A} = \mathbf{I}$ it follows

$$\begin{aligned}\boldsymbol{\mu} &= (2\pi)^4(\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}\,,\\ \mu_0 &= |2\pi\tau|(2\pi)^2\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}\,.\end{aligned}$$

If we consider a sequence (f_n) converging strongly (in particular, this is the case when we consider the homogeneous problem, $\mathsf{f}_n = \mathsf{0}$), it follows that both μ_0 and $\boldsymbol{\mu}$ are null measures, implying that the homogeneous heat equation does not allow a distribution of initial disturbances. In other words, the sequence (γ_n) does not affect macroscopic energy terms. Also, it implies the following result.

Corollary 5 *Let* (u_n) *be a sequence of solutions of problems (6), with* $\mathsf{f}_n \to 0$ *strongly in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^+;\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d))$. *Then* $\nabla u_n \longrightarrow 0$ *strongly in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^+\times\mathbf{R}^d)$, *as well as* $v_n^0 = \sqrt{\partial_t}u_n \longrightarrow 0$ *in the same topology.*

∎

For the special case of equations with constant coefficients (constant matrix $\mathbf{A}$), the corollary could have also been proved by using the regularity of homogeneous heat equation. The above approach generalises this result to the equations with variable coefficients. A similar property also holds for the advection-diffusion equation [6].

The above calculation also enables us to express the distributional limit of the time derivative of energy $\frac{d}{dt}E_n(t)$ associated to (6). More precisely, for $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$ we consider

$$E_n^\varphi(t) := \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n^2(t,\mathbf{x}) d\mathbf{x}\,,$$

having the time derivative (up to a L^2 compact term) equal to

$$\frac{d}{dt}E_n^\varphi(t) = -2\Bigl(\int_{\mathbf{R}^d}\varphi(\mathbf{x})\mathbf{A}(t,\mathbf{x})\nabla u_n(t,\mathbf{x})\cdot\nabla u_n(t,\mathbf{x})d\mathbf{x} + \int_{\mathbf{R}^d}\varphi(\mathbf{x})\nabla u_n(t,\mathbf{x})\cdot\mathsf{f}_n(t,\mathbf{x})d\mathbf{x}\Bigr)\,.$$

By passing to the limit, for a $\theta \in \mathrm{C}_c(\mathbf{R}^+)$ Theorem 3 gives us

$$\lim_n \int \theta(t)\frac{d}{dt}E_n^\varphi(t)dt = -2\Bigl(\langle\boldsymbol{\mu}, \mathbf{A}\theta\varphi\boxtimes 1\rangle + \langle \mathsf{tr}\boldsymbol{\mu}_{12}, \theta\varphi\boxtimes 1\rangle\Bigr),$$

which, by means of the above calculation for parabolic H-measures, can be expressed as

$$\begin{aligned}-2\Bigl\langle \frac{(2\pi)^2}{\tau^2+(2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}(\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi})\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} - \frac{2\pi i\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}}{\tau+2\pi i\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi}}, \theta\varphi\boxtimes 1\Bigr\rangle \\ = 2\Bigl\langle \frac{2\pi i\tau\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}}{\tau^2+(2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}, \theta\varphi\boxtimes 1\Bigr\rangle = 0\,,\end{aligned}$$

as scalar measure $\nu = \tau\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}$ is antipodally antisymmetric on P^d, i.e. $\nu(\tau,\boldsymbol{\xi}) = -\nu(-\tau,-\boldsymbol{\xi})$.

Thus $\frac{d}{dt}E_n^\varphi(t) \rightharpoonup 0$ in $\mathcal{D}'(\mathbf{R}^+)$ and $\frac{d}{dt}(u_n^2) \rightharpoonup 0$ in $\mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^d)$.

Application to the Schrödinger equation

Let us examine the applicability of the introduced parabolic H-measures to the situations governed by the Schrödinger equation, which is of a similar form as it is the heat equation, but which is not hypoelliptic. We take $\mathbf{A} \in \mathrm{L}^\infty(\mathbf{R}_0^+\times\mathbf{R}^d;\mathrm{M}_{d\times d}(\mathbf{C}))$ to be a hermitian matrix field such that $\mathbf{A}\mathsf{v}\cdot\mathsf{v} \geqslant \alpha|\mathsf{v}|^2$ (a.e.) for an $\alpha > 0$. Furthermore, we suppose $\mathbf{A}(\cdot,\mathbf{x}) \in \mathrm{C}^1([0,T];\mathrm{M}_{d\times d}(\mathbf{C}))$ for some $T \in \mathbf{R}^+$, and consider the initial value problem:

$$(17)\qquad \begin{cases} i\partial_t u + \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ \qquad\qquad u_n(0,\cdot) = u_0\,. \end{cases}$$

The existence and the properties of solutions for the above problems are derived from the theory presented in [14, Ch. XVIII, §7.1] (a similar presentation can also be found in [28]), where a comparison to the heat (diffusion) equation can be found as well. Note that in physically relevant situations one has $\mathbf{A} = \mathbf{I}$, but our primary goal is to test the new tool, in a more general situation.

Theorem 7 (existence and regularity) *Let* $u_0 \in \mathrm{H}^1(\mathbf{R}^d)$ *and* $f \in W(0,T;\mathrm{L}^2(\mathbf{R}^d),\mathrm{H}^{-1}(\mathbf{R}^d))$. *Then under above assumptions on* $\mathbf{A}$ *there exists a unique solution* $u \in \mathrm{C}([0,T];\mathrm{H}^1(\mathbf{R}^d))$ *of (17) such that* $\partial_t u \in \mathrm{C}([0,T];\mathrm{H}^{-1}(\mathbf{R}^d))$. *Furthermore, the following bound holds*

$$\|u\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^1(\mathbf{R}^d))} + \|\partial_t u\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^{-1}(\mathbf{R}^d))} \leqslant C\left(\|u(0)\|_{\mathrm{L}^2(\mathbf{R}^d)} + \|f\|_W\right)\,,$$

for some constant $C \in \mathbf{R}^+$. *Also, for the homogeneous equation* ($f = 0$) *the conservation law* $\|u(t)\|_{\mathrm{L}^2} = \|u(0)\|_{\mathrm{L}^2}$ *is valid for any* $t \in [0,T]$.

∎

In the sequel we consider a sequence of initial value problems

$$\left\{\begin{aligned} i\partial_t u_n + \mathsf{div}\,(\mathbf{A}\nabla u_n) &= f_n \\ u_n(0,\cdot) &= u_n^0\,, \end{aligned}\right. \tag{18}$$

where $u_n^0 \rightharpoonup 0$ in $\mathrm{H}^1(\mathbf{R}^d)$, while $f_n \rightharpoonup 0$ in $W(0,T;\mathrm{L}^2(\mathbf{R}^d),\mathrm{H}^{-1}(\mathbf{R}^d))$.

As in the previous subsection we would like to apply the localisation principle in order to examine parabolic H-measures associated to (derivatives of) solutions of (18). The procedure can be carried out in a similar way as for the heat equation, but we choose to apply a slightly different approach here.

By means of Theorem 7 and the results on fractional derivatives (Theorem 6), similarly as it was done in the previous subsection, it can be shown that the sequence of (localised in time) functions $\tilde{\mathsf{v}}_n = (v_n^0, \mathsf{v}_n)^\top := (\sqrt{\partial_t}u_n, \nabla u_n)^\top$ converges weakly to zero in $\mathrm{L}^2(\mathbf{R}^{1+d};\mathbf{R}^{1+d})$. Thus a parabolic H-measure $\tilde{\boldsymbol{\mu}}$ of the form

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \mu_0 & \boldsymbol{\mu}_{01} \\ \boldsymbol{\mu}_{10} & \boldsymbol{\mu} \end{bmatrix}$$

can be associated to a chosen subsequence, where μ_0 and $\boldsymbol{\mu}$ denote measures associated to (sub)sequences of functions $v_n^0 = \sqrt{\partial_t}u_n$ and $\mathsf{v}_n = \nabla u_n$, respectively.

Theorem 8 *The traces of both parabolic H-measures $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}$ satisfy the following relation*

$$Q(\mathsf{tr}\tilde{\boldsymbol{\mu}}) = Q(\mathsf{tr}\boldsymbol{\mu}) = 0\,,$$

where $Q(t,\mathbf{x};\tau,\boldsymbol{\xi}) := 2\pi\tau + (2\pi)^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}$. Thus both $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}$ are supported at points (with $\tau,\boldsymbol{\xi}$ projection lying in the South hemisphere of P^d) of the form $2\pi\tau = -4\pi^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}$.

Dem. First, we shall prove that $\tilde{\boldsymbol{\mu}}$ is of the form

$$\tilde{\boldsymbol{\mu}} = \overline{\frac{\mathsf{p}\otimes\mathsf{p}}{|\mathsf{p}|^2}}\,\tilde{\nu}, \tag{19}$$

where $\mathsf{p} = (\sqrt{2\pi i\tau}, 2\pi i\boldsymbol{\xi})^\top$, while $\tilde{\nu} := \mathsf{tr}\tilde{\boldsymbol{\mu}}$ is a scalar measure.

Let $\mathsf{P} = (\sqrt{\partial_t},\nabla)^\top$ be the differential operator with symbol p defined above. As $P_j v_k^n = P_j P_k u_n = P_k v_j^n$, the localisation principle implies

$$p_j\tilde{\mu}^{mk} = p_k\tilde{\mu}^{mj} \quad j,k,m\in 0..d\,. \tag{20}$$

Specially, by taking $m = j$, summation with respect to j gives

$$\tilde{\boldsymbol{\mu}}^\top\mathsf{p} = (\mathsf{tr}\tilde{\boldsymbol{\mu}})\,\mathsf{p}\,.$$

By using the hermitian character of parabolic H-measures, the multiplication of (20) by $\overline{p}_j$, after summation over j gives

$$|\mathsf{p}|^2\tilde{\mu}^{mk} = (\tilde{\boldsymbol{\mu}}\overline{\mathsf{p}})_m\,p_k = \mathsf{tr}\tilde{\boldsymbol{\mu}}\overline{p}_m p_k\,,$$

which proves (19).

On the other hand, the (localised in time) sequence (f_n) is bounded in $\mathrm{L}^2(\mathbf{R}^{1+d})$, which implies its strong convergence in $\mathrm{H}_{\mathrm{loc}}^{-\frac{1}{2},-1}(\mathbf{R}^{1+d})$. Thus an application of the Localisation principle to the equation (18_1) gives

$$\tilde{\boldsymbol{\mu}}\begin{bmatrix} i\sqrt{2\pi i\tau} \\ 2\pi i\mathbf{A}^\top\boldsymbol{\xi}\end{bmatrix} = \frac{\tilde{\nu}}{|\mathsf{p}|^2}\left(\begin{bmatrix} i\sqrt{2\pi i\tau} \\ 2\pi i\mathbf{A}^\top\boldsymbol{\xi}\end{bmatrix}\cdot\overline{\mathsf{p}}\right)\overline{\mathsf{p}} = 0\,.$$

As $\mathsf{p}\neq 0$ on P^d we get

$$\left(\begin{bmatrix} i\sqrt{2\pi i\tau} \\ 2\pi i\mathbf{A}^\top\boldsymbol{\xi}\end{bmatrix}\cdot\overline{\mathsf{p}}\right)\tilde{\nu} = -(2\pi\tau + 4\pi^2\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})\mathsf{tr}\tilde{\boldsymbol{\mu}} = 0.$$

Similarly, as $\nu = \frac{(2\pi|\boldsymbol{\xi}|)^2}{|\mathsf{p}|^2}\tilde{\nu}$, the same holds also for ν. The positivity of parabolic H-measures implying that their support is contained in the support of their trace completes the proof. **Q.E.D.**

Corollary 6 *In the case* $\mathbf{A} = \mathbf{I}$*, the above description of the support of* $\boldsymbol{\mu}$ *implies the equipartition of energy*

$$\mu_0 = \mathsf{tr}\boldsymbol{\mu} \ .$$

Dem. According to (19)

$$\mu_0 = \frac{|2\pi\tau|}{|\mathsf{p}|^2}\tilde{\nu} = \frac{|2\pi\tau|}{|2\pi\boldsymbol{\xi}|^2}\mathsf{tr}\boldsymbol{\mu} \ ,$$

and the localisation principle implies that the last fraction is equal to 1. **Q.E.D.**

We conclude this section with additional remark which will be used later when we apply the Propagation principle to the Schrödinger equation.

Remark. Let us first notice that a relation similar to (19) holds when considering parabolic H-measure $\boldsymbol{\mu}$. Indeed, after taking into account (20), and repeating the proof of (19) with summations starting from 1, one obtains

$$\boldsymbol{\mu} = \frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\,\nu \ .$$

We can also describe the off-diagonal terms of the parabolic H-measure associated to sequence $(\mathsf{v}_n, f_n)^\top$:

$$\begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \mu_f \end{bmatrix} ,$$

where μ_f is a parabolic H-measure associated to a subsequence of (f_n).

The identities $P_j v_k^n = P_k v_j^n$ and an application of the Localisation principle to the above block matrix measure for $m = d + 1$ yields

$$\xi_i \mu_{21}^j = \xi_j \mu_{21}^i \quad i, j \in 1..d \ .$$

Multiplication of the above relation by ξ_i and summation give us

$$|\boldsymbol{\xi}|^2 \mu_{21}^j = \xi_j \left(\boldsymbol{\mu}_{21}\cdot\boldsymbol{\xi}\right) ,$$

implying that $\boldsymbol{\mu}_{21}$ is of the form $\boldsymbol{\xi}\,\nu_{21}$, where ν_{21} is a scalar measure $\frac{(\boldsymbol{\mu}_{21}\cdot\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}$. A similar result holds for $\boldsymbol{\mu}_{12}$ as well:

$$\boldsymbol{\mu}_{12} = \boldsymbol{\xi}\,\nu_{12} \ ,$$

where, due to the hermitian property of H-measures, ν_{12} is a scalar measure equal to $\bar{\nu}_{21}$.

∎

Application to the vibrating plate equation

Our next example, while motivated by real physical problems, is oversimplified. We consider the equation modelling the vibration of a thin infinite elastic plate (by this we avoid the discussion of boundary conditions).

As our approach allows it, we consider the density ϱ of the material and the fourth rank tensor M, describing the elastic properties, to depend both on time t and the space variable $\mathbf{x}$. Note that M is assumed to have the following symmetries: $M_{klij} = M_{ijkl} = M_{jikl} = M_{ijlk}$.

Regarding the notation, we mainly follow [3, 2], where the interested reader can also find more details regarding the physical origin of the problem, as well as some further references. Presently, our goal is limited to testing the applicability of the new tool, the parabolic H-measures. The intricacies of realistic physical situations will require a separate study, which we plan to undertake in the future.

Therefore, we consider the initial value problem

$$
\begin{cases} \partial_t(\varrho\partial_t u) + \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u) = f \\ u_n(0) = u_0 \\ \partial_t u_n(0) = u_1, \end{cases} \tag{21}
$$

where $\varrho \in \mathrm{C}^1(\mathbf{R}^+; \mathrm{L}^\infty(\mathbf{R}^d))$ is a real function, coercive in the sense that $\varrho \geqslant \varrho_0$, where $\varrho_0 \in \mathbf{R}^+$ is a given constant. On the other hand, $\mathbf{M}$ is a real symmetric tensor field of order four (which we consider as an operator on symmetric rank-two tensors, i.e. symmetric matrices) such that (for given $\alpha > 0$)

$$
\begin{aligned}
&\mathbf{M}(\cdot, \mathbf{x}) \in \mathrm{C}^1(\mathbf{R}^+; \mathcal{L}(\mathrm{M}_{d\times d})) \\
&\mathbf{M}, \partial_t\mathbf{M} \in \mathrm{C}_b(\mathbf{R}^+ \times \mathbf{R}^d; \mathcal{L}(\mathrm{M}_{d\times d})) \\
&\mathbf{MA}\cdot\mathbf{A} \geqslant \alpha\mathbf{A}\cdot\mathbf{A}, \quad \mathbf{A} \in \mathrm{M}_{d\times d}\,.
\end{aligned}
$$

The existence of a solution to (21) is given by the following theorem (derived from the theory presented in [14, Ch. XVIII, §5]; see also [id., §6, sect. 5.6]).

Theorem 9 *Let $f \in \mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d), u_0 \in \mathrm{H}^2(\mathbf{R}^d)$ and $u_1 \in \mathrm{L}^2(\mathbf{R}^d)$. Then under the above assumptions on coefficients $\mathbf{M}$ there exists a unique solution $u \in \mathrm{C}([0,T]; \mathrm{H}^2(\mathbf{R}^d))$ of (21) such that $u' \in \mathrm{L}^2([0,T]; \mathrm{H}^2(\mathbf{R}^d)) \cap \mathrm{C}([0,T]; \mathrm{L}^2(\mathbf{R}^d))$. Furthermore, the following bound holds (C being a positive constant)*

$$
\begin{aligned}
\|u\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^2(\mathbf{R}^d))} &+ \|\partial_t u\|_{\mathrm{L}^2([0,T];\mathrm{H}^2(\mathbf{R}^d))} + \|\partial_t u\|_{\mathrm{L}^\infty([0,T];\mathrm{L}^2(\mathbf{R}^d))} \\
&\leq C\left(\|u_0\|_{\mathrm{H}^2(\mathbf{R}^d)} + \|u_1\|_{\mathrm{L}^2(\mathbf{R}^d)} + \|f\|_{\mathrm{L}^2(\mathbf{R}^+\times\mathbf{R}^d)}\right).
\end{aligned}
$$

∎

In the sequel we consider a sequence of initial value problems

$$
\begin{cases} \partial_t(\varrho\partial_t u_n) + \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u_n) = f_n \\ u_n(0) = u_n^0 \\ \partial_t u_n(0) = u_n^1, \end{cases} \tag{22}
$$

where $u_n^0 \longrightarrow 0$ in $\mathrm{H}^2(\mathbf{R}^d)$, $u_n^1 \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d)$ and $f_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^+ \times \mathbf{R}^d)$.

By means of the last theorem, a sequence (u_n) of solutions to (22) converges weakly to zero in $\mathrm{L}^\infty([0,T]; \mathrm{H}^2(\mathbf{R}^d))$, and $\partial_t u_n \longrightarrow 0$ in $\mathrm{L}^\infty([0,T]; \mathrm{L}^2(\mathbf{R}^d))$. For that reason, the macroscopic limit of the plate energy $\int \left(\varrho|\partial_t u_n|^2 + \mathbf{M}\nabla\nabla u_n \cdot \nabla\nabla u_n\right) d\mathbf{x}$ can be described by the associated parabolic H-measure.

The above estimates on the solutions yield that for a test function $\theta \in \mathrm{C}_c^\infty(\mathbf{R}^+)$ the sequence of localised (in time) functions (θu_n) satisfies the equation (22_1), up to a term converging strongly to 0 in $\mathrm{H}^{-1,-2}(\mathbf{R}^{1+d})$. Thus, as in preceding applications, multiplication by θ does not affect the application of the Localisation principle, and in the sequel we shall not distinguish the sequence of solutions (u_n) from the sequence of localised functions (θu_n).

Prior to the calculation, we have to address the algebraic difficulty of working with rank-four tensors. We should associate a parabolic H-measure to the sequence $(\partial_t u_n, \nabla\nabla u_n)^\top$, with $1 + d^2$ scalar components (at the moment we shall ignore the fact that, due to the Schwarz symmetry of second derivatives, there are only $1 + d(d+1)/2$ independent scalar components). Therefore the corresponding H-measure will be a $(1+d^2) \times (1+d^2)$ matrix measure, which we can write as

$$
\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \mu_0 & \boldsymbol{\mu}_{01} \\ \boldsymbol{\mu}_{10} & \boldsymbol{\mu} \end{bmatrix}.
$$

We have a similar result as for the Schrödinger equation.

Theorem 10 *The traces of parabolic H-measures* $\tilde{\boldsymbol{\mu}}$ *and* $\boldsymbol{\mu}$ *both satisfy the following relation*

$$Q(\mathsf{tr}\tilde{\boldsymbol{\mu}}) = Q(\mathsf{tr}\boldsymbol{\mu}) = 0 \,,$$

where $Q(t,\mathbf{x};\tau,\boldsymbol{\xi}) := -(2\pi\tau)^2\varrho(t,\mathbf{x}) + (2\pi)^4\mathbf{M}(t,\mathbf{x})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})$*. Thus both* $\tilde{\boldsymbol{\mu}}$ *and* $\boldsymbol{\mu}$ *are supported at points of* $\mathbf{R}^{1+d}\times \mathrm{P}^d$ *where* $(2\pi\tau)^2\varrho(t,\mathbf{x}) = (2\pi)^4\mathbf{M}(t,\mathbf{x})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})$*.*

Dem. Let $\mathbf{P} = (\partial_t, \nabla\nabla)^\top$ be the differential operator with symbol $\mathbf{p} = (2\pi i\tau, -(2\pi)^2(\boldsymbol{\xi}\otimes\boldsymbol{\xi}))^\top$. By means of Schwarz relations, similarly as it was done in the previous section for the Schrödinger equation, one shows that

$$\tilde{\boldsymbol{\mu}} = \frac{\overline{\mathbf{p}\otimes\mathbf{p}}}{|\mathbf{p}|^2}\,\tilde{\nu}\,, \tag{23}$$

where $\tilde{\nu}$ is a scalar measure. Note that here we have to take the complex conjugate, as the symbol $\mathbf{p}$ is not real, as it was for the heat equation.

Instead of rewriting the symbols and matrices as one-dimensional vectors, we shall keep their original form, and treat the components of $\boldsymbol{\mu}$ as if it were a rank-four tensor.

The application of the localisation principle to the equation (22_1) gives

$$\tilde{\boldsymbol{\mu}}\begin{bmatrix} 2\pi i\tau\varrho \\ -4\pi^2\mathbf{M}(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\end{bmatrix} = \mathbf{0},$$

which combined with (23) implies

$$\frac{\tilde{\nu}}{|\mathbf{p}|^2}\Big((2\pi i\tau\varrho, -4\pi^2\mathbf{M}(\boldsymbol{\xi}\otimes\boldsymbol{\xi}))\cdot\overline{\mathbf{p}}\Big)\,\overline{\mathbf{p}} = \mathbf{0}\,.$$

As $\mathbf{p}\neq 0$ on P^d we get $\Big((2\pi i\tau\varrho, -4\pi^2\mathbf{M}(\boldsymbol{\xi}\otimes\boldsymbol{\xi}))\cdot\overline{\mathbf{p}}\Big)\tilde{\nu} = 0$, i.e. $\tilde{\boldsymbol{\mu}}$ is supported at points satisfying

$$(2\pi\tau)^2\varrho = (2\pi)^4\,\mathbf{M}(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\,.$$

As $\nu = \frac{(2\pi|\boldsymbol{\xi}|)^4}{|\mathbf{p}|^2}\tilde{\nu}$, the same holds also for $\nu = \mathsf{tr}\boldsymbol{\mu}$, and the positivity of above parabolic H-measures now implies the result on their support. **Q.E.D.**

Finally, as in the case of the Schrödinger equation, one can easily prove the following corollary.

Corollary 7 *In the case* $\mathbf{M}$ *is an identity, while* $\varrho = 1$*, the equipartition of energy* $\mu_0 = \mathsf{tr}\boldsymbol{\mu}$ *holds.*

∎

4. Propagation principle

Function spaces and symbols

For the purpose of proving the Second commutation lemma, we need a parabolic version of Tartar spaces X^m [40]. For $m \in \mathbf{N}_0$, X^m is defined as a Banach space containing functions such that their derivatives up to order $m \in \mathbf{N}_0$ belong to $\mathcal{F}\mathrm{L}^1(\mathbf{R}^d)$, i.e. their Fourier transform is an L^1 function. The norm on $X^m(\mathbf{R}^d)$ is given by (we have chosen to replace Tartar's $1 + |2\pi\boldsymbol{\xi}|^m$ by $k^m(\boldsymbol{\xi}) := ((1 + |2\pi\boldsymbol{\xi}|^2)^{m/2}$, which is equivalent)

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} k^m |\mathcal{F}w| \, d\boldsymbol{\xi} \,.$$

Clearly, $\mathcal{F}\mathrm{L}^1(\mathbf{R}^d)$ is a Banach algebra for multiplication of functions, as $\mathrm{L}^1(\mathbf{R}^d)$ is under convolution, while $\bar{\mathcal{F}}(\varphi\psi) = \bar{\mathcal{F}}\varphi * \bar{\mathcal{F}}\psi$.

These spaces happen again to be a particular case of Hörmander's spaces $B_{1,k}$ (v. [25, 10.1]; in fact, $X^m = B_{1,k^m}$). In the same vein as at the beginning of previous section, we shall restate their main properties in our notation, while referring the reader to Hörmander's book for the proofs.

The parabolic variant of Tartar's spaces X^m is defined as (here we again use $k_p(\tau, \boldsymbol{\xi}) := \sqrt[4]{1 + (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}$)

$$\mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d}) : = \Big\{ b \in \mathcal{S}' : k_p^m \, \hat{b} \in \mathrm{L}^1(\mathbf{R}^{1+d}) \Big\} \,.$$

$\mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d})$ is a vector space, and supplied with the norm

$$\|b\|_{\mathrm{X}^{\frac{m}{2},m}} := \int_{\mathbf{R}^{1+d}} k_p^m \, |\hat{b}| \, d\tau d\boldsymbol{\xi}$$

it becomes a Banach space. Furthermore, $\mathcal{S} \hookrightarrow \mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d}) \hookrightarrow \mathcal{S}'$, the inclusions being dense and continuous.

As the Fourier transform (and its inverse) maps L^1 into C_0, the spatial derivatives up to order m of a function from $\mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d})$, as well as its time derivatives up to order $m/2$, belong to $\mathrm{C}_0(\mathbf{R}^{1+d})$. Furthermore, for $s \in \mathbf{R}$, $s > m + d/2 + 1$, the following embedding holds

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) \hookrightarrow \mathrm{X}^{\frac{m}{2},m}(\mathbf{R}^{1+d}).$$

Indeed, for $v \in \mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ we have

$$\|v\|_{\mathrm{X}^{\frac{m}{2},m}} = \int_{\mathbf{R}^{1+d}} k_p^m \, |\hat{v}| \, d\tau d\boldsymbol{\xi} \leqslant \|k_p^{-(s-m)}\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \|v\|_{\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})} \,.$$

The statement follows as function k_p^{-r} belongs to $\mathrm{L}^2(\mathbf{R}^{1+d})$ for $r > d/2+1$. The above embedding corresponds to a similar one in the classical case: $\mathrm{H}^s(\mathbf{R}^d) \hookrightarrow \mathrm{X}^m(\mathbf{R}^d)$ for $s > m + d/2$ (cf. [40, p. 205]).

In order to relate our results to the classical theory of pseudodifferential operators, we shall also need some nonstandard symbols [25, 18.1]. Let $k(\tau, \boldsymbol{\xi}) := \sqrt{1 + 4\pi^2(\tau^2 + |\boldsymbol{\xi}|^2)}$ (in accordance with our earlier definition on $\mathbf{R}^d$); then for $m \in \mathbf{R}$, $\rho \in \langle 0, 1]$ and $\delta \in [0, 1\rangle$ we define $\mathrm{S}^m_{\rho,\delta}$ as the set:

$$\Big\{ a \in \mathrm{C}^\infty(\mathbf{R}^{1+d} \times \mathbf{R}^{1+d}) : (\forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{N}_0^{1+d})(\exists\, C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0) \quad |\partial_{\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} a| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} k^{m - \rho|\boldsymbol{\alpha}| + \delta|\boldsymbol{\beta}|} \Big\} \,.$$

It can easily be checked that $\mathrm{S}^m_{\rho,\delta}$ is a vector space, and its elements are called *symbols* of order m and type ρ, δ. The best possible constant $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is taken as the value of corresponding seminorm of a in $\mathrm{S}^m_{\rho,\delta}$, and in such a way these seminorms make $\mathrm{S}^m_{\rho,\delta}$ into a Fréchet space. While $\mathrm{S}^m_{1,0}$ is the space of classical symbols (sometimes denoted only by S^m), here we shall have to use $\mathrm{S}^m_{\frac{1}{2},0}$. We would like to mention that in a related context these symbols have been used in [12].

Let us mention in passing that there are also local versions of these symbol classes ([25, 7.9] or [36, 1.19]), for which the same notation $\mathrm{S}^m_{\rho,\delta}$ is used, possibly causing some confusion. We shall use only the above defined global classes of symbols.

The symbols are used to define corresponding operators on $\mathcal{S}$ (and by transposition, also on $\mathcal{S}'$) by:

$$a(\cdot; D)\varphi := \bar{\mathcal{F}}(a\hat{\varphi}) \,.$$

This is a generalisation of earlier considered operators M_ϕ and P_ψ.

For such operators and symbols, some calculus rules are valid; in particular we shall need the fact [25, Theorem 18.1.8] that if $a \in \mathrm{S}^k_{\frac{1}{2},0}$ and $b \in \mathrm{S}^m_{\frac{1}{2},0}$, then the symbol of composition operator $a(\cdot, D)b(\cdot, D)$ is in $\mathrm{S}^{k+m}_{\frac{1}{2},0}$, and it is given by asymptotic expansion:

$$\sum_{|\boldsymbol{\alpha}|\geqslant 0} \frac{1}{\boldsymbol{\alpha}!}(\partial_{\boldsymbol{\alpha}} a)(D^{\boldsymbol{\alpha}} b) \,. \tag{24}$$

For the precise meaning of the above asymptotic sum cf. [25, Proposition 18.1.3].

Let us denote by $\psi^p = \psi \circ p$ the parabolic extension of function $\psi \in \mathrm{C}^\infty(\mathrm{P}^d)$ to $\mathbf{R}^{1+d}_*$. This gives us

$$\begin{aligned}
\nabla^{\tau,\boldsymbol{\xi}}\psi^p(\tau,\boldsymbol{\xi}) &= \frac{1}{\rho^2\sqrt{(\boldsymbol{\xi}/2)^4+\tau^2}}\begin{bmatrix} \left|\frac{\boldsymbol{\xi}}{2}\right|^2 & -\frac{1}{2}\frac{\tau\boldsymbol{\xi}}{\rho} \\ -\frac{\tau\boldsymbol{\xi}}{2} & \rho\left(\sqrt{|\boldsymbol{\xi}/2|^4+\tau^2}\,\mathbf{I} - \frac{\boldsymbol{\xi}}{2}\otimes\frac{\boldsymbol{\xi}}{2}\right)\end{bmatrix}\nabla^{\tau,\boldsymbol{\xi}}\psi^p(\tau_0,\boldsymbol{\xi}_0), \\
&= \frac{1}{\rho^2}\begin{bmatrix} 1-(\alpha\tau_0)^2 & -\alpha^2\tau_0\frac{\boldsymbol{\xi}_0}{2} \\ -\rho\alpha^2\tau_0\frac{\boldsymbol{\xi}_0}{2} & \rho(\mathbf{I}-\alpha^2\frac{\boldsymbol{\xi}_0}{2}\otimes\frac{\boldsymbol{\xi}_0}{2})\end{bmatrix}\nabla^{\tau,\boldsymbol{\xi}}\psi(\tau_0,\boldsymbol{\xi}_0), \\
&= \begin{bmatrix} \frac{1}{\rho^2}\partial^\tau_T\psi(\tau_0,\boldsymbol{\xi}_0) \\ \frac{1}{\rho}\nabla^{\boldsymbol{\xi}}_T\psi(\tau_0,\boldsymbol{\xi}_0)\end{bmatrix} \,,
\end{aligned} \tag{25}$$

where $\nabla_T = \nabla - \mathsf{n}\,\partial^{\mathsf{n}}$ stands for a tangential gradient, while $\mathsf{n} = \alpha\begin{bmatrix}\tau \\ \frac{\boldsymbol{\xi}}{2}\end{bmatrix}$ denotes the outward unit normal on P^d. Of course, both ρ and α are functions of τ and $\boldsymbol{\xi}$, but we have suppressed it in writing.

Furthermore, let us introduce a smooth function

$$\tilde{\psi} := (1-\theta)\psi^p, \tag{26}$$

where $\theta \in \mathrm{C}^\infty_c(\mathbf{R}^{1+d})$ equals 1 on a neighbourhood of the origin, while it vanishes for points with the parabolic distance from origin $\rho \geqslant 1$.

Lemma 9 *$\tilde{\psi}$ belongs to $S^0_{\frac{1}{2},0}$, while $\rho^{-m} \in S^{-\frac{m}{2}}_{\frac{1}{2},0}$ for $m \geqslant 0$.*

Dem. As $\tilde{\psi}$ is of class C^∞, it is enough to provide estimates outside of a compact set defined by $\rho(\tau,\boldsymbol{\xi}) \leqslant 1$.

Following the steps in (25), by induction it easily follows that for arbitrary multiindex $\boldsymbol{\alpha} = (\alpha_0, \boldsymbol{\alpha}') \in \mathbf{N}^{1+d}_0$ there exists a smooth function $\psi_{\boldsymbol{\alpha}}$ on P^d such that

$$\partial^{\alpha_0}_\tau \partial^{\boldsymbol{\alpha}'}_{\boldsymbol{\xi}}\psi^p(\tau,\boldsymbol{\xi}) = \frac{1}{\rho^{\alpha_0+|\boldsymbol{\alpha}|}(\tau,\boldsymbol{\xi})}\psi_{\boldsymbol{\alpha}}(\tau_0,\boldsymbol{\xi}_0).$$

As ρ behaves as k_p for large $(\tau, \boldsymbol{\xi})$, and $k^{\frac{1}{2}}(\tau, \boldsymbol{\xi}) \leqslant 2k_p(\tau, \boldsymbol{\xi})$, we get the required bound

$$\left|\partial_\tau^{\alpha_0} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}'} \psi^p(\tau, \boldsymbol{\xi})\right| \leqslant C_{\boldsymbol{\alpha}} k^{-\frac{|\boldsymbol{\alpha}|}{2}}(\tau, \boldsymbol{\xi}) \,.$$

Similarly, one shows that $\partial_\tau^{\alpha_0} \partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}'} \rho^{-m}$ behaves as $\rho^{-m-\alpha_0-|\boldsymbol{\alpha}|}$, which is of order $k^{-\frac{m+|\boldsymbol{\alpha}|}{2}}$. **Q.E.D.**

Second commutation lemma

Lemma 10 *Let P_ψ and M_ϕ be the Fourier and pointwise multiplier operators on $\mathrm{L}^2(\mathbf{R}^{1+d})$ defined in (2.2) and (2.3) above, with associated symbols $\psi \in \mathrm{C}^1(\mathrm{P}^d)$ and $\phi \in \mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})$ respectively. Then the following results on the commutator $K = [P_\psi, M_\phi] = P_\psi M_\phi - M_\phi P_\psi$ hold.*
a) *The commutator K is a continuous operator from $\mathrm{L}^2(\mathbf{R}^{1+d})$ to $\mathrm{H}^{\frac{1}{2},1}(\mathbf{R}^{1+d})$.*
b) *For the parabolic extension $\psi^p = \psi \circ p$ (as defined in Sect. 2), we have for $j \in 1..d$, up to a compact operator on $\mathrm{L}^2(\mathbf{R}^{1+d})$,*

$$\partial_j K = \partial_j \left(P_\psi M_\phi - M_\phi P_\psi\right) = P_{\xi_j \nabla_{\boldsymbol{\xi}} \psi^p} M_{\nabla_{\mathbf{x}} \phi} \,. \tag{27}$$

A similar expression also holds for $\sqrt{\partial_t}\Big(P_\psi M_\phi - M_\phi P_\psi\Big)$, with ξ_j replaced by $\sqrt{\frac{\tau}{2\pi i}}$.

Dem. **a)** After applying the Fourier transform we have

$$\mathcal{F}\Big(\partial_j(Ku)\Big)(\tau, \boldsymbol{\xi}) = 2\pi i \xi_j \int\limits_{\mathbf{R}^{1+d}} \Big(\psi(\tau_0, \boldsymbol{\xi}_0) - \psi(\sigma_0, \boldsymbol{\eta}_0)\Big) \hat{\phi}(\tau - \sigma, \boldsymbol{\xi} - \boldsymbol{\eta}) \hat{u}(\sigma, \boldsymbol{\eta}) d\sigma d\boldsymbol{\eta}.$$

Denoting by L the Lipschitz constant of function ψ on P^d, Lemma 1 gives us the bound

$$\Big|\psi(\tau_0, \boldsymbol{\xi}_0) - \psi(\sigma_0, \boldsymbol{\eta}_0)\Big| \leqslant CL \frac{\rho(\tau - \sigma, \boldsymbol{\xi} - \boldsymbol{\eta})}{\rho(\tau, \boldsymbol{\xi})}.$$

Thus

$$\left|\mathcal{F}\Big(\partial_j(Ku)\Big)(\tau, \boldsymbol{\xi})\right| \leqslant 2\pi CL \frac{|\xi_j|}{\rho(\tau, \boldsymbol{\xi})} \int\limits_{\mathbf{R}^{1+d}} \rho(\tau - \sigma, \boldsymbol{\xi} - \boldsymbol{\eta}) \Big|\hat{\phi}(\tau - \sigma, \boldsymbol{\xi} - \boldsymbol{\eta})\Big| |\hat{u}(\sigma, \boldsymbol{\eta})| d\sigma d\boldsymbol{\eta}.$$

As $\frac{|\xi_j|}{\rho(\tau, \boldsymbol{\xi})} \leqslant \sqrt{2}$, the Plancherel formula and the Young inequality finally give us

$$\begin{aligned}
\|\partial_j(Ku)\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} &\leqslant \sqrt{2}\, 2\pi CL \Big\| |\rho\hat{\phi}| * |\hat{u}| \Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \\
&\leqslant \sqrt{2}\, 2\pi CL \|\rho\hat{\phi}\|_{\mathrm{L}^1(\mathbf{R}^{1+d})} \|u\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \leqslant \sqrt{2}\, 2\pi CL \|\phi\|_{\mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})} \|u\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}.
\end{aligned}$$

Analogously

$$\|\sqrt{\partial_t}(Ku)\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \leqslant 2\pi CL \|\phi\|_{\mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})} \|u\|_{\mathrm{L}^2(\mathbf{R}^{1+d})},$$

where the bound $\frac{\sqrt{|\tau|}}{\rho(\tau, \boldsymbol{\xi})} \leqslant 1$ has been used. These upper bounds imply that $P_\psi M_\phi - M_\phi P_\psi$ is a continuous operator from $\mathrm{L}^2(\mathbf{R}^{1+d})$ into $\mathrm{H}^{\frac{1}{2},1}(\mathbf{R}^{1+d})$, with the norm bounded by $\tilde{C}\|\psi\|_{\mathrm{W}^{1,\infty}(\mathrm{P}^d)}$ $\|\phi\|_{\mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})}$.

b) For the second part of the lemma, we prove it first under additional assumptions on ϕ, and in the second step make an approximation argument.

I. Assume additionally that $\phi \in \mathcal{S}(\mathbf{R}^{1+d})$ (actually, we shall only use that it is of class C^1, not just in $\mathbf{x}$, but also in t), and such that its Fourier transform is compactly supported.

Our goal is to show that (27) holds up to a compact operator. First, let us replace K by $\tilde{K} := [P_{\tilde{\psi}}, M_\phi]$, where $\tilde{\psi}$ is defined by (26). We have

$$\mathcal{F}\Big(\partial_j\Big((K-\tilde{K})u\Big)\Big) = 2\pi i\xi_j \int_{\mathbf{R}^{1+d}} \Big((\theta\psi^p)(\tau,\boldsymbol{\xi}) - (\theta\psi^p)(\sigma,\boldsymbol{\eta})\Big)\hat{\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})\hat{u}(\sigma,\boldsymbol{\eta})d\sigma d\boldsymbol{\eta}\,.$$

As $2\pi i\xi_j\Big((\theta\psi^p)(\tau,\boldsymbol{\xi}) - (\theta\psi^p)(\sigma,\boldsymbol{\eta})\Big)\hat{\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})$ is a bounded function with compact support, it is a kernel of a Hilbert-Schmidt operator, which is compact.

Similarly, we can replace $P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p}M_{\nabla_{\mathbf{x}}\phi}$ by $P_{\xi_j\nabla^{\boldsymbol{\xi}}\tilde{\psi}}M_{\nabla_{\mathbf{x}}\phi}$, as the very same argument gives that $P_{\xi_j\nabla^{\boldsymbol{\xi}}(\psi^p-\tilde{\psi})}M_{\nabla_{\mathbf{x}}\phi}$ is a compact operator as well.

Thus it is enough to prove that $\partial_j\tilde{K}$ and $P_{\xi_j\nabla^{\boldsymbol{\xi}}\tilde{\psi}}M_{\nabla_{\mathbf{x}}\phi}$ are equal modulo a compact operator.

To this end, first note that by the Mean value theorem for $\tilde{\psi}$ one has:

$$\tilde{\psi}(\tau,\boldsymbol{\xi}) = \tilde{\psi}(\sigma,\boldsymbol{\eta}) + \partial^0\tilde{\psi}(\vartheta,\boldsymbol{\zeta})(\tau-\sigma) + \nabla^{\boldsymbol{\xi}}\tilde{\psi}(\vartheta,\boldsymbol{\zeta})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\;,$$

where $(\vartheta,\boldsymbol{\zeta}) = (1-\theta)(\tau,\boldsymbol{\xi}) + \theta(\sigma,\boldsymbol{\eta})$, for some $\theta\in\langle 0,1\rangle$.

Now we can write

$$\begin{aligned}\mathcal{F}\Big(\Big(\partial_j\tilde{K} - P_{\xi_j\nabla^{\boldsymbol{\xi}}\tilde{\psi}}M_{\nabla_{\mathbf{x}}\phi}\Big)u\Big)(\tau,\boldsymbol{\xi}) &= \int_{\mathbf{R}^{1+d}} \xi_j\Big(2\pi i\Big(\tilde{\psi}(\tau,\boldsymbol{\xi}) - \tilde{\psi}(\sigma,\boldsymbol{\eta})\Big)\hat{\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta}) \\ &\qquad - \nabla^{\boldsymbol{\xi}}\tilde{\psi}(\tau,\boldsymbol{\xi})\cdot\widehat{\nabla_{\mathbf{x}}\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})\Big)\hat{u}(\sigma,\boldsymbol{\eta})d\sigma d\boldsymbol{\eta} \\ &= \int_{\mathbf{R}^{1+d}} \xi_j\Big(\Big(\nabla^{\boldsymbol{\xi}}\tilde{\psi}(\vartheta,\boldsymbol{\zeta}) - \nabla^{\boldsymbol{\xi}}\tilde{\psi}(\tau,\boldsymbol{\xi})\Big)\cdot\widehat{\nabla_{\mathbf{x}}\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta}) \\ &\qquad + \partial_\tau\tilde{\psi}(\vartheta,\boldsymbol{\zeta})\widehat{\partial_t\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})\Big)\hat{u}(\sigma,\boldsymbol{\eta})d\sigma d\boldsymbol{\eta}\,.\end{aligned}$$

In order to estimate the last integral, for each $m\in\mathbf{N}$ we introduce a compact set

$$S^m := \Big\{(\tau,\boldsymbol{\xi},\sigma,\boldsymbol{\eta})\in\mathbf{R}^{2(1+d)} : (\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})\in\operatorname{supp}\hat{\phi},\ \rho(\tau,\boldsymbol{\xi})\leqslant m\Big\}\;,$$

and divide the integral into two parts

$$\begin{aligned}E = &\int_{\mathbf{R}^{1+d}} \mathsf{A}^m(\tau,\boldsymbol{\xi},\sigma,\boldsymbol{\eta})\cdot\widehat{\nabla_{t,\mathbf{x}}\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})\hat{u}(\sigma,\boldsymbol{\eta})d\sigma d\boldsymbol{\eta} \\ &+ \int_{\mathbf{R}^{1+d}} \mathsf{B}^m(\tau,\boldsymbol{\xi},\sigma,\boldsymbol{\eta})\cdot\widehat{\nabla_{t,\mathbf{x}}\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})\hat{u}(\sigma,\boldsymbol{\eta})d\sigma d\boldsymbol{\eta},\end{aligned} \tag{28}$$

where

$$\begin{aligned}\mathsf{A}^m(\tau,\boldsymbol{\xi},\sigma,\boldsymbol{\eta}) &:= \xi_j\chi_{S^m}\begin{bmatrix}\partial_\tau\tilde{\psi}(\vartheta,\boldsymbol{\zeta}) \\ \nabla^{\boldsymbol{\xi}}\tilde{\psi}(\vartheta,\boldsymbol{\zeta}) - \nabla^{\boldsymbol{\xi}}\tilde{\psi}(\tau,\boldsymbol{\xi})\end{bmatrix}\;, \\ \mathsf{B}^m(\tau,\boldsymbol{\xi},\sigma,\boldsymbol{\eta}) &:= \xi_j(1-\chi_{S^m})\begin{bmatrix}\partial_\tau\tilde{\psi}(\vartheta,\boldsymbol{\zeta}) \\ \nabla^{\boldsymbol{\xi}}\tilde{\psi}(\vartheta,\boldsymbol{\zeta}) - \nabla^{\boldsymbol{\xi}}\tilde{\psi}(\tau,\boldsymbol{\xi})\end{bmatrix}\;.\end{aligned}$$

As $\mathsf{A}^m(\tau,\boldsymbol{\xi},\sigma,\boldsymbol{\eta})\cdot\widehat{\nabla_{t,\mathbf{x}}\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})$ is a bounded function with compact support, it is a kernel of a Hilbert-Schmidt operator on $\mathrm{L}^2(\mathbf{R}^{1+d})$, which is compact.

In order to estimate the second term in (28), we use Lemma 1 and the triangle inequality (for ρ). Denoting by T, S and Q the points $(\tau,\boldsymbol{\xi}), (\sigma,\boldsymbol{\eta})$ and $(\vartheta,\boldsymbol{\zeta})$ respectively, we have

$$\rho(Q) = \rho\Big(T-\theta(T-S)\Big) \geqslant \rho(T) - \rho\Big(\theta(T-S)\Big) \geqslant \rho(T) - \rho(T-S),$$

and we can take m large enough such that $\tilde{\psi}$ can be replaced by ψ in B^m, and that inequality $\rho(Q)\geqslant\frac{\rho(T)}{2}$ is valid on the strip defined by $T-S\in\operatorname{supp}\hat{\phi}$ and $\rho(T)>m$.

Using the characterisation (25) for derivatives of ψ we have

$$\begin{aligned}\left|\nabla^{\boldsymbol{\xi}}\psi^p(Q)-\nabla^{\boldsymbol{\xi}}\psi^p(T)\right| &= \left|\frac{1}{\rho(Q)}\nabla_T^{\boldsymbol{\xi}}\psi(Q_0)-\frac{1}{\rho(T)}\nabla_T^{\boldsymbol{\xi}}\psi(T_0)\right| \\ &\leqslant \left|\nabla_T^{\boldsymbol{\xi}}\psi(Q_0)\right|\left|\frac{1}{\rho(Q)}-\frac{1}{\rho(T)}\right|+\frac{1}{\rho(T)}\left|\nabla_T^{\boldsymbol{\xi}}\psi(Q_0)-\nabla_T^{\boldsymbol{\xi}}\psi(T_0)\right| \\ &= \left|\nabla_T^{\boldsymbol{\xi}}\psi(Q_0)\right|\left|\frac{\rho(T)-\rho(Q)}{\rho(Q)\rho(T)}\right|+\frac{1}{\rho(T)}\left|\nabla_T^{\boldsymbol{\xi}}\psi(Q_0)-\nabla_T^{\boldsymbol{\xi}}\psi(T_0)\right|.\end{aligned}$$

As $\nabla_T^{\boldsymbol{\xi}}\psi$ is uniformly continuous on P^d

$$(\forall\,\varepsilon>0)(\exists\,\delta\in\langle 0,\varepsilon\rangle)(\forall\,Q_0,T_0\in\mathrm{P}^d)\qquad \left|Q_0T_0\right|\leqslant\delta \quad\Longrightarrow\quad \left|\nabla_T^{\boldsymbol{\xi}}(Q_0)-\nabla_T^{\boldsymbol{\xi}}(T_0)\right|<\varepsilon\,.$$

Restricting our attention only to the points in the strip $T-S\in\operatorname{supp}\hat{\phi}$, the last bound remains valid if we take $\rho(T)$ large enough (i.e. $\rho(T)>\frac{CC_\phi}{\delta}$, where $C=2(2+\sqrt{2\sqrt{2}})$ is the constant from Lemma 1 and $C_\phi=\sup\{\rho(T):T\in\operatorname{supp}\hat{\phi}\}$), as

$$\left|Q_0T_0\right|\leqslant C\frac{\rho(T-Q)}{\rho(T)}\leqslant C\frac{\rho(T-S)}{\rho(T)}<\delta.$$

Thus we get the bound

$$\left|\nabla^{\boldsymbol{\xi}}\psi^p(Q)-\nabla^{\boldsymbol{\xi}}\psi^p(T)\right|\leqslant\frac{\varepsilon}{\rho(T)}\left(\frac{2}{C}\|\psi\|_{\mathrm{W}^{1,\infty}(\mathrm{P}^d)}+1\right).$$

Similarly, we have

$$\left|\xi_j\partial_\tau\psi(Q)\right|\leqslant\|\psi\|_{\mathrm{W}^{1,\infty}(\mathrm{P}^d)}\frac{|\xi_j|}{\rho^2(Q)}\leqslant\frac{4}{CC_\phi}\|\psi\|_{\mathrm{W}^{1,\infty}(\mathrm{P}^d)}\varepsilon.$$

The Young inequality now furnishes the proof, as (the L^2 norms are taken of functions in variables $(\tau,\boldsymbol{\xi})$, which also appear in $(\vartheta,\boldsymbol{\zeta})=(1-\theta)(\tau,\boldsymbol{\xi})+\theta(\sigma,\boldsymbol{\eta})$, where θ is fixed)

$$\begin{aligned}\Big\|\xi_j\int_{\mathbf{R}^{1+d}}(1-\chi_{S^m})(\tau,\boldsymbol{\xi};\sigma,\boldsymbol{\eta})\Big(\Big(\nabla^{\boldsymbol{\xi}}\tilde{\psi}(\vartheta,\boldsymbol{\zeta})-\nabla^{\boldsymbol{\xi}}\tilde{\psi}(\tau,\boldsymbol{\xi})\Big)\cdot\widehat{\nabla_{\mathbf{x}}\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta}) \\ +\partial_\tau\tilde{\psi}(\vartheta,\boldsymbol{\zeta})\widehat{\partial_t\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})\Big)\hat{u}(\sigma,\boldsymbol{\eta})d\sigma d\boldsymbol{\eta}\Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \\ \leqslant\varepsilon\tilde{C}\Big\|\int_{\mathbf{R}^{1+d}}|\widehat{\nabla_{t,\mathbf{x}}\phi}(\tau-\sigma,\boldsymbol{\xi}-\boldsymbol{\eta})||\hat{u}(\sigma,\boldsymbol{\eta})|d\sigma d\boldsymbol{\eta}\Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \\ \leqslant\varepsilon\tilde{C}\Big\||\widehat{\nabla_{t,\mathbf{x}}\phi}|\Big\|_{\mathrm{L}^1(\mathbf{R}^{1+d})}\Big\|\hat{u}\Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})},\end{aligned}$$

new constant $\tilde{C}$ depending on (the support of) function ϕ, and the Lipschitz constant of ψ.

Therefore $\partial_j\tilde{K}-P_{\xi_j\nabla^{\boldsymbol{\xi}}\tilde{\psi}}M_{\nabla_{\mathbf{x}}\phi}$ is a limit (in the uniform operator topology of $\mathcal{L}(\mathrm{L}^2)$) of compact operators, and thus compact itself.

II. Having an arbitrary function $\phi\in\mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})$, we can approximate it in the same space by a sequence of functions (ϕ_n) from $\mathcal{S}(\mathbf{R}^{1+d})$, such that their Fourier transform is compactly supported. Indeed, we first choose a sequence of functions $f_n\in\mathcal{S}(\mathbf{R}^{1+d})$ such that $f_n\to\phi$ in $\mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})$. On the other hand, $\hat{f}_n\in\mathcal{S}(\mathbf{R}^{1+d})\hookrightarrow\mathrm{L}^1_{k_p}(\mathbf{R}^{1+d})$ where the latter is the L^1 space with the weight $k_p(\tau,\boldsymbol{\xi})=\sqrt[4]{1+(2\pi\tau)^2+(2\pi|\boldsymbol{\xi}|)^4}$. As $\mathrm{C}^\infty_c(\mathbf{R}^{1+d})$ is dense in $\mathrm{L}^1_{k_p}(\mathbf{R}^{1+d})$, to each f_n we can associate a sequence of C^∞_c functions $g_n^k\to\hat{f}_n$ in $\mathrm{L}^1_{k_p}(\mathbf{R}^{1+d})$, implying $(g_n^k)^\vee\to f_n\to\phi$ in $\mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^{1+d})$. The claim follows by means of a diagonalisation procedure.

The associated sequence of commutators $\partial_j C_n := \partial_j(P_\psi M_{\phi_n} - M_{\phi_n} P_\psi)$ converges in norm to the operator $\partial_j C = \partial_j(P_\psi M_\phi - M_\phi P_\psi)$:

$$\begin{aligned}\Big\|\partial_j(C_n - C)\Big\|_{\mathcal{L}(\mathrm{L}^2)} &= \Big\|\partial_j\Big(P_\psi(M_{\phi_n} - M_\phi) - (M_{\phi_n} - M_\phi)P_\psi\Big)\Big\|_{\mathcal{L}(\mathrm{L}^2)} \\ &\leqslant \tilde{C}\left(\|\psi\|_{\mathrm{W}^{1,\infty}}\|\phi_n - \phi\|_{\mathrm{X}^{\frac{1}{2},1}}\right) \longrightarrow 0.\end{aligned}$$

In the same manner, by using that

$$\Big\|P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p} M_{\nabla_{\mathbf{x}}\phi} u\Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \Big\|\xi_j\nabla^{\boldsymbol{\xi}}\psi^p \cdot \widehat{\nabla_{\mathbf{x}}\phi u}\Big\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} \leqslant \|\xi_j\nabla^{\boldsymbol{\xi}}\psi^p\|_{\mathrm{L}^\infty}\|\widehat{\nabla_{\mathbf{x}}\phi}\|_{\mathrm{L}^1}\|u\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}$$

we show that $P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p} M_{\nabla_{\mathbf{x}}\phi_n} \longrightarrow P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p} M_{\nabla_{\mathbf{x}}\phi}$ in $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^{1+d}))$.

Therefore for a sequence of functions $u_m \rightharpoonup 0$ weakly in $\mathrm{L}^2(\mathbf{R}^{1+d})$ we have

$$\begin{aligned}\Big\|\partial_j C u_m - P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p} M_{\nabla_{\mathbf{x}}\phi} u_m\Big\|_{\mathrm{L}^2} &\leqslant \Big\|\partial_j C u_m - \partial_j C_n u_m\Big\|_{\mathrm{L}^2} + \Big\|\partial_j C_n u_m - P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p} M_{\nabla_{\mathbf{x}}\phi_n} u_m\Big\|_{\mathrm{L}^2} \\ &+ \Big\|P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p} M_{\nabla_{\mathbf{x}}\phi_n} u_m - P_{\xi_j\nabla^{\boldsymbol{\xi}}\psi^p} M_{\nabla_{\mathbf{x}}\phi} u_m\Big\|_{\mathrm{L}^2}\end{aligned}$$

where, according to the first part of the proof, the penultimate term converges to 0. **Q.E.D.**

Remark. The classical symbolic calculus provides a straightforward, but only partial proof of the second part of the last lemma.

Indeed, if ψ is taken from $\mathrm{C}^\infty(\mathrm{P}^d)$, then the symbol $\tilde{\psi}$ belongs to the Hörmander class $S^0_{\frac{1}{2},0}$, as well as $\phi \in \mathcal{S}(\mathbf{R}^{1+d})$. In that case the standard theory of pseudodifferential operators, namely expression (4.1), gives us that $\partial_j\tilde{K}$ has asymptotic expansion

$$2\pi i\xi_j\sigma(\tilde{K}) + \partial_j\sigma(\tilde{K}) = \sum_{|\boldsymbol{\alpha}|\geqslant 1}\frac{1}{\boldsymbol{\alpha}!}D^{\boldsymbol{\alpha}}\tilde{\psi}\,\partial_{\boldsymbol{\alpha}}\left(2\pi i\xi_j\phi + \partial_j\phi\right), \tag{29}$$

where D stands for the operator $\frac{1}{2\pi i}\partial$, while $\boldsymbol{\alpha} = (\alpha_0, \boldsymbol{\alpha}')$ as in the proof of Lemma 9.

In the proof of Lemma 9 we found that $\partial^{\boldsymbol{\alpha}}\tilde{\psi}$ behaves as $\rho^{-(\alpha_0+|\boldsymbol{\alpha}|)}$ for large $(\tau,\boldsymbol{\xi})$. On the other hand, partial derivatives (with respect to t and $\mathbf{x}$) of ϕ remain in $S^0_{\frac{1}{2},0}$, implying that the symbol of commutator $\tilde{K}$ belongs to $S^{-\frac{1}{2}}_{\frac{1}{2},0}$, as well as its derivative $\partial_j\sigma(\tilde{K})$.

For this reason, the terms in (29) of the form $\xi_j\partial^{\boldsymbol{\alpha}}\tilde{\psi}$ belong to $S^{-\frac{1}{2}}_{\frac{1}{2},0}$ for $\alpha_0 \geqslant 1$ or $|\boldsymbol{\alpha}'| \geqslant 2$, and as the principal symbol of $\partial_j\tilde{K}$ it remains only $\xi_j\nabla^{\boldsymbol{\xi}}\tilde{\psi}^p\,\nabla_{\mathbf{x}}\phi$. As this term is parabolicly homogeneous (at least outside the compact set $\rho \leqslant 1$), according to Lemma 9 it belongs to $S^0_{\frac{1}{2},0}$ and corresponds to a continuous operator on $\mathrm{L}^2(\mathbf{R}^{1+d})$.

The symbol of $\partial_j\tilde{K} - P_{\xi_j\nabla^{\boldsymbol{\xi}}\tilde{\psi}} M_{\nabla_{\mathbf{x}}\phi}$ thus belongs to $S^{-\frac{1}{2}}_{\frac{1}{2},0}$, and it corresponds to a continuous operator from $\mathrm{L}^2(\mathbf{R}^{1+d})$ to $\mathrm{H}^{\frac{1}{2}}(\mathbf{R}^{1+d})$. In the Second commutation lemma we showed a stronger result, by proving that it is a compact operator on $\mathrm{L}^2(\mathbf{R}^{1+d})$. Also, our proof of that lemma requires lower regularity assumptions on symbols.

∎

Remark. The Propagation principle can be generalised to parabolicly homogeneous symbols of order m, i.e.

$$\psi^{m,p}(\tau,\boldsymbol{\xi}) = \rho^m(\tau,\boldsymbol{\xi})\psi(\tau_0,\boldsymbol{\xi}_0)\,,$$

where $\psi \in \mathrm{C}(\mathrm{P}^d)$. In the case $m \in \mathbf{R}^+$ the Fourier multiplier P^m_ψ defined by $\mathcal{F}(P^m_\psi u) = \rho^m(\psi\circ p)\hat{u}$

is a continuous operator from $\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ to $\mathrm{H}^{\frac{s-m}{2},s-m}(\mathbf{R}^{1+d})$:

$$\begin{aligned}\|P_\psi^m u\|_{\mathrm{H}^{\frac{s-m}{2},s-m}} &= \Big\|\Big(\sqrt[4]{1+\rho(\tau,\boldsymbol{\xi})^4}\Big)^{s-m}\widehat{P_\psi^m u}\Big\|_{\mathrm{L}^2}\\ &= \Big\|\Big(\sqrt[4]{1+\rho(\tau,\boldsymbol{\xi})^4}\Big)^{s-m}\rho^m\psi\,\hat u\Big\|_{\mathrm{L}^2}\\ &= \Big\|\Big(\frac{\rho}{\sqrt[4]{1+\rho(\tau,\boldsymbol{\xi})^4}}\Big)^{m}\psi\,\Big(\sqrt[4]{1+\rho(\tau,\boldsymbol{\xi})^4}\Big)^{s}\hat u\Big\|_{\mathrm{L}^2}\leqslant\|\psi\|_{\mathrm{L}^\infty}\|\hat u\|_{\mathrm{H}^{\frac{s}{2},s}}.\end{aligned}$$

A generalisation of the second commutation lemma can be stated as follows:

For $\psi\in\mathrm{C}^1(\mathrm{P}^d)$ let $\psi^{m,p}=\rho^m(\psi\circ p)$ be its parabolicly homogeneous extension of order $m\in\mathbf{N}$, and $\phi\in\mathrm{X}^{\frac{s+1}{2},s+1}(\mathbf{R}^{1+d})$ for $s\geqslant m-1$. Then $K=[P_\psi^m,M_\phi]=P_\psi^m M_\phi-M_\phi P_\psi^m$ is a continuous operator from $\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d})$ to $\mathrm{H}^{\frac{s-m+1}{2},s-m+1}(\mathbf{R}^{1+d})$.

In particular, for the case $s=0$ and $m=1$, the commutator K, up to the compact operator, equals

$$P_{\frac{1}{2\pi i}\nabla^{\boldsymbol{\xi}}\psi^{1,p}}M_{\nabla_{\mathbf{x}}\phi}\,.$$

The proof is technical, but follows along the same lines as the proof of the Second commutation lemma.

∎

Application to the Schrödinger equation

We reconsider a sequence (3.14) of initial value problems

$$(30)\qquad\begin{cases} i\partial_t u_n+\mathsf{div}\,(\mathbf{A}\nabla u_n)=f_n\\ u_n(0,\cdot)=u_n^0\,,\end{cases}$$

where, as before, $u_n^0\rightharpoonup 0$ in $\mathrm{H}^1(\mathbf{R}^d)$, $f_n\rightharpoonup 0$ in $\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbf{R}^d))$, with $(\partial_t f_n)$ being bounded in $\mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^d))$.

These assumptions assure that (on a subsequence) $\begin{bmatrix}\nabla u_n\\ f_n\end{bmatrix}$ determines the parabolic H-measure of the block form (cf. the Remark following Corollary 6)

$$\begin{bmatrix}\boldsymbol{\mu} & \boldsymbol{\mu}_{12}\\ \boldsymbol{\mu}_{21} & \mu_f\end{bmatrix},$$

where $\boldsymbol{\mu}=\frac{\boldsymbol{\xi}\otimes\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\nu$, while $\boldsymbol{\mu}_{12}=\boldsymbol{\xi}\nu_{12}$. The localisation principle also gives that $\boldsymbol{\mu}$ is supported within the closed set of $\mathbf{R}^{1+d}\times\mathrm{P}^d$ determined by the relation $Q(t,\mathbf{x};\tau,\boldsymbol{\xi})=0$, which is disjoint with the set where $\boldsymbol{\xi}=\mathsf{0}$. Recall that $Q(t,\mathbf{x};\tau,\boldsymbol{\xi})=2\pi\tau+4\pi^2\mathbf{A}(t,\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi}$ is the symbol of the Schrödinger operator introduced in Section 3.

With the aid of Second commutation lemma we are able to prove the following result.

Theorem 11 *Under the above assumptions and notation, if we additionally assume that $\mathbf{A}\in\mathrm{X}^{\frac12,1}(\mathbf{R}_0^+\times\mathbf{R}^d;\mathrm{M}_{d\times d})$ (i.e. that $\mathbf{A}\in\mathrm{C}^1\cap\mathrm{X}^{\frac12,1}$), the trace of parabolic H-measure $\nu=\mathsf{tr}\boldsymbol{\mu}$ satisfies the equation*

$$(31)\qquad\langle\nu,\{\Psi,Q\}\rangle+\Big\langle\nu,\Psi\,\frac{\alpha^2}{4}\frac{3-\alpha^2}{\alpha^2-1}\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}Q\Big\rangle=\langle 2\mathsf{Re}\,\nu_{12},4\pi^2|\boldsymbol{\xi}|^2\Psi\rangle\,,$$

where $\Psi=\phi\boxtimes\psi$, $\phi\in\mathrm{C}_c^1(\mathbf{R}^+\times\mathbf{R}^d)$ and $\psi\in\mathrm{C}^1(\mathrm{P}^d)$.

In the formula above we have used the Poisson bracket *(only in $\mathbf{x}$ and $\boldsymbol{\xi}$, not in t and τ):*

$$\{\Psi,Q\}:=\nabla^{\boldsymbol{\xi}}\Psi\cdot\nabla_{\mathbf{x}}Q-\nabla_{\mathbf{x}}\Psi\cdot\nabla^{\boldsymbol{\xi}}Q\,,$$

while $\alpha^2=\frac{4}{4-|\boldsymbol{\xi}|^2}$ on P^d (cf. the calculation of mean curvature on P^d preceding Lemma 2).

Dem. First we assume that all u_n have their supports in a fixed compact set in $\mathbf{R}^+ \times \mathbf{R}^d$.

Let M_ϕ and P_ψ be scalar pseudodifferential operators, associated to $\phi \in \mathrm{C}^1_c(\mathbf{R}^+ \times \mathbf{R}^d)$ and $\psi \in \mathrm{C}^1(\mathrm{P}^d)$ respectively. Applying them to equation (30_1), and then taking a scalar product in $\mathrm{L}^2(\mathbf{R}^{1+d})$ with $\partial_m u_n$, for an $m \in 1..d$ we obtain

$$\Big\langle\, i\phi P_\psi\, \partial_t u_n + \phi P_\psi\, \mathsf{div}\,(\mathbf{A}\nabla u_n) \;\Big|\; \partial_m u_n \,\Big\rangle = \langle\, \phi P_\psi\, f_n \mid \partial_m u_n \,\rangle. \tag{32}$$

Let us temporarily assume that ∇u_n is in H^1, so that $\mathsf{div}\,(\mathbf{A}\nabla u_n)$ is in L^2. This will, by equation (30_1), imply that $\partial_t u_n$ is in L^2 as well.

Partial integration with respect to time and space variables of the first term in the last equation gives (as ∂_m, ∂_t and P_ψ mutually commute):

$$\begin{aligned}\langle\, i\phi P_\psi \partial_t u_n \mid \partial_m u_n \,\rangle &= -\langle\, i(\partial_m\phi) P_\psi\, \partial_t u_n \mid u_n \,\rangle - \langle\, i\phi P_\psi\, \partial_t\partial_m u_n \mid u_n \,\rangle \\ &= -\langle\, i(\partial_m\phi) P_\psi\, \partial_t u_n \mid u_n \,\rangle + \langle\, i(\partial_t\phi) P_\psi\, \partial_m u_n \mid u_n \,\rangle - \langle\, \phi P_\psi \partial_m u_n \mid i\partial_t u_n \,\rangle\,.\end{aligned}$$

Similarly, for the second term in (32) we have

$$\Big\langle\, \phi\mathsf{div}\,(P_\psi \mathbf{A}\nabla u_n) \;\Big|\; \partial_m u_n \,\Big\rangle = -\Big\langle\, P_\psi \mathbf{A}\nabla u_n \cdot \nabla\bar\phi \;\Big|\; \partial_m u_n \,\Big\rangle - \Big\langle\, \phi P_\psi \mathbf{A}\nabla u_n \;\Big|\; \partial_m \nabla u_n \,\Big\rangle\,.$$

The last term we can further rewrite:

$$\begin{aligned}-\Big\langle \phi\Big([P_\psi, \mathbf{A}] + \mathbf{A}P_\psi\Big)\nabla u_n \Big| \partial_m \nabla u_n \Big\rangle &= \Big\langle (\partial_m\phi)[P_\psi, \mathbf{A}]\nabla u_n + \phi\, \partial_m([P_\psi, \mathbf{A}]\nabla u_n) \;\Big|\; \nabla u_n \,\Big\rangle \\ &\quad + \Big\langle \partial_m(\phi\mathbf{A}) P_\psi \nabla u_n \;\Big|\; \nabla u_n \Big\rangle + \Big\langle \phi P_\psi \nabla\partial_m u_n \;\Big|\; \mathbf{A}\nabla u_n \,\Big\rangle,\end{aligned}$$

where we have used the assumption that $\mathbf{A}$ is an hermitian matrix.

We still need to rewrite the last term above, taking into account the equation (30_1):

$$\Big\langle P_\psi \nabla\partial_m u_n \;\Big|\; \bar\phi\mathbf{A}\nabla u_n \,\Big\rangle = -\Big\langle\, P_\psi \partial_m u_n \;\Big|\; \mathbf{A}\nabla u_n \cdot \nabla\phi + \bar\phi f_n - \bar\phi i \partial_t u_n \,\Big\rangle\,.$$

By summing up the above expressions (two terms with $\partial_t u_n$ cancel out), one can rewrite (32) as:

$$\begin{aligned}-\Big\langle\, i(\partial_m\phi) P_\psi\, \partial_t u_n \;\Big|\; u_n \,\Big\rangle &+ \Big\langle\, i(\partial_t\phi) P_\psi\, \partial_m u_n \;\Big|\; u_n \,\Big\rangle - \Big\langle\, P_\psi \mathbf{A}\nabla u_n \;\Big|\; \nabla\bar\phi(\partial_m u_n) \,\Big\rangle \\ &+ \Big\langle\, (\partial_m\phi)[P_\psi, \mathbf{A}]\nabla u_n \;\Big|\; \nabla u_n \,\Big\rangle + \Big\langle\, \phi\, \partial_m[P_\psi, \mathbf{A}]\nabla u_n \;\Big|\; \nabla u_n \,\Big\rangle \\ &+ \Big\langle\, \partial_m(\phi\mathbf{A}) P_\psi \nabla u_n \;\Big|\; \nabla u_n \,\Big\rangle - \Big\langle\, (\mathbf{A}\nabla\phi) P_\psi \partial_m u_n \;\Big|\; \nabla u_n \,\Big\rangle \\ = \Big\langle\, \phi P_\psi\, f_n \;\Big|\; \partial_m u_n \,\Big\rangle &+ \Big\langle\, \phi P_\psi \partial_m u_n \;\Big|\; f_n \,\Big\rangle.\end{aligned}$$

If we interpret the first term as the duality between $\mathrm{L}^2([0,T]; \mathrm{H}^{-1}(\mathbf{R}^{1+d}))$ and $\mathrm{L}^2([0,T]; \mathrm{H}^1(\mathbf{R}^{1+d}))$ (i.e. as $-\overline{\langle i(\partial_m\phi) P_\psi\, \partial_t u_n, u_n\rangle}$), we can use the density argument to conclude that the above equality holds without additional assumption of ∇u_n being in H^1 (as all the terms in this equality indeed make sense under the original assumptions of the theorem).

As $u_n \longrightarrow 0$ in L^2 (for this conclusion we have to use the *Compactness lemma* [43, Lect. 24], which is often associated with the name of Jean-Pierre Aubin), the second term in the equality above converges to 0. Similarly, $[P_\psi, \mathbf{A}]$ is a compact operator by the First commutation lemma, so the fourth term tends to 0 as well.

The limits of the remaining terms on the left, except for the first one, can be expressed by parabolic H-measure $\boldsymbol{\mu}$, determined by (a sub)sequence (of) (∇u_n) (cf. (2.4)). For the fifth term we have to make use of the Second commutation lemma.

The limit of right hand side, also by (2.4), involves parabolic H-measure $\boldsymbol{\mu}_{12}$, determined by both sequences (∇u_n) and (f_n). Thus we have got (after taking into account the form of these parabolic H-measures):

$$
\begin{aligned}
(33)\qquad & -\Big\langle \frac{\xi_m \nu}{|\boldsymbol{\xi}|^2}, \psi \mathbf{A}^\top \nabla\phi\cdot\boldsymbol{\xi}\Big\rangle + \Big\langle \frac{\xi_m \nu}{|\boldsymbol{\xi}|^2}, \phi\nabla^{\boldsymbol{\xi}}\psi\cdot\nabla_{\mathbf{x}}(\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})\Big\rangle + \Big\langle \frac{\nu}{|\boldsymbol{\xi}|^2}, \psi\partial_m(\phi\mathbf{A})\boldsymbol{\xi}\cdot\boldsymbol{\xi}\Big\rangle \\
& - \Big\langle \frac{\xi_m \nu}{|\boldsymbol{\xi}|^2}, \psi\mathbf{A}\nabla\phi\cdot\boldsymbol{\xi}\Big\rangle = \langle \xi_m 2\mathsf{Re}\,\nu_{12}, \psi\phi\rangle + \lim_n \overline{\langle i(\partial_m\phi)P_\psi\,\partial_t u_n, u_n\rangle}\,.
\end{aligned}
$$

In order to express the last term of the above equation in terms of measure $\boldsymbol{\mu}$, we apply once again the above procedure, but this time applying $M_{\partial_m\phi}$ first (instead of M_ϕ) and taking the scalar product of (30_1) with u_n (instead of $\partial_m u_n$). After integration by parts with respect to space variables in the second term, using the same density argument as above, we obtain

$$
\langle\, i(\partial_m\phi)P_\psi\partial_t u_n \mid u_n\,\rangle - \Big\langle\, P_\psi\mathbf{A}\nabla u_n \,\Big|\, u_n\nabla(\partial_m\bar{\phi})\,\Big\rangle - \langle\,(\partial_m\phi)P_\psi\mathbf{A}\nabla u_n \mid \nabla u_n\,\rangle = 0\,.
$$

Passing to the limit, the second term converges to 0 and one finds that $\lim_n \overline{\langle i(\partial_m\phi)P_\psi\partial_t u_n, u_n\rangle} = \langle\boldsymbol{\mu}, (\partial_m\phi)\psi\,\mathbf{A}\rangle$.

By means of the last result one obtains from (33) the relation satisfied by the parabolic H-measure ν

$$
\begin{aligned}
& -2\Big\langle \frac{\xi_m \nu}{|\boldsymbol{\xi}|^2}, \psi\,(\mathsf{Re}\,\mathbf{A})\;\nabla\phi\cdot\boldsymbol{\xi}\Big\rangle + \Big\langle \frac{\xi_m \nu}{|\boldsymbol{\xi}|^2}, \phi\nabla^{\boldsymbol{\xi}}\psi\cdot\nabla_{\mathbf{x}}(\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})\Big\rangle \\
& \qquad + \Big\langle \frac{\nu}{|\boldsymbol{\xi}|^2}, \psi\phi(\partial_m\mathbf{A})\boldsymbol{\xi}\cdot\boldsymbol{\xi}\Big\rangle = \langle \xi_m 2\mathsf{Re}\,\nu_{12}, \psi\phi\rangle\,.
\end{aligned}
$$

Using the symbol Q of the Schrödinger operator, the relation can be rewritten in the form

$$
(34)\qquad \Big\langle \frac{\xi_m \nu}{|\boldsymbol{\xi}|^2}, \nabla^{\boldsymbol{\xi}}\Psi\cdot\nabla_{\mathbf{x}}Q - \nabla_{\mathbf{x}}\Psi\cdot\nabla^{\boldsymbol{\xi}}Q\Big\rangle + \Big\langle \frac{\nu}{|\boldsymbol{\xi}|^2}, \Psi\,\partial_m Q\Big\rangle = \Big\langle 4\pi^2\xi_m 2\mathsf{Re}\,\nu_{12}, \Psi\Big\rangle,
$$

where $\Psi = \phi\boxtimes\psi$.

Before the next step we have to compute

$$
\nabla^{\boldsymbol{\xi}}\left(\frac{\xi_m}{\rho}\right) = \frac{\sqrt{\left|\frac{\boldsymbol{\xi}}{2}\right|^4 + \tau^2}\,\mathsf{e}_m - \frac{\xi_m}{2}\frac{\boldsymbol{\xi}}{2}}{\rho\sqrt{\left|\frac{\boldsymbol{\xi}}{2}\right|^4 + \tau^2}} = \mathsf{e}_m - \alpha^2\frac{\xi_m}{2}\frac{\boldsymbol{\xi}}{2}\,,
$$

where in the second equality we used the fact that on surface P^d one has $\rho = 1$ and $\sqrt{\left|\frac{\boldsymbol{\xi}}{2}\right|^4 + \tau^2} = \tau^2 + \left|\frac{\boldsymbol{\xi}}{2}\right|^2 = \frac{\mathsf{n}^2}{\alpha^2}$, with $\mathsf{n} = \alpha\begin{bmatrix}\tau\\ \boldsymbol{\xi}/2\end{bmatrix}$ denoting the unit normal on P^d.

Now we can replace ψ by $\tilde{\psi} = \frac{\xi_m}{\rho}\psi$, where, as before, ψ is an arbitrary symbol from $\mathrm{C}^1(\mathrm{P}^d)$. Thus the relation (34) becomes

$$
\begin{aligned}
\langle 4\pi^2\xi_m^2 2\mathsf{Re}\,\nu_{12}, \Psi\rangle &= \Big\langle \frac{\xi_m\nu}{|\boldsymbol{\xi}|^2}, \Big(\Psi\Big(\mathsf{e}_m - \frac{\alpha^2}{4}\xi_m\boldsymbol{\xi}\Big) + \xi_m\nabla^{\boldsymbol{\xi}}\Psi\Big)\cdot\nabla_{\mathbf{x}}Q - \xi_m\nabla_{\mathbf{x}}\Psi\cdot\nabla^{\boldsymbol{\xi}}Q\Big\rangle \\
&\quad + \Big\langle \frac{\nu}{|\boldsymbol{\xi}|^2}, \xi_m\Psi\,\partial_m Q\Big\rangle \\
&= \Big\langle \nu, \Psi\frac{1}{|\boldsymbol{\xi}|^2}\xi_m\partial_m Q - \Psi\frac{\alpha^2}{4}\frac{\xi_m^2}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}Q\Big\rangle \\
&\quad + \Big\langle \frac{\xi_m^2\nu}{|\boldsymbol{\xi}|^2}, \nabla^{\boldsymbol{\xi}}\Psi\cdot\nabla_{\mathbf{x}}Q - \nabla_{\mathbf{x}}\Psi\cdot\nabla^{\boldsymbol{\xi}}Q\Big\rangle + \Big\langle \frac{\nu}{|\boldsymbol{\xi}|^2}, \xi_m\Psi\,\partial_m Q\Big\rangle.
\end{aligned}
$$

The summation with respect to $m \in 1..d$ gives

$$\left\langle \nu, \Psi\left(\frac{1}{|\boldsymbol{\xi}|^2} - \frac{\alpha^2}{4}\right)\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}Q\right\rangle + \left\langle \nu, \nabla^{\boldsymbol{\xi}}\Psi\cdot\nabla_{\mathbf{x}}Q - \nabla_{\mathbf{x}}\Psi\cdot\nabla^{\boldsymbol{\xi}}Q\right\rangle + \left\langle \nu, \Psi\,\frac{\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}Q}{|\boldsymbol{\xi}|^2}\right\rangle = \left\langle 2\mathsf{Re}\,\nu_{12}, 4\pi^2|\boldsymbol{\xi}|^2\Psi\right\rangle .$$

As $\frac{\alpha^2}{4} = \frac{1}{4-|\boldsymbol{\xi}|^2}$, we get the relation (31).

If we take an arbitrary solution of (30_1) which does not have a compact support, we take a smooth real function w with compact support and write the Schrödinger equation for wu_n:

$$i\partial_t(wu_n) + \mathsf{div}\,(\mathbf{A}\nabla(wu_n)) = wf_n + 2\,(\mathsf{Re}\,\mathbf{A})\,\nabla w\cdot\overline{\nabla u_n} + \Big(i\partial_t(w) + \mathsf{div}\,(\mathbf{A}\nabla w)\Big)u_n\,.$$

Now we can repeat the preceding analysis with ν replaced by $w^2\nu$ and ν_{12} replaced by

$$w^2\nu_{12} + w\nabla w\cdot 2\frac{\nu}{|\boldsymbol{\xi}|^2}\,(\mathsf{Re}\,\mathbf{A})\,\boldsymbol{\xi} = w^2\nu_{12} + (\nabla_{\mathbf{x}}(w^2)\cdot\nabla^{\boldsymbol{\xi}}Q)\frac{\nu}{|\boldsymbol{\xi}|^2},$$

obtaining (31) with Ψ replaced by $w^2\Psi$. **Q.E.D.**

Next, we would like to perform integration by parts in (31) in order to get a *transport* equation for ν. For that purpose we use Corollary 1, with q taken to be of the form $(0, \nu\nabla_{\mathbf{x}}Q)^\top$, while p is replaced by Ψ, a parabolicly homogeneous test function in $(\tau, \boldsymbol{\xi})$ (for the integration by parts on P^d, t and $\mathbf{x}$ are only parameters).

In order to simplify the computation, we assume that ν is absolutely continuous with respect to the Lebesgue measure

$$d\nu = \nu(t, \mathbf{x}, \tau, \boldsymbol{\xi})dt\,d\mathbf{x}\,d\tau\,d\boldsymbol{\xi}.$$

The calculations could be carried out in the case of a general measure ν as well, because all functions that are applied to ν are continuous, and all computations thus could be performed in a distributional sense.

The equation from Corollary 1 reads now

(35)

$$\begin{aligned}\int_{\mathrm{P}^d}\nu\nabla^{\boldsymbol{\xi}}\Psi\cdot\nabla_{\mathbf{x}}Q dA &= \int_{\mathrm{P}^d}\Psi\left(\frac{\alpha^2}{4}(\alpha^2+d-1)\nu\nabla_{\mathbf{x}}Q\cdot\boldsymbol{\xi} - \mathsf{div}_{\boldsymbol{\xi}}(\nu\nabla_{\mathbf{x}}Q) + \frac{\alpha}{2}\Big(\nabla^{\tau,\boldsymbol{\xi}}(\nu\nabla_{\mathbf{x}}Q)\Big)\mathsf{n}\cdot\boldsymbol{\xi}\right)dA\\ &= \int_{\mathrm{P}^d}\Psi\nu\left(\frac{\alpha^2}{4}(\alpha^2+d-1)\nabla_{\mathbf{x}}Q\cdot\boldsymbol{\xi} - \mathsf{div}^{\boldsymbol{\xi}}\nabla_{\mathbf{x}}Q + \frac{\alpha^2}{4}\Big((\nabla^{\boldsymbol{\xi}}\otimes\nabla_{\mathbf{x}})Q\Big)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\right)dA\\ &\quad - \int_{\mathrm{P}^d}\Psi\nabla_{\mathbf{x}}Q\cdot\left(\nabla^{\boldsymbol{\xi}}\nu - \frac{\alpha}{2}(\mathsf{n}\cdot\nabla^{\tau,\boldsymbol{\xi}}\nu)\boldsymbol{\xi}\right)dA\;,\end{aligned}$$

where we have used a simple observation that $\partial_\tau\nabla_{\mathbf{x}}Q = 0$.

The spatial derivatives of symbol Q are homogeneous of degree 2 with respect to $\boldsymbol{\xi}$ (as $\nabla_{\mathbf{x}}Q$ is independent of both t and τ, we do not write them as variables), i.e.

$$\nabla_{\mathbf{x}}Q(\mathbf{x}; s\boldsymbol{\xi}) = s^2\nabla_{\mathbf{x}}Q(\mathbf{x}; \boldsymbol{\xi}).$$

Taking the derivative of the last expression with respect to parameter s and inserting $s = 1$ yields

$$\Big((\nabla_{\mathbf{x}}\otimes\nabla^{\boldsymbol{\xi}})Q\Big)(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\xi} = 2\nabla_{\mathbf{x}}Q(\mathbf{x}, \boldsymbol{\xi}).$$

Thus we get for (35)

$$\begin{aligned}\int_{\mathrm{P}^d}\nu\nabla^{\boldsymbol{\xi}}\Psi\cdot\nabla_{\mathbf{x}}Q dA &= \int_{\mathrm{P}^d}\Psi\nu\left(\frac{\alpha^2}{4}(\alpha^2+d+1)\nabla_{\mathbf{x}}Q\cdot\boldsymbol{\xi} - \mathsf{div}^{\boldsymbol{\xi}}\nabla_{\mathbf{x}}Q\right)dA\\ &\quad - \int_{\mathrm{P}^d}\Psi\nabla_{\mathbf{x}}Q\cdot\left(\nabla^{\boldsymbol{\xi}}\nu - \frac{\alpha}{2}(\mathsf{n}\cdot\nabla^{\tau,\boldsymbol{\xi}}\nu)\boldsymbol{\xi}\right)dA.\end{aligned}$$

On the other hand

$$-\int_{\mathrm{P}^d} \nu \nabla_{\mathbf{x}} \Psi \cdot \nabla^{\boldsymbol{\xi}} Q dA = \int_{\mathrm{P}^d} \Psi \left(\nabla_{\mathbf{x}} \nu \cdot \nabla^{\boldsymbol{\xi}} Q + \nu \mathsf{div}_{\mathbf{x}} \nabla^{\boldsymbol{\xi}} Q \right) dA \,,$$

and the relation (31) becomes

$$\begin{aligned} \langle 2\mathsf{Re}\, \nu_{12}, 4\pi^2 |\boldsymbol{\xi}|^2 \Psi \rangle = & \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} \nu \cdot \nabla^{\boldsymbol{\xi}} Q dA + \int_{\mathrm{P}^d} \Psi \nu \frac{\alpha^2}{4} \left(\alpha^2 + d + 1 + \frac{3-\alpha^2}{\alpha^2 - 1} \right) \nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi} dA \\ & - \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} Q \cdot \left(\nabla^{\boldsymbol{\xi}} \nu - \frac{\alpha}{2} (\mathsf{n} \cdot \nabla_{\tau,\boldsymbol{\xi}} \nu) \boldsymbol{\xi} \right) dA. \end{aligned}$$

Finally, differentiating the localisation principle for the Schrödinger equation we have $\nu \partial_i Q = -(\partial_i \nu) Q$ and we get

$$\begin{aligned} \langle 2\mathsf{Re}\, \nu_{12}, 4\pi^2 |\boldsymbol{\xi}|^2 \Psi \rangle = & \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} \nu \cdot \left(\nabla^{\boldsymbol{\xi}} Q - \frac{\alpha^2}{4} \left(\alpha^2 + d + \frac{2}{\alpha^2 - 1} \right) Q \boldsymbol{\xi} \right) dA \\ & - \int_{\mathrm{P}^d} \Psi \begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q \end{bmatrix} \cdot \left(\nabla^{\tau,\boldsymbol{\xi}} \nu - (\nabla^{\tau,\boldsymbol{\xi}} \nu \cdot \mathsf{n}) \mathsf{n} \right) dA. \end{aligned}$$

As Ψ is an arbitrary compactly supported test function on $\mathbf{R}^+_0 \times \mathbf{R}^d \times \mathrm{P}^d$ we have proved the following theorem.

Theorem 12 *For any parabolic H-measure $\boldsymbol{\mu}$ associated to (a subsequence of) ∇u_n, where (u_n) is a sequence of solutions to (30) with $\mathbf{A} \in \mathrm{C}^1(\mathbf{R}^+_0 \times \mathbf{R}^d; \mathrm{M}_{d\times d}) \cap \mathrm{X}^{\frac{1}{2},1}(\mathbf{R}^+_0 \times \mathbf{R}^d; \mathrm{M}_{d\times d})$, the trace $\nu = \mathsf{tr} \boldsymbol{\mu}$ satisfies the transport equation*

$$\nabla_{\mathbf{x}} \nu \cdot \left(\nabla^{\boldsymbol{\xi}} Q - \frac{\alpha^2}{4} \left(\alpha^2 + d + \frac{2}{\alpha^2 - 1} \right) Q \boldsymbol{\xi} \right) - \nabla^{\tau,\boldsymbol{\xi}} \nu \cdot \left(\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q \end{bmatrix} - \left(\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q \end{bmatrix} \cdot \mathsf{n} \right) \mathsf{n} \right) = |2\pi \boldsymbol{\xi}|^2 \, 2\mathsf{Re}\, \nu_{12}.$$

∎

Remark. Let us consider the characteristics of the above transport equation, i.e. let us discuss the system of ordinary differential equations

$$\begin{aligned} \frac{d}{ds} \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} &= \begin{bmatrix} 0 \\ \nabla^{\boldsymbol{\xi}} Q - \frac{\alpha^2}{4} \left(\alpha^2 + d + \frac{2}{\alpha^2 - 1} \right) Q \boldsymbol{\xi} \end{bmatrix} \\ \frac{d}{ds} \begin{bmatrix} \tau \\ \boldsymbol{\xi} \end{bmatrix} &= -\left(\mathbf{I} - \mathsf{n} \otimes \mathsf{n} \right) \begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q \end{bmatrix} , \end{aligned}$$

with initial conditions

$$t(0) = t_0 \,, \quad \mathbf{x}(0) = \mathbf{x}_0 \,, \qquad \tau(0) = \tau_0 \,, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \,.$$

This system has a solution, which might not be unique (if we have additional smoothness of $\mathbf{A}$, like $\mathbf{A} \in \mathrm{C}^1(\mathbf{R}^+_0 \times \mathbf{R}^d; \mathrm{M}_{d\times d})$, the solution will be unique).

As we are interested in frequencies $(\tau, \boldsymbol{\xi})$ belonging to P^d, we should restrict ourselves to $(\tau_0, \boldsymbol{\xi}_0) \in \mathrm{P}^d$. In that case, multiplication of the second ordinary differential equation above by $\mathsf{n}/\alpha = (\tau, \boldsymbol{\xi}/2)$ leads to

$$\frac{1}{2} \frac{d}{ds} (\tau^2 + |\boldsymbol{\xi}|^2/2) = -(1 - \mathsf{n}^2) \begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q \end{bmatrix} \cdot \frac{\mathsf{n}}{\alpha} = 0 \,,$$

as n is a unit vector. Thus, if the initial value $(\tau_0, \boldsymbol{\xi}_0)$ belongs to P^d, then $(\tau, \boldsymbol{\xi})$ remains on P^d over the interval of existence.

Taking into account Theorem 12, we conclude that in the case of a homogeneous equation (i.e. when $\nu_{12}=0$), measure ν remains constant along the integral curves on $\mathbf{R}^{1+d}\times \mathrm{P}^d$.

Furthermore, if the initial conditions satisfy

$$Q(t_0,\mathbf{x}_0;\tau_0,\boldsymbol{\xi}_0)=0\,,$$

then along any characteristic we have $q(s):=Q(t(s),\mathbf{x}(s);\tau(s),\boldsymbol{\xi}(s))=0$, for $s\geqslant 0$. Indeed,

$$\begin{aligned}\frac{dq}{ds}&=\nabla_{t,\mathbf{x}}Q\cdot\begin{bmatrix}0\\ \nabla^{\boldsymbol{\xi}}Q-\frac{\alpha^2}{4}\Big(\alpha^2+d+\frac{2}{\alpha^2-1}\Big)Q\boldsymbol{\xi}\end{bmatrix}-\nabla^{\tau,\boldsymbol{\xi}}Q\cdot\Big(\mathbf{I}-\mathsf{n}\otimes\mathsf{n}\Big)\begin{bmatrix}0\\ \nabla_{\mathbf{x}}Q\end{bmatrix}\\ &=-\frac{\alpha^2}{4}\Big(\alpha^2+d-2+\frac{2}{\alpha^2-1}\Big)(\nabla_{\mathbf{x}}Q\cdot\boldsymbol{\xi})q\,,\end{aligned}$$

where we have used the identity $\nabla^{\tau,\boldsymbol{\xi}}Q\cdot\mathsf{n}=\alpha Q$. This can be considered as an ordinary differential equation for q with initial condition $q(0)=0$, which has a unique solution $q=0$.

This result generalises the localisation principle $Q\nu=0$, and shows that Q vanishes along integral curves that pass through the support of ν.

∎

Application to the vibrating plate equation

As a second illustration of the propagation principle, we consider a sequence of initial value problems (3.18) for the vibrating plate equation.

We follow the steps in the proof of Theorem 11 by applying pseudodifferential operators M_ϕ and P_ψ, associated to $\phi\in \mathrm{C}^1_c(\mathbf{R}^+\times\mathbf{R}^d)$ and $\psi\in\mathrm{C}^1(\mathrm{P}^d)$ respectively, to (3.18$_1$) and after taking the L^2 scalar product with $\partial_m u_n$, for an $m\in 1..d$:

$$\langle\,\phi P_\psi\,\partial_t(\varrho\,\partial_t u_n)\mid\partial_m u_n\,\rangle+\Big\langle\,\phi P_\psi\,\mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u_n)\;\Big|\;\partial_m u_n\,\Big\rangle=\langle\,\phi P_\psi\, f_n\mid\partial_m u_n\,\rangle.$$

Similarly to what was done in the previous subsection, partial integration of the first term yields:

$$\begin{aligned}\langle\,\phi P_\psi\,\partial_t(\varrho\,\partial_t u_n)\mid\partial_m u_n\,\rangle&=-\langle\,(\partial_t\phi)P_\psi(\varrho\,\partial_t u_n)\mid\partial_m u_n\,\rangle+\langle\,\partial_m(\phi[P_\psi\,,\varrho]\,\partial_t u_n)\mid\partial_t u_n\,\rangle\\ &+\langle\,(\partial_m(\phi\varrho))\,P_\psi\,\partial_t u_n\mid\partial_t u_n\,\rangle-\langle\,(\partial_t\phi)\varrho\,P_\psi\,\partial_m u_n\mid\partial_t u_n\,\rangle-\langle\,\phi P_\psi\,\partial_m u_n\mid\partial_t(\varrho\,\partial_t u_n)\,\rangle\,.\end{aligned}$$

Following the analogous procedure, with some additional technical details we suppress here, for the second term one gets

$$\begin{aligned}&\Big\langle\,\phi P_\psi\,\mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u_n)\;\Big|\;\partial_m u_n\,\Big\rangle\\ &\quad=\Big\langle\,(\nabla\nabla\phi)\cdot\overline{P_\psi(\mathbf{M}\nabla\nabla u_n)}\;\Big|\;\partial_m u_n\,\Big\rangle+2\Big\langle\,P_\psi(\mathbf{M}\nabla\nabla u_n)\nabla\phi\;\Big|\;\partial_m\nabla u_n\,\Big\rangle\\ &\quad-\Big\langle\,(\partial_m(\phi\mathbf{M}))P_\psi\,\nabla\nabla u_n\;\Big|\;\nabla\nabla u_n\,\Big\rangle-\Big\langle\,\partial_m\phi[P_\psi,\mathbf{M}]\nabla\nabla u_n\;\Big|\;\nabla\nabla u_n\,\Big\rangle\\ &\quad+\Big\langle\,P_\psi(\nabla\partial_m u_n)\otimes\nabla\bar{\varphi}\;\Big|\;\mathbf{M}\nabla\nabla u_n\,\Big\rangle+\Big\langle\,(\nabla\nabla\phi)P_\psi\,\partial_m u_n\;\Big|\;\mathbf{M}\nabla\nabla u_n\,\Big\rangle\\ &\quad+\Big\langle\,\nabla\phi\otimes\overline{P_\psi\,\nabla\partial_m u_n}\;\Big|\;\mathbf{M}\nabla\nabla u_n\,\Big\rangle-\Big\langle\,\phi P_\psi\,\partial_m u_n\;\Big|\;\mathsf{div}\,\mathsf{div}\,\mathbf{M}(\nabla\nabla u_n)\,\Big\rangle.\end{aligned}$$

After passing to the limit, while using the form (3.19) for the parabolic H-measure associated to sequence $(\partial_t u_n,\nabla\nabla u_n)$ we get

$$\begin{aligned}&\Big\langle\frac{(2\pi\tau)^2\tilde{\nu}}{|\mathsf{p}|^2},\xi_m\phi\nabla^{\boldsymbol{\xi}}\psi\cdot\nabla_{\mathbf{x}}\varrho\Big\rangle+\Big\langle\frac{(2\pi\tau)^2\tilde{\nu}}{|\mathsf{p}|^2},\psi\partial_m(\phi\varrho)\Big\rangle+4\Big\langle\frac{(2\pi)^4\tilde{\nu}}{|\mathsf{p}|^2},\xi_m\psi(\mathbf{M}\boldsymbol{\xi}\otimes\boldsymbol{\xi})\boldsymbol{\xi}\cdot\nabla_{\mathbf{x}}\bar{\phi}\Big\rangle\\ &-\Big\langle\frac{(2\pi)^4\tilde{\nu}}{|\mathsf{p}|^2},\psi\partial_m(\phi\mathbf{M})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\Big\rangle-\Big\langle\frac{(2\pi)^4\tilde{\nu}}{|\mathsf{p}|^2},\phi\xi_m\nabla^{\boldsymbol{\xi}}\psi\cdot(\nabla_{\mathbf{x}}\mathbf{M})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\Big\rangle=0\,.\end{aligned}$$

Again we have used the Aubin compactness lemma, which provides the strong convergence of (u_n) in $\mathrm{L}^2([0,T];\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^d))$.

In the terms of symbol $Q(t, \mathbf{x}; \tau, \boldsymbol{\xi}) := -(2\pi\tau)^2 \varrho(t, \mathbf{x}) + (2\pi)^4 \mathbf{M}(t, \mathbf{x})(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \cdot (\boldsymbol{\xi} \otimes \boldsymbol{\xi})$ for the vibrating plate equation this can be rewritten as

$$\Big\langle \frac{\tilde{\nu}}{|\mathsf{p}|^2}, \xi_m \phi \nabla^{\boldsymbol{\xi}} \psi \cdot \nabla_{\mathbf{x}} Q - \xi_m \psi \nabla^{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}} \bar{\phi} \Big\rangle + \Big\langle \frac{\tilde{\nu}}{|\mathsf{p}|^2}, \psi \partial_m (\phi Q) \Big\rangle = 0 ,$$

a relation analogous to formula (34).

Replacing ψ by $\tilde{\psi}_i = \frac{\xi_m \xi_i^2}{\rho^3} \psi$, where, as before, ψ is an arbitrary symbol from $\mathrm{C}^1(\mathrm{P}^d)$, the summation with respect to i and m yields (recall that $\alpha = \sqrt{2/(\tau^2+1)}$ denotes the quantity defined when we computed the curvature of P^d, in the calculations immediately preceding Lemma 2)

$$\Big\langle \frac{|\boldsymbol{\xi}|^4 \tilde{\nu}}{|\mathsf{p}|^2}, \nabla^{\boldsymbol{\xi}} \Psi \cdot \nabla_{\mathbf{x}} Q - \nabla_{\mathbf{x}} \Psi \cdot \nabla^{\boldsymbol{\xi}} Q \Big\rangle + \Big\langle \frac{|\boldsymbol{\xi}|^4 \tilde{\nu}}{|\mathsf{p}|^2}, 3\Psi \left(\frac{1}{|\boldsymbol{\xi}|^2} - \frac{\alpha^2}{4} \right) \boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} Q \Big\rangle + \Big\langle \frac{|\boldsymbol{\xi}|^2 \tilde{\nu}}{|\mathsf{p}|^2}, \Psi \, \boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} Q \Big\rangle = 0 ,$$

where $\Psi = \phi \boxtimes \psi$.

As $\nu := \mathsf{tr} \boldsymbol{\mu} = \frac{(2\pi|\boldsymbol{\xi}|)^4}{|\mathsf{p}|^2} \tilde{\nu}$, where $\boldsymbol{\mu}$ is a parabolic H-measure associated to a subsequence of $(\nabla\nabla u_n)$, the last relation can be rewritten as

$$\Big\langle \nu, \nabla^{\boldsymbol{\xi}} \Psi \cdot \nabla_{\mathbf{x}} Q - \nabla_{\mathbf{x}} \Psi \cdot \nabla^{\boldsymbol{\xi}} Q \Big\rangle + \Big\langle \nu, \Psi \left(\frac{4}{|\boldsymbol{\xi}|^2} - \frac{3}{4} \alpha^2 \right) \boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} Q \Big\rangle = 0 . \tag{36}$$

Integration by parts (Corollary 1) gives

$$\begin{aligned} &\int_{\mathrm{P}^d} \nu \nabla^{\boldsymbol{\xi}} \Psi \cdot \nabla_{\mathbf{x}} Q dA = - \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} Q \cdot \left(\nabla^{\boldsymbol{\xi}} \nu - \frac{\alpha}{2} (\mathsf{n} \cdot \nabla^{\tau, \boldsymbol{\xi}} \nu) \boldsymbol{\xi} \right) dA \\ &\quad + \int_{\mathrm{P}^d} \Psi \nu \left(\frac{\alpha^2}{4} (\alpha^2 + d - 1) \nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi} - \mathsf{div}^{\boldsymbol{\xi}} \nabla_{\mathbf{x}} Q + \frac{\alpha^2}{2} \tau \partial_\tau \nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi} + \frac{\alpha^2}{4} \Big((\nabla^{\boldsymbol{\xi}} \otimes \nabla_{\mathbf{x}}) Q \Big) \boldsymbol{\xi} \cdot \boldsymbol{\xi} \right) dA. \end{aligned} \tag{37}$$

The symbol Q, as well as its derivatives $\partial_j Q$, is parabolicly homogeneous of degree 4 with respect to dual variables, i.e.

$$\nabla_{\mathbf{x}} Q(t, \mathbf{x}; s^2 \tau, s\boldsymbol{\xi}) = s^4 \nabla_{\mathbf{x}} Q(t, \mathbf{x}, \tau, \boldsymbol{\xi}).$$

Differentiating the above expressions with respect to parameter s, after taking $s = 1$, yields

$$2 \Big(\partial_\tau \nabla_{\mathbf{x}} Q \Big)(t, \mathbf{x}; \tau, \boldsymbol{\xi}) \tau + \Big((\nabla_{\mathbf{x}} \otimes \nabla^{\boldsymbol{\xi}}) Q \Big)(t, \mathbf{x}; \tau, \boldsymbol{\xi}) \boldsymbol{\xi} = 4 \nabla_{\mathbf{x}} Q(t, \mathbf{x}; \tau, \boldsymbol{\xi}).$$

Thus the formula (37) becomes

$$\begin{aligned} \int_{\mathrm{P}^d} \nu \nabla^{\boldsymbol{\xi}} \Psi \cdot \nabla_{\mathbf{x}} Q dA = &- \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} Q \cdot \left(\nabla^{\boldsymbol{\xi}} \nu - \frac{\alpha}{2} \boldsymbol{\xi} (\mathsf{n} \cdot \nabla_{\tau, \boldsymbol{\xi}} \nu) \right) dA, \\ &+ \int_{\mathrm{P}^d} \Psi \nu \left(\frac{\alpha^2}{4} (\alpha^2 + d + 3) \nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi} - \mathsf{div}^{\boldsymbol{\xi}} \nabla_{\mathbf{x}} Q \right) dA \end{aligned}$$

and, as $\frac{1}{|\boldsymbol{\xi}|^2} = \frac{\alpha^2}{4(\alpha^2 - 1)}$, the relation (36) reads

$$\begin{aligned} \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} \nu \cdot \nabla^{\boldsymbol{\xi}} Q dA + \int_{\mathrm{P}^d} \Psi \nu \frac{\alpha^2}{4} \left(\alpha^2 + d + 3 + \frac{7 - 3\alpha^2}{\alpha^2 - 1} \right) \nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi} dA& \\ - \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} Q \cdot \left(\nabla^{\boldsymbol{\xi}} \nu - \frac{\alpha}{2} \boldsymbol{\xi} (\mathsf{n} \cdot \nabla^{\tau, \boldsymbol{\xi}} \nu) \right) dA = 0.& \end{aligned}$$

Finally, as by the localisation principle $\nu \partial_i Q = -(\partial_i \nu) Q$ we get

$$\begin{aligned} \int_{\mathrm{P}^d} \Psi \nabla_{\mathbf{x}} \nu \cdot \left(\nabla^{\boldsymbol{\xi}} Q - \frac{\alpha^2}{4} \left(\alpha^2 + d + \frac{4}{\alpha^2 - 1} \right) Q \boldsymbol{\xi} \right) dA& \\ - \int_{\mathrm{P}^d} \Psi \begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q \end{bmatrix} \cdot \Big(\nabla^{\tau, \boldsymbol{\xi}} \nu - (\nabla^{\tau, \boldsymbol{\xi}} \nu \cdot \mathsf{n}) \mathsf{n} \Big) dA = 0.& \end{aligned}$$

Thus we have obtained the following theorem.

Theorem 13 *For any parabolic H-measure $\boldsymbol{\mu}$ associated to (a subsequence of) (∇u_n), where (u_n) is a sequence of solutions to (3.18) with additional assumption that ϱ and $\mathbf{M}$ are in $\mathrm{C}^1 \cap \mathrm{X}^{\frac{1}{2},1}$, the trace $\nu = \mathsf{tr}\boldsymbol{\mu}$ satisfies the transport equation*

$$\nabla_{\mathbf{x}}\nu \cdot \left(\nabla^{\boldsymbol{\xi}} Q - \frac{\alpha^2}{4}\left(\alpha^2 + d + \frac{4}{\alpha^2-1}\right) Q\boldsymbol{\xi}\right) - \nabla^{\tau,\boldsymbol{\xi}}\nu \cdot \left(\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q\end{bmatrix} - \left(\begin{bmatrix} 0 \\ \nabla_{\mathbf{x}} Q\end{bmatrix}\cdot \mathsf{n}\right)\mathsf{n}\right) = 0\,.$$

∎

Remark. Similarly as it was done for the Schrödinger equation, we can consider the characteristics of the above transport equation. If we start from a frequency $(\tau_0, \boldsymbol{\xi}_0) \in \mathrm{P}^d$ as above, we can conclude that the characteristics remain in P^d over the interval of their existence, so H-measure ν in Theorem 13 remains constant along such integral curves on $\mathbf{R}^{1+d} \times \mathrm{P}^d$.

Similarly we also get that $q(s) := Q(t(s), \mathbf{x}(s); \tau(s), \boldsymbol{\xi}(s)) = 0$ for $s \geqslant 0$, if we start from $q(0) = 0$. Indeed, here we can use the identity $\nabla^{\tau,\boldsymbol{\xi}} Q \cdot \mathsf{n} = 2\alpha Q$ to obtain

$$\frac{dq}{ds} = -\frac{\alpha^2}{4}\left(\alpha^2 + d - 4 + \frac{4}{\alpha^2 - 1}\right)(\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}) q\,,$$

and then we can conclude as before.

∎

Concluding remarks

We hope that it has been demonstrated that the introduced choice of surface P^d is appropriate for expected applications of parabolic H-measures. Also, after the model equations we treated, it is hoped that other, more realistic applications, are now feasible.

A number of possible generalisations have been mentioned in the paper; a detailed development of such tools should not be difficult along the same lines but, in our opinion, should only be undertaken with a precise application as an ultimate goal.

At a formal level, we can unify the obtained results. In the paper, up to the nonhomogeneous term, we considered equations of the form

$$P_0 \varrho P_0 u_n + \mathsf{P}_1 \cdot \mathbf{A} \mathsf{P}_1 u_n = 0\,,$$

where P_0 and P_1 stand for (pseudo)differential operators in time and space variables, respectively. Denoting their (principal) symbols by p_0 and p_1, and by $Q = \varrho p_0^2 + \mathbf{A}\mathsf{p}_1 \cdot \mathsf{p}_1$ the symbol of the differential operator defining the left-hand side of the above equation, we can state the results obtained for the parabolic H-measure $\tilde{\boldsymbol{\mu}}$ associated to the sequence $(P_0 u_n, \mathsf{P}_1 u_n)$, converging weakly in L^2 to $\mathsf{0}$.

The measure $\tilde{\boldsymbol{\mu}}$ is of the form

$$\tilde{\boldsymbol{\mu}} = \frac{\overline{\mathsf{p} \otimes \mathsf{p}}}{|\mathsf{p}|^2}\tilde{\nu}\,, \tag{38}$$

where $\tilde{\nu} := \mathsf{tr}\tilde{\boldsymbol{\mu}}$ is a scalar measure, and the localisation principle reads

$$Q\tilde{\nu} = 0\,. \tag{39}$$

Finally, the propagation principle states

$$\left\langle \frac{\xi_m \tilde{\nu}}{|\mathsf{p}|^2}, \{\phi, Q\}\right\rangle + \left\langle \frac{\tilde{\nu}}{|\mathsf{p}|^2}, \phi\, \partial_m Q\right\rangle = 0\,. \tag{40}$$

Let us note that the above results are also valid for the original *H*-measures when applied to the wave equation. Thus (38) and (39) are found in [40, Lemma 3.10], while the propagation principle given in [40, Theorem 3.12], when reduced to the homogeneous problem becomes a special case of (40) for $m = 0$ and time independent coefficients.

This provides additional motivation for further development of the theory of generalised H-measures, applicable to equations of different types, eventually enabling the unification of results valid for particular types of problems.

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