PULLBACK EXPONENTIAL ATTRACTORS FOR EVOLUTION PROCESSES IN BANACH SPACES: THEORETICAL RESULTS

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Abstract. We construct exponential pullback attractors for time continuous asymptotically compact evolution processes in Banach spaces and derive estimates on the fractal dimension of the attractors. We also discuss the corresponding results for autonomous processes.

1. Introduction

The study of the longtime dynamics of semigroups acting in infinite dimensional spaces can often be reduced to the study of the dynamics on the global attractor. It is a strictly invariant compact subset, which attracts all bounded subsets of the phase space. To be more precise, let $T(t) : X \to X$, $t \geq 0$, be operators in a metric space (X, d_X). We call the family $\{T(t)|\ t \geq 0\}$ a ***semigroup*** in X if it satisfies the properties

$$
\begin{aligned}
&T(t) \circ T(s) = T(t+s), \quad t, s \geq 0,\\
&T(0) = Id, \quad \text{and}\\
&\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X \quad \text{is continuous,}
\end{aligned}
$$

where Id denotes the identity operator in X. The subset $\mathcal{A} \subset X$ is the ***global attractor*** for the semigroup $\{T(t)|\ t \geq 0\}$ if $\mathcal{A} \neq \emptyset$ is compact, strictly invariant, that is $T(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and for every bounded subset $D \subset X$

$$\lim_{t\to\infty} \operatorname{dist}_{\mathrm{H}}(T(t)D, \mathcal{A}) = 0.$$

Here, $\operatorname{dist}_{\mathrm{H}}(\cdot, \cdot)$ is the Hausdorff semidistance in X; that is, $\operatorname{dist}_{\mathrm{H}}(A, B) = \sup_{a\in A} \inf_{b\in B} d_X(a, b)$. The global attractor is unique, minimal within the family of closed subsets that attract all bounded sets and the maximal bounded invariant subset of the phase space. In most cases the fractal dimension of the global attractor is finite. The rate of convergence however can be

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arbitrarily slow and the global attractor is in general sensitive to perturbations. Due to these drawbacks the notion of an exponential attractor was introduced in [6] proposing to consider a larger set, which is still finite dimensional and attracts all bounded subsets at an exponential rate. This concept requires to weaken the invariance property of the attracting set.

Definition 1. *We call $\mathcal{M} \subset X$ an* **exponential attractor** *for the semigroup $\{T(t)|\ t \geq 0\}$ if $\mathcal{M} \neq \emptyset$ is compact, of finite fractal dimension $\dim_{\mathrm{f}}(\mathcal{M}) < \infty$, semi-invariant, that is $T(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$, and for every bounded subset $D \subset X$ there exists a constant $\omega > 0$ such that*

$$\lim_{t\to\infty} e^{\omega t} \mathrm{dist}_{\mathrm{H}}(T(t)D, \mathcal{A}) = 0,$$

where $\dim_{\mathrm{f}}(A) = \limsup_{\epsilon\to 0} \log_{\frac{1}{\epsilon}} N_\epsilon^X(A)$, and $N_\epsilon^X(A)$ denotes the minimal number of ϵ-balls in the space X with centers in A needed to cover the subset $A \subset X$.

Thanks to the exponential rate of attraction exponential attractors are more robust under perturbations than the global attractor. Furthermore, if a semigroup possesses an exponential attractor, the global attractor is contained in the exponential attractor, which immediately implies its existence and finite-dimensionality. However, exponential attractors are only semi-invariant under the action of the semigroup and consequently not unique. Indeed, if $\mathcal{M}$ is an exponential attractor then $T(t)\mathcal{M}$ yields an exponential attractor for the semigroup, for each $t \geq 0$.

The existence proof and method for the construction of the exponential attractor in [6] is non-constructive, applicable for semigroups acting in Hilbert spaces and based on the squeezing property of the semigroup. In [7] an alternative method and explicit algorithm for the construction of exponential attractors was developed for discrete semigroups acting in Banach spaces. It is based on the compact embedding of the phase space into an auxiliary normed space and uses the regularizing or smoothing property of the semigroup. The rate of convergence and the bound on the fractal dimension of the exponential attractor can explicitly be estimated in terms of the entropy properties of this embedding.

While the notion of attractors in the autonomous setting is well established and well understood, its counterpart for non-autonomous evolution equations is not as well established and less understood. In non-autonomous problems the rule of evolution from the instant s to the instant $t \geq s$ depends not only on the elapsed time $t - s$ as in the autonomous case, but also on the starting time s. The rules of evolution in the non-autonomous setting are dictated by what is called an evolution process.

Definition 2. *The two-parameter family* $\{U(t,s)|\ t,s\in\mathbb{R},\ t\geq s\}$ *of continuous operators from* X *into itself is called an* ***evolution process*** *in* X *if it satisfies the properties*

$$U(t,s)\circ U(s,r)=U(t,r),\quad t\geq s\geq r$$
$$U(t,t)=Id,\quad t\in\mathbb{R}\quad \textit{and}$$
$$\mathcal{T}\times X\ni(t,s,x)\mapsto U(t,s)x\in X,\qquad \textit{is continuous},$$

where $\mathcal{T}:=\{(t,s)\in\mathbb{R}\times\mathbb{R}|\ t\geq s\}$.

Evolution processes extend the definition of semigroups. Indeed, if $\{T(t)|\ t\geq 0\}$ is a semigroup, then $U(t,s)=T(t-s)$ is an autonomous evolution process. There are different approaches to generalize the notion of global attractors of semigroups to non-autonomous evolution processes (cf. [1], [3], [4]). In our work we use the notion of so-called pullback attractors (or kernel sections in [4]).

Remark 1. *We will reserve the letter* T (*the letter* U) *to denote the semigroup* (*the evolution process*) *under consideration. When we need to decompose the semigroup* $\{T(t)|\ t\geq 0\}$ (*the evolution process* $\{U(t,s)|\ t\geq s\}$) *as a sum of two operators, we will always write* $T(t)=S(t)+C(t)$ $(U(t,s)=S(t,s)+C(t,s))$.

Definition 3. *The family of nonempty subsets* $\{\mathcal{A}(t)|\ t\in\mathbb{R}\}$ *of* X *is called a* ***global pullback attractor for the process*** $\{U(t,s)|\ t\geq s\}$ *if* $\mathcal{A}(t)$ *is compact for all* $t\in\mathbb{R}$, *the family* $\{\mathcal{A}(t)|\ t\in\mathbb{R}\}$ *is strictly invariant, that is*

$$U(t,s)\mathcal{A}(s)=\mathcal{A}(t)\qquad \text{for all } t\geq s,$$

it pullback attracts all bounded subsets of X, *that is for every bounded* $D\subset X$ *and* $t\in\mathbb{R}$

$$\lim_{s\to\infty}\mathrm{dist_H}(U(t,t-s)D,\mathcal{A}(t))=0,$$

and the family is minimal within the families of closed subsets that pullback attract all bounded subsets of X.

Comparing with the notion of global attractors of semigroups the minimality is an additional property needed to ensure uniqueness of the pullback attractor, since non-autonomous invariance is a weaker concept than the invariance of a fixed set in the autonomous case. If we replace the pullback attraction in the above definition by the forwards convergence; that is, for every bounded set $D\subset X$ and $t\in\mathbb{R}$

$$\lim_{s\to\infty}\mathrm{dist_H}(U(t+s,t)D,\mathcal{A}(t+s))=0,$$

we call the family $\{\mathcal{A}(t)|\ t\in\mathbb{R}\}$ a ***forwards attractor*** for the process $\{U(t,s)|\ t\geq s\}$. If the pullback (forwards) convergence to the attractor holds uniformly in $t\in\mathbb{R}$, it implies the forwards (pullback) convergence, and the attractors coincide. However, these concepts are not related in general (cf. [3]).

The construction of exponential attractors for discrete semigroups in [7] was further developed in [8] and also extended to non-autonomous problems using the concept of forwards attractors. An explicit algorithm for discrete evolution processes was presented and in an application to non-autonomous reaction-diffusion systems also an exponential attractor of the generated time continuous process was constructed. Based on these results the construction has recently been modified considering the pullback approach, and the algorithm has been extended to time continuous evolution processes in [5] and [10]. Applying the pullback approach and generalizing the concept of exponential attractors for evolution processes we obtain the following definition (cf. [5] and [8]).

Definition 4. *Let $\{U(t,s)|\ t\geq s\}$ be an evolution process in a metric space X. We call the family $\mathcal{M}=\{\mathcal{M}(t)|\ t\in\mathbb{R}\}$ a* ***pullback exponential attractor for the evolution process*** *$\{U(t,s)|\ t\geq s\}$ in X if*

(i) *the subsets $\mathcal{M}(t)\subset X$ are non-empty and compact in X for all $t\in\mathbb{R}$,*

(ii) *the family is positively semi-invariant, that is*

$$U(t,s)\mathcal{M}(s)\subset\mathcal{M}(t)\qquad\text{for all } t\geq s,$$

(iii) *the fractal dimension in X of the sections $\mathcal{M}(t)$, $t\in\mathbb{R}$, is uniformly bounded and*

(iv) *the family $\{\mathcal{M}(t)|\ t\in\mathbb{R}\}$ exponentially pullback attracts bounded subsets of X; that is, there exists a positive constant $\omega>0$ such that for every bounded subset $D\subset X$ and $t\in\mathbb{R}$*

$$\lim_{s\to\infty}e^{\omega s}\mathrm{dist}_{\mathrm{H}}(U(t,t-s)D,\mathcal{M}(t))=0.$$

The constructions in [5] and [10] are similar, require strong regularity assumptions on the process and restrictive assumptions with respect to the pullback attraction. Based on these articles we propose a different construction for time continuous evolution processes and generalize the results to asymptotically compact processes. In particular, we show the existence of a pullback exponential attractor under significantly weaker hypothesis and obtain better estimates for the fractal dimension of the sections $\mathcal{A}(t)$ of the pullback attractor. The constructions in [5] and [10] are based on the existence of a fixed bounded pullback absorbing set. This allows the pullback attractor to be unbounded in the future, but it is always bounded in the past. Instead of a fixed bounded absorbing set we consider a family of time-dependent absorbing sets, which can even grow in the past, and obtain an exponential pullback attractor with sections, that are not necessarily uniformly bounded in the past. If the pullback exponential attractor exists, it contains the global pullback attractor and immediately implies its existence and the finite dimensionality of its sections. Existence proofs for global pullback attractors of asymptotically compact processes often require the boundedness of the global pullback attractor in the past (cf. [1]), and the finite dimensionality of global pullback attractors that are not uniformly bounded in the past has been an open problem (see [10] or Remark 3.2 in [11]). Hence, our main theorem also yields existence results for global pullback attractors, that are possibly unbounded in the past, and implies their finite fractal dimension.

The outline of our paper is as follows. For convenience of the reader in Section 2 we summarize some known results about the existence of pullback attractors and the construction of exponential attractors for evolution processes.

In Section 3.1 we present the construction of an exponential pullback attractor for discrete evolution processes. The method is based on the results in [5], [8] and [10], but generalizes the construction for asymptotically compact processes. Moreover, instead of a fixed pullback absorbing set we consider a time-dependent family of absorbing sets, which can grow in the past.

We then propose an alternative construction for time continuous processes in Section 3.2. Our method does not require the strong regularity properties in time of the process as in [5] and [10], which are typical for parabolic problems. It yields better estimates for the fractal dimension of the sections of the attractor and is applicable for a larger class of evolution problems. If the conditions in the cited articles are satisfied our pullback exponential attractor is contained in the pullback exponential attractor constructed in [5] and [10].

We also formulate the corresponding corollaries in the autonomous case, that is for semigroups. The results in the discrete case generalize the results in [2], [5], [7] and [8]. The invariance of sets in the non-autonomous setting is a strictly weaker concept than the invariance of a fixed set under the action of a semigroup. Our main result applied to time continuous semigroups does not yield an exponential attractor in the usual sense, the attractor lacks the property of semi-invariance. In [5] and [10] the union over a certain time interval of the image of the discrete attractor is taken to construct the exponential attractor for continuous time processes. Applied to autonomous evolution processes the construction yields an exponential attractor for the semigroup, which is semi-invariant. However, to ensure the finite dimensionality of the sections it requires strong regularity in time of the process and leads to an artificial increase in the dimension estimates for the continuous attractor, when compared to the discrete attractor. We suggest to weaken the semi-invariance property and introduce the notion of exponential pullback attractors for time continuous semigroups. Exponential pullback attractors exist under weaker hypothesis and satisfy the same dimension estimates as exponential attractors of the associated discrete semigroup.

In the final stage of our work we became aware that a similar construction for time-continuous exponential attractors was developed in [9] based on the concept of forwards attractors. However, the setting is different, the process is asymptotically compact in the weaker space, and the construction is based on the existence of a fixed bounded absorbing set for the process. The aim in this article was not to show the existence of forwards attractors in general but to construct exponential forwards attractors if the uniform attractor (or kernel of the process, cf. [4]) exists. The stated hypothesis immediately imply the boundedness of the exponential forwards attractor and the existence and boundedness of the pullback attractor.

The unboundedness of pullback attractors is essential when considering unbounded non-autonomous terms in the equation or random attractors. In a forthcoming paper we apply our abstract results to an initial value problem for a non-autonomous Chaffee-Infante equation and an evolution process generated by a non-linear damped wave equation.

2. Existence of Pullback Attractors and the Pullback Exponential Attractor Constructed in [5] and [10]

2.1. **Global Pullback Attractors.** Let X be a metric space and $\{U(t,s)|\ t\geq s\}$ be an evolution process in X as defined in the introduction. The following theorem characterizes the evolution processes possessing a global pullback attractor (see [1]).

Theorem 1. *Let $\{U(t,s)|\ t\geq s\}$ be an evolution process in a complete metric space X. Then, the following statements are equivalent:*

(a) *The evolution process $\{U(t,s)|\ t\geq s\}$ possesses a global pullback attractor.*
(b) *There exists a family of compact subsets $\{K(t)|\ t\in\mathbb{R}\}$ of X such that for all $t\in\mathbb{R}$ the set $K(t)$ pullback attracts all bounded subsets of X at time t.*

Furthermore, the pullback global attractor is given by

$$\mathcal{A}(t)=\overline{\bigcup_{\substack{D\,\subset\,X\\ \text{bdd}}}\omega(D,t)},$$

where $\omega(D,t)$ denotes the pullback ω-limit set of $D\subset X$ at time instant $t\in\mathbb{R}$.

The pullback ω-limit set of $D\subset X$ at time instant $t\in\mathbb{R}$ is defined as

$$\omega(D,t):=\bigcap_{r\geq 0}\overline{\bigcup_{s\geq r}U(t,t-s)D}.$$

Note that by Theorem 1 the existence of a pullback exponential attractor immediately implies the existence of the global pullback attractor, which is contained in the pullback exponential attractor and possesses finite dimensional sections.

To show the existence of global (pullback) attractors in applications one generally derives a-priori estimates to prove the existence of a bounded absorbing set. For evolution processes, which are not eventually compact, it is then generally difficult to apply Theorem 1 directly. To conclude the existence of the global pullback attractor in problems with asymptotically compact processes it is often shown that the process satisfies a stronger pullback absorbing property (cf. [1]).

Definition 5. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in a metric space X. A family of bounded subsets $\{B(t)|\ t \in \mathbb{R}\}$ is said to be* ***strongly pullback absorbing all bounded subsets*** *of X, if for all bounded $D \subset X$ and $s \leq t$ there exits $T_{D,s} \geq 0$ such that*

$$U(s, s-r)D \subset B(t) \qquad \text{for all } r \geq T_{D,s}.$$

Processes possessing a family of bounded strongly pullback absorbing subsets are called ***pullback strongly bounded dissipative****.*

In other words, at a given time $t \in \mathbb{R}$ the absorbing set $B(t)$ is also a pullback absorbing set for all earlier times $s \leq t$. Processes that are pullback asymptotically compact and pullback strongly bounded dissipative possess a global pullback attractor (see [1]).

Definition 6. *An evolution process $\{U(t,s)|\ t \geq s\}$ in a metric space X is called* ***pullback asymptotically compact*** *if for every $t \in \mathbb{R}$, every sequence $\{s_n\}_{n\in\mathbb{N}}, s_n \geq 0$, and bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that*

$$\lim_{n\to\infty} s_n = \infty \quad \text{and} \quad \{S(t, t-s_n)x_n|\ n \in \mathbb{N}\} \text{ is bounded}$$

the sequence $\{S(t, t-s_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

Theorem 2. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in a complete metric space X that is pullback asymptotically compact. If there exists a family $\{B(t)|\ t \in \mathbb{R}\}$ of bounded sets that strongly pullback absorb all bounded subsets of X, then there exists the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$. Moreover, $\bigcup_{s\leq t} \mathcal{A}(s)$ is bounded for all $t \in \mathbb{R}$ and the global pullback attractor is given by*

$$\mathcal{A}(t) = \omega(B(t), t).$$

This theorem extends the respective result for semigroups (cf. Theorem 1.1, Chapter 1 in [13]). The global pullback attractor of strongly bounded dissipative processes however, is always bounded in the past. To be more precise, for every $t \in \mathbb{R}$ the union

$$\bigcup_{s\leq t} \mathcal{A}(s)$$

is bounded.

2.2. **Exponential Pullback Attractors.** In the sequel we shortly summarize the results in [5] and [10], where pullback exponential attractors for time continuous processes, which satisfy a certain smoothing property, were constructed. We assume $\{U(t,s)|\ t \geq s\}$ is an evolution process in V and the phase space $(V, \|\cdot\|_V)$ is a Banach space. The construction of the exponential attractor is based on the compact embedding of the phase space into an auxiliary normed space.

(H_0) We assume $(W, \|\cdot\|_W)$ is another normed space such that the embedding $V \hookrightarrow\hookrightarrow W$ is compact and

$$\|v\|_W \leq \mu \|v\|_V \qquad \text{for all} \ \ v \in V,$$

where the constant $\mu > 0$.

Moreover, it was assumed that the process is strongly bounded dissipative. To be more precise, for some $t_0 \in \mathbb{R}$ the following assumptions were made:

(H_1) For the process $\{U(t,s)|\ t \geq s\}$ there exists a bounded set $B \subset V$, that uniformly pullback absorbs all bounded subsets of V for all $t \leq t_0$: For all bounded subsets $D \subset V$ there exists $T_D \geq 0$ such that

$$\bigcup_{t \leq t_0} U(t, t-s)D \subset B \qquad \text{for all } s \geq T_D.$$

(H_2) The evolution process $\{U(t,s)|\ t \geq s\}$ satisfies within the absorbing set the following smoothing property

$$\|U(t, t-T_B)u - U(t, t-T_B)v\|_V \leq \kappa \|u - v\|_W \qquad \text{for all } u, v \in B, t \leq t_0,$$

where T_B denotes the absorbing time in (H_1) corresponding to the absorbing set B.

(H_3) The evolution process $\{U(t,s)|\ t \geq s\}$ is Lipschitz continuous: For every $t \in \mathbb{R}$ and $s \leq t$ exists a constant $L_{t,s} \geq 0$ such that

$$\|U(t,s)u - U(t,s)v\|_V \leq L_{t,s} \|u - v\|_V \qquad \text{for all } u, v \in B.$$

(H_4) The evolution process is Hölder continuous in time: There exist constants $\lambda_1, \lambda_2 > 0$ and exponents $0 < \theta_1, \theta_2 \leq 1$ such that

$$\sup_{t \leq t_0} \|U(t, t-T_B)u - U(t-s, t-s-T_B)u\|_W \leq \lambda_1 s^{\theta_1} \qquad \text{for all } s \in [0, T_B], u \in B,$$

$$\sup_{t \leq t_0} \|U(t, t-s_1)u - U(t, t-s_2)u\|_W \leq \lambda_2 |s_1 - s_2|^{\theta_2} \qquad \text{for all } s_1, s_2 \in [T_B, 2T_B], u \in B.$$

Remark 2.

- *Hypothesis (H_1) not only implies that the process is pullback strongly bounded dissipative, but also that the absorbing time corresponding to a bounded subset $D \subset V$ is independent of the time instant $t \leq t_0$. We generalize these uniform assumptions in* *the next section. The pullback exponential attractor may be unbounded in the future, but it is always bounded in the past, which in turn implies the boundedness of the global pullback attractor. Indeed, for every $t \in \mathbb{R}$, the union*

$$\bigcup_{s \leq t} \mathcal{A}(t) \subset \bigcup_{s \leq t} \mathcal{M}(t)$$

is bounded.

- *The smoothing property (H_2) implies the (eventual) compactness of the evolution process in V. Moreover, by Theorem 2 the global pullback attractor exists and for all $t \leq t_0$*

$$\mathcal{A}(t) = \omega(B, t).$$

- *The Hölder continuity in time of the process was needed for the construction of the time continuous attractor and is typical for parabolic problems. It is a restrictive assumption and generally not satisfied, for instance for hyperbolic equations. Moreover, to apply the result to hyperbolic problems requires the extension of the construction to asymptotically compact processes.*
- *The assumptions $(H_0) - (H_4)$ are taken from* [5]. *In* [10] *the hypothesis are similar, but less general (for instance, the process is assumed to be Hölder continuous in the metric of V).*
- *The absorbing time T_B, the smoothing time in (H_2) and the interval, where the process is Hölder continuous coincide. This is not necessary for the construction of the exponential attractor (cf. Section 3).*

For further details and the proof of the following result, see [5] and [10]. In the sequel we denote by $B_r^X(a)$ the ball of radius $r > 0$ and center $a \in X$ in a metric space X.

Theorem 3. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in a Banach space V and the assumptions $(H_0) - (H_4)$ be satisfied. Then, for every $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$, and the fractal dimension of its sections is uniformly bounded*

$$\sup_{t\in\mathbb{R}} \dim_{\mathrm{f}}(\mathcal{M}^\nu(t)) \leq \max\{\frac{1}{\theta_1}, \frac{1}{\theta_2}\}(1 + \log_{\frac{1}{2\nu}}(1+\mu\kappa)) + \log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\right),$$

where $N_\epsilon^X(A)$ denotes the minimal number of ϵ-balls in a metric space X with centers in A needed to cover the subset $A \subset X$.

3. Pullback Exponential Attractors for Asymptotically Compact Evolution Processes

We assume $(V, \|\cdot\|_V)$ is a Banach space and $\{U(t,s)|\ t \geq s\}$ is an evolution process in V. The construction of the exponential attractor is based on the compact embedding (H_0) of the phase space into an auxiliary normed space. Moreover, we assume that the process U can be represented as $U = S + C$, where $\{S(t,s)|\ t \geq s\}$ and $\{C(t,s)|\ t \geq s\}$ are families of operators satisfying the following properties:

$(\mathcal{H}_1)$ For the process $\{U(t,s)|\ t \geq s\}$ there exists a family of bounded sets $B(t) \subset V$, $t \in \mathbb{R}$, that pullback absorbs all bounded subsets of V; that is, for all bounded subsets $D \subset V$ and all $t \in \mathbb{R}$ there exists $T_{D,t} > 0$ such that

$$U(t, t-s)D \subset B(t) \qquad \text{for all } s \geq T_{D,t}.$$

$(\mathcal{H}_2)$ There exists $\tilde{t} > 0$ such that $\{S(t,s)|\ t \geq s\}$ satisfies within the absorbing set the following smoothing property

$$\|S(t, t-\tilde{t})u - S(t, t-\tilde{t})v\|_V \leq \kappa \|u - v\|_W \qquad \text{for all } u, v \in B(t-\tilde{t}),\ t \in \mathbb{R}.$$

$(\mathcal{H}_3)$ The family of operators $\{C(t,s)|\ t \geq s\}$ are contractions within the absorbing sets,

$$\|C(t, t-\tilde{t})u - C(t, t-\tilde{t})v\|_V \leq \lambda \|u - v\|_V \qquad \text{for all } u, v \in B(t-\tilde{t}),\ t \in \mathbb{R},$$

where the contraction constant $0 \leq \lambda < \frac{1}{2}$.

$(\mathcal{H}_4)$ The process $\{U(t,s)|\ t \geq s\}$ is Lipschitz continuous within the absorbing sets; that is, for all $t \in \mathbb{R}$ and $s \in]t, t+\tilde{t}]$ there exists a constant $L_{t,s} > 0$ such that

$$\|U(s,t)u - U(s,t)v\|_V \leq L_{t,s} \|u - v\|_V \qquad \text{for all } u, v \in B(t),\ t \in \mathbb{R}.$$

In order to construct the pullback exponential attractor we need to impose additional assumptions regarding the uniformity of the absorbing time in hypothesis $(\mathcal{H}_1)$.

(A_1) The family of absorbing sets is positively semi-invariant for the process $\{U(t,s)|\ t \geq s\}$, that is

$$U(t,s)B(s) \subset B(t) \qquad \text{for all } t \geq s, t, s \in \mathbb{R}.$$

(A_2) For a bounded subset $D \subset V$ and $t \in \mathbb{R}$ the corresponding absorbing times are bounded in the past; that is, for all $t \in \mathbb{R}$ there exists $T_{D,t} > 0$ such that

$$U(s, s-r)D \subset B(s) \qquad \text{for all } s \leq t,\ r \geq T_{D,t}.$$

Our main result is the following.

Theorem 4. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in V and the assumptions (H_0), $(\mathcal{H}_1)$- $(\mathcal{H}_4)$, (A_1) and (A_2) be satisfied. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)\}$ growths at most sub-exponentially in the past. Then, for any $\nu \in$*

$(0, \frac{1}{2} - \lambda)$ *there exists a pullback exponential attractor* $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\} \equiv \{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ *for the evolution process* $\{U(t,s)|\ t \geq s\}$*, and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0)) \right) \qquad \text{for all } t \in \mathbb{R}.$$

Remark 3.

- *The uniform pullback absorbing assumption* (H_1) *in the previous section immediately implies hypothesis* $(\mathcal{H}_1)$*,* (A_1) *and* (A_2)*.*
- *For our construction the Hölder continuity in time* (H_4) *of the process is not required. Moreover, we essentially improve the dimension estimates in Theorem 3.*
- *We generalized the construction for asymptotically compact operators in the space* V*. A similar result for discrete evolution processes has been obtained in* [8]*, however, under hypothesis that are difficult to verify in applications (cf. Theorem 2.3). In the articles* [2]*,* [7] *and* [9] *processes that are asymptotically compact in the weaker space* W *were considered; that is, the operator* C *is a contraction in* W*, not in* V*.*
- *Time-dependent absorbing sets were also considered in* [9]*, where forwards exponential attractors were constructed. However, it was assumed that the diameter of the absorbing sets* $B(t)$ *is uniformly bounded and the absorbing time is independent of the time instant. Hence, the union* $\bigcup_{t\in\mathbb{R}} B(t)$ *is a bounded absorbing set for the process and satisfies the uniform hypothesis in Section 2. Furthermore, the aim was not to prove the existence of forwards attractors in general, but knowing the existence of the uniform attractor, to show the existence of time-dependent forwards exponential attractors (or kernel sections).*

We remark that in applications the family of contraction operators often forms an evolution process in V. In this case, and if the contraction property $(\mathcal{H}_3)$ is globally satisfied, the smoothing time and the contraction time can be arbitrary, and it suffices that the evolution process C is a strict contraction. To be more precise we could replace Assumptions $(\mathcal{H}_2)$-$(\mathcal{H}_4)$ by the following:

$(\widetilde{\mathcal{H}}_2)$ The family $\{S(t,s)|\ t \geq s\}$ satisfies the following smoothing property within the absorbing sets: There exists $\tilde{t} > 0$ such that for all $s \geq \tilde{t}$

$$\|S(t+s,t)u - S(t+s,t)v\|_V \leq \kappa_s \|u-v\|_W \qquad \text{for all } u, v \in B(t),\ t \in \mathbb{R},$$

for some constant $\kappa_s > 0$.

$(\widetilde{\mathcal{H}}_3)$ The family $\{C(t,s)|\ t \geq s\}$ is an evolution process and a strict contraction in V: There exists $\hat{t} > 0$ such that

$$\|C(t+\hat{t},t)u - C(t+\hat{t},t)v\|_V \leq \lambda \|u-v\|_V \qquad \text{for all } u, v \in V,\ t \in \mathbb{R},$$

where the contraction constant $0 \leq \lambda < 1$.

($\widetilde{\mathcal{H}}_4$) The evolution process $\{U(t,s)|\ t \geq s\}$ satisfies the Lipschitz continuity in $(\mathcal{H}_4)$ for all $t \in \mathbb{R}$ and $t \leq s \leq t + \hat{t}$.

Indeed, let $k \in \mathbb{N}$ be such that $\lambda^k < \frac{1}{2}$ and $k\hat{t} \geq \tilde{t}$. Then, Property $(\widetilde{\mathcal{H}}_3)$ implies

$$\|C(t+\hat{t}k,t)u - C(t+\hat{t}k,t)v\|_V \quad \leq \quad \lambda^k \|u-v\|_V \qquad \text{for all } u,v \in B(t),\ t \in \mathbb{R}.$$

Furthermore, by the smoothing property $(\widetilde{\mathcal{H}}_2)$ follows

$$\|S(t+\hat{t}k,t)u - S(t+\hat{t}k,t)v\|_V \quad \leq \quad \kappa \|u-v\|_W \qquad \text{for all } u,v \in B(t),\ t \in \mathbb{R},$$

where $\kappa := \kappa_{\hat{t}k}$. Consequently, the assumptions $(\mathcal{H}_2)$-$(\mathcal{H}_3)$ are satisfied if we replace $\tilde{t}$ by $\hat{t}k$ and the smoothing and contraction constants by $\tilde{\lambda} = \lambda^k$ and $\tilde{\kappa} = \kappa_{\hat{t}k}$.

3.1. **Discrete Evolution Processes.** We first construct exponential pullback attractors for discrete processes. Let $\{U(n,m)|\ m,n \in \mathbb{Z},\ n \geq m\}$ be a discrete evolution process in a Banach space V. Without loss of generality we assume that $\tilde{t} = 1$ in assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. In this case, the discrete process is certainly Lipschitz continuous in V by properties $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, and assumption $(\mathcal{H}_4)$ is automatically satisfied.

Theorem 5. *Let $\{U(n,m)|\ n \geq m\}$ be a discrete evolution process in V and the assumptions (H_0), $(\mathcal{H}_1)$ - $(\mathcal{H}_3)$, $(A1)$ and $(A2)$ with $\tilde{t} = 1$ be satisfied for discrete times $t, s \in \mathbb{Z}$. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)\}$, $t \in \mathbb{Z}$, growths at most sub-exponentially in the past. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}(k)|\ k \in \mathbb{Z}\} \equiv \{\mathcal{M}^\nu(k)|\ k \in \mathbb{Z}\}$ for the evolution process $\{U(n,m)|\ n \geq m\}$, and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}(k)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0)) \right) \qquad \text{for all } k \in \mathbb{Z}.$$

Proof. **Coverings of** $U(k,k-n)B(k-n)$

Let $\nu \in (0, \frac{1}{2} - \lambda)$ be fixed, $R_k > 0$ and $v_k \in B(k)$ be such that $B(k) \subset B^V_{R_k}(v_k)$ for all $k \in \mathbb{Z}$. Moreover, we choose elements $w_1, \ldots w_N \in V$ such that

$$B^V_1(0) \subset \bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa}}(w_i),$$

where $N := N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$. We define $W^0(k) := \{v_k\}$ and construct by induction in $n \in \mathbb{N}$ sets $W^n(k), n \in \mathbb{N}$, that depend on the time instant k and satisfy for all $k \in \mathbb{Z}$

(W1) $W^n(k) \subset U(k,k-n)B(k-n) \subset B(k)$,

(W2) $\sharp W^n(k) \leq N^n$,

(W3) $U(k,k-n)B(k-n) \subset \bigcup_{u \in W^n(k)} B^V_{(2(\nu+\lambda))^n R_{k-n}}(u)$.

To construct a covering of the image $U(k,k-1)B(k-1)$, $k\in\mathbb{Z}$, we note that $v\in B^V_{R_{k-1}}(v_{k-1})$ implies

$$\frac{1}{R_{k-1}}(v-v_{k-1})\in B^V_1(0)\subset\bigcup_{i=1}^N B^W_{\frac{\nu}{\kappa}}(w_i)$$

and consequently,

$$B^V_{R_{k-1}}(v_{k-1})\subset\bigcup_{i=1}^N B^W_{R_{k-1}\frac{\nu}{\kappa}}(R_{k-1}w_i+v_{k-1}).$$

Due to the smoothing property $(\mathcal{H}_2)$ we obtain

$$\|S(k,k-1)\tilde{u}-S(k,k-1)\tilde{v}\|_V\le\kappa\|\tilde{u}-\tilde{v}\|_W<2\nu R_{k-1}$$

for all $\tilde{u},\tilde{v}\in B^W_{R_{k-1}\frac{\nu}{\kappa}}(R_{k-1}w_i+v_{k-1})\cap B(k-1)$, which yields

$$S(k,k-1)\big(B^V_{R_{k-1}}(v_{k-1})\cap B(k-1)\big)\subset\bigcup_{i=1}^N B^V_{2R_{k-1}\nu}(z_i),$$

for some $z_1,\dots,z_N\in S(k,k-1)B(k-1)$. In particular, we can choose $y_1,\dots,y_N\in B(k-1)$ such that $z_i=S(k,k-1)y_i$, where $i=1,\dots,N$. For $u\in B(k-1)$ the contraction property $(\mathcal{H}_3)$ now implies

$$\|C(k,k-1)u-C(k,k-1)y_i\|_V\le\lambda\|u-y_i\|_V<2\lambda R_{k-1},$$

for all $i=1,\dots,N$, and we conclude

$$C(k,k-1)B(k-1)\subset B^V_{2\lambda R_{k-1}}(C(k,k-1)y_i).$$

Finally, we obtain the covering

$$\begin{aligned}U(k,k-1)B(k-1)&=(S(k,k-1)+C(k,k-1))B(k-1)\\&\subset\bigcup_{i=1}^N B^V_{2\nu R_{k-1}}\Big((S(k,k-1)y_i)\cup B^V_{2\lambda R_{k-1}}(C(k,k-1)y_i)\Big)\\&\subset\bigcup_{i=1}^N B^V_{2(\nu+\lambda)R_{k-1}}(U(k,k-1)y_i),\end{aligned}$$

with centres $U(k,k-1)y_i\in U(k,k-1)B(k-1)$, $i=1,\dots,N$. Denoting the new set of centers by $W^1(k)$ follows

$$U(k,k-1)B(k-1)\subset\bigcup_{u\in W^1(k)}B^V_{2(\nu+\lambda)R_{k-1}}(u),$$

with $W^1(k)\subset U(k,k-1)B(k-1)\subset B(k)$ and $\sharp W^1(k)\le N$.

Let us assume that the sets $W^l(k)$ are already constructed for all $l\le n$ and $k\in\mathbb{Z}$, which yields the coverings

$$U(k,k-n)B(k-n)\subset\bigcup_{u\in W^n(k)}B^V_{(2(\nu+\lambda))^nR_{k-n}}(u)\qquad\text{for } k\in\mathbb{Z}.$$

In order to construct a covering of

$$\begin{aligned} U(k,k-(n+1))B(k-(n+1)) &= U(k,k-1)U(k-1,k-1-n)B(k-1-n) \\ &\subset \bigcup_{u\in W^n(k-1)} U(k,k-1)B^V_{(2(\nu+\lambda))^n R_{k-n-1}}(u) \end{aligned}$$

let $u \in W^n(k-1)$. We proceed as before and use the covering of the unit ball $B^V_1(0)$ by $\frac{\nu}{\kappa}$-balls in W to conclude

$$B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u) \subset \bigcup_{i=1}^N B^W_{(2(\nu+\lambda))^n R_{k-1-n}\frac{\nu}{\kappa}}((2(\nu+\lambda))^n R_{k-1-n} w_i + u).$$

By the smoothing property $(\mathcal{H}_2)$ then follows

$$\begin{aligned} &S(k,k-1)\Big(U(k-1,k-1-n)B(k-1-n) \cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big) \\ &\qquad \subset \bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^n 2\nu R_{k-1-n}}(S(k,k-1)y_i^u), \end{aligned}$$

for some $y_1^u, \dots, y_N^u \in U(k-1,k-1-n)B(k-1-n)$. Furthermore, the contraction property $(\mathcal{H}_3)$ implies

$$\begin{aligned} &C(k,k-1)\Big(U(k-1,k-1-n)B(k-1-n) \cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big) \\ &\qquad \subset B^V_{(2(\nu+\lambda))^n 2\lambda R_{k-1-n}}(C(k,k-1)y_i^u), \end{aligned}$$

for all $i = 1, \dots, N$. Consequently, we obtain the covering

$$\begin{aligned} &U(k,k-1)\Big(U(k-1,k-1-n)B(k-1-n) \cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big) \\ &= \big(S(k,k-1)+C(k,k-1)\big)\Big(U(k-1,k-1-n)B(k-1-n) \cap B^V_{(2(\nu+\lambda))^n R_{k-1-n}}(u)\Big) \\ &\subset \bigcup_{i=1}^N \Big(B^V_{(2(\nu+\lambda))^n 2\nu R_{k-1-n}}(S(k,k-1)y_i^u) + B^V_{(2(\nu+\lambda))^n 2\lambda R_{k-1-n}}(C(k,k-1)y_i^u)\Big) \\ &\subset \bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^{n+1} R_{k-1-n}}\big(S(k,k-1)y_i^u + C(k,k-1)y_i^u\big) \\ &= \bigcup_{i=1}^N B^V_{(2(\nu+\lambda))^{n+1} R_{k-1-n}}\big(U(k,k-1)y_i^u\big), \end{aligned}$$

with centres $U(k,k-1)y_i^u \in U(k,k-1-n)B(k-1-n)$, for $1 = 1, \dots, N$. Constructing in the same way for every $u \in W^n(k-1)$ such a covering by balls with radius $(2(\nu+\lambda))^{n+1}R_{k-1-n}$ in V we obtain a covering of the set $U(k,k-(n+1))B(k-(n+1))$ and denote the new set of centres by $W^{n+1}(k)$. This yields $\sharp W^{n+1}(k) \le N \sharp W^n(k-1) \le N^{n+1}$, by construction the set of centres $W^{n+1}(k) \subset U(k,k-(n+1))B(k-(n+1))$, and

$$U(k,k-(n+1))B(k-(n+1)) \subset \bigcup_{u\in W^{n+1}(k)} B^V_{(2(\nu+\lambda))^{n+1} R_{k-1-n}}(u),$$

which concludes the proof of the properties $(W1)$-$(W3)$.

Definition of the Pullback Exponential Attractor

We define $E^0(k) := W^0(k)$ for all $k \in \mathbb{Z}$, and set $E^n(k) := W^n(k) \cup U(k,k-1)E^{n-1}(k-1)$, $n \in \mathbb{N}$. Then, the family of sets $E^n(k)$, $n \in \mathbb{N}_0$, satisfies for all $k \in \mathbb{Z}$

(E1) $U(k,k-1)E^n(k-1) \subset E^{n+1}(k), \quad E^n(k) \subset U(k,k-n)B(k-n) \subset B(k)$,

(E2) $E^n(k) = \bigcup_{l=0}^n U(k,k-l)W^{n-l}(k-l), \quad \sharp E^n(k) \leq \sum_{l=0}^n N^l$,

(E3) $U(k,k-n)B(k-n) \subset \bigcup_{u \in E^n(k)} B^V_{(2(\nu+\lambda))^n R_{k-n}}(u)$.

These relations are immediate consequences of the definition of the sets $E^n(k)$, the properties of the sets $W^n(k)$ and the semi-invariance of the absorbing family $\{B(k)|\ k \in \mathbb{Z}\}$, and can be proved by induction.

Using the family of sets $E^n(k)$ we define $\widetilde{\mathcal{M}}(k) := \bigcup_{n\in\mathbb{N}_0} E^n(k)$, for all $k \in \mathbb{Z}$, and show that its closure $\{\mathcal{M}(k)|\ k \in \mathbb{Z}\} := \{\overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}\ |\ k \in \mathbb{Z}\}$ yields an exponential pullback attractor for the process $\{U(n,m)|\ n \geq m\}$.

Semi-invariance of the Exponential Attractor

The family $\{\widetilde{\mathcal{M}}(k)|\ k \in \mathbb{Z}\}$ is positively semi-invariant: Indeed, for all $l \in \mathbb{N}, k \in \mathbb{Z}$, we obtain by applying the property $(E1)$

$$U(k+l,k)\widetilde{\mathcal{M}}(k) := \bigcup_{n\in\mathbb{N}_0} U(k+l,k)E^n(k) \subset \bigcup_{n\in\mathbb{N}_0} E^{n+l}(k+l) \subset \bigcup_{n\in\mathbb{N}_0} E^n(k+l) = \widetilde{\mathcal{M}}(k+l).$$

Since the process is continuous follows also semi-invariance of the family $\{\mathcal{M}(k)|\ k \in \mathbb{Z}\}$

$$U(k+l,k)\mathcal{M}(k) = U(k+l,k)\overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V} \subset \overline{U(k+l,k)\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V} \subset \overline{\widetilde{\mathcal{M}}(k+l)}^{\|\cdot\|_V} = \mathcal{M}(k+l),$$

for all $l \in \mathbb{N}, k \in \mathbb{Z}$.

Compactness and Finite Dimensionality of the Exponential Attractor

We first prove that for all $k \in \mathbb{Z}$ the sets $\widetilde{\mathcal{M}}(k)$ are non-empty, precompact and of finite fractal dimension in V. Note, that for any $m \in \mathbb{N}$ and $n \geq m$ holds

$$\begin{aligned} E^n(k) &\subset U(k,k-n)B(k-n) = U(k,k-m)U(k-m,k-n)B(k-n) \\ &\subset U(k,k-m)B(k-m), \end{aligned}$$

since the family of absorbing sets is positively semi-invariant. Consequently, for all $m \in \mathbb{N}$ we obtain

$$\widetilde{\mathcal{M}}(k) = \bigcup_{n=0}^m E^n(k) \cup \bigcup_{n=m+1}^\infty E^n(k) \subset \bigcup_{n=0}^m E^n(k) \cup U(k,k-m)B(k-m).$$

Let $\epsilon > 0$ and $m \in \mathbb{N}$ be sufficiently large such that $(2(\nu+\lambda))^m R_{k-m} \leq \epsilon < (2(\nu+\lambda))^{m-1} R_{k-m+1}$ holds, then

$$U(k,k-m)B(k-m) \subset \bigcup_{u\in W^m(k)} B^V_\epsilon(u).$$

Hence, we can estimate the number of ϵ-balls in V needed to cover $\widetilde{\mathcal{M}}(k)$ by

$$\begin{aligned} N_\epsilon^V(\widetilde{\mathcal{M}}(k)) &\le \sharp(\bigcup_{n=0}^m E^n(k)) + \sharp W^m(k) \le (m+1)\sharp E^m(k) + N^m \\ &\le (m+1)^2 N^m + N^m \le 2(m+1)^2 N^m, \end{aligned}$$

for all $k \in \mathbb{Z}$, where we used properties $(W2)$ and $(E2)$. This proves the precompactness of $\widetilde{\mathcal{M}}(k)$ in V. As V is a Banach space, taking the closure $\mathcal{M}(k) := \overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}$ of the precompact sets $\widetilde{\mathcal{M}}(k)$ we obtain a family of compact subsets in V.

For the fractal dimension of the sets $\widetilde{\mathcal{M}}(k)$ we conclude

$$\begin{aligned} \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}(k)) &= \limsup_{\epsilon\to 0} \frac{\ln(N_\epsilon^V(\widetilde{\mathcal{M}}(k)))}{\ln\frac{1}{\epsilon}} \\ &\le \limsup_{\epsilon\to 0} \frac{\ln(2) + 2\ln(m+1) + m\ln(N)}{\ln\frac{1}{\epsilon}} \le \log_{\frac{1}{2(\nu+\lambda)}}(N). \end{aligned}$$

Consequently, the fractal dimension of the sections $\mathcal{M}(k)$ is uniformly bounded by the same value, since

$$\dim_{\mathrm{f}}^V(\mathcal{M}(k)) = \dim_{\mathrm{f}}^V(\overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}) = \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}(k)).$$

Exponential Pullback Attraction

We are left to show that the set $\mathcal{M}(k)$ exponentially pullback attracts all bounded subsets of V at time $k \in \mathbb{Z}$. By assumptions $(\mathcal{H}_1)$ and (A_2) for any bounded subset $D \subset V$ and $k \in \mathbb{Z}$ there exists an $n_{D,k} \in \mathbb{N}$ such that $U(l, l-n)D \subset B(l)$ for all $n \ge n_{D,k}$ and $l \le k$.

If $n \ge n_{D,k} + 1$, that is $n = n_{D,k} + n_0$ with some $n_0 \in \mathbb{N}$, then

$$\begin{aligned} \mathrm{dist}_{\mathrm{H}}^V(U(k,k-n)D, \widetilde{\mathcal{M}}(k)) &\le \mathrm{dist}_{\mathrm{H}}^V(U(k,k-n_0)U(k-n_0, k-n_0-n_{D,k})D, \bigcup_{n=0}^\infty E^n(k)) \\ &\le \mathrm{dist}_{\mathrm{H}}^V(U(k,k-n_0)B(k-n_0), \bigcup_{n=0}^\infty E^n(k)) \\ &\le \mathrm{dist}_{\mathrm{H}}^V(U(k,k-n_0)B(k-n_0), E^{n_0}(k)) \\ &\le (2(\nu+\lambda))^{n_0} R_{k-n_0} \le c e^{-\omega n}, \end{aligned}$$

for some constants $c \ge 0$ and $\omega > 0$. Finally, the sets $\mathcal{M}(k) = \overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}$ certainly pullback attract all bounded subsets $D \subset V$ at time $k \in \mathbb{Z}$ exponentially, since

$$\begin{aligned} \mathrm{dist}_{\mathrm{H}}^V(U(k,k-n)D, \mathcal{M}(k)) &= \mathrm{dist}_{\mathrm{H}}^V(U(k,k-n)D, \overline{\widetilde{\mathcal{M}}(k)}^{\|\cdot\|_V}) \\ &\le \mathrm{dist}_{\mathrm{H}}^V(U(k,k-n)D, \widetilde{\mathcal{M}}(k)), \end{aligned}$$

and the exponential pullback attraction property of $\{\mathcal{M}(k) |\ k \in \mathbb{Z}\}$ follows from the exponential pullback attraction property of the family $\{\widetilde{\mathcal{M}}(k) |\ k \in \mathbb{Z}\}$.

This shows that $\{\mathcal{M}(k)|\ k \in \mathbb{Z}\}$ is an exponential pullback attractor for the evolution process $\{U(n,m)|\ n \geq m\}$ with compact sections $\mathcal{M}(k)$ in V. □

3.2. **Continuous Time Evolution Processes.** Using the results of the previous section we now construct pullback exponential attractors for time continuous asymptotically compact processes in V.

Proof of Theorem 4. We consider the discrete process $\{\widetilde{U}(n,m)|\ n \geq m\}$ in V, where $\tilde{U}(n,m) := U(n\tilde{t}, m\tilde{t})$ for all $n \geq m$, which satisfies the assumptions (H_0) and $(\mathcal{H}_1) - (\mathcal{H}_3)$ of the previous subsection. Theorem 5 implies the existence of a pullback exponential attractor $\{\mathcal{M}^d(k)|\ k \in \mathbb{Z}\}$ for the discrete process $\{\widetilde{U}(n,m)|\ n \geq m\}$, where $\mathcal{M}^d(k) = \overline{\widetilde{\mathcal{M}}^d(k)}^{\|\cdot\|_V}$.

To obtain a pullback exponential attractor for the continuous time process we define

$$\widetilde{\mathcal{M}}(t) := U(t, k\tilde{t})\widetilde{\mathcal{M}}^d(k) \qquad \text{for } t \in [k\tilde{t}, (k+1)\tilde{t}[,$$

and take its closure in V, $\mathcal{M}(t) := \overline{\widetilde{\mathcal{M}}(t)}^{\|\cdot\|_V}$ for all $t \in \mathbb{R}$. Due to the Lipschitz-continuity of the process, the sets $\widetilde{\mathcal{M}}(t)$ are compact in V, and we obtain the same (uniform) bound on the fractal dimension of the sections $\mathcal{M}(t)$,

$$\dim_{\mathrm{f}}^V(\mathcal{M}(t)) = \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}(t)) = \dim_{\mathrm{f}}^V(U(k\tilde{t}+t, k\tilde{t})\widetilde{\mathcal{M}}^d(k)) \leq \dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}^d(k)) \qquad t \in [k\tilde{t}, (k+1)\tilde{t}[.$$

Moreover, $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ is positively semi-invariant: Let $t, s \in \mathbb{R}$ and $t \geq s$. Then, $s = k\tilde{t} + s_1$ and $t = l\tilde{t} + t_1$ for some $k, l \in \mathbb{Z}$, $k \leq l$ and $s_1, t_1 \in [0, \tilde{t}[$.

If $l \geq k+1$ we observe

$$\begin{aligned} U(t,s)\widetilde{\mathcal{M}}(s) &= U(l\tilde{t}+t_1, k\tilde{t}+s_1)\widetilde{\mathcal{M}}(k\tilde{t}+s_1) = U(l\tilde{t}+t_1, k\tilde{t}+s_1)U(k\tilde{t}+s_1, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) \\ &= U(l\tilde{t}+t_1, l\tilde{t})U(l\tilde{t}, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) \subset U(l\tilde{t}+t_1, l\tilde{t})\widetilde{\mathcal{M}}(l\tilde{t}) = \widetilde{\mathcal{M}}(l\tilde{t}+t_1) = \widetilde{\mathcal{M}}(t), \end{aligned}$$

where we used the semi-invariance of the family $\{\mathcal{M}(k\tilde{t})|\ k \in \mathbb{Z}\}$ under the action of the process $\{\widetilde{U}(n,m)|\ n \geq m\}$.

On the other hand, if $l = k$, then $s = k\tilde{t} + s_1$ and $t = k\tilde{t} + t_1$ for some $s_1, t_1 \in [0, \tilde{t}[$ and

$$\begin{aligned} U(t,s)\widetilde{\mathcal{M}}(s) &= U(k\tilde{t}+t_1, k\tilde{t}+s_1)\widetilde{\mathcal{M}}(k\tilde{t}+s_1) = U(k\tilde{t}+t_1, k\tilde{t}+s_1)U(k\tilde{t}+s_1, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) \\ &= U(k\tilde{t}+t_1, k\tilde{t})\widetilde{\mathcal{M}}(k\tilde{t}) = \widetilde{\mathcal{M}}(k\tilde{t}+t_1) = \widetilde{\mathcal{M}}(t). \end{aligned}$$

By the continuity of the process we obtain the semi-invariance of the family $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$.

Finally, the set $\mathcal{M}(t)$ exponentially pullback attracts all bounded subsets of V at time $t \in \mathbb{R}$. This follows immediately from the exponential pullback attracting property of the discrete attractor $\{\mathcal{M}(k\tilde{t})|\ k \in \mathbb{Z}\}$.

□

An immediate consequence is the existence and finite dimensionality of the global pullback attractor. If the hypothesis of Theorem 4 are satisfied, the existence of the global pullback attractor actually follows from Corollary 6 in [12], where the assumptions are even weaker. However, our result also implies that the fractal dimension of the sections of the pullback attractor is uniformly bounded, which has been an open problem for global pullback attractors that are not uniformly bounded in the past (cf. [10] and [11])

Corollary 1. *Under the assumptions of Theorem 4, the global pullback attractor* $\{\mathcal{A}(t)|\ t\in\mathbb{R}\}$ *of the evolution process* $\{U(t,s)|\ t\geq s\}$ *exists, is contained in the pullback exponential attractor constructed in Theorem 4, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^{V}(\mathcal{A}(t))\leq\log_{\frac{1}{2(\nu+\lambda)}}\left(N_{\frac{\nu}{\kappa}}^{W}(B_1^V(0))\right)\qquad\text{for all } t\in\mathbb{R}.$$

Proof. This follows immediately from Theorem 4 and the characterization of processes possessing a global pullback attractor in Theorem 1. □

Theorem 4 in the special case that $\lambda=0$ yields the result for processes, which satisfy the smoothing property with respect to the spaces V and W.

Corollary 2. *Let* $\{S(t,s)|\ t\geq s\}$ *be an evolution process in* V *and the assumptions* (H_0) *and* $(\mathcal{H}_2)$ *be satisfied. Moreover, we assume that properties* $(\mathcal{H}_1)$, $(\mathcal{H}_4)$, (A_1) *and* (A_2) *hold with* $\{U(t,s)|\ t\geq s\}$ *replaced by* $\{S(t,s)|\ t\geq s\}$, *where it suffices that the absorbing family is bounded in the metric of* W, *and the diameter of the family of absorbing sets* $\{B(t)\}$ *growths at most sub-exponentially in the past. Then, for any* $\nu\in(0,\frac{1}{2})$ *there exists a pullback exponential attractor* $\{\mathcal{M}(t)|\ t\in\mathbb{R}\}\equiv\{\mathcal{M}^{\nu}(t)|\ t\in\mathbb{R}\}$ *for the evolution process* $\{S(t,s)|\ t\geq s\}$, *and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^{V}(\mathcal{M}(t))\leq\log_{\frac{1}{2\nu}}\left(N_{\frac{\nu}{\kappa}}^{W}(B_1^V(0))\right)\qquad\text{for all } t\in\mathbb{R}.$$

We could also consider processes in the weaker phase space W as analyzed in [2] and [7] for semigroups and in [9] for evolution processes. Such attractors are also called bi-space attractors or (V,W) attractors.

Theorem 6. *Let* $\{U(t,s)|\ t\geq s\}$ *be an evolution process in* W *and the assumptions* (H_0), $(\mathcal{H}_2)$, *(A1) and (A2) be satisfied. Moreover, we assume that* $(\mathcal{H}_1)$ *holds for a bounded pullback absorbing family in* W, *properties* $(\mathcal{H}_3)$ *and* $(\mathcal{H}_4)$ *are satisfied with* V *replaced by* W *and the diameter of the family of absorbing sets* $\{B(t)\}$ *growths at most sub-exponentially in the past (with respect to the metric of* W*). Then, for any* $\nu\in(0,\frac{1}{2}-\lambda)$ *there exists a pullback exponential attractor* $\{\mathcal{M}^{\nu}(t)|\ t\in\mathbb{R}\}\equiv\{\mathcal{M}(t)|\ t\in\mathbb{R}\}$ *for the evolution process* $\{U(t,s)|\ t\geq s\}$ *in* W, *and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^{W}(\mathcal{M}(t))\leq\log_{\frac{1}{2(\nu+\lambda)}}\left(N_{\frac{\nu}{\kappa}}^{W}(B_1^V(0))\right)\qquad\text{for all } t\in\mathbb{R}.$$

The pullback exponential attractor for time continuous processes constructed in the proof of Theorem 4 differs from the pullback exponential attractor constructed in [5] and [10]. In both articles the union over a certain time interval of the image of the discrete attractor is taken to construct the continuous attractor. To obtain finite-dimensionality of the sections it requires strong regularity assumptions of the process. We take the time evolution instead and prove under significantly weaker assumptions the existence of a pullback exponential attractor. If the assumptions of [5] and [10] are satisfied, the pullback exponential attractor of Theorem 4 is contained in the pullback exponential attractor constructed in [5] and [10].

4. A Pullback Exponential Attractor for Time Continuous Semigroups

We also formulate the corresponding results for semigroups, that is for autonomous processes. The results are similar and generalize the results in [8] for discrete semigroups, differ however in the time continuous case. The invariance property in the non-autonomous setting is a weaker concept than the invariance of a (fixed) set under the action of a semigroup. In [2], [5] and [10] the union over a certain time interval of the image of the discrete attractor is taken to obtain semi-invariance of the exponential attractor for continuous processes, semigroups respectively. It requires certain regularity properties of the process in time and leads to weaker estimates for the fractal dimension.

Instead of constructing exponential attractors for continuous time semigroups we consider pullback exponential attractors, or non-autonomous exponential attractors (forwards and pullback convergence are equivalent in this case). They coincide with exponential attractors in the discrete case, and the pullback exponential attractors for time continuous semigroups satisfy the same dimension estimates as exponential attractors of discrete semigroups. In other words, weakening the semi-invariance property of the exponential attractor we avoid the artificial increase in the fractal dimension of the attractor. Moreover, we do not need such strong regularity assumptions as Hölder continuity in time of the semigroup (cf. [8], [5], [2]) to obtain finite dimensionality of the attractor.

Let V be a Banach space and $\{U(t,s)|\ t \geq s\}$ be an autonomous evolution process in V, that is the operators $U(t,s) : V \to V$ only depend on the elapsed time $t-s$. In this case, the family of operators $T(t-s) := U(t-s,0)$, $t \geq s$, forms a semigroup in V. The definition of an exponential attractor for semigroups was given in the introduction.

Due to the mentioned drawbacks of exponential attractors for time continuous semigroups we propose to weaken the semi-invariance property of the exponential attractor and to consider pullback exponential attractors instead. Applied to autonomous processes Definition 4 leads to the following:

Definition 7. *We call the family $\mathcal{M} = \{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ a* ***pullback exponential attractor for the semigroup*** *$\{T(t)|\ t \geq 0\}$ in V if there exists $0 < a < \infty$ such that $\mathcal{M}(t) = \mathcal{M}(a+t)$ for all $t \in \mathbb{R}$,*

(i) *the subsets $\mathcal{M}(t) \subset V$ are non-empty and compact in V for all $t \in \mathbb{R}$,*
(ii) *the family is positively semi-invariant, that is*

$$T(t)\mathcal{M}(s) \subset \mathcal{M}(t+s) \qquad \text{for all } t \geq 0, s \in \mathbb{R},$$

(iii) *the fractal dimension of the sets $\mathcal{M}(t)$, $t \in \mathbb{R}$, is uniformly bounded and*
(iv) *the family exponentially attracts all bounded subsets of V uniformly, that is there exists a positive constant $\omega > 0$ such that for any bounded subset $D \subset V$*

$$\lim_{s\to\infty} \sup_{t\in[0,a]} e^{\omega s} \mathrm{dist}_{\mathrm{H}}^{V}(T(s)D, \mathcal{M}(t)) = 0.$$

Definition 7 implies that $\widetilde{\mathcal{M}} = \mathcal{M}(a)$ is an exponential attractor of the associated discrete semigroup $\widetilde{T}(n) := T(na)$. Any member $\mathcal{M}(t)$ of the family satisfies all the properties of an exponential attractor except the semi-invariance.

Remark 4. *As in the non-autonomous case, if the exponential attractor exists, the global attractor $\mathcal{A}$ is contained in the exponential attractor of the semigroup and has finite fractal dimension. The same holds in the case of pullback exponential attractors for semigroups: Any member of the family $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ contains the global attractor $\mathcal{A}$.*

4.1. **Discrete Semigroups.** We consider semigroups that satisfy the smoothing property asymptotically, that is, semigroups that can eventually be represented as a sum $T = S + C$, where T satisfies the smoothing property and C is a contraction. To be more precise, let $\{T(n)|\ n \in \mathbb{N}\}$ be a discrete semigroup in V such that $T(n) = S(n) + C(n)$, where $\{S(n)|\ n \in \mathbb{N}\}$ is a family (need not to be a semigroup) and $\{C(n)|\ n \in \mathbb{N}\}$ is a semigroup which satisfy:

($S1$) There exists a bounded absorbing set $B \subset V$ for the semigroup $\{T(n)|\ n \in \mathbb{N}\}$; that is, for all bounded subsets $D \subset V$ there exists $n_D \in \mathbb{N}$ such that

$$T(n)D \subset B \qquad \text{for all } n \geq n_D.$$

($S2$) The family $\{S(n)|\ n \in \mathbb{N}\}$ satisfies within the absorbing set the smoothing property: There exists $\tilde{n} \in \mathbb{N}$ such that

$$\|S(n)u - S(n)v\|_V \leq \kappa_n \|u - v\|_W \qquad \text{for all } u, v \in B,\ n \geq \tilde{n},$$

where the constant $\kappa_n > 0$.

($S3$) The family $\{C(n)|\ n \in \mathbb{N}\}$ forms a semigroup in V. Moreover, there exists $\hat{n} \in \mathbb{N}$ such that $C(n)B \subset B$ for all $n \geq \hat{n}$ and

$$\|C(\hat{n})u - C(\hat{n})v\|_V \leq \lambda \|u - v\|_V \qquad \text{for all } u, v \in B,$$

where the constant $0 \leq \lambda < \frac{1}{2}$.

Lemma 1.

(i) *If* $\{T(n)|\ n \in \mathbb{N}\}$ *is a discrete semigroup in* V *such that property* $(S1)$ *is satisfied, then there exists a bounded absorbing set* $\widetilde{B}$ *which is positively semi-invariant and properties* $(S2)$ *and* $(S3)$ *hold with* B *replaced by* $\widetilde{B}$*.*

(ii) *Without loss of generality we can assume that the smoothing time in* $(S2)$ *and the contraction time in* $(S3)$ *coincide, that is* $\tilde{n} = \hat{n}$*.*

(iii) *Taking iterates it suffices in assumption* $(S3)$ *that* $C(n)$ *is (eventually) a contraction with contraction constant* $\lambda \in [0, 1)$*.*

Proof. (i) Indeed, taking

$$\widetilde{B} := \bigcup_{k=0}^{n_B - 1} T(n_B + k)B,$$

it is a bounded absorbing set for the semigroup with corresponding absorbing time $n_{\widetilde{B}} = 1$. It is bounded since $T(n_B + k)B \subset B$ for all $k \in \mathbb{N}_0$, by Property $(S1)$. Moreover, if $D \subset V$ is bounded, there exists $n_D \in \mathbb{N}$ such that $T(n)D \subset B$ for all $n \geq n_D$. Hence, for all $n \geq n_D + n_B$ we obtain

$$T(n)D = T(n - n_D - n_B)T(n_B)T(n_D)D \subset T(n - n_D - n_B)T(n_B)B \subset \widetilde{B}$$

and

$$T(1)\widetilde{B} = \bigcup_{k=0}^{n_B - 1} T(1)T(n_B + k)B \subset \widetilde{B}.$$

Properties $(S2)$ and $(S3)$ certainly hold for all $u, v \in \widetilde{B} \subset B$.

(ii) Let $n_0 = l\hat{n}$ and $l \in \mathbb{N}$ be such that $l\hat{n} \geq \tilde{n}$. By the smoothing property $(S2)$ follows

$$\|S(n_0)u - S(n_0)v\|_V = \|S(\hat{n}l)u - S(\hat{n}l)v\|_V \leq \kappa_{\hat{n}l}\|u - v\|_W \qquad \text{for all } u, v \in B,$$

and Property $(S3)$ implies

$$\|C(n_0)u - C(n_0)v\|_V = \|C(\hat{n}l)u - C(\hat{n}l)v\|_V \leq \lambda^l \|u - v\|_V \qquad \text{for all } u, v \in B.$$

Hence, modifying the constants we can assume $\hat{n} = \tilde{n}$.

(iii) Taking iterates, the semigroup $\{C(n)|\ n \in \mathbb{N}\}$ is a contraction with contraction constant less than $\frac{1}{2}$, which leads to a modification of $\hat{n}$ and the constant λ in $(S3)$. □

The following theorem generalizes the results in [8], where exponential attractors for discrete semigroups, that satisfy the smoothing property, were constructed under the assumption that the absorbing time and smoothing time coincide and $n_B = \tilde{n} = 1$. Theorem 1.3 of the cited article yields the existence of exponential attractors for asymptotically compact semigroups, but under different and more restrictive assumptions, which are difficult to verify in applications.

Theorem 7. *If the conditions* (H_0), $(S1)$, $(S2)$ *and* $(S3)$ *are satisfied, then, for any* $\nu \in (0, \frac{1}{2}-\tilde{\lambda})$ *there exists an exponential attractor* $\mathcal{M} \equiv \mathcal{M}^\nu$ *in* V *for the semigroup* $\{T(n) |\ n \in \mathbb{N}\}$, *and its fractal dimension can be estimated by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu) \leq \log_{\frac{1}{2(\nu+\tilde{\lambda})}} \left(N^W_{\frac{\nu}{\tilde{\kappa}}}(B^V_1(0)) \right),$$

where $\tilde{\lambda} := \lambda^l$, $\tilde{\kappa} := \kappa_{l\hat{n}}$ *and* $l \in \mathbb{N}$ *is the smallest integer such that* $l\hat{n} \geq \tilde{n}$.

Proof. If necessary we replace $\hat{n}$ and $\tilde{n}$ by $n_0 = l\hat{n}$ and λ and $\kappa_{\tilde{n}}$ by $\tilde{\lambda}$ and $\tilde{\kappa}$. Then, due to Lemma 1 without loss of generality we can assume that the absorbing set is positively semi-invariant and the contraction time in $(S2)$ and smoothing time in $(S3)$ coincide, that is $\tilde{n} = \hat{n}$. We only give a sketch of the proof and indicate, where the proof of Theorem 5 has to be modified, or where it simplifies in the autonomous setting.

Coverings of $T(n\tilde{n})B$

Let $\nu \in (0, \frac{1}{2} - \lambda)$ be fixed, $R > 0$ and $v_0 \in B$ be such that $B \subset B^V_R(v_0)$. Moreover, we choose $w_1, \ldots w_N \in V$ such that

$$B^V_1(0) \subset \bigcup_{i=1}^{N} B^W_{\frac{\nu}{\kappa}}(w_i),$$

where $N := N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$ and $\kappa = \kappa_{\tilde{n}}$. We define $W^0 := \{v_0\}$ and inductively construct sets $W^n, n \in \mathbb{N}$, such that

(W1') $W^n \subset T(n\tilde{n})B \subset B$,
(W2') $\sharp W^n \leq N^n$,
(W3') $T(n\tilde{n})B \subset \bigcup_{u\in W^n} B^V_{(2(\nu+\lambda))^n R}(u)$.

As in the proof of Theorem 5 the coverings of the iterates $T(n\tilde{n})B$ are constructed using the properties $(S2)$ and $(S3)$ and the covering of the unit ball $B^V_1(0)$ in W. The sets of centers W^n are now independent of time.

Definition of the Exponential Attractor

In order to obtain semi-invariance of the exponential attractor, we modify the construction of the sets E^n and define

$$\begin{aligned} E^0 &:= W^0, \quad E^1 := W^1 \cup T(1)W^0 \cup T(2)W^0 \cup \cdots \cup T(\tilde{n})W^0 \\ E^2 &:= W^2 \cup T(1)W^1 \cup \cdots \cup T(\tilde{n})W^1 \cup T(\tilde{n}+1)W^0 \cup \cdots \cup T(2\tilde{n})W^0 \\ &\vdots \\ E^n &:= W^n \cup T(1)W^{n-1} \cup \cdots \cup T(\tilde{n})W^{n-1} \cup \cdots \cup T(\tilde{n}(n-1)+1)W^0 \cup \cdots \cup T(\tilde{n}n)W^0 \\ &= W^n \cup \bigcup_{k=1}^{n} \bigcup_{l=1}^{\tilde{n}} T((k-1)\tilde{n}+l)W^{n-k}. \end{aligned}$$

Due to the semi-invariance of the absorbing set B follows

$$T(n)B \subset T(m)B \qquad \text{for all } n \geq m,$$

and for all $n \in \mathbb{N}$ the sets E^n satisfy the properties

(E1') $T(1)E^n \subset E^n \cup E^{n+1}$, $E^0 \subset B$, $E^n \subset T((n-1)\tilde{n})B \subset B$,

(E2') $\sharp E^n \leq \tilde{n}(n+1)N^n$,

(E3') $T(n\tilde{n})B \subset \bigcup_{u \in E^n} B^V_{(2(\nu+\lambda))^n R}(u)$.

From the first relation immediately follows $T(k)E^n \subset E^n \cup E^{n+1} \cup \cdots \cup E^{n+k}$, for all $k \in \mathbb{N}$. We now define

$$\mathcal{M} := \bigcup_{n \in \mathbb{N}_0} E^n.$$

Semi-invariance, Pre-compactness and Finite-dimensionality

For all $k \in \mathbb{N}$ we obtain by using property $(E1')$

$$T(k)\mathcal{M} := \bigcup_{n \in \mathbb{N}_0} T(k)E^n \subset \bigcup_{n \in \mathbb{N}_0} (E^n \cup \cdots \cup E^{n+k}) \subset \bigcup_{n \in \mathbb{N}_0} E^n = \mathcal{M}.$$

Furthermore, as $E^n \subset T((m-1)\tilde{n})B$ for all $n \geq m$ we conclude

$$\mathcal{M} = \bigcup_{n=0}^{m} E^n \cup \bigcup_{n=m+1}^{\infty} E^n \subset \bigcup_{n=0}^{m} E^n \cup T(m\tilde{n})B.$$

Properties (E2') and (W3') now imply

$$\sharp(\bigcup_{n=0}^{m} E^n) \leq (m+1)\sharp E^m \leq (m+1)^2 \tilde{n} N^m$$

and $T(m\tilde{n})B \subset \bigcup_{u \in W^m} B^V_{(2(\nu+\lambda))^m R}(u)$. For arbitrary $\epsilon > 0$ we choose m sufficiently large such that

$$(2(\nu+\lambda))^m R \leq \epsilon < (2(\nu+\lambda))^{m-1} R$$

holds. An estimate for the number of ϵ-balls needed to cover $\mathcal{M}$ is then given by

$$N_\epsilon^V(\mathcal{M}) \leq \sharp(\bigcup_{n=0}^{m} E^n) + \sharp W^m \leq (m+1)^2 \tilde{n} N^m + \tilde{n} N^m \leq 2(m+1)^2 \tilde{n} N^m,$$

where we used properties (W2') and (E2'). This proves the pre-compactness of $\mathcal{M}$ in V, and the fractal dimension of $\mathcal{M}$ can be estimated as in the proof of Theorem 5.

We are left to show that $\mathcal{M}$ exponentially attracts all bounded subsets of V. If $D \subset V$ is bounded there exists $n_D \in \mathbb{N}$ such that $T(n)D \subset B$ for all $n \geq n_D$. If $n \geq n_D + \tilde{n}$, then $n = n_D + \tilde{n}k_0 + k$ for some $k_0 \in \mathbb{N}$, $k \in \mathbb{N}_0$ and

$$\begin{aligned} \mathrm{dist}_{\mathrm{H}}^V(T(n)D, \mathcal{M}) &= \mathrm{dist}_{\mathrm{H}}^V(T(k_0\tilde{n})T(n_D + k)D, \bigcup_{n=0}^{\infty} E^n) \leq \mathrm{dist}_{\mathrm{H}}^V(T(k_0\tilde{n})B, \bigcup_{n=0}^{\infty} E^n) \\ &\leq \mathrm{dist}_{\mathrm{H}}^V(T(k_0\tilde{n})B, W^{k_0}) \leq (2(\nu+\lambda))^{k_0} R = (2(\nu+\lambda))^{\frac{n-n_D-k}{\tilde{n}}} R = e^{-\omega n} c, \end{aligned}$$

for some constant $c \geq 0$ and all $n \geq n_D + \tilde{n}$, where $\omega := \ln(\frac{1}{2(\nu+\lambda)})^{\frac{1}{\tilde{n}}}$.

Compactness of the Exponential Attractor

As V is a Banach space and the semigroup is continuous, taking the closure $\hat{\mathcal{M}} := \overline{\mathcal{M}}^{\|\cdot\|_V}$ of the precompact set $\mathcal{M}$ we obtain a compact subset of V. The semi-invariance, finite fractal dimension and exponential attraction property of $\hat{\mathcal{M}}$ can be shown as in the non-autonomous case. □

4.2. **Time Continuous Semigroups.** We now construct pullback exponential attractors for time continuous semigroups. In addition to the hypothesis of the previous section we assume Lipschitz continuity of the semigroup.

$(S1)'$ There exists a bounded absorbing set $B \subset V$ for the semigroup $\{T(t)|\ t \geq 0\}$, that is for all bounded subsets $D \subset V$ there exists $T_D \geq 0$ such that

$$T(s)D \subset B \qquad \text{for all } s \geq T_D.$$

$(S2)'$ The family of operators $\{S(t)|\ t \geq 0\}$ satisfies within the absorbing set the smoothing property: There exists $\tilde{t} > 0$ such that

$$\|S(\tilde{t})u - S(\tilde{t})v\|_V \leq \kappa \|u - v\|_W \qquad \text{for all } u, v \in B,$$

where the constant $\kappa > 0$.

$(S3)'$ The family $\{C(t)|\ t \geq 0\}$ is a contraction within the absorbing set:

$$\|C(\tilde{t})u - C(\tilde{t})v\|_V \leq \lambda \|u - v\|_V \qquad \text{for all } u, v \in B,$$

where the constant $0 \leq \lambda < \frac{1}{2}$.

$(S4)'$ The semigroup $\{T(t)|\ t \geq 0\}$ is Lipschitz continuous in V within the absorbing set: There exists $s_0 \geq 0$ such that for all $t \geq s_0$

$$\|T(t)u - T(t)v\|_V \leq L_t\|u - v\|_V \qquad \text{for all } u, v \in B,$$

for some constant $L_t > 0$.

Remark 5. *In order to apply Theorem 4 directly we assume that the absorbing and contracting times coincide. However, if the family of contractions forms a semigroup in V we could generalize the hypothesis similarly as in the discrete case and replace $\tilde{t}$ in $(S4)'$ by an arbitrary contraction time $\hat{t} \neq \tilde{t}$. Moreover, it suffices that C is (eventually) a strict contraction with contraction constant $\lambda \in [0, 1)$ (cf. the remark in Section 3).*

Theorem 8. *If the conditions (H_0), $(S1)'$ - $(S4)'$ are satisfied, then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\mathcal{M} \equiv \mathcal{M}^\nu \equiv \{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ for the semigroup $\{T(t)|\ t \geq 0\}$, the sections are compact subsets of V for all $t \in \mathbb{R}$, and their fractal dimension can be estimated by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right).$$

Proof. As in the discrete case without loss of generality we can assume that the absorbing set is positively semi-invariant. Indeed, if necessary we replace B by the bounded absorbing set

$$\widetilde{B} := \bigcup_{s\in[0,T_B]} T(T_B + s)B.$$

The absorbing assumptions $(\mathcal{H}_1)$, (A_1) and (A_2) in Section 3.2 are certainly satisfied. Theorem 4 applied to the autonomous process $U(t, s) := T(t - s)$ yields the existence of a pullback exponential attractor for the semigroup $\{T(t)|\ t \geq 0\}$, which satisfies the properties in Definition 7 with $a = \tilde{t}$.

□

Corollary 3. *Under the assumptions of Theorem 8 the global attractor $\mathcal{A}$ of the semigroup $\{T(t)|\ t \geq 0\}$ is contained in any member of the pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\}$, and its fractal dimension is bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{A}) \leq \log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right).$$

Proof. This follows by Theorem 1.1, Chapter 1 in [13]. □

Like in the non-autonomous case, as a corollary we obtain the existence of exponential pullback attractors for semigroups that satisfy the smoothing property.

Corollary 4. *Let $\{S(t)|\ t \geq 0\}$ be a Lipschitz continuous semigroup in V and the assumptions (H_0) and $(S2)'$ be satisfied. Moreover, we assume that $(S1)'$ holds with $\{T(t)|\ t \geq 0\}$ replaced by $\{S(t)|\ t \geq 0\}$. Here, it suffices that the absorbing set is bounded in the metric of W. Then, for any $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\{\mathcal{M}(t)|\ t \in \mathbb{R}\} \equiv \{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$ for the semigroup $\{S(t)|\ t \geq 0\}$, and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}(t)) \leq \log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \qquad \text{for all } t \in \mathbb{R}.$$

Proof. This is a direct consequence of Corollary 2. □

We could also consider asymptotically compact semigroups in the weaker phase space W, and obtain the existence of exponential pullback attractors in W as studied in [2], and for discrete semigroups in [7].

Theorem 9. *Let $\{T(t)|\ t \geq 0\}$ be a semigroup in W and the assumptions (H_0) and $(S2)'$ be satisfied. Moreover, we assume that $(S1)'$ holds for a bounded absorbing set in W and properties $(S3)'$ and $(S4)'$ are satisfied with V replaced by W. Then, for any $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\} \equiv \{\mathcal{M}(t)|\ t \in \mathbb{R}\}$ for the semigroup $\{T(t)|\ t \geq 0\}$ in W, and the fractal dimension of its sections can be estimated by*

$$\dim_{\mathrm{f}}^W(\mathcal{M}(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \qquad \text{for all } t \in \mathbb{R}.$$

Proof. It follows from Theorem 6 applied to the autonomous process $U(t,s) := T(t-s)$, $t \geq s$. □

Remark 6.

- *The construction of exponential attractors for discrete semigroups in the proof of Theorem 7 slightly generalizes the construction in [8] for semigroups satisfying the smoothing property and for the case that $n_B = \tilde{n} = 1$.*
- *To construct the exponential attractor for time continuous semigroups $\{T(t)|\ t \geq 0\}$ generally the union*

$$\widetilde{\mathcal{M}} := \bigcup_{t \in [\tilde{t}, 2\tilde{t}]} T(t)\mathcal{M}_d,$$

is taken, where $\mathcal{M}_d$ denotes the exponential attractor for the associated discrete semigroup $\{T(n\tilde{t})|\ n \in \mathbb{N}\}$ (cf. [2], [5]). If the semigroup is Hölder continuous in time,

$$\|T(t_1)u - T(t_2)u\|_V \leq c|t_1 - t_2|^\theta \qquad \text{for all } \ t_1, t_2 \in [\tilde{t}, 2\tilde{t}], u \in B$$

for some constant $c \geq 0$ and $0 < \theta \leq 1$, the exponential attractor $\widetilde{\mathcal{M}}$ is finite dimensional, and a bound for its fractal dimension is given by

$$\dim_{\mathrm{f}}^V(\widetilde{\mathcal{M}}) \leq \frac{1}{\theta} + \dim_{\mathrm{f}}^V(\mathcal{M}_d).$$

This is a stronger estimate than in Corollary 2.6 in [5]*, but can easily be deduced from the construction. The bounds on the fractal dimension of the attractors in* [2] *are weaker.*

Certainly, any member of the pullback exponential attractor for time continuous semigroups constructed in Theorem 8 is contained in the exponential attractor, $\mathcal{M}(t) \subset \widetilde{\mathcal{M}}$ *for all* $t \in \mathbb{R}$*.*

- *The exponential pullback attractor constructed in* [5] *and* [10] *(cf. Section 2) applied to non-autonomous processes yields an exponential attractor in the original sense, that is it satisfies all properties in Definition 1.*

5. Concluding Remarks

We constructed pullback exponential attractors for asymptotically compact evolution processes in Banach spaces assuming that the process possesses a family of time-dependent pullback absorbing sets that possibly grow in the past. In a forthcoming paper we discuss properties and consequences of the construction and apply the theoretical results to prove the existence of pullback exponential attractors in two applications. In both cases, previous results are not applicable and the generalizations we developed in this article are required.

First, we consider a non-autonomous Chafee-Infante initial boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$,

$$\begin{aligned} u_t(x,t) &= \triangle u(x,t) + \lambda u(x,t) - \beta(t)\big(u(x,t)\big)^3 && x \in \Omega,\ t > s, \\ u(x,t) &= 0 && x \in \partial\Omega,\ t \geq s, \\ u(x,s) &= u^s(x) && x \in \Omega, \end{aligned}$$

where $s \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and the initial data $u^s \in C_0(\overline{\Omega})$. The non-autonomous term $\beta : \mathbb{R} \to \mathbb{R}_+$ is strictly positive, continuously differentiable, bounded when time t tends to ∞ and vanishes when t goes to $-\infty$. We show that the generated evolution process satisfies the smoothing property and possesses a semi-invariant family of pullback absorbing sets. The diameter of the absorbing sets grows in the past since the non-autonomous term β vanishes when t tends to $-\infty$. Our results yield the existence of a pullback exponential attractor, which in turn implies the existence of the global pullback attractor for the generated evolution process. In particular, we obtain an example for an unbounded pullback attractor of finite fractal dimension, which has been an open problem (see the introduction in [10] and Remark 3.2 in [11]).

Another application is the non-autonomous dissipative wave equation

$$
\begin{aligned}
&u_{tt}(x,t) + \beta(t)u_t(x,t) = \Delta u(x,t) + f(u(x,t)) && x \in \Omega,\ t > s,\\
&u(x,t) = 0 && x \in \partial\Omega,\ t \geq s,\\
&u(x,s) = u^s(x) && x \in \Omega,\\
&u_t(x,s) = v^s(x) && x \in \Omega,
\end{aligned}
$$

where $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 3$, is a bounded domain. We assume that the non-linearity $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and is of sub-critical growth. Furthermore, the function $\beta : \mathbb{R} \to \mathbb{R}_+$ is Hölder continuous and bounded from above and below by positive constants.

The initial value problem generates an asymptotically compact evolution process U in the space $V := H_0^1(\Omega) \times L^2(\Omega)$. We prove that the evolution process can be represented as sum $U = S + C$, where the family of operators S satisfies the smoothing property with respect to V and an auxiliary normed space W such that $V \hookrightarrow\hookrightarrow W$, and C is a family of contractions in the stronger space V. Our main result implies the existence of a pullback exponential attractor. This setting was not considered in previous constructions of exponential attractors, where it was always assumed that the family C is a contraction in the weaker space W (among others see [7], [8] and [9]). Moreover, former existence results for pullback exponential attractors require the Hölder continuity in time of the evolution process (cf. [5] and [10]).

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