Optimized Overlapping Domain Decomposition: Convergence Proofs

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1 Introduction

During the last two decades many domain decomposition algorithms have been constructed and lot of techniques have been developed to prove the convergence of the algorithms at the continuous level. Among the techniques used to prove the convergence of classical Schwarz algorithms, the first technique is the maximum principle used by Schwarz. Adopting this technique M. Gander and H. Zhao proved a convergence result for n-dimensional linear heat equation in Gander and Zhao [2002]. The second technique is that of the orthogonal projections, used by P. L. Lions in Lions [1988], and his convergence results are for linear Laplace equation and linear Stokes equation. In the same paper, P. L. Lions also proved that the Schwarz sequences for linear elliptic equations are related to classical minimization methods over product spaces and this technique was then used by L. Badea in Badea [1991] for nonlinear monotone elliptic problems. Another technique is the Fourier and Laplace transforms used in the papers Giladi and Keller [2002], Gander and Stuart [1998] for some 1-dimensional evolution equations, with constant coefficients. In Lui [2002], Lui [2001], S. H. Lui used the idea of upper-lower solutions methods to study the convergence problem for some PDEs, with initial guess to be an upper or lower solution of the equations and monotone iterations.

For nonoverlapping optimized Schwarz methods, P. L. Lions in Lions [1989] proposed to use an energy estimate argument to study the convergence of the algorithm. The energy estimate technique was then developed in Benamou and Desprès [1997] for Helmholtz equation and it has then become a very powerful tool to study nonoverlapping problems. J.-H. Kimn in Kimn [2005] proved the convergence of an overlapping optimized Schwarz method for Poisson's equation with Robin boundary data and S. Loisel and D. B.

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Szyld in Loisel and Szyld [2010] extended the technique of J.-H. Kimn to linear symmetric elliptic equation. Another technique is to use semiclassical analysis, which works for overlapping optimized Schwarz methods with rectangle subdomains, linear advection diffusion equations on the half plane (see Nataf and Nier [1998]).

This paper is devoted to the study of the convergence of Schwarz methods at the continuous level. We give a sketch of the proof of the convergence of optimized Schwarz methods for semilinear parabolic equations, with multiple subdomains. Complete convergence proofs for both classical and optimized Schwarz methods, both semilinear parabolic and elliptic equations, with multiple subdomains could be found in Tran [2012].

2 Convergence for Semilinear Parabolic Equations

Consider the following parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}(x,t) \\ \qquad\qquad + c(x)u(x,t) = F(x,t,u(x,t)), \text{ in } \Omega \times (0,\infty), \\ u(x,t) = g(x,t), \text{ on } \partial\Omega \times (0,\infty), \\ u(x,0) = g(x,0), \text{ on } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded and smooth enough domain in $\mathbb{R}^n$. The following conditions are imposed on (1).
(A1) For all i,j in $\{1,\dots,I\}$, $a_{i,j}(x) = a_{j,i}(x)$. There exist strictly positive numbers λ, Λ such that $A = (a_{i,j}(x)) \geq \lambda I$ in the sense of symmetric positive definite matrices and $a_{i,j}(x) < \Lambda$ in Ω.
(A2) The functions $a_{i,j}$, b_i, c are in $C^\infty(\mathbb{R}^n)$ and g is in $C^\infty(\mathbb{R}^{n+1})$.
(A3) There exists $C > 0$, such that
$\forall\ t \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^n$, $|F(x,t,z) - F(x,t,z')| \leq C|z - z'|$, $\forall\ z,\ z' \in \mathbb{R}$.

We now describe the way that we decompose the domain Ω: The domain Ω is divided into I smooth overlapping subdomains $\{\Omega_l\}_{l\in\{1,I\}}$:

$$(\partial\Omega_l \backslash \partial\Omega) \cap (\partial\Omega_{l'} \backslash \partial\Omega) = \emptyset, \quad \forall \quad l, l' \in \{1,\dots,I\}, \quad l \neq l';$$

$$\forall l \in \{1,\dots,I\}, \forall l', l'' \in J_l, l'' \neq l', \quad \Omega_{l'} \cap \Omega_{l''} = \emptyset,$$

where

$$J_l = \{l' | \Omega_{l'} \cap \Omega_l \neq \emptyset\};$$

$$\cup_{l=1}^n \Omega_l = \Omega.$$

This decomposition means that we do not consider cross-points in this paper. Denote by $\Gamma_{l,l'}$, for $l' \in J_l$, the set $(\partial\Omega_l \backslash \partial\Omega) \cap \overline{\Omega}_{l'}$. The transmission operator

$\mathfrak{B}_{l,l'}$ is of Robin type $\mathfrak{B}_{l,l'}v = \sum_{i,j=1}^{n} a_{i,j}\frac{\partial v}{\partial x_i} n_{l,l',j} + p_{l,l'}v$ and $n_{l,l',j}$ is the j-th component of the outward unit normal vector of $\Gamma_{l,l'}$; $p_{l,l'}$ is positive and belongs to $L^\infty(\Gamma_{l,l'})$. The iterate $\#k$ in the l-th domain, denoted by u_l^k of the Schwarz waveform relaxation algorithm is defined by:

$$\begin{cases} \frac{\partial u_l^k}{\partial t} - \sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2 u_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u_l^k}{\partial x_i} + c u_l^k = F(t,x,u_l^k), \text{ in } \Omega_l \times (0,\infty), \\ \\ \mathfrak{B}_{l,l'} u_l^k = \mathfrak{B}_{l,l'} u_{l'}^{k-1}, \text{ on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l, \end{cases} \tag{2}$$

where

$$u_l^k(x,t) = g(x,t) \text{ on } (\partial\Omega_l \cap \partial\Omega) \times (0,\infty), \quad u_l^k(x,0) = g(x,0) \text{ in } \Omega_l.$$

The initial guess u^0 is bounded in $C^\infty(\overline{\Omega \times (0,\infty)})$; and at step 0, the Equations (2) is solved with boundary data

$$\mathfrak{B}_{l,l'} u_l^1(x,t) = u^0(x,t) \text{ on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l.$$

A compatibility condition on $u^0(x,t)$ is also assumed

$$\mathfrak{B}_{l,l'} g(x,0) = u^0(x,0) \text{ on } \Gamma_{l,l'}, \forall l' \in J_l.$$

By an induction argument, the algorithm is well-posed. Let e_l^k be $u_l^k - u$

$$\begin{cases} \frac{\partial e_l^k}{\partial t} - \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2 e_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial e_l^k}{\partial x_i} \\ \qquad + c(x) e_l^k = F(t,x,u_l^k) - F(t,x,u), \text{ in } \Omega_l \times (0,\infty), \\ \mathfrak{B}_{l,l'} e_l^k(x,t) = \mathfrak{B}_{l,l'} e_{l'}^{k-1}(x,t), \qquad \text{on } \Gamma_{l,l'} \times (0,\infty), \forall l' \in J_l. \end{cases} \tag{3}$$

Moreover,

$$e_l^k(x,t) = 0 \text{ on } (\partial\Omega_l \cap \partial\Omega) \times (0,\infty), \quad e_l^k(x,0) = 0 \text{ in } \Omega_l.$$

For any function f in $L^2(0,\infty)$, define

$$\int_0^\infty f(x)\exp(-yx)dx.$$

For any fixed positive number α, define

$$|f|_\alpha = \sup_{\alpha' > \alpha} \left[\int_{\alpha'}^{\alpha'+1} \left(\int_0^\infty f(x)\exp(-yx)dx \right)^2 dy \right]^{\frac{1}{2}},$$

and

$$\mathbb{L}_\alpha^2(0,\infty) = \{ f \ : \ f \in L^2(0,\infty), |f|_\alpha < \infty \}.$$

Thus $(\mathbb{L}^2_\alpha(0,\infty), |.|_\alpha)$ is a normed subspace of $L^2(0,\infty)$.

Theorem 1. *Consider the Schwarz algorithm with Robin transmission conditions and the initial guess u^0 in $C_c^\infty(\overline{\Omega\times(0,\infty)})$. There exists a constant α large enough such that*

$$\lim_{k\to\infty}\sum_{l=1}^{I}\int_{\Omega_l}|e_l^k|_\alpha^2 dx = 0.$$

Proof. Let g_l be a function bounded and greater than 1 in $C^\infty(\mathbb{R}^n,\mathbb{R})$, α be a positive constant, we define

$$\Phi_l^k(x) := \left(\int_0^\infty e_l^k \exp(-\alpha t)dt\right) g_l(x),$$

then $\Phi_l^k(x)$ belongs to $H^1(\Omega_l)$.

Let B_i^l and C^l be functions in $L^\infty(\mathbb{R}^n)$ defined by

$$B_i^l := b_i + \sum_{j=1}^n \left(a_{i,j}\frac{\partial_j g_l}{g_l}\right),$$

$$C^l = \left[\frac{\alpha}{2} + \sum_{i,j=1}^n\left(-a_{i,j}\frac{2\partial_i g_l \partial_j g_l}{(g_l)^2} - \partial_j a_{i,j}\frac{\partial_i g}{g} + a_{i,j}\frac{\partial_{i,j}g_l}{g_l}\right) - \sum_{i=1}^n b_i\frac{\partial_i g_l}{g_l}\right].$$

Define

$$\begin{aligned}\mathfrak{L}_{lR}(\Phi_l^k) = &-\sum_{i,j=1}^n \partial_j(a_{i,j}\partial_i\Phi_l^k) + \sum_{i=1}^n B_i^l\partial_i\Phi_l^k + C^l\Phi_l^k\\ &+\left\{\int_0^\infty \left[\left(\frac{\alpha}{2}+c\right)e_l^k - F(u_l^k) + F(u)\right]\exp(-\alpha t)dt\right\}g_l.\end{aligned}$$

It is possible to suppose α to be large such that C^l belongs to $(\frac{\alpha}{4},\alpha)$.

Lemma 1. *Choose g_l, $g_{l'}$ such that $\nabla g_l = \nabla g_{l'} = 0$ on $\Gamma_{l,l'}$ and $\frac{g_{l'}}{g_l} > 1$ on $\Gamma_{l,l'}$, for all l' in J_l. Φ_l^k is then a solution of the following equation*

$$\begin{cases}\mathfrak{L}_{lR}(\Phi_l^k) = 0, & \text{in } \Omega_l\times(0,\infty),\\ \beta_l\mathfrak{B}_{l,l'}(\Phi_l^k) = \mathfrak{B}_{l,l'}(\Phi_{l'}^{k-1}) & \text{on } \Gamma_{l,l'}\times(0,\infty), \forall l'\in J_l.\end{cases} \tag{4}$$

where $\beta_l = \frac{g_{l'}}{g_l}$ on $\Gamma_{l,l'}$, for all l' in J_l.

For all l in $\{1,I\}$, denote by $\tilde{\Omega}_l$ the open set $\Omega_l\backslash\overline{\cup_{l'\in J_l}\Omega_{l'}}$. For all l in I such that $\varphi_l^{k+1} = \varphi_{l'}^k$ on $\Gamma_{l,l'}$ for all l' in J_l, let φ_l^k and φ_l^{k+1} be functions in $H^1(\tilde{\Omega}_l)$ and $H^1(\Omega_l)$. Use the test functions φ_l^{k+1} and φ_l^k, and take the sum (with respect to l in $\{1,I\}$) of $\int_{\tilde{\Omega}_l}\mathfrak{L}_{lR}(\Phi_l^{k+1})\varphi_l^{k+1}$ and $\int_{\tilde{\Omega}_l}\mathfrak{L}_{lR}(\Phi_l^k)\varphi_l^k$ to get

$$
\begin{aligned}
&-\sum_{l=1}^{I}\Big\{\int_{\tilde\Omega_l} C^l\Phi_l^k\varphi_l^k dx+\\
&+\int_{\tilde\Omega_l}\sum_{i,j=1}^{n} a_{i,j}\partial_i\Phi_l^k\partial_j\varphi_l^k dx+\sum_{i=1}^{n}\int_{\tilde\Omega_l} B_i^l\partial_i\Phi_l^k\varphi_l^k dx-\sum_{l'\in J_l}\int_{\Gamma_{l',l}} p_{l',l}\Phi_l^k\varphi_l^k d\sigma\\
&+\int_{\tilde\Omega_l}\Big\{\int_0^\infty\Big[\Big(\frac{\alpha}{2}+c\Big)e_l^k-F(u_l^k)+F(u)\Big]\exp(-\alpha t)dt\Big\}g_l\varphi_l^k dx\Big\} \qquad (5)\\
=&\sum_{l=1}^{I}\beta_l\Big\{\int_{\Omega_l} C^l\Phi_l^{k+1}\varphi_l^{k+1}dx+\\
&+\int_{\Omega_l}\sum_{i,j=1}^{n} a_{i,j}\partial_i\Phi_l^{k+1}\partial_j\varphi_l^{k+1}dx+\sum_{l'\in J_l}\int_{\Gamma_{l,l'}} p_{l,l'}\Phi_l^{k+1}\varphi_l^{k+1}d\sigma\\
&+\int_{\Omega_l}\sum_{i=1}^{n} B_i^l\partial_i\Phi_l^{k+1}\varphi_l^{k+1}dx+\\
&+\int_{\Omega_l}\Big\{\int_0^\infty\Big[\Big(\frac{\alpha}{2}+c\Big)e_l^{k+1}-F(u_l^{k+1})+F(u)\Big]\exp(-\alpha t)dt\Big\}g_l\varphi_l^{k+1}dx\Big\}.
\end{aligned}
$$

In (5), choose φ_l^{k+1} to be Φ_l^{k+1}, then there exists φ_l^k, such that for all l' in J_l $\varphi_l^k=\varphi_{l'}^{k+1}$ on $\Gamma_{l,l'}$ and

$$
||\varphi_l^k||_{H^1(\Omega_l)}\le C\sum_{l'\in J_l}||\varphi_{l'}^{k+1}||_{H^1(\Omega_{l'})} \text{ and } ||\varphi_l^k||_{L^2(\Omega_l)}\le C\sum_{l'\in J_l}||\varphi_{l'}^{k+1}||_{L^2(\Omega_{l'})},
$$

where C is a positive constant.
The right hand side of (5) is then greater than or equal to

$$
\begin{aligned}
&\sum_{l=1}^{I}\beta_l\Big\{\int_{\Omega_l}\lambda|\nabla\Phi_l^{k+1}|^2dx-\sum_{i=1}^{n}\int_{\Omega_l}||B_i^l||_{L^\infty(\Omega_l)}\left|\partial_i\Phi_l^{k+1}\right||\Phi_l^{k+1}|dx\Big\}.\\
\ge&\sum_{l=1}^{I}\beta_l\Big\{\int_{\Omega_l}\frac{\lambda}{2}|\nabla\Phi_l^{k+1}|^2dx+\frac{\alpha}{8}\int_{\Omega_l}|\Phi_l^{k+1}|^2\Big\}. \qquad (6)
\end{aligned}
$$

Similarly, the left hand side of (5) is less than or equal to

$$\begin{aligned}
&\sum_{l=1}^{I}\left\{\int_{\tilde{\Omega}_l} \Lambda|\nabla\Phi_l^k||\nabla\varphi_l^k|dx+\sum_{i=1}^{n}\int_{\tilde{\Omega}_l}||B_i^l||_{L^\infty(\tilde{\Omega}_l)}\left|\partial_i\Phi_l^k\right||\varphi_l^k|dx\right.\\
&\quad\left.+\sum_{l'\in J_l}||p_{l',l}||_{L^\infty(\Gamma_{l',l})}(||\Phi_l^k||^2_{H^1(\tilde{\Omega}_l)}+||\varphi_l^k||^2_{H^1(\tilde{\Omega}_l)})\right\}\\
&\leq\sum_{l=1}^{I}M_1\left\{\frac{1}{2}(||\nabla\Phi_l^k||^2_{L^2(\tilde{\Omega}_l)}+(\max_{i\in\{1,I\}}||B_i^l||_{L^\infty(\tilde{\Omega}_l)})^2||\varphi_l^k||^2_{L^2(\tilde{\Omega}_l)})\right.\\
&\quad+\int_{\tilde{\Omega}_l}2\alpha|\Phi_l^k||\varphi_l^k|dx+\sum_{l'\in J_l}\int_{\Gamma_{l',l}}p_{l',l}|\Phi_l^k||\varphi_l^k|d\sigma \qquad (7)\\
&\quad\left.+\Lambda\left(||\nabla\Phi_l^k||^2_{L^2(\tilde{\Omega}_l)}+||\nabla\varphi_l^k||^2_{L^2(\tilde{\Omega}_l)}\right)+\frac{\alpha}{2}||\Phi_l^k||^2_{L^2(\tilde{\Omega}_l)}+\frac{\alpha}{2}||\varphi_l^k||^2_{L^2(\tilde{\Omega}_l)}\right\},
\end{aligned}$$

where M_1 depends only on $\{\Omega_l\}_{l\in\{1,I\}}$ and the equation (3). Choose α such that $\alpha > (\max_{i\in\{1,I\}}||B_i^l||_{L^\infty(\tilde{\Omega}_l)})^2$, there exists M_2 positive, depending only on $\{\Omega_l\}_{l\in\{1,I\}}$ and (3) such that the right hand side of (7) is dominated by

$$\begin{aligned}
&\sum_{l=1}^{I}M_2\left\{\int_{\tilde{\Omega}_l}\left(\frac{\lambda}{2}|\nabla\Phi_l^k|^2dx+\frac{\alpha}{8}|\Phi_l^k|^2+\frac{\lambda}{2}|\nabla\Phi_l^{k+1}|^2+\frac{\alpha}{8}|\Phi_l^{k+1}|^2\right)dx\right\} \qquad (8)\\
&\leq\sum_{l=1}^{I}M_2\left(\frac{\lambda}{2}||\nabla\Phi_l^k||^2_{L^2(\Omega_l)}+\frac{\alpha}{8}||\Phi_l^k||^2_{L^2(\Omega_l)}+\frac{\lambda}{2}||\nabla\Phi_l^{k+1}||^2_{L^2(\Omega_l)}+\frac{\alpha}{8}||\Phi_l^{k+1}||^2_{L^2(\Omega_l)}\right).
\end{aligned}$$

Define

$$E_k:=\sum_{l=1}^{I}\left(\frac{\lambda}{2}||\nabla\Phi_l^k||^2_{L^2(\Omega_l)}+\frac{\alpha}{8}||\Phi_l^k||^2_{L^2(\Omega_l)}\right), \qquad (9)$$

then (6), (7) and (8) imply

$$(\beta-M_2)E_{k+1}\leq M_2E_k, \qquad (10)$$

where $\beta=\min\{\beta_1,\dots,\beta_I\}$.
Since M_2 depends only on $\{\Omega_l\}_{l\in\{1,I\}}$ and (3), β can be chosen such that

$$M_3:=\frac{M_2}{\beta-M_2}<1.$$

We get

$$\begin{aligned}
E_k&\leq M_3^kE_0\\
&\leq M_3^k\sum_{l=1}^{I}\left(\frac{\lambda}{2}||\nabla\Phi_l^0||^2_{L^2(\Omega_l)}+\frac{\alpha}{8}||\Phi_l^0||^2_{L^2(\Omega_l)}\right).
\end{aligned}$$

That deduces

$$||\Phi_l^k||^2_{L^2(\Omega_l)} \leq M_3^k \sum_{l=1}^{I} \left(\frac{4\lambda}{\alpha} ||\nabla \Phi_l^0||^2_{L^2(\Omega_l)} + ||\Phi_l^0||^2_{L^2(\Omega_l)} \right). \tag{11}$$

Since (11) still holds if M_3 and λ are fixed, and α is replaced by $y > \alpha$, then

$$\begin{aligned} &\sum_{l=1}^{I} \int_{\Omega_l} \left(\int_0^\infty e_l^k \exp(-yt) dt g_l \right)^2 dx \\ &\leq M_3^k \left[\frac{4\lambda}{y} \sum_{l=1}^{I} \int_{\Omega_l} \left(\int_0^\infty |\nabla e_l^0| \exp(-yt) dt \right)^2 g_l^2 dx \right. \\ &\quad + \frac{4\lambda}{y} \sum_{l=1}^{I} \int_{\Omega_l} \left(\int_0^\infty e_l^0 \exp(-yt) dt \right)^2 |\nabla g_l|^2 dx \\ &\quad \left. + \sum_{l=1}^{I} \int_{\Omega_l} \left(\int_0^\infty e_l^0 \exp(-yt) dt \right)^2 g_l^2 dx \right]. \end{aligned} \tag{12}$$

Let α' be a constant larger than or equal to α, (12) implies

$$\begin{aligned} &\sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left(\int_0^\infty e_l^k \exp(-yt) dt \right)^2 g_l^2 dy dx \\ &\leq M_3^k \left[\sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \frac{4\lambda}{y} \left(\int_0^\infty |\nabla e_l^0| \exp(-yt) dt \right)^2 g_l^2 dy dx \right. \\ &\quad + \sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \frac{4\lambda}{y} \left(\int_0^\infty e_l^0 \exp(-yt) dt \right)^2 |\nabla g_l|^2 dy dx \\ &\quad \left. + \sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left(\int_0^\infty e_l^0 \exp(-yt) dt \right)^2 g_l^2 dy dx \right]. \end{aligned} \tag{13}$$

Since u^0 belongs to $C_c^\infty(\overline{\Omega \times (0,\infty)})$, the right hand side of (13) is bounded by a constant $M_3^k M_4(\alpha)$. The fact that g_l is greater than 1 implies

$$\sum_{l=1}^{I} \int_{\Omega_l} \int_{\alpha'}^{\alpha'+1} \left(\int_0^\infty e_l^k \exp(-yt) dt \right)^2 dy dx \leq M_3^k M_4(\alpha). \tag{14}$$

Inequality (14) deduces

$$\lim_{k \to \infty} \sum_{l=1}^{I} \int_{\Omega_l} |e_l^k|^2_\alpha dx = 0. \tag{15}$$

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