Mathematical models of incompressible fluids as singular limits of complete fluid systems

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Abstract

A rigorous justification of several well-known mathematical models of incompressible fluid flows can be given in terms of singular limits of the scaled Navier-Stokes-Fourier system, where some of the characteristic numbers become small or large enough. We discuss the problem in the framework of global-in-time solutions for both the primitive and the target system.

Key words:
Scale analysis, Navier-Stokes-Fourier system, incompressible limit

Contents

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1 Introduction

Fluid dynamics is a part of fluid mechanics that deals with fluid flows. Its applicability includes prediction of weather patterns in meteorology, numerous engineering problems involving fluids in motion, understanding complicated dynamics of gaseous stars and the interstellar space in astrophysics, and even certain traffic problems as soon the traffic can be viewed as a continuous fluid, to name only a few. While most of these applications concern compressible or at least slightly compressible fluids, the prevailing part of theoretical studies is devoted to mathematical models of idealized *incompressible fluids*. The classical *Navier-Stokes system* of equations describing the motion of a viscous incompressible fluid has become one of the prototype examples of a simple nonlinear problem in the theory of partial differential equations, the complete solution of which is still highly open despite a concerted and long lasting effort of generations of excellent mathematicians, see the survey paper of Fefferman [38].

Simplified mathematical models, like those based on the concept of incompressibility, may be viewed, and in many cases can be formally derived, as limits of more complex systems, where certain characteristic parameters become negligible or predominant. In a series of seminal papers, Ebin [32] and Klainerman, Majda [57], [58] set forth a rigorous basis of a mathematical theory of singular limits, in particular in the so-called low-Mach-number regime, where the speed of sound in a compressible medium dominates the characteristic speed of the fluid and the latter is driven to incompressibility. More recently, Klein et al. [59], [60], [61], [62], exploiting the same idea, proposed several numerical methods for solving complex problems in fluid dynamics in the singular limit regimes. Besides the evident benefit of solving more efficiently and with less numerical effort a given problem, the ultimate goal of these studies was to shed some light on the complex behavior of solutions to the original system in the singular regime. At a purely theoretical level, Hagstrom and Lorentz [52] established existence of global-in-time classical solutions to the compressible Navier-Stokes system in a two dimensional physical space provided the Mach number is small enough. Similar ideas were exploited by Sideris and Thomases [96] in the context of nonlinear elastodynamics.

The standard *primitive equations* in mathematical fluid dynamics reflect the basic physical principles: conservation of mass, momentum, and energy, together with general thermodynamic relations between the corresponding macroscopic state variables. In order to reveal characteristic properties of a specific fluid flow, the system must be written in terms of dimensionless variables scaled by a suitable system of characteristic units. In such a way, a system of equations determined to describe the mass flow rate of petroleum in a pipe line is clearly distinguished from its counterpart designed to model the evolution of a gaseous star. Typical time, length and other scales may differ drastically in problems of weather forecasting from those in astrophysics. A spectacular example of interaction of different scales are the gigantic eruptions observed on the Sun, where, however, the medium viscosity of the fluid is the same as that of honey - a material for which any kind of turbulent behavior under *normal conditions* is hardly expected.

Following a simple scheme

$$\boxed{\text{primitive system}} \longrightarrow \text{(singular limit)} \longrightarrow \boxed{\text{target system}}$$

we discuss several rather theoretical aspects of problems in continuum fluid mechanics arising as physically grounded singular limits of complete, meaning *energetically closed*, fluid systems. In particular, we focus on the following basic issues:

- *Solvability* of the primitive system in the framework of physically admissible data, existence of solutions on a given, possibly large, time interval.
- *Stability* of given families of solutions to the primitive system with respect to the singular parameters, pre-compactness in suitable topologies.
- *Convergence* toward the target system in the process of singular limit, identification of possible sources of instabilities, with the perspective of future implementation of numerical methods.

As the target problem in our analysis typically includes the *Navier-Stokes* system describing the motion of an incompressible fluid in the physically relevant three-dimensional space, any rigorous mathematical theory in necessarily based on the concept of *weak solution* introduced in the pioneering work of Leray [67], and later developed by Hopf [54], Ladyzhenskaya [65], among many others. Indeed strong or classical solutions to this system are known to exist only on possibly short time intervals, or for specific classes of (small) initial data. Here "small" is to be understood close to a (known) regular solution. Although regular solutions exist for a generic class of initial data, and the set of (hypothetical) singularities is very "small" (see Caffarelli, Kohn, and Nirenberg [18]), we have to rely on the weak solutions as long as our analysis includes large data and solutions defined globally in time. Accordingly, solutions of the primitive problem are considered in the same framework. A recent analogy of Leray's theory for the compressible barotropic fluids was developed by P.-L.Lions [71], and later extended to physically more realistic state equations in [39]. Needless to say we can hardly expect to get better results for the apparently more complicated primitive system in the case of a complete fluid, where the balance of the total energy must be taken into account. Although the limit process is confined to relatively weak topologies, we can still clearly identify the principal difficulties that are essentially independent of the regularity properties of the solutions, namely the presence of rapidly oscillating *acoustic waves*.

To conclude this introduction, we point out that an alternative approach to singular limits, leaning on the concept of classical solutions to primitive systems defined on possibly short time intervals, has been developed for both viscous and inviscid fluids. We refer the reader to the papers of Alazard [1], [2], [3], Danchin [23], [24], Gallagher [44], Hoff [53], Métivier and Schochet [83], Schochet [92], [93], or the monograph by Chemin et al. [21], to name only a few, for various aspects of this method.

2 Energetically closed fluid systems

In this study, we concentrate on *energetically closed* fluid systems, that means, the energy is neither supplied from the outer world nor lost within the system, and the physical boundary of the fluid is energetically insulated. In accordance with *First law of thermodynamics*, the total energy of such a system is constant in time. Although we are primarily concerned with *continuum* fluid mechanics, the basic state variables - the fluid mass density and internal energy as well as the basic thermodynamic relations - are better understood in terms of statistical mechanics, see Gallavotti [45].

2.1 Thermostatic state variables, entropy

We assume that the state of a fluid at rest is fully determined by two fundamental quantities: the mass *density* ϱ, and the *internal energy* e. Moreover, *Second law of thermodynamics* postulates the existence of another state variable - the *entropy* s. The entropy is an increasing function of the internal energy, the relation

$$\frac{\partial s}{\partial e} = \frac{1}{\vartheta} > 0$$

defines the *absolute temperature* ϑ. According to Callen [19], the entropy enjoys the following remarkable properties:

- *Third law of thermodynamics.* The entropy tends to zero provided $\vartheta \to 0$, cf. Belgiorno [12], [13].
- The entropy production rate is non-negative in any admissible physical process.
- Equilibrium states of the system minimize the entropy production.
- Equilibrium states maximize the total entropy among all admissible states with the same mass and energy.

Another thermodynamic quantity considered in this study is the pressure p. The functions e, s, and p are interrelated through *Gibbs' equation*

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \tag{2.1}$$

see Callen [19]. In what follows, the *thermodynamic functions* $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$, $p = p(\varrho, \vartheta)$ are considered as given functions of the density ϱ and the absolute temperature ϑ.

The specific relations $(\varrho, \vartheta) \mapsto p(\varrho, \vartheta)$, $(\varrho, \vartheta) \mapsto e(\varrho, \vartheta)$ are termed *equations of state*, see Eliezer, Ghatak, and Hora [34]. A simple and illustrative example is the equation of state of a general monoatomic gas, where

$$p = \frac{2}{3}\varrho e, \tag{2.2}$$

see [34]. Combining (2.1), (2.2) we easily deduce that

$$p = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \text{ for a certain function } P. \tag{2.3}$$

The associated entropy reads

$$s = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \text{ where } S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2}.$$

The simplest and most natural choice is to take $P(Z) = aZ$, $a > 0$, for which (2.3) reduces to the standard Boyle-Marriot law of a perfect gas. However, it is interesting to note that this assumption violates *Third law of thermodynamics* as $s \to -\infty$ for $\vartheta \to 0$. We may conclude that P in (2.3) cannot be linear at least for *real gases* in a so-called degenerate regime $\varrho/\vartheta^{3/2} >> 1$.

2.2 Thermodynamic stability, equilibrium states

The real fluids are rarely found in equilibrium. In accordance with the previous discussion, we still assume that the instantaneous state of a fluid can be described by the thermostatic variables ϱ and ϑ that are now measurable functions of the space variable x in the underlying physical space $\Omega \subset \mathbb{R}^3$. We also assume that the thermodynamic functions p, e, and s can be used to describe the system *out of equilibrium.* We introduce the *total mass*

$$M = \int_\Omega \varrho \, \mathrm{d}x$$

and the total energy

$$E = \int_\Omega \varrho e(\varrho, \vartheta) - \varrho F \, \mathrm{d}x,$$

where F represent a potential of a conservative force acting on the fluid.

Accordingly, the equilibrium state is characterized by a constant temperature $\overline{\vartheta} > 0$ and a static density distribution $\tilde{\varrho}$ satisfying

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F \text{ on } \Omega. \tag{2.4}$$

Our goal is to show that the equilibrium state is *uniquely* determined by the total mass M and the total energy E. We exploit the general principle that any equilibrium state maximizes the total entropy

$$S = \int_\Omega \varrho s(\varrho, \vartheta) \, \mathrm{d}x$$

among all states with the same M and E. To see this, we introduce *hypothesis of thermodynamic stability*:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0, \tag{2.5}$$

cf. Bechtel, Rooney and Forest [11]. The former condition in (2.5) says that *compressibility* of the fluid is always positive, while the latter means that *specific heat at constant volume* is positive.

Next, we introduce a quantity called *Helmholtz function*,

$$H(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta), \tag{2.6}$$

reminiscent of the Helmholtz free energy $\varrho e - \vartheta \varrho s$. Since e and s satisfy Gibbs' relation (2.1), we compute that

$$\frac{\partial^2 H(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho}$$

and

$$\frac{\partial H(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta} (\vartheta - \overline{\vartheta}) \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}.$$

In view of hypothesis of thermodynamic stability (2.6), we have

- $\varrho \mapsto H(\varrho, \overline{\vartheta})$ is strictly convex,
- $\vartheta \mapsto H(\varrho, \vartheta)$ is decreasing for $\vartheta < \overline{\vartheta}$ and increasing for $\vartheta > \overline{\vartheta}$ for any fixed ϱ.

The next observation is that the static density distribution $\tilde{\varrho}$ satisfies

$$\frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} = F + \text{const in } \Omega$$

provided $\tilde{\varrho}$ is *strictly positive*. We point out that such a conclusion *may be false* if $\tilde{\varrho}$ vanishes on some part of Ω. On the other hand, strict positivity of $\tilde{\varrho}$ is guaranteed as soon as $\partial_\varrho p(0, \overline{\vartheta}) > 0$, see [42] for more information on the structure of static solutions.

By virtue of the previous arguments, the function

$$(\varrho, \vartheta) \mapsto H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta})$$

attains its strict global minimum (zero) at $\tilde{\varrho}$, $\overline{\vartheta}$. On the other hand, for all functions ϱ, ϑ such that

$$M = \int_\Omega \varrho \, \mathrm{d}x = \int_\Omega \tilde{\varrho} \, \mathrm{d}x, \ E = \int_\Omega \varrho e(\varrho, \vartheta) - \varrho F \, \mathrm{d}x = \int_\Omega \tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) - \tilde{\varrho} F \, \mathrm{d}x,$$

we deduce that

$$\int_\Omega \Big(H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \Big) \, \mathrm{d}x = \overline{\vartheta} \int_\Omega \Big(\tilde{\varrho} s(\tilde{\varrho}, \overline{\vartheta}) - \varrho s(\varrho, \vartheta) \Big) \, \mathrm{d}x.$$

Thus, in accordance with *Second law of thermodynamics*, the static (equilibrium) solution indeed maximizes the total entropy among all admissible states of the system having the same total mass and energy. In particular, the static solution is unique provided that $\tilde{\varrho}$ is strictly positive.

It seems interesting to see the impact of hypothesis of thermodynamic stability on the monoatomic state equation introduced in (2.3). Besides the obvious observation that $P' > 0$ in (2.3), the latter condition in (2.5) gives rise to

$$\frac{\frac{5}{3} P(Z) - P'(Z) Z}{Z} > 0,$$

in particular, the function $Z \mapsto P(Z)/Z^{5/3}$ is decreasing, and we set

$$p_\infty = \lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}}.$$

The function $p_c(\varrho) = p_\infty \varrho^{5/3}$ is termed *cold pressure* as it corresponds to the limit of p for $\vartheta \to 0$. As a matter of fact, most real gases in the degenerate regime $\varrho / \vartheta^{3/2} >> 1$ behave like Fermi gas for which $p_\infty > 0$, see Eliezer, Ghatak and

Hora [34]. Accordingly, the static distribution of the density related to the value $\overline{\vartheta} = 0$ satisfies

$$p_\infty \nabla_x \tilde{\varrho}^{5/3} = \tilde{\varrho} \nabla_x F. \tag{2.7}$$

Integrating (2.7) we easily obtain that

$$\varrho^{2/3} = \frac{2F}{5p_\infty} + c \text{ for a certain constant } c.$$

Thus if the prescribed total mass M is small enough, solutions $\tilde{\varrho}$ of (2.7) must vanish on a certain part of Ω. In other words, the degenerate state of zero temperature admit equilibrium solutions containing vacuum! On the other hand, the state for which the temperature vanishes identically is possibly not *reachable* by any physically relevant time evolution of the fluid.

2.3 Time evolution of fluid systems

Fluids out of equilibrium evolve (change) in time. We consider two basic concepts of transport in a fluid: *convection* and *diffusion*.

2.3.1 Kinematics of fluid motion

Convection is responsible for the transfer of *mass* and *energy* by means of bulk macroscopic currents of molecules within the fluid. By its proper definition, convection is always accompanied by mass transport characterized by a macroscopic *velocity field* $\vec{u}$. Given a fixed volume $B \subset \Omega$, the physical principle of *mass conservation* reads:

$$\int_B \varrho(t_2, x) \ \mathrm{d}x - \int_B \varrho(t_1, x) \ \mathrm{d}x = - \int_{t_1}^{t_2} \int_B \varrho(t, x) \vec{u}(t, x) \cdot \vec{n}(x) \ \mathrm{dS}_x, \tag{2.8}$$

where $t \in \mathbb{R}$ denotes the time, and $\vec{n}$ is the outer normal vector to the boundary ∂B. Relation (2.8) should hold on any time interval $[t_1, t_2]$ and any set B.

A priori, the density ϱ need not be a continuous function of t and x. It is the total mass of the fluid contained in B that should change continuously in time,

$$t \mapsto \int_B \varrho(t, x) \ \mathrm{d}x \ \in \ C[t_1, t_2].$$

Accordingly, it is convenient to replace (2.8) by

$$\lim_{\delta \to 0} \int_{t_1}^{t_2} \int_B \Big(\varrho(t, x) \partial_t \varphi_\delta(t, x) + \varrho(t, x) \vec{u}(t, x) \cdot \nabla_x \varphi_\delta(t, x) \Big) \mathrm{d}x \ \mathrm{d}t = 0, \tag{2.9}$$

where φ_δ is a family of smooth functions vanishing for $t = t_1, t_2$ and $x \in \partial B$, and such that

$$0 \le \varphi_\delta \le 1, \ \varphi_\delta \nearrow 1 \text{ as } \ \delta \to 0.$$

Apparently, relation (2.9) is *stronger* than (2.8) if the velocity and the density are continuous functions, and, on the other hand, it makes sense provided ϱ and $\vec{u}$ are merely locally integrable.

The next step is to replace (2.9) by a more restrictive stipulation

$$\int_0^T \int_\Omega \Big(\varrho(t,x)\partial_t \varphi(t,x) + \varrho(t,x)\vec{u}(t,x) \cdot \nabla_x \varphi(t,x) \Big) \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{2.10}$$

for any $\varphi \in C_c^\infty((0,T) \times \Omega)$, where $(0,T)$ is a reference time interval. It is easy to check that relation (2.10) gives rise to the standard form of *equation of continuity*

$$\partial_t \varrho(t,x) + \mathrm{div}_x(\varrho(t,x)\vec{u}(t,x)) = 0 \tag{2.11}$$

provided ϱ and $\vec{u}$ are continuously differentiable. Relation (2.10) is called *weak formulation* of (2.11), where the derivatives are understood in the sense of distributions. The weak solutions discussed in this study are based on weak formulation of basic balance law in the spirit of (2.10).

2.3.2 Mass transport

Smooth velocity fields determine the *streamlines* - trajectories of (hypothetical) individual fluid particles. These are defined as solutions of a system of ordinary differential equations

$$\vec{X}'(t) = \vec{u}(t, \vec{X}(t)), \ \vec{X}(0) = \vec{X}_0,$$

where $\vec{X}_0$ denotes the initial position of a given material point. A description of a fluid related to the moving particles $\vec{X}(t)$ is called *Lagrangean*, in contrast with the *Eulerian description* used above, where the reference coordinate system is related to points in the physical space.

For the streamlines to be uniquely defined, the vector field $\vec{u}$ must be Lipschitz continuous with respect to the $x-$variable. Unfortunately, the velocity fields obtained in the framework of weak solutions to problems in fluid mechanics do not (are not known to) enjoy this property. Pursuing the philosophy of the previous section, we may say that the motion of individual (non-existing) particles is irrelevant and concentrate on the *volume transport*. A natural question to ask is therefore what are the minimal regularity properties of the velocity field $\vec{u}$ for ϱ to be uniquely determined by equation (2.11).

To avoid problems with the boundary of the physical space, assume that $\Omega = \mathbb{R}^3$. Following DiPerna and Lions [30] we regularize (2.11) by means of convolution with a family of regularizing kernels κ_δ obtaining

$$\partial_t \varrho_\delta + \mathrm{div}_x(\varrho_\delta \vec{u}) = r_\delta \equiv \mathrm{div}_x(\varrho_\delta \vec{u} - (\varrho\vec{u})_\delta),$$

where we have denoted $v_\delta = v \star \kappa_\delta$. Note that the same treatment can be applied to the weak solutions satisfying (2.10). Since the left-hand side can be written in the form

$$\partial_t \varrho_\delta + \mathrm{div}_x(\varrho_\delta \vec{u}) = \partial_t \varrho_\delta + \nabla_x \varrho_\delta \cdot \vec{u} + \varrho_\delta \mathrm{div}_x \vec{u},$$

the solution ϱ_δ is uniquely determined by the initial data as soon as

$$\mathrm{div}_x \vec{u} \in L^1(0,T;L^\infty(\Omega)). \tag{2.12}$$

The last step is to let $\delta \to 0$ to recover the original solution ϱ. To this end, we need to show that the commutator r_δ vanishes in the limit. It turns out that this is indeed the case provided

$$\varrho \in L^p(0,T;L^q(\Omega)),\ \vec{u} \in L^{p'}(0,T;W^{1,q'}(\Omega;\mathbb{R}^3)),\ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1,$$

see DiPerna and Lions [30]. Although the above procedure has been recently generalized to a vast class of "densities" and velocity fields by Ambrosio [5], condition (2.12) remains essentially unchanged.

Solutions of (2.11) resulting from the regularizing procedure delineated above are usually termed *renormalized solutions.* It is easy to check that they satisfy, in the weak sense, *renormalized equation of continuity*

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\vec{u}) + \Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x\vec{u} = 0, \tag{2.13}$$

where b is an arbitrary (non-linear) function. Similarly to the concept of *entropy solution* to scalar conservation laws introduced by Kruzhkov [64], equation (2.13) can be taken as in *intrinsic* definition of renormalized solutions.

Let us point out that validity of (2.12) in the class of weak solutions to compressible fluid models represents a major open problem. As a matter of fact, condition (2.12) plays a role of important *regularity criterion* for the compressible Navier-Stokes system (see Fan, Jiang and Ou [37]), similar to the celebrated condition

$$\mathrm{curl}\ \vec{u} \in L^1(0,T;L^\infty(\Omega;\mathbb{R}^3))$$

of Beale, Kato and Majda [10] for the incompressible fluids.

Violation of (2.12) may result in two kinds of severe singularities for the density ϱ, namely (i) the *mass collapse* $\varrho = \infty$ or (ii) the *vacuum* $\varrho = 0$. As the Navier-Stokes system as well as many other related equations in fluid dynamics apply only to *non-dilute* fluids, the appearance of a vacuum would certainly indicate a serious defect of the model.

2.3.3 Newton's second law, equation of linear momentum

A rather vague definition of a fluid asserts that fluid is a material that can flow. More specifically, fluids can be characterized by *Stokes' law*:

$$\mathbb{T} = \mathbb{S} - p\mathbb{I},$$

where $\mathbb{T}$ is the Cauchy stress, p is the thermostatic pressure introduced in Section 2.1, and the symbol $\mathbb{S}$ denotes the *viscous stress tensor.* Accordingly, the classical formulation of *Newton's second law,* or *conservation of linear momentum,* reads

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p = \mathrm{div}_x\mathbb{S} + \varrho\nabla_x F, \tag{2.14}$$

or, in the weak form similar to (2.10),

$$\int_0^T \int_\Omega \Big(\varrho\vec{u}\cdot\partial_t\varphi + \varrho(\vec{u}\otimes\vec{u}) : \nabla_x\varphi + p\mathrm{div}_x\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.15}$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - \varrho\nabla_x F\cdot\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any compactly supported function φ ranging in $\mathbb{R}^3$.

In the absence of viscous stresses, meaning $\mathbb{S}\equiv 0$, we deduce from (2.11), (2.15) that

$$\partial_t\Big(\frac{1}{2}\varrho|\vec{u}|^2 - \varrho F\Big) + \mathrm{div}_x\Big(\frac{1}{2}\varrho|\vec{u}|^2\vec{u}\Big) + \mathrm{div}_x(p\vec{u}) = p\mathrm{div}_x\vec{u},$$

where $\frac{1}{2}\varrho|\vec{u}|^2$ is the *kinetic energy*. If, in addition, we assume that p depends only on ϱ, the term $p\mathrm{div}_x\vec{u}$ may be expressed by means of (2.13) yielding

$$\partial_t\Big(\varrho Q(\varrho)\Big) + \mathrm{div}_x\Big(\varrho Q(\varrho)\vec{u}\Big) + p(\varrho)\mathrm{div}_x\vec{u} = 0,$$

where we have set

$$Q(\varrho) = \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

Thus we have deduced *conservation of total energy* in the form

$$\partial_t\Big(\underbrace{\frac{1}{2}\varrho|\vec{u}|^2}_{\text{kinetic energy}} + \underbrace{\varrho Q(\varrho)}_{\text{potential energy}} - \varrho F\Big) + \mathrm{div}_x\Big(\Big(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho Q(\varrho)\Big)\vec{u}\Big)$$

$$+\mathrm{div}_x(p\vec{u}) = 0.$$

However, any real evolutionary process based on principles of classical thermodynamics is bound by *Second law of thermodynamics* to dissipate a part of its mechanical energy into heat. Note that a “ghost effect” of this principle seems to be somehow incorporated even in formally conservative systems like (2.11), (2.14), with $\mathbb{S}\equiv 0$, because of the inevitable occurrence of shocks in their solutions, see Serre [94].

Dissipative effects in (2.14) are produced exclusively by the viscous stress $\mathbb{S}$. *Viscosity* is always related to motion and as such depends on the velocity gradient $\nabla_x\vec{u}$. Note that $\mathbb{S}$ cannot depend on $\vec{u}$ itself as the viscosity vanishes for the velocity fields related to *rigid motions*.

For the sake of simplicity, we assume that $\mathbb{S}$ is a linear function of $\nabla_x\vec{u}$. It can be shown, by virtue of the universal principle of material frame indifference, that $\mathbb{S}$ is given by *Newton's rheological law*

$$\mathbb{S} = \mu\Big(\nabla_x\vec{u} + \nabla_x\vec{u}^t - \frac{2}{3}\mathrm{div}_x\vec{u}\mathbb{I}\Big) + \eta\mathrm{div}_x\vec{u}\ \mathbb{I}, \tag{2.16}$$

where μ is the *shear viscosity coefficient* and η the *bulk viscosity coefficient* that may depend on the thermostatic variables ϱ and ϑ. It is interesting to note that the velocities, for which the tensor

$$\nabla_x \vec{u} + \nabla_x \vec{u}^t - \frac{1}{2} \mathrm{div}_x \vec{u} \mathbb{I}$$

vanishes, form a finite dimensional space of *conformal Killing fields*, see Reshetnyak [90].

In the presence of viscosity, the *balance of kinetic energy* reads

$$\partial_t \Big(\frac{1}{2} \varrho |\vec{u}|^2 - \varrho F \Big) + \mathrm{div}_x \Big(\frac{1}{2} \varrho |\vec{u}|^2 \vec{u} \Big) + \mathrm{div}_x (p \vec{u}) - \mathrm{div}_x (\mathbb{S} \vec{u}) = p \mathrm{div}_x \vec{u} - \mathbb{S} : \nabla_x \vec{u}. \tag{2.17}$$

Since the total energy of the fluid is a conserved quantity, we formally write down a balance law for the internal energy:

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \vec{u}) + \nabla_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \mathrm{div}_x \vec{u}, \tag{2.18}$$

where $\vec{q}$ is the internal energy *diffusive* flux specified in the next section.

2.3.4 Irreversibility of time, mechanical energy dissipation, entropy production

From the analytical point of view, equation (2.18) is not very convenient as the terms on the right-hand side are difficult to handle in view of the available *a priori* bounds. Fortunately, we can use Gibbs' relation (2.1) to rewrite (2.18) in the form of *entropy balance*

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \sigma, \tag{2.19}$$

with the *entropy production rate*

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{2.20}$$

In agreement with *Second law of thermodynamics*, σ is non-negative for any physically admissible process. This yields, in accordance with (2.16),

$$\mu \left| \nabla_x \vec{u} + \nabla_x \vec{u}^t - \frac{2}{3} \mathrm{div}_x \vec{u} \; \mathbb{I} \right|^2 \geq 0, \;\; \eta |\mathrm{div}_x \vec{u}|^2 \geq 0,$$

and

$$-\vec{q} \cdot \nabla_x \vartheta \geq 0. \tag{2.21}$$

In particular, the viscosity coefficients μ and η must be non-negative.

Relation (2.21) suggests that the internal energy flux should take place in the opposite direction to the temperature gradient. The simplest solution is provided by *Fourier's law*

$$\vec{q} = -\kappa \nabla_x \vartheta, \tag{2.22}$$

where $\kappa \geq 0$ is the *heat conductivity coefficient* that may depend on ϱ and ϑ. Thus the transport of heat takes place even if the fluid is at rest, meaning if the velocity field vanishes, and continues as long as ϑ is not homogeneous in space. The heat transfer by means of $\vec{q}$ is an example of *diffusion transport* in fluid dynamics - energy transfer caused by random Brownian motion of individual molecules. As pointed out, heat diffusion takes place even if the mass transfer caused by convection vanishes. Being of random nature, diffusion processes are *macroscopically* irreversible. In such a way, equation (2.21) and positivity of σ indicate the irreversibility of time in classical fluid mechanics.

2.3.5 Total energy balance, boundary conditions

Putting equations (2.17), (2.18) together, we obtain the total energy balance in the form

$$\partial_t \left(\underbrace{\frac{1}{2}\varrho|\vec{u}|^2}_{\text{kinetic energy}} + \underbrace{\varrho e(\varrho,\vartheta)}_{\text{internal energy}} - \varrho F \right) \tag{2.23}$$

$$+\mathrm{div}_x \left(\underbrace{\left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) + p(\varrho,\vartheta)\right)\vec{u}}_{\text{convective energy flux}} \right) + \mathrm{div}_x \left(\underbrace{\vec{q} - \mathbb{S}\vec{u}}_{\text{diffusive energy flux}} \right) = 0.$$

As our aim is to study energetically insulated systems, the total energy must be independent of time,

$$\partial_t \int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\,\mathrm{d}x = 0. \tag{2.24}$$

Accordingly, the flux of the energy through the boundary $\partial\Omega$ should vanish at any time. To this end, we assume that the boundary is *impermeable*,

$$\vec{u}\cdot\vec{n}|_{\partial\Omega} = 0 \tag{2.25}$$

and *thermally insulated*,

$$\vec{q}\cdot\vec{n}|_{\partial\Omega} = 0. \tag{2.26}$$

Condition (2.25) may supplemented with

$$(\mathbb{S}\vec{u})\cdot\vec{n}|_{\partial\Omega} = 0. \tag{2.27}$$

As the viscous stress tensor $\mathbb{S}$ is symmetric, we have $(\mathbb{S}\vec{u})\cdot\vec{n} = (\mathbb{S}\vec{n})\cdot\vec{u}$; whence (2.25), (2.27) give rise to the *complete slip boundary condition*

$$(\mathbb{S}\vec{n})\times\vec{n}|_{\partial\Omega} = 0. \tag{2.28}$$

Note that (2.25), (2.27) are obviously satisfied provided the fluid adheres completely to the boundary, specifically

$$\vec{u}|_{\partial\Omega} = 0 \tag{2.29}$$

provided $\partial\Omega$ is at reast. As evidenced by vast amount of literature, the *no-slip* boundary conditions (2.29) are the most popular for models of *viscous* fluids. However, some recent studies of nano-fluids and related phenomena suggest a compromise between (2.28), (2.29) provided by the so-called *Navier's boundary condition*

$$\beta[\vec{u}]_{\rm tangent} + [\mathbb{S}\vec{n}]_{\rm tangent}|_{\Omega} = 0, \tag{2.30}$$

where $\beta \geq 0$ plays a role of a friction parameter (see Bulíček, Málek, and Rajagopal [15], Priezjev and Troian [88], among others). For the total energy to be conserved, condition (2.30) must be supplemented with

$$\vec{q} \cdot \vec{n} + \beta|\vec{u}|^2|_{\partial\Omega} = 0 \tag{2.31}$$

replacing (2.26).

2.3.6 Entropy production in the framework of weak solutions

Weak solutions do not provide sufficient control over the kinetic energy transfer. Such a phenomenon is well known in the theory of (formally) conservative non-linear balance laws, where the mechanical energy is dissipated by means of shock waves, whereas the entropy is being produced on sets of zero measure that cannot be captured by the classical framework. Accordingly, the kinetic energy balance (2.17) may become an *inequality*

$$\partial_t\left(\frac{1}{2}\varrho|\vec{u}|^2 - \varrho F\right) + {\rm div}_x\left(\frac{1}{2}\varrho|\vec{u}|^2\vec{u}\right) + {\rm div}_x(p\vec{u}) - {\rm div}_x(\mathbb{S}\vec{u}) - p{\rm div}_x\vec{u} + \mathbb{S} : \nabla_x\vec{u} \leq 0$$

due to (hypothetical) occurrence of concentrations in the dissipative term $\mathbb{S} : \nabla_x\vec{u}$. Although such a phenomenon would certainly indicate a serious drawback of the model, the problem is still open even in the context of incompressible fluids, see Caffarelli, Kohn and Nirenberg [18], Duchon and Robert [31], Eyink [36], Nagasawa [85] for various interesting related issues. Very roughly indeed, we can say that a part of the dissipated energy expressed by the quadratic expression $\mathbb{S} : \nabla_x\vec{u}$ may "disappears" attaining higher and higher Fourier modes in a finite time.

The fact that kinetic energy may be lost in an uncontrollable way was observed in a recent paper by Ma, Ukai and Yang [75], where time periodic solutions were constructed for the full Navier-Stokes-Fourier system on the whole space, albeit in dimension 5. Formally, the growth of entropy precludes such a possibility, therefore there must be a part of energy "spread" quickly to very large spatial regions. This example also shows that unbounded domains may be, in a certain sense, unphysical.

Accepting the hypothetical possibility of uncontrollable dissipation of the kinetic energy, the entropy production σ must be enhanced accordingly in order to keep the total energy constant in time. Thus we replace (2.20) by

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right), \tag{2.32}$$

where σ is understood as a non-negative measure on the space-time cylinder $[0,T] \times \overline{\Omega}$. Consequently, a proper weak formulation of entropy balance (2.19) reads

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \frac{\vec{q}}{\vartheta} \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t + < \sigma, \varphi >= 0 \tag{2.33}$$

for any $\varphi \in C_c^\infty((0,T) \times \overline{\Omega})$, where $\sigma \in \mathcal{M}^+([0,T] \times \overline{\Omega})$, and $<,>$ denotes the duality pairing between $\mathcal{M}$ and the space C of continuous functions. Note that the weak formulation implicitly includes satisfaction of the no-flux boundary condition (2.26).

2.4 Navier-Stokes-Fourier system

In order to conclude the introductory part of this study, we formulate the primitive system to be rescaled and analyzed in the forthcoming sections. It is more convenient to write down the equations in the "classical" form although they should be understood in the weak sense specified above. On the basis of historical background, the primitive system is called *Navier-Stokes-Fourier system* consisting of the following field equations (see Gallavotti [46]) :

Conservation of Mass

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0; \tag{2.34}$$

Balance of Momentum

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F; \tag{2.35}$$

Balance of Entropy

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \sigma; \tag{2.36}$$

supplemented with the boundary conditions:

Impermeability

$$\vec{u} \cdot \vec{n}|_{\partial\Omega} = 0; \tag{2.37}$$

Navier's Slip Condition

$$\beta[\vec{u}]_{\text{tangent}} + [\mathbb{S}\vec{n}]_{\text{tangent}}|_{\Omega} = 0, \ \beta \in [0, \infty], \tag{2.38}$$

where $\beta = \infty$ represents the no-slip condition (2.29);

Flux Condition

$$\vec{q} \cdot \vec{n} + \beta |\vec{u}|^2|_{\partial\Omega} = 0, \ \beta < \infty, \ \vec{q} \cdot \vec{n}|_{\partial\Omega} = 0 \text{ for } \beta = 0, \infty. \tag{2.39}$$

In addition, we have

Total Energy Conservation

$$\partial_t \int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e - \varrho F \right) (t, \cdot) \ \mathrm{d}x = 0, \tag{2.40}$$

where the thermodynamic functions p, s, and e are interrelated through

GIBBS' EQUATION

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{2.41}$$

Finally, the entropy production satisfies

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{2.42}$$

It is not difficult to check that (2.42) necessarily becomes an *equality* in the framework of smooth solutions to the system. As a matter of fact, a bit more tedious but still mathematically quite standard argument guarantees equality in (2.42) provided a weak solution ϱ, ϑ, $\vec{u}$ is know to belong to the following regularity class:

- the density and the absolute temperature are measurable functions satisfying
$$0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho}, \ 0 < \underline{\vartheta} < \vartheta < \overline{\vartheta} \text{ a.a. on } (0,T) \times \Omega;$$
- the velocity is bounded, specifically,
$$\|\vec{u}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq \overline{u};$$
- the density is square integrable,
$$\nabla_x \varrho \in L^2((0,T) \times \Omega; \mathbb{R}^3).$$

The last condition seems quite restrictive but still inevitable in passing from the entropy to internal energy balance equation.

3 Scale analysis of the primitive system

We introduce a dimensionless version of the primitive equations represented by the Navier-Stokes-Fourier system. We identify the characteristic numbers and discuss the associated singular limit problems, where one or more of these parameters tends to zero or become extremely large. Finally, we show that similar scaling procedure may be imposed also on the shape and diameter of the physical domain as well as some parameters appearing in the boundary conditions.

3.1 Scaled equations

For each physical quantity X, we introduce its characteristic value $X_{\rm char}$ and replace X with its dimensionless analogue $X/X_{\rm char}$ in the Navier-Stokes-Fourier system introduced in Section 2.4. As a result we obtain a scaled system of equations (see Klein et al. [62]):

$$[{\rm Sr}]\partial_t \varrho + {\rm div}_x(\varrho\vec{u}) = 0, \tag{3.1}$$

$$[{\rm Sr}]\partial_t(\varrho\vec{u}) + {\rm div}_x(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{[{\rm Ma}]^2}\nabla_x p = \frac{1}{[{\rm Re}]}{\rm div}_x\mathbb{S} + \frac{1}{[{\rm Fr}]^2}\varrho\nabla_x F, \tag{3.2}$$

$$[{\rm Sr}]\partial_t(\varrho s) + {\rm div}_x(\varrho s\vec{u}) + \left(\frac{p_{\rm char}}{\varrho_{\rm char}e_{\rm char}}\right)\frac{1}{[{\rm Pe}]}{\rm div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \left(\frac{p_{\rm char}}{\varrho_{\rm char}e_{\rm char}}\right)\sigma, \tag{3.3}$$

where

$$\sigma \geq \frac{1}{\vartheta}\Big(\frac{[{\rm Ma}]^2}{[{\rm Re}]}\mathbb{S}:\nabla_x\vec{u} - \frac{1}{[{\rm Pe}]}\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\Big), \tag{3.4}$$

together with the total energy balance

$$\frac{{\rm d}}{{\rm d}t}\int_\Omega\left([{\rm Ma}]^2\frac{1}{2}\varrho|\vec{u}|^2 + \left(\frac{\varrho_{\rm char}e_{\rm char}}{p_{\rm char}}\right)\varrho e - \frac{[{\rm Ma}]^2}{[{\rm Fr}]^2}\varrho F\right)\ {\rm d}x = 0. \tag{3.5}$$

The characteristic numbers are defined as follows:

Symbol	Definition	Name
Sr	$L_{\rm char}/(T_{\rm char}U_{\rm char})$	Strouhal number
Ma	$U_{\rm char}/\sqrt{p_{\rm char}/\varrho_{\rm char}}$	Mach number
Re	$\varrho_{\rm char}U_{\rm char}L_{\rm char}/\nu_{\rm char}$	Reynolds number
Fr	$U_{\rm char}/\sqrt{F_{\rm char}}$	Froude number
Pe	$p_{\rm char}L_{\rm char}U_{\rm char}/(\vartheta_{\rm char}\kappa_{\rm char})$	Péclet number

The symbols $T_{\rm char}$, $L_{\rm char}$, $U_{\rm char}$, and $\nu_{\rm char}$ denote the characteristic time, length, velocity, and viscosity, respectively. Note that there are two viscosity coefficients, namely μ and η that may yield, in general, two Reynolds numbers related to the shear and bulk viscosity, respectively. In practical situation when the compressible medium is a gas, it is customary to adopt *Stokes' hypothesis* $\eta = 0$ and set $\nu_{\rm char} = \mu_{\rm char}$.

Two qualitatively different scalings should be distinguished: (i) *process* scaling, where the material remains the same but the kinematic variables $T_{\rm char}$, $L_{\rm char}$, $U_{\rm char}$ vary, (ii) *constitutive* scaling, where we change the material properties of the fluid. In the process scaling the choice of the characteristic numbers is basically arbitrary, while the constitutive scaling is restricted by natural thermodynamic relations. For instance, the quantity

$$\left(\frac{\varrho_{\rm char}e_{\rm char}}{p_{\rm char}}\right) \approx 1$$

should be of order 1 for gases. Similarly, changes in $\vartheta_{\rm char}$ should be reflected in $\nu_{\rm char}$ provided the latter is temperature dependent.

The reader may consult the monographs of Chemin et al. [21], Majda [77], Zeytounian [101], [103] for other possibilities of the choice of scaling parameters.

3.2 Incompressible limits

In the process of incompressible limits, the pressure p becomes (formally) constant or at least a given function of x, independent of time. Obviously this is the case when the Mach number Ma becomes small, say, ${\rm Ma} = \varepsilon \to 0$. This means that the characteristic speed $U_{\rm char}$ is dominated by the quantity $\sqrt{p_{\rm char}/\varrho_{\rm char}}$, where the latter is the *speed of sound*. In addition, we are interested in processes, where the limit velocity field is non-zero. Consequently, in order to control the velocity, the entropy production σ/ε^2 must remain bounded uniformly for $\varepsilon \to 0$. Keeping the Péclet number of order 1 we therefore conclude that $|\nabla_x \vartheta| \approx \varepsilon$, meaning, the temperature necessarily becomes spatially homogeneous in the limit process.

Accordingly, the static solutions $\vartheta = \overline{\vartheta} > 0$ - a positive constant, and $\tilde\varrho = \tilde\varrho(x)$, solving

$$\nabla_x p(\tilde\varrho, \overline{\vartheta}) = \tilde\varrho \nabla_x F,$$

will play a crucial role in the forthcoming analysis of the incompressible limits. As a consequence of hypothesis of thermodynamic stability (2.5), we expect the static solution to be globally stable, meaning, attracting all global-in-time trajectories generated by solutions to the Navier-Stokes-Fourier system. In order to see that this is indeed the case, we revoke the Helmholtz function $H(\varrho, \vartheta)$ introduced in (2.6). Using arguments similar to Section 2.2, we deduce a so-called *total dissipation balance* in the form

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec u|^2 + H(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde\varrho) - H(\tilde\varrho,\overline{\vartheta}) \right)(\tau_2,\cdot)\ {\rm d}x \tag{3.6}$$

$$+\overline{\vartheta}\sigma\Big([\tau_1,\tau_2]\times\overline{\Omega}\Big) =$$

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec u|^2 + H(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde\varrho) - H(\tilde\varrho,\overline{\vartheta}) \right)(\tau_1,\cdot)\ {\rm d}x$$

for a.a. $\tau_1 < \tau_2$, provided Ω is a bounded domain and the static density $\tilde\varrho$ is normalized so that

$$\int_\Omega \tilde\varrho \ {\rm d}x = \int_\Omega \varrho(t,\cdot)\ {\rm d}x.$$

As we have seen in Section 2.2, the quantity

$$H(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde\varrho) - H(\tilde\varrho,\overline{\vartheta})$$

behaves like

$$(\varrho - \tilde\varrho)^2 + (\vartheta - \tilde\vartheta)^2$$

at least for ϱ, ϑ that are closed to the equilibrium state $\tilde{\varrho}$, $\overline{\vartheta}$. Thus the integral

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + H(\varrho,\vartheta) - \frac{\partial H(\tilde{\varrho},\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho},\overline{\vartheta}) \right)(t,\cdot)\ \mathrm{d}x$$

can be used to "measure" the distance of any trajectory $(\varrho(t,\cdot),\vartheta(t,\cdot),\vec{u}(t,\cdot))$ to the equilibrium state $(\tilde{\varrho},\overline{\vartheta},0)$. It follows from (3.6) that this distance is decreasing in time.

Identity (3.6) holds for the unscaled system. Assume now that Ma $= \varepsilon$, Fr $= \varepsilon^{\alpha/2}$ for a certain parameter $\alpha \in [0,2]$, while the other characteristic numbers in the scaled system (3.1 - 3.5) are supposed to be of order 1. Thus the scaled version of (3.6) reads

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho,\vartheta) - \frac{\partial H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon,\overline{\vartheta}) \right)\right)(\tau_2,\cdot)\ \mathrm{d}x \tag{3.7}$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\sigma\Big([\tau_1,\tau_2]\times\overline{\Omega}\Big) =$$

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho,\vartheta) - \frac{\partial H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon,\overline{\vartheta}) \right)\right)(\tau_1,\cdot)\ \mathrm{d}x,$$

where

$$\nabla_x p(\tilde{\varrho}_\varepsilon,\overline{\varrho}) = \varepsilon^{2-\alpha}\tilde{\varrho}_\varepsilon \nabla_x F,$$

and

$$\sigma \geq \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{3.8}$$

Consequently, in order to control the norm of global-in-time solutions in the low Mach number regime, we have to make sure that the *initial data* ϱ_0, ϑ_0, $\vec{u}_0$ are chosen in such a way that the expression

$$\int_\Omega \left(\frac{1}{2}\varrho_0|\vec{u}_0|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho_0,\vartheta_0) - \frac{\partial H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial \varrho}(\varrho_0 - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon,\overline{\vartheta}) \right)\right)(\tau_1,\cdot)\ \mathrm{d}x,$$

remains bounded for $\varepsilon \to 0$. This leads to the concept of *prepared* data discussed in the next section.

3.2.1 Ill and well prepared initial data

Assume that the initial state of the fluid governed by the scaled Navier-Stokes-Fourier system, with Ma $= \varepsilon$, Fr $= \varepsilon^\alpha$, is determined by the data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon},\ \vec{u}(0,\cdot) = \vec{u}_{0,\varepsilon}.$$

Going back to the scaled dissipation balance (3.7), we say that the data are *prepared* as soon as the integral

$$\int_\Omega \left(\frac{1}{2}\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - \frac{\partial H(\tilde{\varrho}_\varepsilon,\overline{\vartheta})}{\partial \varrho}(\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon,\overline{\vartheta}) \right)\right)\ \mathrm{d}x$$

is bounded uniformly for $\varepsilon \to 0$. Writing

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}$$

we observe the *prepared* data correspond to $\varrho^{(1)}_{0,\varepsilon}$, $\vartheta^{(1)}_{0,\varepsilon}$ bounded in certain function spaces, most typically in $L^2(\Omega) \cap L^\infty(\Omega)$. Rather inconsistently, such a choice is usually termed *ill-prepared* initial data in the literature.

On the other hand, *well-prepared* data enjoy the property

$$\varrho^{(1)}_{0,\varepsilon} \to 0,\ \vartheta^{(1)}_{0,\varepsilon} \to 0 \text{ as } \varepsilon \to 0,$$

together with the "incompressibility" of the initial velocity field

$$\mathrm{div}_x \vec{u}_{0,\varepsilon} = 0.$$

As we shall see below, the *ill-prepared* data create rapidly oscillating *acoustic waves* in the asymptotic limit $\varepsilon \to 0$. Although these waves may be effectively damped by the physical boundary or decay, at least locally, to zero on unbounded physical domains, their presence create the major technical problems in the analysis of incompressible limits and may lead to instabilities in numerical schemes.

3.3 Acoustic waves

Following the idea of Lighthill [68], [69] we rewrite the scaled Navier-Stokes-Fourier system in the form of an *acoustic equation.* To this end, we first consider a simple situation, where $\mathrm{Ma} = \varepsilon$ and $F \equiv 0$, while the other characteristic numbers scale to one. Accordingly, the equilibrium density $\tilde{\varrho} = \overline{\varrho}$ is constant in space and we may linearize the system around the equilibrium solution $(\overline{\varrho}, \overline{\vartheta})$.

Formally, we have

$$\varepsilon \partial_t \left(\frac{\varrho - \overline{\varrho}}{\varepsilon} \right) + \mathrm{div}_x (\varrho \vec{u}) = 0,$$

$$\varepsilon \partial_t (\varrho \vec{u}) + \nabla_x \left(\frac{p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) = -\varepsilon \mathrm{div}_x (\varrho \vec{u} \otimes \vec{u}) + \varepsilon \mathrm{div}_x \mathbb{S},$$

$$\varepsilon \partial_t \left(\frac{\varrho s(\varrho, \vartheta) - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) + s(\overline{\varrho}, \overline{\vartheta}) \mathrm{div}_x (\varrho \vec{u})$$

$$= \varepsilon \mathrm{div}_x \left(\varrho \frac{s(\overline{\varrho}, \overline{\vartheta}) - s(\varrho, \vartheta)}{\varepsilon} \vec{u} \right) + \varepsilon \mathrm{div}_x \left(\frac{\kappa(\vartheta)}{\vartheta} \frac{\nabla_x \vartheta}{\varepsilon} \right) + \varepsilon \frac{\sigma}{\varepsilon}.$$

Moreover, linearization of thermodynamic functions yields:

$$\frac{p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \approx \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \left(\frac{\varrho - \overline{\varrho}}{\varepsilon} \right) + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \left(\frac{\vartheta - \overline{\vartheta}}{\varepsilon} \right),$$

$$\frac{\varrho s(\varrho,\vartheta)-\overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\approx\frac{\partial(\varrho s)(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\left(\frac{\varrho-\overline{\varrho}}{\varepsilon}\right)+\frac{\partial(\varrho s)(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\left(\frac{\vartheta-\overline{\vartheta}}{\varepsilon}\right);$$

whence, determining constants A, B so that

$$A\frac{\partial(\varrho s)(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}+B=\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho},\ A\frac{\partial(\varrho s)(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}=\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta},$$

we may rewrite the acoustic equation in a concise form

$$\varepsilon\partial_t r+\omega\mathrm{div}_x\vec{V}=\varepsilon A\left(\mathrm{div}_x\left(\varrho\frac{s(\overline{\varrho},\overline{\vartheta})-s(\varrho,\vartheta)}{\varepsilon}\vec{u}\right)+\mathrm{div}_x\left(\frac{\kappa(\vartheta)}{\vartheta}\frac{\nabla_x\vartheta}{\varepsilon}\right)+\frac{\sigma}{\varepsilon}\right), \tag{3.9}$$

$$\varepsilon\partial_t\vec{V}+\nabla_x r=\varepsilon\left(-\mathrm{div}_x(\varrho\vec{u}\otimes\vec{u})+\mathrm{div}_x\mathbb{S}\right) \tag{3.10}$$

$$\varepsilon\nabla_x\left(A\frac{\varrho s(\varrho,\vartheta)-\overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}+B\left(\frac{\varrho-\overline{\varrho}}{\varepsilon^2}\right)-\left(\frac{p(\varrho,\vartheta)-p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right)\right),$$

where

$$\omega=As(\overline{\varrho},\overline{\vartheta})+B=\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}+\left|\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\right|^2\left(\overline{\varrho}^2+\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\right)^{-1},$$

and

$$r=A\frac{\varrho s(\varrho,\vartheta)-\overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}+B\left(\frac{\varrho-\overline{\varrho}}{\varepsilon}\right),\ \vec{V}=\varrho\vec{u}.$$

Note that, thanks to the specific choice of the parameters A, B, the right-hand sides of (3.9), (3.10) are bounded in terms of the quantities $(\varrho-\overline{\varrho})/\varepsilon$, $(\vartheta-\overline{\vartheta})/\varepsilon$.

As we have seen in Section 2.3.6, the entropy production rate σ at the level of weak solutions is represented by a non-negative *measure* on $[0,T]\times\overline{\Omega}$. Consequently, to avoid time discontinuities in solutions of (3.9), (3.10),we introduce a measure-valued function Σ,

$$<\Sigma(\tau),\varphi>=-<\sigma,[0,\tau]\varphi>\ \text{ for any }\varphi\in C(\overline{\Omega}).$$

Clearly, $-\Sigma(\tau)\in\mathcal{M}^+(\overline{\Omega})$ for any $\tau\in[0,T]$, and the mapping $\tau\mapsto -<\Sigma,\varphi>$ is non-decreasing in τ for any fixed $\varphi\in C(\overline{\Omega})$, $\varphi\geq 0$. Finally, we can check that

$$\partial_t\Sigma=-\sigma\ \text{in the sense of generalized derivatives.}$$

Thus we arrive at the final form of the acoustic equation:

$$\varepsilon\partial_t R+\omega\mathrm{div}_x\vec{V}=\varepsilon F_1, \tag{3.11}$$

$$\varepsilon\partial_t\vec{V}+\nabla_x R=\varepsilon\vec{F}_2, \tag{3.12}$$

with

$$R=A\frac{\varrho s(\varrho,\vartheta)-\overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}+B\left(\frac{\varrho-\overline{\varrho}}{\varepsilon}\right)+A\frac{\Sigma}{\varepsilon},$$

$$F_1 = A\left(\mathrm{div}_x\left(\varrho\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho,\vartheta)}{\varepsilon}\vec{u}\right) + \mathrm{div}_x\left(\frac{\kappa(\vartheta)}{\vartheta}\frac{\nabla_x\vartheta}{\varepsilon}\right)\right),$$

and

$$\vec{F}_2 = (-\mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \mathrm{div}_x\mathbb{S})$$

$$+\nabla_x\left(A\frac{\varrho s(\varrho,\vartheta) - \overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2} + B\left(\frac{\varrho-\overline{\varrho}}{\varepsilon^2}\right) - \left(\frac{p(\varrho,\vartheta) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right)\right) + A\nabla_x\frac{\Sigma}{\varepsilon^2}.$$

3.3.1 Acoustic propagator

Consider the homogeneous unscaled acoustic equation

$$\partial_t R + \omega\mathrm{div}_x\vec{V} = 0,\ \partial_t\vec{V} + \nabla_x R = 0 \text{ in } (0,T)\times\Omega, \tag{3.13}$$

where

$$\vec{V}\cdot\vec{n}|_{\partial\Omega} = 0. \tag{3.14}$$

The first observation is that (3.13) admits *finite speed of propagation* $\sqrt{\omega}$. Indeed a direct application of Gauss-Green formula yields

$$\int_{\Omega\cap B(\tau)}\left(|R|^2 + |\vec{V}|^2\right)\,\mathrm{d}x \leq \int_{\Omega\cap B}\left(|R|^2 + |\vec{V}|^2\right)\,\mathrm{d}x \text{ for any } \tau\geq 0, \tag{3.15}$$

where B is a ball, and

$$B(\tau) = \{x\in B\cap\Omega\ |\ \mathrm{dist}[x,\partial B] > \sqrt{\omega}\tau\}.$$

The second observation is that the linear space $[R,\vec{V}]$, R constant, $\mathrm{div}_x\vec{V} = 0$, is invariant under the action of the evolution group generated by (3.14), more specifically, it coincides with the kernel of its generator.

Finally, we write $\vec{V}$ as

$$\vec{V} = \vec{H}[\vec{V}] + \nabla_x\Phi,$$

where $\vec{V}$ denotes the Helmholtz projection onto the space of solenoidal functions, specifically,

$$\Delta\Phi = \mathrm{div}_x\vec{V} \text{ in } \Omega;\ \nabla_x\Phi\cdot\vec{n}|_{\partial\Omega} = \vec{V}\cdot\vec{n}|_{\partial\Omega}(=0).$$

Rewriting (3.13), (3.14) in terms of the potential Φ we arrive at a *wave equation*

$$\partial_t R + \omega\Delta_x\Phi = 0,\ \partial_t\Phi + R = 0 \text{ in } (0,T)\times\Omega, \tag{3.16}$$

$$\nabla_x\Phi\cdot\vec{n}|_{\partial\Omega} = 0. \tag{3.17}$$

Solutions of (3.16), (3.17) may be written explicitly as

$$\Phi(t) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\omega\Delta_N}t\right)\left[\Phi_0 + \frac{\mathrm{i}}{\sqrt{-\omega\Delta_N}}R_0\right] \tag{3.18}$$

$$\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\omega\Delta_N}t\right)\left[\Phi_0 - \frac{\mathrm{i}}{\sqrt{-\omega\Delta_N}}R_0\right]$$

provided

$$\Phi(0,\cdot) = \Phi_0,\ R_{0,\cdot} = R_0.$$

The symbol $-\Delta_N$ denotes the self-adjoint extension of the Neumann Laplacean on the Hilbert space $L^2(\Omega)$.

3.3.2 Duhamel's formula

Our aim is to express a general solution of (3.11), (3.12) by means of (3.18) and the standard *Duhamel's formula*. To this end, we have to apply the Helmholtz projection to $\vec{F}_2$. This formal manipulation is justified provided R and F_2 satisfy certain compatibility conditions on the boundary. This is the case, for instance, when the velocity field satisfies the complete slip condition (2.38), with $\beta = 0$. This stipulation corresponds to the acoustically *hard* boundary conditions $\nabla_x \Phi \cdot \vec{n}|_{\partial\Omega} = 0$. Writing

$$\vec{F}_2 = \vec{H}[\vec{F}_2] + \nabla_x H_2,$$

we get

$$\begin{aligned}\Phi(t) = &\frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\omega\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 + \frac{\mathrm{i}}{\sqrt{-\omega\Delta_N}}R_0\right] \\ &\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\omega\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 - \frac{\mathrm{i}}{\sqrt{-\omega\Delta_N}}R_0\right] \\ &+\frac{1}{2}\int_0^t \exp\left(\mathrm{i}\sqrt{-\omega\Delta_N}\frac{(t-s)}{\varepsilon}\right)\left[H_2(s) + \frac{\mathrm{i}}{\sqrt{-\omega\Delta_N}}F_1(s)\right]\ \mathrm{d}s \\ &+\frac{1}{2}\int_0^t \exp\left(-\mathrm{i}\sqrt{-\omega\Delta_N}\frac{(t-s)}{\varepsilon}\right)\left[H_2(s) - \frac{\mathrm{i}}{\sqrt{-\omega\Delta_N}}F_1(s)\right]\ \mathrm{d}s.\end{aligned} \tag{3.19}$$

Formula (3.19) reveals immediately the principal difficulty of the acoustically hard boundary enforced by the complete slip condition, namely, fast oscillations of the gradient component of the velocity field, with frequencies proportional to $1/\varepsilon$. This phenomenon is basically independent of the smoothness of the solution of the primitive system and persistent provided the physical domain Ω is bounded. The gradient component $\nabla_x \Phi$ should vanish in the incompressible limit, however, the convergence takes place *only* in the weak topology, meaning, in the sense of time averages. Strong convergence can be expected in the following two cases:

- The underlying physical domain is unbounded, or its diameter is proportional to the speed of the sound waves, meaning, to $1/\varepsilon$. The dispersion then dominates and provides the desired local decay. This idea was used Desjardins and Grenier [28] to show time decay of acoustic waves on the whole space $\mathbb{R}^3$ by means of the so-called Strichartz estimates.
- The underlying spatial domain is bounded, but the velocity field satisfies the no-slip boundary condition $\vec{u}|_{\partial\Omega} = 0$. In such a case, formula (3.19) is no longer valid, and the effect of the viscous stress must be incorporated in the acoustic equation. The resulting system exhibits, under certain geometrical restrictions imposed on Ω, a viscous boundary layer that resulting in the decay of the gradient part, see Desjardins et al. [29].

3.3.3 Stratified fluids

The operator $-\omega\Delta_N$ in (3.19) can be replaced by any self-adjoint non-negative operator A in order to obtain an abstract system

$$\varepsilon\partial_t R - A[\Phi] = \varepsilon F_1 \ , \ \varepsilon\partial_t \Phi + R = \varepsilon H_2, \ R(0) = R_0, \ \Phi(0) = \Phi_0.$$

This kind of problem arises in the study of *strongly stratified flows*, where the Laplacean is replaced by

$$A[\Phi] = -\frac{1}{\tilde{\varrho}} \mathrm{div}_x (\tilde{\varrho} \nabla_x \Phi) \text{ in } \Omega, \ \tilde{\varrho} \nabla_x \Phi \cdot \vec{n}|_{\partial\Omega} = 0,$$

where $\tilde{\varrho}$ is a non-constant solution of the static problem, see Masmoudi [81] or [41, Chapter 6]. Note that A can be extended as a self-adjoint operator on the weighted Lebesgue space

$$L^2_{\tilde{\varrho}}(\Omega) \text{ endowed with the scalar product } < v, w >= \int_\Omega vw \ \tilde{\varrho} \ \mathrm{d}x,$$

see Wilcox [100]. Spectral properties of the operator A under various hypotheses on the specific form of $\tilde{\varrho}$ were studied by DeBièvre and Pravica [25], [26], among others.

3.4 Incompressible limits in the framework of weak solutions

As we observed in Section 3.2, the uniform bounds on the family of solutions of the scaled system are provided by the total dissipation balance (3.7). Under hypothesis of thermodynamic stability, the Helmholtz function H enjoys the following coercivity properties:

$$H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \geq c(K)\Big(|\varrho - \tilde{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2\Big) \tag{3.20}$$

for all (ϱ, ϑ), $(\tilde{\varrho}, \overline{\vartheta})$ belonging to a compact set $K \subset (0, \infty)^2$, whereas

$$H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \geq c(K)\Big(1 + \varrho|s(\varrho, \vartheta)| + \varrho e(\varrho, \vartheta)\Big) \tag{3.21}$$

otherwise. This motivates a decomposition of each measurable function $h = h(t, x)$ as a sum of its *essential* part $[h]_{\mathrm{ess}}$ and *residual* part $[h]_{\mathrm{res}}$, where

$$[h]_{\mathrm{ess}} = \chi(\varrho, \vartheta)h, \ [h]_{\mathrm{res}} = (1 - \chi(\varrho, \vartheta))h,$$

$$\chi \in C^\infty_c(0, \infty)^2, \ 0 \leq \chi \leq 1, \chi \equiv 1 \text{ in an open neighborhood of } (\tilde{\varrho}, \overline{\vartheta}),$$

and ϱ, ϑ are solutions of the scaled Navier-Stokes-Fourier system, cf. [41, Chapter 4].

Now, let $\{\varrho_\varepsilon, \vartheta_\varepsilon, \vec{u}_\varepsilon\}_{\varepsilon>0}$ be a family of (weak) solutions to the scaled Navier-Stokes-Fourier system, with $\text{Ma} = \varepsilon$. The total dissipation balance established in (3.7) provides the following uniform bounds:

$$\left\| \left[\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon} \right]_{\text{ess}} (t, \cdot) \right\|_{L^2(\Omega)} + \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\text{ess}} (t, \cdot) \right\|_{L^2(\Omega)} \leq c$$

$$\| [\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\text{res}} (t, \cdot) \|_{L^1(\Omega)} + \| [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\text{res}} (t, \cdot) \|_{L^1(\Omega)} \leq \varepsilon^2 c,$$

and

$$\left\| \varrho |\vec{u}_\varepsilon|^2 (t, \cdot) \right\|_{L^1(\Omega)} \leq c,$$

where the constants are independent of ε and $t \in [0, T]$. Thus it is the *essential* part of each quantity that contains the decisive piece of information necessary for the limit process, while the *residual* component vanishes in the asymptotic limit $\varepsilon \to 0$.

Moreover, by the same token,

$$\| \sigma_\varepsilon \|_{\mathcal{M}([0,T] \times \overline{\Omega}} \leq \varepsilon^2 c. \tag{3.22}$$

Quite remarkably, the entropy production rate σ_ε, represented by a measure that is, in general, very difficult to describe, disappears in the incompressible limit. *The ambiguous inequality sign in the entropy balance (3.3) becomes equality in the incompressible limit, even if the equation may be scaled by* $1/\varepsilon$. This remarkable feature of the low Mach number limits allows us to identify several well-known systems in fluid mechanics as an incompressible limit of the full Navier-Stokes-Fourier system using the framework of weak solutions.

Finally, as a byproduct of (3.22), we may infer that

$$\int_0^T \int_\Omega \left(|\nabla_x \vec{u}_\varepsilon|^2 + \left| \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right| \right)^2 \, \mathrm{d}x \, \mathrm{d}t \leq c, \tag{3.23}$$

meaning, the diffusive terms provide certain *compactness* in the space variable for $\vec{u}_\varepsilon$ and ϑ_ε. As we have already observed in Section 3.3, compactness with respect to the *time* variable is a more delicate issue, and, as a matter of fact, fails in many cases of practical interest.

With help of the previous estimates, it is easy to check that

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ and } \vartheta_\varepsilon \to \overline{\vartheta} \text{ a.a. in } (0, T) \times \Omega.$$

Note that pointwise (a.a.) convergence is enough to pass to the limit in all non-linear terms appearing in the weak formulation of the Navier-Stokes-Fourier system introduced in Section 2.4. Thus we can perform successfully the limit passage as soon as we can established pointwise convergence of the velocities $\{\vec{u}_\varepsilon\}_{\varepsilon>0}$. However, as the velocity $\vec{u}_\varepsilon \approx \varrho_\varepsilon \vec{u}_\varepsilon = \vec{V}_\varepsilon$ appears in the acoustic equation (3.11), (3.12), pointwise convergence may be spoiled by rapid oscillations of the gradient component. The situation can be saved in three rather different ways:

- The main problem is to control the quadratic convective term $\mathrm{div}_x(\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon)$ in the momentum equation. Fortunately, it can be shown that this term may be written in the form

$$\mathrm{div}_x(\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon) = \text{compact terms} + \text{a gradient}.$$

  Thus all possible oscillations are supported by the gradient term, and, as the latter is irrelevant in the incompressible limit, the passage $\varepsilon \to 0$ can be accomplished. As a matter of fact, this argument works independently of the boundary conditions imposed on $\vec{u}_\varepsilon$, see Lions and Masmoudi [72], [73], Masmoudi [79], [80].

- Taking *well prepared* initial data (cf. Section 3.2.1) we may achieve strong convergence to zero of the gradient part of the velocity without much effort. However, the scope of possible applications becomes rather restricted.

- The problem is considered on an unbounded or sufficiently "large" domain, the acoustic waves being eliminated locally by dispersion. We will discuss the issue in the forthcoming section.

3.4.1 Remarks on two-scale convergence

Since solutions of the scaled acoustic equation on a *bounded* domain are expected to develop fast time oscillations with the frequency proportional to $1/\varepsilon$, it seems natural to investigate the asymptotic behavior of solutions with respect to both the real (slow) time t and the fast time $\tau = t/\varepsilon$. To this end, the concept of *two scale convergence* introduced by Allaire [4] and Nguetseng [86] can be adapted. The limits characterizing the behavior of oscillating solutions are then described in the spirit of the theory of homogenization. The reader may consult the review paper by Visintin [99] for more information on the recent development of the two-scale calculus.

3.5 Problems on large spatial domains

The influence of acoustic waves on the fluid motion close to the incompressible regime is negligible, at least in most real world applications. The standard argument asserts that the underlying physical space is practically *unbounded* or, more correctly, sufficiently *large* when compared to the sound speed in the material in question, see Klein [59], [60]. If $\Omega = \mathbb{R}^3$, the expected *local* decay of the acoustic energy follows immediately from the dispersive estimates. Desjardins and Grenier [28] exploited this idea combined with the non-trivial Strichartz estimates for the acoustic equation in order to show strong (pointwise) convergence of the velocity field in the low Mach number limit for a barotropic fluid flow in the whole physical space $\mathbb{R}^3$. A similar approach was adapted in [43] to the complete Navier-Stokes-Fourier system considered on "large" spatial domains, on which the Strichartz estimates were replaced by global integrability of the local energy established by Burq [16], Smith and Sogge [97]. Note that the concept of the so-called radiation boundary conditions, amply used in numerical analysis, is based on the same physical principle, see Engquist and Majda [35].

In contrast with the simple geometry of the whole space $\mathbb{R}^3$, where the efficient mathematical methods based on Fourier analysis are at hand, any *real* problem of wave propagation inevitably includes the influence of the boundary representing a rigid wall in the physical space. As is well-known, the Strichartz estimates become much more delicate and usually require severe geometrical restrictions to be imposed on the boundary. For instance, if the fluid domain is exterior to a compact obstacle, the latter must be starshaped or at least *non-trapping*, see Burq [16], Metcalfe [82], Smith and Sogge [97]), and the references cited therein. In what follows, we propose a simple and rather versatile method to study the local decay of acoustic waves based on the celebrated *RAGE theorem* and a result of Kato [56].

3.5.1 Reduction to unbounded domains

All available results concerning local decay of acoustic waves hold on unbounded spatial domains. On the other hand, as we observed in Section 3.3.1, the acoustic equation admits a finite speed of propagation proportional to the $1/\varepsilon$, where ε is the Mach number. As we are interested in *local decay* of the acoustic energy, we can therefore assume that the physical space depends on ε in such a way that

$$\Omega = \Omega_\varepsilon \supset (\Omega \cap B_{r_\varepsilon}), \text{ where } B_{r_\varepsilon} = \{x \mid |x| \leq r_\varepsilon\}$$

$$\varepsilon r_\varepsilon \to \infty \text{ as } \varepsilon \to 0.$$

Accordingly, since the acoustic waves are considered on a *compact* space-time cylinder $[0,T] \times K$, $K \subset \Omega$, we may assume that solutions are defined on the (unbounded) domain Ω.

3.5.2 RAGE theorem

Recalling Duhamel's formula (3.19) we claim that a necessary condition for the local energy decay of acoustic waves represented by the potential Φ is the absence of eigenvalues of the operator $-\Delta_N$ in Ω. RAGE theorem will tell us that this condition is also sufficient provided the bounded operator

$$v \mapsto \chi G(-\Delta_N)[v] : L^2(\Omega) \to L^2(\Omega) \text{ is absolutely continuous}$$

for

$$\chi \in C_c^\infty(\Omega), \; G \in C_c^\infty(0,\infty).$$

Clearly to see that, it is enough to observe that $\mathcal{D}(-\Delta_N)$ consists of functions that possess two generalized derivatives locally integrable in Ω.

The celebrated *RAGE theorem* (see Cycon et al. [22, Theorem 5.8]) reads as follows:

Theorem 3.1 *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto the space of continuity H_c of A, specifically,*

$$H = H_c \oplus \mathrm{cl}_H \Big\{ \mathrm{span}\{ w \in H \mid w \text{ an eigenvector of } A \} \Big\}.$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tA) C P_c \exp(\mathrm{i}tA)\ \mathrm{d}t \right\|_{\mathcal{L}(H)} \to 0 \text{ as } \tau \to \infty. \tag{3.24}$$

Interesting generalizations of this result to Banach spaces were obtained by Kreulich [63]. Theorem 3.1 will be applied to the solution group $\exp(\mathrm{i}\sqrt{-\omega\Delta_N} t)$, associated to the acoustic equation. As already pointed, the result is optimal as it requires only that $H_c = H$, meaning, the absence of eigenvalues of the acoustic generator.

Our aim is to apply Theorem 3.1 to $H = L^2(\Omega)$, $A = \sqrt{-\omega\Delta_N}$, $C = \chi^2 G(-\Delta_N)$, with $\chi \in C_c^\infty(\Omega)$, $\chi \geq 0$, $G \in C_c^\infty(0,\infty)$, $0 \leq G \leq 1$. Taking $\tau = 1/\varepsilon$ in (3.24) we obtain

$$\int_0^T \left\langle \exp\left(-\mathrm{i}\frac{t}{\varepsilon} \sqrt{-\omega\Delta_N} \right) \chi^2 G(-\Delta_N) \exp\left(\mathrm{i}\frac{t}{\varepsilon} \sqrt{-\omega\Delta_N} \right) X; Y \right\rangle \ \mathrm{d}t$$
$$\leq \zeta(\varepsilon) \|X\|_{L^2(\Omega)} \|Y\|_{L^2(\Omega)},$$

where $\zeta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus for $Y = G(-\Delta_N)[X]$ we deduce that

$$\int_0^T \left\| \chi G(-\Delta_N) \exp\left(\mathrm{i}\sqrt{-\omega\Delta_N}\frac{t}{\varepsilon} \right) [X] \right\|^2_{L^2(\Omega)} \ \mathrm{d}t \tag{3.25}$$

$$\leq \zeta(\varepsilon) \|X\|^2_{L^2(\Omega)} \text{ for any } X \in L^2(\Omega),\ \zeta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Similarly, the integrals in (3.19) can be handled as follows:

$$\left\| \chi \int_0^T G(-\Delta_N) \exp\left(\mathrm{i}\frac{t-s}{\varepsilon} \sqrt{-\omega\Delta_N} \right) [Y(s)]\ \mathrm{d}s \right\|^2_{L^2((0,T)\times\Omega)} = \tag{3.26}$$

$$\int_0^T \left(\left\| \int_0^T \chi G(-\Delta_N) \exp\left(\mathrm{i}\frac{t-s}{\varepsilon} \sqrt{-\omega\Delta_N} \right) [Y(s)]\ \mathrm{d}s \right\|^2_{L^2(\Omega)} \right) \ \mathrm{d}t$$

$$\leq \int_0^T \int_0^T \left\| \chi G(-\Delta_N) \exp\left(\mathrm{i}\frac{t-s}{\varepsilon} \sqrt{-\omega\Delta_N} \right) [Y(s)] \right\|^2_{L^2(\Omega)} \ \mathrm{d}t\ \mathrm{d}s$$

$$\leq \zeta(\varepsilon) \int_0^T \left\| \exp\left(-\mathrm{i}\frac{s}{\varepsilon} \sqrt{-\omega\Delta_N} \right) [Y(s)] \right\|^2_{L^2(\Omega)} \ \mathrm{d}s = \zeta(\varepsilon) \int_0^T \|Y(s)\|^2_{L^2(\Omega)}\ \mathrm{d}s.$$

The above estimates provide decay in $L^2((0,T) \times K)$ of the projection of the acoustic potential on a compact part of the spectrum of $-\Delta_N$. However, as the velocity gradient is bounded in $L^2((0,T)\times\Omega)$ (see (3.23)), this piece of information is sufficient to establish the desired decay $\nabla_x\Phi\to 0$ in $L^2((0,T)\times K;\mathbb{R}^3)$, see [40].

The method based on RAGE theorem is rather simple but does not provide any qualitative information on the *rate* of the local decay determined by the function ζ. By the same token, the method is rather unstable with respect to possible small perturbations of the physical boundary. In the following section, we propose another method based on an abstract result of Kato [56].

3.5.3 Kato's theorem

An alternative method to study the local decay of acoustic waves is based on an abstract result of Kato [56] (see also Burq et al. [17], Reed and Simon [89, Theorem XIII.25 and Corollary]):

Theorem 3.2 *Let C be a closed densely defined linear operator and A a self-adjoint densely defined linear operator in a Hilbert space H. For $\lambda\notin\mathbb{R}$, let $R_A[\lambda]=(A-\lambda\mathrm{Id})^{-1}$ denote the resolvent of A. Suppose that*

$$\Gamma=\sup_{\lambda\notin\mathbb{R},\ v\in\mathcal{D}(C^*),\ \|v\|_H=1}\|C\circ R_A[\lambda]\circ C^*[v]\|_H<\infty. \tag{3.27}$$

Then

$$\sup_{w\in X,\ \|w\|_H=1}\frac{\pi}{2}\int_{-\infty}^{\infty}\|C\exp(-\mathrm{i}tA)[w]\|_X^2\ \mathrm{d}t\le\Gamma^2.$$

Similarly to the preceding part, we take $H=L^2(\Omega)$, $A=\sqrt{-\omega\Delta_N}$, and

$$C[v]=\chi G(-\Delta_N)[v],\ \chi\in C_c^\infty(\Omega),\ G\in C_c^\infty(0,\infty).$$

Taking, for a moment, the conclusion of Theorem 3.2 for granted, we obtain

$$\int_0^T\left\|\chi G(-\Delta_N)\exp\left(\mathrm{i}\sqrt{-\omega\Delta_N}\frac{t}{\varepsilon}\right)[X]\right\|_{L^2(\Omega)}^2\ \mathrm{d}t \tag{3.28}$$

$$\le\varepsilon\int_{-\infty}^{\infty}\left\|\chi G(-\Delta_N)\exp\left(\mathrm{i}\sqrt{-\omega\Delta_N}t\right)[X]\right\|_{L^2(\Omega)}^2\ \mathrm{d}t\le\varepsilon\Gamma^2\|X\|_{L^2(\Omega)}^2.$$

Consequently, Kato's theorem yields the same conclusion as RAGE theorem, with a specific decay rate $\zeta(\varepsilon)=\varepsilon\Gamma^2$. In addition, Kato's result does not require compactness of the operator C.

It remains to clarify the meaning of hypothesis (3.27). In the present setting, we have

$$C\circ\frac{1}{A-\lambda I}\circ C^*=\chi G(-\Delta_N)\frac{1}{\sqrt{-\omega\Delta_N}-\lambda}G(-\Delta_N)\chi;$$

whence, for G given, it is enough to consider λ in a bounded rectangle

$$0<\alpha\le\mathrm{Re}[\lambda]\le\beta<\infty,\ 0<|\mathrm{Im}[\lambda]|<\delta. \tag{3.29}$$

Furthermore, we have

$$\chi G(-\Delta_N)\frac{1}{\sqrt{-\omega\Delta_N}-\lambda}G(-\Delta_N)\chi = \chi\frac{1}{(-\omega\Delta_N)-\lambda^2}M(\lambda,-\Delta_N)\chi,$$

where we have set

$$M(\lambda,-\Delta_N) = G^2(-\Delta_N)((-\omega\Delta_N)+\lambda).$$

The operator $-\Delta_N$ satisfies the limiting absorption principle (see Eidus [33], Leis [66, Chapter 4.6], Vainberg [98, Chapter VIII.2]), specifically,

$$\sup_{\mu\in\mathbb{C};\alpha<\mathrm{Re}[\mu]<\beta;\mathrm{Im}[\mu]\neq 0}\left\|\mathcal{V}\circ\frac{1}{(-\Delta_N)-\mu}\circ\mathcal{V}\right\|_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]} \leq c(\alpha,\beta,\varphi)<\infty \tag{3.30}$$

for any choice of $0<\alpha<\beta<\infty$,

$$\mathcal{V}(x) = (1+|x|^2)^{-\frac{s}{2}},\ s>1,$$

provided Ω is an *exterior* domain. Thus hypothesis (3.27) will be satisfied provided (i) the operator $(-\Delta_N)$ satisfies the limiting absorption principle, and (ii)

$$\left\|\mathcal{V}^{-1}\circ M(\lambda,-\Delta_N)\circ\chi\right\|_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]}\leq c \tag{3.31}$$

for any λ belonging to the set (3.29).

In order to see (3.31), we follow Isozaki [55] writing

$$M(\lambda,-\Delta_N) = H(\sqrt{-\Delta_N}) = \int_{-\infty}^{\infty}\exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)\tilde{H}(t)\ \mathrm{d}t, \tag{3.32}$$

where $\tilde{H}$ is the Fourier transform of H. On the other hand,

$$\left\|(1+|x|^2)^{s/2}\exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)[\chi g]\right\|^2_{L^2(\Omega)}$$
$$=\int_\Omega(1+|x|^2)^s\left|\exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)[\chi g]\right|^2\ \mathrm{d}x.$$

However, since $w=\exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)[\chi g]$ solves the wave equation

$$\partial_t^2 w-\Delta_N w = 0$$

admitting a finite speed of propagation of order 1, we have

$$\int_\Omega(1+|x|^2)^s\left|\exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)[\chi g]\right|^2\ \mathrm{d}x$$
$$=\int_{|x|\leq t+r}(1+|x|^2)^s\left|\exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)[\chi g]\right|^2\ \mathrm{d}x,$$

where r is the radius of support of χ. Thus we may infer that

$$\int_{|x|\leq r+t} (1+|x|^2)^s \left| \exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)[\chi g]\right|^2 \,\mathrm{d}x$$

$$\leq c(1+t^{2s}) \left\| \exp\left(\mathrm{i}\sqrt{-\Delta_N}t\right)[\chi g]\right\|^2_{L^2(\Omega)},$$

which, together with (3.32), yields (3.31).

Thus validity of hypothesis (3.27) depends essentially on two factors:

- The speed of propagation in the (unscaled) acoustic equations.
- The limiting absorption principle (LAP). Several extensions of LAP to other classes of unbounded domains and more general elliptic operators are available, see Dermenjian and Guillot [27], Shimizu [95], among others.

3.6 Rescaled boundary conditions

A proper choice of boundary conditions plays an important role in fluid mechanics, cf. Section 2.3.5. This issue have been subjected to discussion for over two centuries by many distinguished scientists who developed the foundations of fluid mechanics, including Bernoulli, Coulomb, Navier, Couette, Poisson, Stokes, to mention only a few significant names. The commonly accepted hypothesis asserts that there is no relative motion between a *viscous* fluid, described by a velocity field $\vec{u}$ and the solid wall $\partial\Omega$, meaning,

$$[\vec{u}]_{\mathrm{tangential}}|_{\partial\Omega} = 0,$$

and

$$\vec{u}\cdot\vec{n}|_{\partial\Omega} = 0,$$

provided the wall is impermeable. These so-called *no-slip* boundary condition turned out to be extremely successful in reproducing the velocity profiles for macroscopic flows, in particular for incompressible fluids (liquids).

On the other hand, Navier suggested to replace the no-slip conditions by a hypothesis that

$$\beta[\vec{u}]_{\mathrm{tangential}} + [\mathbb{S}\vec{n}]_{\mathrm{tangential}} = 0 \text{ on } \partial\Omega, \tag{3.33}$$

discussed in Section 2.3.5. In the presence of slip, the fluid motion is opposed by a force proportional to the relative velocity between the fluid and the solid wall. Hypothesis (3.33) may be viewed as a convenient alternative to the no-slip condition whenever the rate of flow is sufficiently strong (turbulent regimes) and the medium is a compressible gas of low viscosity, typical for meteorological models.

Adopting (3.33) as a suitable condition for compressible viscous fluids, we may perform the incompressible limit taking the friction coefficient inversely proportional to a certain power of the Mach number, specifically,

$$[\vec{u}]_{\mathrm{tangential}} + \varepsilon^{\alpha}[\mathbb{S}\vec{n}]_{\mathrm{tangential}} = 0 \text{ on } \partial\Omega,\ \alpha > 0.$$

Accordingly, the no-slip boundary condition

$$\vec{u}|_{\partial\Omega} = 0$$

may be recovered in the incompressible limit as expected. Of course, the parameter $\alpha > 0$ must be chosen sufficiently small not to spoil the analysis of acoustic waves discussed in the preceding section.

Similarly, we can impose a "friction" force directly, adding a term proportional to

$$\frac{1}{\varepsilon^\alpha} 1_K \varrho \vec{u}$$

as a driving force in the momentum equation (2.35), where $K \subset \Omega$ is a closed set. Accordingly, the fluid motion vanishes on the set K in the asymptotic limit.

4 Applications

The last section presents a sample of well-established mathematical models that can be identified as incompressible limits of the full Navier-Stokes-Fourier system by means of the methods specified in the preceding part of this paper.

4.1 Oberbeck-Boussinesq approximation

One of the simplest ways to filter acoustic waves from the equations governing fluid motion is through the *Oberbeck-Boussinesq approximation* (OBA) (see Boussinesq [14], Oberbeck [87]), which is widely used to facilitate both theoretical analysis and numerical computation, see Zeytounian [102]. Very roughly indeed, we may say that the mechanical fluid properties in (OBA) are assumed to be temperature independent, apart from the density for which a linear temperature dependence is assumed. The thermal conductivity, viscosity, specific heat remain constant while the density is allowed to vary linearly with temperature in the term representing buoyancy.

The *Oberbeck-Boussinesq approximation* reads:

$$\mathrm{div}_x \vec{U} = 0, \tag{4.1}$$

$$\overline{\varrho}\Big(\partial_t \vec{U} + \mathrm{div}_x(\vec{U} \otimes \vec{U})\Big) + \nabla_x \Pi = \mu \Delta \vec{U} + r \nabla_x F, \tag{4.2}$$

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Big[\partial_t \Theta + \mathrm{div}_x(\Theta \vec{U})\Big] - \kappa \Delta \Theta - \overline{\varrho}\,\overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) \mathrm{div}_x(F \vec{U}) = 0, \tag{4.3}$$

where we have set

$\alpha(\overline{\varrho}, \overline{\vartheta}) = \frac{1}{\overline{\varrho}} \frac{\partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})}$	coefficient of thermal expansion,
$c_p(\overline{\varrho}, \overline{\vartheta}) = \partial_\vartheta e(\overline{\varrho}, \overline{\vartheta}) + \alpha(\overline{\varrho}, \overline{\vartheta}) \frac{\overline{\vartheta}}{\overline{\varrho}} \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta})$	specific heat at constant pressure.

Finally, r and Θ are interrelated through *Boussinesq equation*

$$r + \overline{\varrho}\alpha(\overline{\varrho}, \overline{\vartheta})\Theta = 0. \tag{4.4}$$

The fluid is incompressible moving with velocity $\vec{U}$, the symbol Θ denotes a small deviation of the temperature from its reference value $\overline{\vartheta}$, and $\nabla_x F$ represents the gravitational force acting on the fluid. Typically,

$$-\Delta F = d,$$

where d denotes the mass density of the hard core(the Earth or a star) located *outside* the fluid domain Ω. Accordingly, the potential F is harmonic in Ω, and equation (4.3) may be rewritten in terms of a new variable

$$\Theta \approx \Theta - \frac{\overline{\vartheta}\alpha(\overline{\varrho}, \overline{\vartheta})}{c_p(\overline{\varrho}, \overline{\vartheta})} F.$$

Oberbeck-Boussinesq system can be rigorously justified as a singular limit of the scaled Navier-Stokes-Fourier system (3.1 - 3.5) provided $\mathrm{Ma} = \varepsilon$, $\mathrm{Fr} = \sqrt{\varepsilon}$, with the remaining characteristic numbers of order 1. More specifically, if $\{\varrho_\varepsilon, \vartheta_\varepsilon, \vec{u}_\varepsilon\}$ is a family of solutions to the scaled system, then

$$\frac{\varrho_\varepsilon - \tilde{\varrho}_\varepsilon}{\varepsilon} \to r, \ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta$$

and

$$\vec{u}_\varepsilon \to \vec{U} \text{ as } \varepsilon \to 0,$$

where $\tilde{\varrho}_\varepsilon$ are solutions of the scaled static equation

$$\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon \tilde{\varrho}_\varepsilon \nabla_x F \text{ in } \Omega,$$

see [41, Chapter 5]. Convergence takes place in different topologies, depending on the geometry of the domain and a particular choice of the initial data.

It may seem that Oberbeck-Boussinesq system "leaks" energy. Indeed, setting all parameters to be one and assuming insulating boundary conditions, we derive easily the total "energy balance" in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \left(|\vec{U}|^2 + |\Theta|^2 \right) \ \mathrm{d}x + \int_\Omega \left(\mu |\nabla_x \vec{U}|^2 + \kappa |\nabla_x \Theta|^2 \right) \ \mathrm{d}x = 0.$$

However, we have to keep in mind that Θ should be interpreted as a deviation from an equilibrium temperature rather than the temperature itself.

4.2 Anelastic approximation

There are numerous applications of mathematical fluid dynamics motivated by problems arising in *astrophysics*. However, investigations in this field are hampered by both theoretical and observational problems. The vast range of different scales that extend in the case of stars from the stellar radius to 10^2 m or even less entirely prevents a complex numerical as well as analytical solution. Progress in this field therefore calls for a combination of physical intuition with rigorous analysis of highly simplified mathematical models.

A typical example is the flow dynamics observed in stellar radiative zones that represents a major challenge of the current theory of stellar interiors. Under these circumstances, the fluid behaves as a plasma characterized by the following features:

- A strong *radiative transport* predominates the molecular one. This is due to extremely hot and energetic radiation fields prevailing the plasma. Accordingly, the Péclet number Pe is vanishingly small.
- Strong *stratification effects* prevail because of the enormous gravitational potential of gaseous celestial bodies.
- The convective motions are much slower than the speed of sound; whence the Mach number Ma is small. The fluid is therefore almost *incompressible*, whereas the density variations can be simulated via a so-called *anelastic approximation*, see Gilman and Glatzmaier [47], [48], Gough [51], Lipps and Hemler [74].

Chandrasekhar [20] proposed a simple alternative to Oberbeck-Boussinesq approximation discussed in the previous section in the case when both Froude and Péclet numbers are small. More recently, Lignières [70] identified a similar system as a suitable model of flow dynamics in stellar radiative zones. The system consists of the following equations:

- *hydrostatic balance equation*

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F;$$

- *anelastic constraint*

$$\mathrm{div}_x (\tilde{\varrho} \vec{U}) = 0$$

- *balance of linear momentum*

$$\partial_t (\tilde{\varrho} \vec{U}) + \mathrm{div}_x (\tilde{\varrho} \vec{U} \otimes \vec{U}) + \tilde{\varrho} \nabla_x \Pi$$
$$= \mu \mathrm{div}_x \Big(\nabla_x \vec{U} + \nabla_x \vec{U}^t - \frac{2}{3} \mathrm{div}_x \vec{U} \mathbb{I} \Big) + \eta \nabla_x \mathrm{div}_x \vec{U} - \frac{\Theta}{\overline{\vartheta}} \tilde{\varrho} \nabla_x F,$$

where Θ and $\vec{U}$ are interrelated through

$$\tilde{\varrho} \nabla_x F \cdot \vec{U} + \kappa(\overline{\vartheta}) \Delta \Theta = 0.$$

The system can be identified as a singular limit of the complete Navier-Stokes-Fourier system, with $\mathrm{Ma} = \mathrm{Fr} = \varepsilon$, $\mathrm{Pe} = \varepsilon^2$. Specifically,

$$\varrho_\varepsilon \to \tilde{\varrho}, \ \vec{u}_\varepsilon \to \vec{U},$$

and

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} \to \Theta$$

as $\varepsilon \to 0$, see [41, Chapter 6] for the proof in special geometries.

The anelastic approximation possesses essentially the same structure as the standard incompressible Navier-Stokes system, however, the number of theoretical studies devoted to this model is considerably lower. Besides other interesting new features, mathematical analysis of the problem requires a modification of the standard Helmholtz decomposition in the form

$$\vec{v} = \vec{H}_{\tilde{\varrho}}[\vec{v}] + \tilde{\varrho} \nabla_x \Phi,$$

where

$$\mathrm{div}_x(\tilde{\varrho} \nabla_x \Phi) = \mathrm{div}_x \vec{v} \text{ in } \Omega, \ \ \tilde{\varrho} \nabla_x \Phi \cdot \vec{n} = \vec{v} \cdot \vec{n} \text{ on } \partial\Omega.$$

Moreover, as pointed out in Section 3.3.3, the propagation of sound is governed by a modified wave equation with variable sound speed.

Numerous examples of models of fluids under stratification can be found in Majda's monograph [77].

5 Conclusion

We have surveyed some recent results on scale analysis of the full Navier-Stokes-Fourier system endowed with conservative boundary conditions. Our approach was based on the concept of weak solutions in the spirit of Leray [67] and P.-L.Lions [71], developed in the context of complete fluid systems in the monographs [39], [41]. Although the class of solutions seems very general, in particular, in view of very mild restrictions imposed on the entropy production rate, we have seen that the theory is sufficiently robust with respect to various kinds of singular limits. In contrast with the more standard approach, based on the concept of strong solutions (see Klainerman and Majda [57], [58]), the present framework does not impose any essential restrictions on the size of the data and the length of the relevant time interval. This property is useful when studying real world applications and their numerical analysis. Of course, we focused on several particular problems and left apart many other interesting aspects of this area of continuum fluid mechanics. In particular, it is interesting to note apparent similarity with the arguments used in the passage between discrete (kinetic) and continuous fluid models, see Bardos et al. [7], [8], [9], Golse and Saint-Raymond [49], [50].

The major drawback of the method is its dependence on the standard energy estimates for the Navier-Stokes system based on the control of the entropy production rate in the whole process of incompressible limits. In particular, the initial distribution of both the density and the temperature must be close to the (global) equilibrium state. There are several interesting problems arising in low Mach number combustion, where the variations of the temperature and the density are arbitrary and only the pressure remains close to a constant, see Majda [76], Majda and Sethian [78].

Last but not least, we ignored completely in this study the enormous amount of results concerning the inviscid *inviscid fluid* modeled by the Euler system. The interested reader may consult the seminal paper by Klainerman and Majda [57], as well as the later studies by Métivier and Schochet [83], [84], Schochet [91], [92], and the references therein.

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