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GLOBAL STABILITY OF SIR EPIDEMIC MODELS WITH A WIDE CLASS OF NONLINEAR INCIDENCE RATES AND DISTRIBUTED DELAYS

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Abstract. In this paper, we establish the global asymptotic stability of equilibria for an SIR model of infectious diseases with distributed time delays governed by a wide class of nonlinear incidence rates. We obtain the global properties of the model by proving the permanence and constructing a suitable Lyapunov functional. Under some suitable assumptions on the nonlinear term in the incidence rate, the global dynamics of the model is completely determined by the basic reproduction number R_0 and the distributed delays do not influence the global dynamics of the model.

1. **Introduction.** Mathematical models which describe the dynamics of infectious diseases have played a crucial role in the disease control in epidemiological aspect. In order to understand the mechanism of disease transmission, many authors have proposed various kinds of epidemic models (see [1]-[23] and the references therein).

One of the basic SIR epidemic models is given as follows (see Hethcote [8]).

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu N(t) - \dfrac{\beta S(t)I(t)}{N(t)} - \mu S(t), \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t)I(t)}{N(t)} - (\mu+\sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

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where $N(t) \equiv S(t) + I(t) + R(t)$. The initial condition of system (1.1) is $S(0) \geq 0$, $I(0) \geq 0$ and $R(0) \geq 0$ with $N(0) = S(0) + I(0) + R(0) \equiv N_0 > 0$. For system (1.1), since $N'(t) = 0$ holds for all $t \geq 0$, we have that $N(t) \equiv N_0$ for all $t \geq 0$.

$S(t)$, $I(t)$ and $R(t)$ denote the proportions of the population susceptible to the disease, of infective members and of members who have been removed from the possibility of infection, respectively. Hence, $N(t)$ denotes the total population size. μ represents the birth rate of the population and the death rates of susceptibles, infected and recovered individuals. We assume that all newborns are susceptibles. σ represents the recovery rate of infectives, and β represents the product of the average number of contacts of an individual per unit time. All the coefficients μ, σ and β are assumed to be positive. For system (1.1), individuals leave the susceptible class at a rate $\frac{\beta S(t)I(t)}{N(t)}$, which is called standard incidence rate. By defining

$$\tilde{S}(t) = \frac{S(t)}{N_0},\ \tilde{I}(t) = \frac{I(t)}{N_0},\ \tilde{R}(t) = \frac{R(t)}{N_0}, \tag{1.2}$$

and dividing the equations in (1.1) by the constant total population size N_0 yields the following form ("˜" is dropped for convenience of readers).

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \beta S(t)I(t) - \mu S(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + \sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t). \end{cases} \tag{1.3}$$

On the other hand, many authors have suggested that the bilinear incidence rate should be modified into a nonlinear incidence rate because the effect concerning the nonlinearity of incidence rates has been observed for some disease transmissions. For example, Capasso and Serio [4] studied the cholera epidemic spread in Bari in 1973 and introduced an incidence rate which takes a form $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$, and Brown and Hasibuan [3] studied infection model of the two-spotted spider mites, *Tetranychus urticae* and introduced an incidence rate which takes a form $(S(t)I(t))^b$. In order to study the impact of those nonlinearity, Korobeinikov and Maini [10] considered a variety of models with the incidence rate of the form $F(S(t))G(I(t))$. Later, Korobeinikov [11, 12] obtained the global properties of the following basic SIR epidemic model with more general framework of the incidence rate.

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - f(S(t), I(t)) - \mu S(t), \\ \dfrac{dI(t)}{dt} = f(S(t), I(t)) - (\mu + \sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t). \end{cases} \tag{1.4}$$

However, we consider that the proof of Korobeinikov [12, Theorem 3.3] as not complete, because to apply Lyapunov-LaSalle asymptotic stability theorem (see LaSalle and Lefschetz [15]) to the stability analysis of the positive equilibrium, we need the persistence or permanence result of the model, which is not found in Korobeinikov [11, 12]. Moreover, it is advocated that more realistic models should incorporate time delays, which enable us to investigate the spread of an infectious disease transmitted by a vector (e.g. mosquitoes, rats, etc.) after an incubation time denoting the time during which the infectious agents develop in the vector (see

[2, 5]). This is called the phenomena of time delay effect which now has important biological meanings in epidemic models.

In this paper, we establish the global asymptotic stability of equilibria for an SIR epidemic model with a wide class of nonlinear incidence rates and distributed delays by modifying Lyapunov functional techniques in Huang et al. [9], Korobeinikov [11, 12] and McCluskey [18, 19]. Our results indicate that the global dynamics is fully determined by a single threshold number R_0 independently of time delay effects under some biologically feasible hypotheses on the nonlinearity of the incidence rate.

The organization of this paper is as follows. In Section 2, for an SIR epidemic model with a wide class of nonlinear incidence rates and distributed delays, we establish our main results. In Section 3, we offer a basic result. In Section 4, we show the global stability of the disease-free equilibrium of the system. In Section 5, we show the permanence of the system and establish the global asymptotic stability of the positive equilibrium for the system with using a key lemma (see Lemma 5.1). Finally, we offer a discussion in Section 6.

2. **Main results.** In the present paper, we consider the following SIR epidemic model with a wide class of nonlinear incidence rates and distributed delays:

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \displaystyle\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \mu S(t), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \sigma) I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t), \end{cases} \tag{2.1}$$

with the initial conditions

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad -h \le \theta \le 0, \quad h > 0, \tag{2.2}$$

where $\varphi_i \in C$ $(i = 1, 2, 3)$ such that $\varphi_i(\theta) = \varphi_i(0) \ge 0$ $(-h \le \theta \le 0,\ i = 1, 3)$ and $\varphi_2(\theta) \ge 0$ $(-h \le \theta \le 0)$. C denotes the Banach space $C([-h, 0], \mathbb{R}^3_{+0})$ of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^3_{+0}$ with the supremum norm, where $\mathbb{R}^n_{+0} = \{(x_1, \cdots, x_n) | x_i \ge 0, i = 1, \cdots, n\}$ for $n \ge 1$. From a biological meaning, we assume that $\varphi_i(0) > 0$ for $i = 1, 2, 3$.

h is a maximum time taken to become infectious and the transmission of the infection is governed by an incidence rate $\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau$. Here, $p(\tau)$ denotes the fraction of vector population in which the time taken to become infectious is τ [21]. We assume that $p(\tau)$ is continuous on $[0, h]$ satisfying $\int_0^h p(\tau) d\tau = 1$, and $f : \mathbb{R}^2_{+0} \to \mathbb{R}_{+0}$ is continuously differentiable in the interior of $\mathbb{R}^2_{+0}$ satisfying $f(0, I) = f(S, 0) = 0$ for $S, I \ge 0$ and the following hypotheses.

(H1) $f(S, I)$ is a strictly monotone increasing function of $S \ge 0$, for any fixed $I > 0$, and a monotone increasing function of $I \ge 0$, for any fixed $S \ge 0$,

(H2) $\phi(S, I) = \frac{f(S,I)}{I}$ is a bounded and monotone decreasing function of $I > 0$, for any fixed $S \ge 0$, and $K(S) \equiv \lim_{I \to +0} \phi(S, I)$ is continuous on $S \ge 0$.

We note that $K(S) > 0$ holds for any $S > 0$. The basic reproduction number of system (2.1) becomes

$$R_0 = \frac{K(S_0)}{\mu + \sigma}, \ S_0 = 1. \tag{2.3}$$

R_0 denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population.

System (2.1) always has a disease-free equilibrium $E_0 = (S_0, 0, 0)$. On the other hand, under the hypotheses (H1) and (H2), if $R_0 > 1$, then system (2.1) also admits a unique positive equilibrium $E_* = (S^*, I^*, R^*)$, where $S^*, I^*, R^* > 0$ satisfying the following equations (see Korobeinikov [11, 12]).

$$\begin{cases} \mu - \mu S^* - f(S^*, I^*) = 0, \\ f(S^*, I^*) - (\mu + \sigma) I^* = 0, \\ \sigma I^* - \mu R^* = 0. \end{cases}$$

Our main theorems are as follows.

Theorem 2.1. *Assume that the hypotheses (H1) and (H2) hold. Then the disease-free equilibrium E_0 of system (2.1) is the only equilibrium and globally asymptotically stable, if and only if $R_0 \leq 1$.*

Theorem 2.2. *Assume that the hypotheses (H1) and (H2) hold. Then the positive equilibrium E_* of system (2.1) is globally asymptotically stable, if and only if $R_0 > 1$.*

Under the hypotheses (H1) and (H2), for a class of delayed epidemic models, $f(S, I)$ includes various special incidence rates. If $f(S, I) = \beta SI$, then the incidence rate becomes a bilinear form, which is proposed in [16, 17, 18, 21] and if $f(S, I) = \frac{\beta SI}{1+\alpha I}$, then the incidence rate describes saturated effects of the prevalence of infectious diseases, which is proposed in [4, 19, 23]. In addition, $f(S, I) = F(S)G(I)$, then the incidence rate is of the form proposed in Huang et al. [9].

In this paper, by modifying techniques in Huang et al. [9], Korobeinikov [11, 12] and McCluskey [18, 19], we complete the proof of the global asymptotic stability of the disease-free equilibrium E_0 and positive equilibrium E_* for system (2.1) with a wide class of nonlinear incidence rate $\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau$ under some biologically suitable assumptions on the function f. We note that the monotone properties (H1) and (H2) of the function f for the global stability of equilibria essentially agree with those in Korobeinikov [12, Theorems 3.1-3.3]. Furthermore, it is remarkable that the restriction of time delay is no longer needed in ensuring the global asymptotic stability of the equilibria of the model. This implies that an incubation period does not influence the global dynamics if the infection incidence rate satisfies the hypotheses (H1) and (H2).

3. **Preliminary.**

In this section, we prove the following basic result, which guarantees the existence and uniqueness of the solution $(S(t), I(t), R(t))$ for system (2.1) satisfying initial conditions (3.4).

Lemma 3.1. *The solution $(S(t), I(t), R(t))$ of system (2.1) with initial conditions (3.4) uniquely exists and is positive for all $t \geq 0$. Furthermore, it holds that*

$$\lim_{t \to +\infty} (S(t) + I(t) + R(t)) = 1. \tag{3.1}$$

Proof. We notice that the right hand side of system (2.1) is completely continuous and locally Lipschitzian on C. Then, it follows that the solution $(S(t), I(t), R(t))$ of system (2.1) exists and is unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to prove that $S(t) > 0$ for all $t \in [0, \alpha)$. Indeed, this follows from that $\dot{S}(t) = \mu > 0$ for any $t \in [0, \alpha)$ when $S(t) = 0$. Let us now show that $I(t) > 0$ for all $t \in [0, \alpha)$. Suppose on the contrary that there exists some $t_1 \in (0, \alpha)$ such that $I(t_1) = 0$ and $I(t) > 0$

for $t \in [0, t_1)$. Integrating the second equation of system (2.1) from 0 to t_1, we see that

$$I(t_1) = I(0)\mathrm{e}^{-(\mu+\sigma)t_1} + \int_0^{t_1}\int_0^h p(\tau) f(S(u), I(u-\tau))\mathrm{e}^{-(\mu+\sigma)(t_1-u)} d\tau du > 0.$$

This contradicts $I(t_1) = 0$. From the third equation of system (2.1), we also have that $R(t) > 0$ for all $t \in [0, \alpha)$. Furthermore, for $t \in [0, \alpha)$, we obtain that

$$\begin{aligned} \dot{N}(t) &= \mu - \mu(S(t) + I(t) + R(t)) \\ &= \mu(1 - N(t)), \end{aligned} \tag{3.2}$$

which implies that $(S(t), I(t), R(t))$ is uniformly bounded on $[0, \alpha)$. It follows that $(S(t), I(t), R(t))$ exists and is unique and positive for all $t \geq 0$. From (3.2), we immediately obtain (3.1), which completes the proof. □

Since the variable R does not appear in the first and the second equations of system (2.1), we omit the third equation of system (2.1). Thus, we consider the following 2-dimensional system.

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \displaystyle\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \mu S(t), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \sigma) I(t). \end{cases} \tag{3.3}$$

with initial conditions

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad -h \leq \theta \leq 0, \tag{3.4}$$

where $\varphi_i \in C$ $(i = 1, 2, 3)$ such that $\varphi_i(\theta) = \varphi_i(0) > 0$ $(-h \leq \theta \leq 0,\ i = 1, 3)$ and $\varphi_2(\theta) \geq 0$ $(-h \leq \theta \leq 0)$ with $\varphi_2(0) > 0$.

4. **Global stability of the disease-free equilibrium for $R_0 \leq 1$.**

In this section, we give a proof of the global asymptotic stability of the disease-free equilibrium $E_0 = (S_0, 0, 0)$ of system (2.1) for $R_0 \leq 1$. The following theorem indicates that the disease can be eradicated in the host population if $R_0 \leq 1$.

Theorem 4.1. *Assume that the hypotheses (H1) and (H2) hold. Then the disease-free equilibrium $Q_0 \equiv (S_0, 0)$ of system (3.3) is the only equilibrium and globally asymptotically stable, if and only if $R_0 \leq 1$.*

Proof. From the hypotheses (H1) and (H2), the disease-free equilibrium is the only equilibrium for system (3.3) (see Korobeinikov [12]). We now consider the following Lyapunov functional.

$$U^0(t) = U_1^0(t) + I(t) + U_+^0(t),$$

where

$$\begin{aligned} U_1^0(t) &= \int_{S_0}^{S(t)} \left(1 - \frac{K(S_0)}{K(s)}\right) ds, \\ U_+^0(t) &= \int_0^h p(\tau) \int_{t-\tau}^t f(S(u+\tau), I(u)) \frac{K(S_0)}{K(S(u+\tau))} du d\tau. \end{aligned}$$

We show that $\frac{dU^0(t)}{dt} \leq 0$ for all $t \geq 0$. First, we calculate $\frac{dU_1^0(t)}{dt}$. By using $\mu = \mu S_0$,

$$\begin{aligned}\frac{dU_1^0(t)}{dt} &= \left(1 - \frac{K(S_0)}{K(S(t))}\right)\left(\mu - \int_0^h p(\tau)f(S(t), I(t-\tau))d\tau - \mu S(t)\right)\\ &= -\mu(S(t) - S_0)\left(1 - \frac{K(S_0)}{K(S(t))}\right)\\ &\quad - \left(1 - \frac{K(S_0)}{K(S(t))}\right)\int_0^h p(\tau)f(S(t), I(t-\tau))d\tau.\end{aligned}$$

Second, calculating $\frac{dU_+^0(t)}{dt}$, we get that

$$\frac{dU_+^0(t)}{dt} = \int_0^h p(\tau)\left\{f(S(t+\tau), I(t))\frac{K(S_0)}{K(S(t+\tau))} - f(S(t), I(t-\tau))\frac{K(S_0)}{K(S(t))}\right\}d\tau.$$

Therefore, it follows that

$$\begin{aligned}&\frac{dU^0(t)}{dt}\\ = &-\mu(S(t) - S_0)\left(1 - \frac{K(S_0)}{K(S(t))}\right) - \left(1 - \frac{K(S_0)}{K(S(t))}\right)\int_0^h p(\tau)f(S(t), I(t-\tau))d\tau\\ &+ \int_0^h p(\tau)f(S(t), I(t-\tau))d\tau - (\mu+\sigma)I(t)\\ &+ \int_0^h p(\tau)\left\{f(S(t+\tau), I(t))\frac{K(S_0)}{K(S(t+\tau))} - f(S(t), I(t-\tau))\frac{K(S_0)}{K(S(t))}\right\}d\tau\\ = &-\mu(S(t) - S_0)\left(1 - \frac{K(S_0)}{K(S(t))}\right)\\ &+ \int_0^h p(\tau)\left\{\frac{\phi(S(t+\tau), I(t))}{\mu+\sigma}\cdot\frac{K(S_0)}{K(S(t+\tau))} - 1\right\}(\mu+\sigma)I(t)d\tau.\end{aligned}$$

By the hypothesis (H1), we obtain that

$$-\mu(S(t) - S_0)\left(1 - \frac{K(S_0)}{K(S(t))}\right) \leq 0,$$

where strict equality holds if and only if $S(t) = S_0$. It follows from the hypothesis (H2) that

$$\begin{aligned}\frac{\phi(S(t+\tau), I(t))}{\mu+\sigma}\cdot\frac{K(S_0)}{K(S(t+\tau))} &\leq \frac{K(S(t+\tau))}{\mu+\sigma}\cdot\frac{K(S_0)}{K(S(t+\tau))}\\ &= \frac{K(S_0)}{\mu+\sigma} = R_0.\end{aligned}$$

Therefore, $R_0 \leq 1$ ensures that $\frac{dU^0(t)}{dt} \leq 0$ for all $t \geq 0$, where $\frac{dU^0(t)}{dt} = 0$ holds if $S(t) = S_0$. Hence, it follows from system (3.3) that Q_0 is the largest invariant set in $\{(S(t), I(t)) \in C \times C | \frac{dU^0(t)}{dt} = 0\}$. From the Lyapunov-LaSalle asymptotic stability theorem [13, Theorem 5.3], we obtain that Q_0 is the only equilibrium of system (3.3) and globally asymptotically stable. This completes the proof. □

Proof of Theorem 2.1. By Theorem 4.1, we immediately obtain the conclusion of this theorem. □

Remark 4.1. To establish the global asymptotic stability of the disease-free equilibrium E_0 for $R_0 \leq 1$, the hypothesis of the monotonicity of $f(S, I)$ of $I \geq 0$ for any fixed $S \geq 0$ in (H1) is not necessary.

5. Permanence and global stability of the positive equilibrium for $R_0 > 1$.

In this section, we show the permanence and the global asymptotic stability of the positive equilibrium $E_* = (S^*, I^*, R^*)$ for system (2.1) for $R_0 > 1$.

5.1. Permanence.

In this subsection, we show the permanence of system (2.1) by using techniques in Song et al. [20] and Wang [22]. From (3.1), let us put sufficiently small $\varepsilon_S > 0$ and sufficiently large $T_S > 0$ satisfying $K(S(t)) \leq K(S_0) + \varepsilon_S$ holds for any $t \geq T_S$. The following theorem indicates that the disease eventually persists in the host population if $R_0 > 1$.

Theorem 5.1. *Assume that the hypotheses (H1) and (H2) hold. If $R_0 > 1$, then for any solution of system (2.1), it holds that*

$$\begin{cases} \liminf\limits_{t\to+\infty} S(t) \geq v_1, \\ \liminf\limits_{t\to+\infty} I(t) \geq v_2 := qI^* \exp\left(-(\mu+\sigma)\rho h\right), \\ \liminf\limits_{t\to+\infty} R(t) \geq v_3 := \dfrac{\gamma v_2}{\mu}, \end{cases}$$

where $v_1 > 0$ satisfies $\mu - K(v_1) - \mu v_1 = 0$, and q and ρ satisfy

$$S^* < \frac{\mu - (K(S_0)+\varepsilon_S)qI^*}{\mu}\left(1 - \mathrm{e}^{-\mu\rho h}\right), \ 0 < q < \frac{\mu}{(K(S_0)+\varepsilon_S)I^*}, \ \rho \geq 1. \tag{5.1}$$

Proof. Let $(S(t), I(t), R(t))$ be a solution of system (2.1) with initial condition (3.4). By Lemma 3.1, it follows that $\limsup_{t\to+\infty} I(t) \leq 1$, which implies from the first equation of system (2.1) and the hypothesis (H2) that, for any $\varepsilon_I > 0$, there is an integer $T_I \geq 0$ such that

$$\begin{aligned} \frac{dS(t)}{dt} &= \mu - \int_0^h p(\tau)\frac{f(S(t), I(t-\tau))}{I(t-\tau)} I(t-\tau)d\tau - \mu S(t) \\ &\geq \mu - K(S(t))\int_0^h p(\tau)I(t-\tau)d\tau - \mu S(t) \\ &= \mu - K(S(t))(1+\varepsilon_I) - \mu S(t), \end{aligned}$$

for $t \geq T_I + h$. Let us now consider the auxiliary equation

$$\frac{dS(t)}{dt} = \mu - K(S(t)) - \mu S(t).$$

Then, one can immediately obtain that $\lim_{t\to+\infty} S(t) = v_1 > 0$. Since (5.2) holds for arbitrary $\varepsilon_I > 0$ sufficiently small, it follows that $\liminf_{t\to+\infty} S(t) \geq v_1 > 0$.

We now prove that it is impossible that $I(t) \leq qI^*$ for all sufficiently large t. Suppose on the contrary that there exists a sufficiently large $t_1 \geq T_S$ such that $I(t) \leq qI^*$ holds for all $t \geq t_1$. Then, similar to the above discussion, we have that for any $t \geq t_1 + h$,

$$\begin{aligned} \frac{dS(t)}{dt} &= \mu - \int_0^h p(\tau)\phi(S(t), I(t-\tau))I(t-\tau)d\tau - \mu S(t) \\ &\geq \mu - (K(S_0)+\varepsilon_S)qI^* - \mu S(t), \end{aligned}$$

which yields for $t \geq t_1 + h$,

$$\begin{aligned} S(t) &\geq S(t_1+h)\mathrm{e}^{-\mu(t-t_1-h)} + \mathrm{e}^{-\mu t}\int_{t_1+h}^{t} \mathrm{e}^{\mu s}(\mu - (K(S_0)+\varepsilon_S)qI^*)ds \\ &= S(t_1+h)\mathrm{e}^{-\mu(t-t_1-h)} + \frac{\mu - (K(S_0)+\varepsilon_S)qI^*}{\mu}\left(1-\mathrm{e}^{-\mu(t-t_1-h)}\right). \end{aligned} \tag{5.2}$$

Hence, it follows from (5.2) that for $t \geq t_1 + h + \rho h$,

$$\begin{aligned} S(t) &> \frac{\mu - (K(S_0)+\varepsilon_S)qI^*}{\mu}\left(1-\mathrm{e}^{-\mu\rho h}\right) \\ &= S^{\triangle} > S^*. \end{aligned} \tag{5.3}$$

Now, we define the following functional.

$$V(t) = I(t) + \int_0^h p(\tau)\int_t^{t+\tau} f(S(u), I(u-\tau))du d\tau. \tag{5.4}$$

Calculating the derivative of $V(t)$ along solutions of system (2.1) gives as follows.

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_0^h p(\tau)f(S(t), I(t-\tau))d\tau - (\mu+\sigma)I(t) \\ &\quad + \int_0^h p(\tau)\left\{f(S(t+\tau), I(t)) - f(S(t), I(t-\tau))\right\}d\tau \\ &= \int_0^h p(\tau)f(S(t+\tau), I(t))d\tau - (\mu+\sigma)I(t). \end{aligned}$$

For $t \geq t_1 + h + \rho h$, it follows from (5.3) and the relation $\mu + \sigma = \phi(S^*, I^*)$ that

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_0^h p(\tau)\left\{\phi(S(t+\tau), I(t)) - (\mu+\sigma)\right\}I(t)d\tau \\ &> \int_0^h p(\tau)\left\{\phi(S(t+\tau), I^*) - \phi(S^*, I^*) + \phi(S^*, I^*) - (\mu+\sigma)\right\}I(t)d\tau \\ &= \int_0^h p(\tau)\{\phi(S(t+\tau), I^*) - \phi(S^*, I^*)\}I(t)d\tau \\ &\geq \{\phi(S^{\triangle}, I^*) - \phi(S^*, I^*)\}I(t). \end{aligned} \tag{5.5}$$

Setting

$$\underline{i} = \min_{\theta\in[-h,0]} I(\theta + t_1 + \rho h + 2h),$$

we claim that $I(t) \geq \underline{i}$ for all $t \geq t_1 + h + \rho h$. Otherwise, if there is a $T \geq 0$ such that $I(t) \geq \underline{i}$ for $t_1 + h + \rho h \leq t \leq t_1 + 2h + \rho h + T$, $I(t_1 + 2h + \rho h + T) = \underline{i}$ and $\frac{d}{dt}I(t)|_{t=t_1+2h+\rho h+T} \leq 0$, it follows from the second equation of system (2.1), the hypotheses (H1) and (H2) that for $t_2 = t_1 + 2h + \rho h + T$,

$$\begin{aligned}
\frac{dI(t)}{dt}\Big|_{t=t_2} &= \int_0^h p(\tau)f(S(t_2), I(t_2-\tau))d\tau - (\mu+\sigma)I(t_2) \\
&= \int_0^h p(\tau)\phi(S(t_2), I(t_2-\tau))I(t_2-\tau)d\tau - (\mu+\sigma)I(t_2) \\
&> \int_0^h p(\tau)\phi(S(t_2), I^*)I(t_2-\tau)d\tau - (\mu+\sigma)I(t_2) \\
&\geq \{\phi(S(t_2), I^*) - (\mu+\sigma)\}\, I(t_2) \\
&\geq \left\{\phi(S^{\triangle}, I^*) - (\mu+\sigma)\right\} \underline{i} \\
&> \{\phi(S^*, I^*) - (\mu+\sigma)\}\, \underline{i} = 0.
\end{aligned}$$

This is a contradiction. Therefore $I(t) \geq \underline{i}$ for all $t \geq t_1 + h + \rho h$. It follows from (5.5) that

$$\frac{dV(t)}{dt} > \{\phi(S^{\triangle}, I^*) - \phi(S^*, I^*)\}\underline{i} > 0, \text{ for } t \geq t_1 + 2h + \rho h,$$

which implies that $\lim_{t\to+\infty} V(t) = +\infty$. However, it holds from (3.1) and (5.4) that $\limsup_{t\to+\infty} V(t) < +\infty$. Hence the claim holds.

Thus, we proved that it is impossible that $I(t) \leq qI^*$ for all sufficiently large t. This implies that we are left to consider the following two possibilities.

(i) $I(t) \geq qI^*$ for all t sufficiently large,
(ii) $I(t)$ oscillates about qI^* for all t sufficiently large.

If the first case holds, then we immediately get the conclusion of the proof. If the second case holds, we show that $I(t) \geq qI^* \exp(-(\mu+\sigma)\rho h)$ for all t sufficiently large. Let $t_3 < t_4$ be sufficiently large such that

$$I(t_3) = I(t_4) = qI^*, \; I(t) < qI^*, \; t_3 < t < t_4.$$

If $t_4 - t_3 \leq \rho h$, then it follows from the second equation of system (3.3) that

$$\frac{dI(t)}{dt} > -(\mu+\sigma)I(t),$$

that is,

$$\begin{aligned}
I(t) &> I(t_3)\exp(-(\mu+\sigma)(t-t_3)) \\
&\geq qI^* \exp(-(\mu+\sigma)\rho h) = v_2.
\end{aligned}$$

If $t_4 - t_3 > \rho h$, we obtain from the second equation of system (3.3) that $I(t) \geq v_2$ for $t_3 \leq t \leq t_3 + \rho h$. We now claim that $I(t) \geq v_2$ for all $t_3 + \rho h \leq t \leq t_4$. Otherwise, there is a $T^* > 0$ such that $I(t) \geq v_2$ for $t_3 \leq t \leq t_3 + \rho h + T^* < t_4$, $I(t_3 + \rho h + T^*) = v_2$ and $\frac{dI(t)}{dt}|_{t=t_3+\rho h+T^*} \leq 0$. On the other hand, for $t_0 = t_3 + \rho h + T^*$, it follows from the second equation of system (3.3) and the relation

$\phi(S(t_0), I(t_0)) > \phi(S(t_0), I^*) \geq \phi(S^{\triangle}, I^*) > \phi(S^*, I^*)$ that

$$\begin{aligned} \frac{dI(t)}{dt}\Big|_{t=t_0} &= \int_0^h p(\tau) f(S(t_0), I(t_0-\tau)) d\tau - (\mu+\sigma) I(t_0) \\ &= \int_0^h p(\tau)\phi(S(t_0), I(t_0-\tau)) I(t_0-\tau) d\tau - (\mu+\sigma) I(t_0) \\ &> \{\phi(S(t_0), I^*) - (\mu+\sigma)\} I(t_0) \\ &\geq \left\{\phi(S^{\triangle}, I^*) - (\mu+\sigma)\right\} I(t_0) \\ &> \{\phi(S^*, I^*) - (\mu+\sigma)\} I(t_0) = 0, \end{aligned}$$

which is a contradiction. Hence $I(t) \geq qI^* \exp(-(\mu+\sigma)\rho h) = v_2$ for $t_3 \leq t \leq t_4$. Since the interval $[t_3, t_4]$ is arbitrarily chosen, we conclude that $I(t) \geq v_2$ for all t sufficiently large for the second case. Thus, we obtain that

$$\liminf_{t\to+\infty} I(t) \geq v_2.$$

From the above discussion, one can see immediately that

$$\liminf_{t\to+\infty} R(t) \geq v_3.$$

Hence, this completes the proof. □

5.2. **Global stability of the positive equilibrium.** In this subsection, we give a proof of the global asymptotic stability of the positive equilibrium E_* for $R_0 > 1$.

For a fixed $0 \leq \tau \leq h$, we put

$$y_t = \frac{I(t)}{I^*},\ \tilde{y}_{t,\tau} = \frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}. \tag{5.6}$$

The following lemma plays a key role to obtain Theorems 2.2 and 5.2.

Lemma 5.1. *Assume that system (2.1) has a positive equilibrium E_*. Under the hypotheses (H1) and (H2), it holds that*

$$g(y_t) - g(\tilde{y}_{t,\tau}) \geq 0, \tag{5.7}$$

for all $t \geq 0$ and $0 \leq \tau \leq h$, where $g(x) = x - 1 - \ln x \geq 0$, for $x > 0$.

Proof. By the definitions of y_t and $\tilde{y}_{t,\tau}$, we have that

$$\tilde{y}_{t,\tau} - 1 = \frac{f(S(t+\tau), I(t)) - f(S(t+\tau), I^*)}{f(S(t+\tau), I^*)},$$

and

$$\begin{aligned} y_t - \tilde{y}_{t,\tau} &= \frac{I(t)}{I^*} - \frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)} \\ &= \frac{I(t)}{f(S(t+\tau), I^*)} \{\phi(S(t+\tau), I^*) - \phi(S(t+\tau), I(t))\}. \end{aligned}$$

Then, it follows from the hypotheses (H1) and (H2) that

$$\begin{aligned} (y_t - \tilde{y}_{t,\tau})(\tilde{y}_{t,\tau} - 1) &= \frac{I(t)}{f(S(t+\tau), I^*)^2} \{\phi(S(t+\tau), I^*) - \phi(S(t+\tau), I(t))\} \\ &\quad \times \{(f(S(t+\tau), I(t)) - f(S(t+\tau), I^*)\} \geq 0, \end{aligned}$$

that is, either $y_t \leq \tilde{y}_{t,\tau} \leq 1$ or $y_t \geq \tilde{y}_{t,\tau} \geq 1$ holds for all $t \geq 0$ and $0 \leq \tau \leq h$. Since $g'(x) = 1 - \frac{1}{x}$ for all $x > 0$ and $g'(1) = 0$, it follows that $g(y_t) \geq g(\tilde{y}_{t,\tau}) \geq 0$. This completes the proof. □

Now, we are in a position to prove the global asymptotic stability of the positive equilibrium E_* for $R_0 > 1$, by applying the technique established by Huang et al. [9], Korobeinikov [11, 12] and McCluskey [18, 19].

Theorem 5.2. *Assume that the hypotheses (H1) and (H2) hold. Then the positive equilibrium $Q_* \equiv (S^*, I^*)$ of the reduced system (3.3) is globally asymptotically stable, if and only if $R_0 > 1$.*

Proof. We now define the following functional.

$$U^*(t) = U_1^*(t) + U_+^*(t), \tag{5.8}$$

where

$$\begin{cases} U_1^*(t) = \displaystyle\int_{S^*}^{S(t)} \left(1 - \frac{f(S^*, I^*)}{f(s, I^*)}\right) ds + I(t) - I^* - I^* \ln \frac{I(t)}{I^*}, \\ U_+^*(t) = f(S^*, I^*) \displaystyle\int_0^h p(\tau) \int_{t-\tau}^t g\left(\frac{f(S(u+\tau), I(u))}{f(S(u+\tau), I^*)}\right) du d\tau. \end{cases} \tag{5.9}$$

We here note that $U_1^*(t)$ satisfies $\frac{\partial U_1^*}{\partial S} = 1 - \frac{f(S^*, I^*)}{f(S, I^*)}$ and $\frac{\partial U_1^*}{\partial I} = 1 - \frac{I^*}{I}$, which implies that the point $(S(t), I(t)) = (S^*, I^*)$ is a stational point of the function $U_1^*(t)$ and it is the unique stational point and the global minimum of this function.

Using the relation $\mu = \mu S^* + f(S^*, I^*)$ and $\mu + \sigma = \frac{f(S^*, I^*)}{I^*}$, the time derivative of the function $U_1^*(t)$ along the positive solution of system (3.3) becomes

$$\begin{aligned} &\frac{dU_1^*(t)}{dt} \\ = &\left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right)\left\{\mu - \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \mu S(t)\right\} \\ &+\left(1 - \frac{I^*}{I(t)}\right)\left(\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu+\sigma) I(t)\right) \\ = &\left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right)\left\{\int_0^h p(\tau)\{f(S^*, I^*) - f(S(t), I(t-\tau))\} d\tau - \mu(S(t) - S^*)\right\} \\ &+\left(1 - \frac{I^*}{I(t)}\right)\left(\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \frac{(S^*, I^*)}{I^*} I(t)\right) \\ = &\ \mu S^*\left(1 - \frac{S(t)}{S^*}\right)\left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \\ &+ f(S^*, I^*)\left(1 - \frac{f(S^*, I^*)}{f(S(t), I^*)}\right) \int_0^h p(\tau)\left\{1 - \frac{f(S(t), I(t-\tau))}{f(S^*, I^*)}\right\} d\tau \\ &+ f(S^*, I^*)\left(1 - \frac{I^*}{I(t)}\right) \int_0^h p(\tau)\left\{\frac{f(S(t), I(t-\tau))}{f(S^*, I^*)} - \frac{I(t)}{I^*}\right\} d\tau, \end{aligned} \tag{5.10}$$

and the time derivative of the function $U_+^*(t)$ becomes

$$\frac{dU_+^*(t)}{dt} = f(S^*, I^*) \int_0^h p(\tau)\left\{g\left(\frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}\right) - g\left(\frac{f(S(t), I(t-\tau))}{f(S(t), I^*)}\right)\right\} d\tau. \tag{5.11}$$

From (5.10) and (5.11), we obtain that

$$\begin{aligned}
&\frac{dU^*(t)}{dt}\\
= \;& \mu S^*\left(1-\frac{S(t)}{S^*}\right)\left(1-\frac{f(S^*,I^*)}{f(S(t),I^*)}\right)\\
&+f(S^*,I^*)\int_0^h p(\tau)\left(1-\frac{f(S^*,I^*)}{f(S(t),I^*)}+\frac{f(S(t),I(t-\tau))}{f(S(t),I^*)}\right)d\tau\\
&+f(S^*,I^*)\int_0^h p(\tau)\left(1-\frac{I(t)}{I^*}-\frac{I^*}{I(t)}\frac{f(S(t),I(t-\tau))}{f(S^*,I^*)}\right)d\tau\\
&+f(S^*,I^*)\int_0^h p(\tau)\left\{g\left(\frac{f(S(t+\tau),I(t))}{f(S(t+\tau),I^*)}\right)-g\left(\frac{f(S(t),I(t-\tau))}{f(S(t),I^*)}\right)\right\}d\tau\\
= \;& \mu S^*\left(1-\frac{S(t)}{S^*}\right)\left(1-\frac{f(S^*,I^*)}{f(S(t),I^*)}\right)\\
&+f(S^*,I^*)\int_0^h p(\tau)\left\{g\left(\frac{f(S(t+\tau),I(t))}{f(S(t+\tau),I^*)}\right)-g\left(\frac{I(t)}{I^*}\right)\right\}d\tau\\
&-f(S^*,I^*)\int_0^h p(\tau)\left\{g\left(\frac{f(S^*,I^*)}{f(S(t),I^*)}\right)+g\left(\frac{I^*}{I(t)}\frac{f(S(t),I(t-\tau))}{f(S^*,I^*)}\right)\right\}d\tau. \quad (5.12)
\end{aligned}$$

From the hypothesis (H1), we obtain

$$\left(1-\frac{S(t)}{S^*}\right)\left(1-\frac{f(S^*,I^*)}{f(S(t),I^*)}\right)\le 0,$$

with strict equality holds if and only if $S(t)=S^*$, and using Lemma 5.1, we have

$$g\left(\frac{f(S(t+\tau),I(t))}{f(S(t+\tau),I^*)}\right)-g\left(\frac{I(t)}{I^*}\right)\le 0, \text{ for all } 0\le\tau\le h.$$

This implies that $\frac{dU^*(t)}{dt}\le 0$ holds for all $t\ge 0$ since S^* and $f(S^*,I^*)$ are non-negative. Therefore, it follows from (5.12) that $\frac{dU^*(t)}{dt}=0$ holds if $S(t)=S^*$ and $f(S^*,I(t-\tau))=\frac{f(S^*,I^*)}{I^*}I(t)$ for almost all $\tau\in[0,h]$. By Hale and Lunel [6, Theorem 5.3.1], solutions of system (3.3) limit to M, the largest invariant subset of $\{\frac{dU^*(t)}{dt}=0\}$. We now show that M consists of only the positive equilibrium Q_*. For each element of M, we have $S(t)=S^*$ and, since M is invariant, $\frac{dS(t)}{dt}=0$. Using the first equation of system (3.3) and the relation $\mu=\mu S^*+f(S^*,I^*)$, we obtain that

$$\begin{aligned}
0 &= \frac{dS(t)}{dt}\\
&= \mu-\int_0^h p(\tau)f(S^*,I(t-\tau))d\tau-\mu S^*\\
&= \mu-\frac{f(S^*,I^*)}{I^*}I(t)-\mu S^*\\
&= \mu S^*+f(S^*,I^*)-\frac{f(S^*,I^*)}{I^*}I(t)-\mu S^*\\
&= f(S^*,I^*)\left(1-\frac{I(t)}{I^*}\right).
\end{aligned}$$

Thus, each element of M satisfies $S(t) = S^*$ and $I(t) = I^*$. Since the permanence result (see Lemma 3.1 and Theorem 5.1) for system (3.3) is already known, by an extension of LaSalle invariance principle [13, Corollary 5.2], Q^* is the only equilibrium of system (3.3) on the line and globally asymptotically stable. Hence, the proof is complete. □

Proof of Theorem 2.2. By Lemma 3.1, Theorems 5.1 and 5.2, we immediately obtain the conclusion of this theorem. □

6. **Discussion.** In this paper, we establish the global asymptotic stability of the disease-free equilibrium for $R_0 \leq 1$, and the positive equilibrium for $R_0 > 1$ by modifying Lyapunov functional techniques in Huang et al. [9], Korobeinikov [11, 12] and McCluskey [18, 19]. From a biological motivation, we do not only extend the nondelayed model (1.4) in Korobeinikov [11, 12] to the delayed model (2.1) but also obtain the permanence result and the global properties for (2.1) with distributed time delays governed by a wide class of nonlinear incidence rate $\int_0^h p(\tau)f(S(t), I(t-\tau))d\tau$. It is noteworthy that the global dynamics is completely determined by the basic reproduction number R_0 independently of the length of an incubation period of the diseases as long as the infection rate has a suitable monotone property characterized by the hypotheses (H1) and (H2).

It has been generally considered reasonable to expect that a biologically feasible functional response is associated with monotonicity with respect to the proportion of susceptible and infected individuals, and is concave, or at least nonconvex with respect to the proportion of infective individuals (see, e.g., [4, 10, 11, 12]). Noting that $\phi(S, I) = \frac{f(S,I)}{I}$ denotes the infection force per unit proportion of infective individuals, the hypotheses that $f(S, I)$ is monotone increasing of I and $\phi(S, I)$ is monotone decreasing of I in (H1) and (H2) describe the crowding (saturation) effects (see, e.g., [4, 19, 23]). Thus, one can see that the hypotheses (H1) and (H2) are natural assumptions which have a biological meaning. Our result further indicates that the disease dynamics is fully determined when the saturation effects appear.

Finally, we have to stress that Lemma 5.1 plays a vital role to establish the global asymptotic stability of the positive equilibrium E_* of system (2.1) for $R_0 > 1$. These techniques are also applicable to various kinds of epidemic models (e.g. SIRS models, SEIR models, etc.). These will be our future consideration.

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