Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

ON THE GLOBAL STABILITY OF AN SIRS EPIDEMIC MODEL WITH DISTRIBUTED DELAYS

Yukihiko Nakata

Basque Center for Applied Mathematics
Bizkaia Technology Park, Building 500 E-48160 Derio, Spain

Yoichi Enatsu

Department of Pure and Applied Mathematics
Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo, 169-8555, Japan

Yoshiaki Muroya

Department of Mathematics
Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo, 169-8555, Japan

Abstract. In this paper, we establish the global asymptotic stability of an endemic equilibrium for an SIRS epidemic model with distributed time delays. It is shown that the global stability holds for any rate of immunity loss, if the basic reproduction number is greater than 1 and less than or equals to a critical value. Otherwise, there is a maximal rate of immunity loss which guarantees the global stability. By using an extension of a Lyapunov functional established by [C.C. McCluskey, Complete global stability for an SIR epidemic model with delay-Distributed or discrete, *Nonlinear Anal. RWA.* **11** (2010) 55-59], we provide a partial answer to an open problem whether the endemic equilibrium is globally stable, whenever it exists, or not.

1. **Introduction and main result.** Currently, one of the important issues to us is controlling the transmission dynamics of infectious diseases. Epidemic models by a system of differential equations have been studied in the literature (see [1]-[13] and the references therein). By analyzing such mathematical models, we obtain further knowledge regarding the disease dynamics and this will give us a useful information to determine whether the disease dies out or not in the host population.

One of the basic mathematical models in epidemiology is called SIR (Susceptible-Infected-Recovered) epidemic model (see Hethcote [4]). Based on the model, for vector-borne diseases, Beretta and Takeuchi [1] introduced distributed delays and considered the delay effect on global stability of equilibria of the system. Their

2000 *Mathematics Subject Classification.* Primary: 34K20 and 34K25; Secondary: 92D30.

Key words and phrases. SIRS epidemic model, global asymptotic stability, distributed delays, Lyapunov functional.

The first author was partially supported by the Grant MTM2010-18318 of the MICINN, Spanish Ministry of Science and Innovation. The second author was partially supported by JSPS Fellows, No.237213 of Japan Society for the Promotion of Science. The third author was partially supported by Scientific Research (c), No.21540230 of Japan Society for the Promotion of Science.

model is given by the following system of delay differential equations:

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu S(t) - \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau, \\ \dfrac{dI(t)}{dt} = \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - (\mu+\gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu R(t). \end{cases} \tag{1.1}$$

$S(t)$, $I(t)$ and $R(t)$ denote the population of susceptible, infective and recovered individuals at time t, respectively. It is assumed that all newborns are susceptible. The positive constant B represents the total number of newborns in the host population per unit time and the positive constant μ represents the death rate of the population. The positive constant β is the transmission coefficient and h is a superior limit of incubation times. The incubation period distribution $f(\tau)$, which denotes the fraction of vector population in which the time taken to become infectious is τ, is assumed to be continuous on $[0,h]$ satisfying $\int_0^h f(\tau)d\tau = 1$ and $f(\tau) \geq 0$ for $0 \leq \tau \leq h$. The positive constant γ represents the recovery rate of infectives.

System (1.1) always has a disease-free equilibrium $E_0 = (S_0, 0, 0)$, $S_0 = B/\mu$ corresponding to the extinction of the disease. Beretta and Takeuchi [1] showed that E_0 is globally asymptotically stable for any $h > 0$ when system (1.1) does not have any equilibria other than E_0. If $\frac{B\beta}{\mu(\mu+\gamma)} > 1$, then there exists a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$, where every component is strictly positive. They also showed that E_* is locally asymptotically stable for any $h > 0$. However, on the global stability analysis, they required that h should be small enough to show the global stability of the endemic equilibrium. The global stability of E_* for any $h > 0$ remained unsolved for a long time. For this problem, McCluskey [6] introduced the following Lyapunov functional:

$$U^0(t) = \frac{1}{\beta I^*} U_S(t) + \frac{1}{\beta S^*} U_I(t) + U_+(t), \tag{1.2}$$

where

$$\begin{cases} U_S(t) = g\left(\dfrac{S(t)}{S^*}\right), \ U_I(t) = g\left(\dfrac{I(t)}{I^*}\right), \ U_+(t) = \displaystyle\int_0^h \alpha(\tau) g\left(\dfrac{I(t-\tau)}{I^*}\right) d\tau, \\ g(x) = x - 1 - \ln x \geq g(1) = 0 \text{ for } x > 0, \ \alpha(\tau) = \displaystyle\int_\tau^h f(\sigma) d\sigma. \end{cases} \tag{1.3}$$

Then they proved that E_* is globally asymptotically stable for any $h > 0$ whenever it exists. In particular, $U_+(t)$ in (1.2) plays an important role to obtain that the global stability does not depend on the duration of the incubation period $h > 0$.

On the other hand, system (1.1) is insufficient to characterize the disease transmission dynamics when we take account of an immunity loss of the diseases. In this situation, recovered individual become susceptible again when the temporary immunity fades away and then, individual moves cyclically among three compartments. Such a population dynamics is described by SIRS epidemic models (see Mena-Lorcat and Hethcote [7]). Considering such an immunity loss, we study the following SIRS epidemic model with distributed delays based on system (1.1) (cf.

[11, 12, 13]):

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu S(t) - \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - (\mu + \gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu + \delta) R(t). \end{cases} \tag{1.4}$$

The state variables and parameters except δ have the same meaning as those in system (1.1). $\delta \geq 0$ denotes the rate at which recovered individuals lose immunity and return to the susceptible class. Zhen et al. [13] also studied similar system and added a disease-related death rate to the second equation of (1.4). The initial conditions for system (1.4) take the form

$$\begin{cases} S(\theta) = \phi_1(\theta),\ I(\theta) = \phi_2(\theta),\ R(\theta) = \phi_3(\theta), \\ \phi_i(\theta) \geq 0,\ \theta \in [-h, 0],\ \phi_i(0) > 0,\ \phi_i \in C^+,\ i = 1, 2, 3, \end{cases} \tag{1.5}$$

where C denotes the Banach space $C([-h, 0], \mathbb{R})$ of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}$ equipped with the sup-norm $||\psi|| = \sup_{-h \leq \theta \leq 0} |\psi(\theta)|$. The nonnegative cone of C is defined as $C^+ = C([-h, 0], \mathbb{R}^+)$. Then, system (1.4) has a unique solution $(S(t), I(t), R(t))$ satisfying $S(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for all $t > 0$ (see [12, 13]).

System (1.4) also has the disease-free equilibrium $E_0 = (S_0, 0, 0)$. By Zhen et al. [13], it is proved that E_0 is globally asymptotically stable for any $h > 0$ when there is no equilibrium other than E_0 (see also Theorem 2.1 below). Similar to system (1.1), the basic reproduction number of system (1.4) is given by

$$R_0 = \frac{B\beta}{\mu(\mu + \gamma)} \tag{1.6}$$

and if $R_0 > 1$, then system (1.4) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$, where

$$S^* = \frac{\mu + \gamma}{\beta},\ I^* = \frac{\mu + \delta}{\mu + \delta + \gamma} S_0 \left(1 - \frac{1}{R_0}\right),\ R^* = \frac{\gamma}{\mu + \delta + \gamma} S_0 \left(1 - \frac{1}{R_0}\right). \tag{1.7}$$

By Zhen et al. [13], it is shown that E_* is locally asymptotically stable for any $h > 0$ and their analysis allows considering the disease-related death rate. However, on the global stability analysis, they also required that the length of the incubation period $h > 0$ and the rate of immunity loss $\delta \geq 0$ should be small enough, since their approach is a modified Lyapunov functional from that of Takeuchi [10] for system (1.1). In contrast, for the corresponding SIRS epidemic model without delay, the endemic equilibrium is globally asymptotically stable whenever it exists (see [2, 7]). Therefore, stability results in [2, 6, 7] motivate us to expect that if the endemic equilibrium of system (1.4) exists, it is globally asymptotically stable for any $h > 0$ as well as system (1.1). This is still an open problem now, as the stability problem for system (1.1) had been unsolved. Concerning SIRS epidemic models with distributed delay which has a nonlinear incidence rate, Muroya et al. [8, 9] and Enatsu et al. [3] established conditions for the global stability of the endemic equilibrium. Muroya et al. [8, 9] developed a monotone iterative scheme for the model and Enatsu et al. [3] constructed a Lyapunov functional using the nonlinearity of the incidence form.

In this paper, a partial answer to the open problem is given by constructing a new Lyapunov functional which combines the Lyapunov functional (1.2) used in McCluskey [6] with a quadratic term for the state variable $R(t)$. For a limit system of (1.4), we consider the following functional:

$$U(t) = U^0(t) + \frac{\delta}{\gamma\beta(S^*)^2 I^*} \cdot \frac{(R(t) - R^*)^2}{2},$$

with (1.7). The limit system (see (3.1) below) is derived by using the relations $I(t) = N(t) - S(t) - R(t)$ and $\lim_{t\to+\infty} N(t) = B/\mu$ to $I(t)$ in the third equation of (1.4), where $N(t) = S(t) + I(t) + R(t)$ is the total population. We establish the following result for the global stability of the endemic equilibrium E_* of system (1.4).

Theorem 1.1. *Let $R_0 > 1$. If*

$$\mu S^* - \delta R^* \geq 0, \tag{1.8}$$

then the endemic equilibrium E_* of system (1.4) is globally asymptotically stable for any $h > 0$. In particular, (1.8) is equivalent to

$$\begin{cases} 0 \leq \delta < +\infty, & \text{for} \quad 1 < R_0 \leq 1 + \dfrac{\mu}{\gamma}, \\ 0 \leq \delta \leq \overline{\delta} := \dfrac{\mu}{\frac{R_0}{1+\frac{\mu}{\gamma}} - 1}, & \text{for} \quad R_0 > 1 + \dfrac{\mu}{\gamma}. \end{cases} \tag{1.9}$$

For system (1.4), this is the first global stability result which does not depend on $h > 0$. In addition, since the monotone iterative method in Xu and Ma [11] for a class of delayed SIRS models used the advantage of the nonlinearity on the incidence function and Lyapunov functional technique in Zhen et al. [13] relies on the positivity of the disease-related death rate, their approaches are not directly applicable for system (1.4). Theorem 1.1 is a partial answer to the problem, whether the endemic equilibrium is globally asymptotically stable whenever it exists or not. In particular, we show that the rate of immunity loss δ does not give any influence for the global stability when $1 < R_0 \leq 1 + \frac{\mu}{\gamma}$. Therefore, biologically, if the basic reproduction number lies in the interval determined by the death rate and the recovery rate, then the disease transmission will eventually equilibrate to the endemic steady state for any rate of immunity loss and any duration of the incubation period. In the other cases, we establish the maximal rate of immunity loss which guarantees the global stability of the endemic steady state. Since (1.8) is always true if $\delta = 0$, we extend the result given by McCluskey [6] (see also Figure 1).

The organization of this paper is as follows. In Section 2, we give basic results for system (1.4) by Zhang and Teng [12] and Zhen et al. [13]. One can see that the basic reproduction number R_0 determines whether the disease dies out or uniformly persists in the population. In Section 3, we study the global stability of the endemic equilibrium and give the proof of Theorem 1.1. At first, we introduce a limit system of (1.4) and then construct a Lyapunov functional for the system. The Lyapunov functional, used in the proof of Theorem 1.1, combines the functional given by McCluskey [6] for system (1.1) and a quadratic term for $R(t)$. In Section 4, some examples are given. A numerical simulation may support our conjecture that the endemic equilibrium is globally asymptotically stable even if (1.8) does not hold.

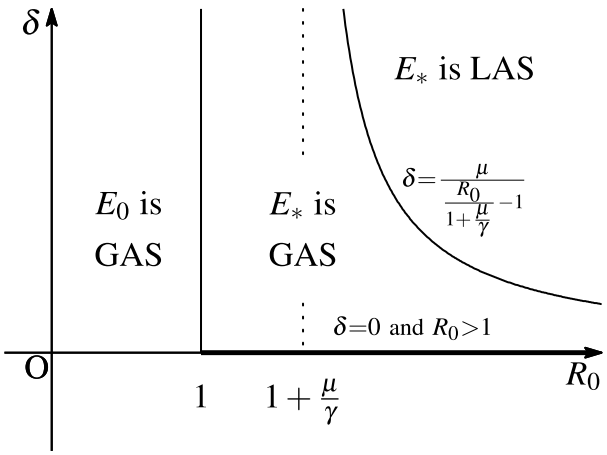

FIGURE 1. Stability region of the disease-free equilibrium and the endemic equilibrium in the parameter space $(R_0,\ \delta)$. The curve denotes the boundary of the global stability region for the endemic equilibrium. LAS and GAS denote locally asymptotically stable and globally asymptotically stable, respectively.

2. Known results.

Lemma 2.1. *For any solution $(S(t), I(t), R(t))$ of system (1.4) with initial conditions (1.5), it holds that $\lim_{t\to+\infty}(S(t)+I(t)+R(t)) = B/\mu$.*

Proof. From system (1.4), it follows that

$$\frac{dN(t)}{dt} = B - \mu(S(t)+I(t)+R(t)) = B - \mu N(t),$$

which implies that $\lim_{t\to+\infty} N(t) = B/\mu$. Hence, this completes the proof. □

The following results from Zhen et al. [13] and Zhang and Teng [12] will be used in the proof of Theorem 1.1, indicating that the disease dies out for $R_0 < 1$ and persists in the population for $R_0 > 1$.

Theorem 2.1. *(Zhen et al.* [13, Theorem 3.1]*.)*

(i) Let $R_0 < 1$. Then the disease-free equilibrium E_0 of system (1.4) is globally asymptotically stable for any $h > 0$.

(ii) Let $R_0 > 1$. Then the disease-free equilibrium E_0 is unstable and the endemic equilibrium E_* of system (1.4) is locally asymptotically stable for any $h > 0$.

Lemma 2.2. *(Zhang and Teng* [12, Propositions 3.2 and 3.3]*.) Let $R_0 > 1$. Then, for any solution of system (1.4) with initial condition (1.5), it holds that*

$$\begin{cases} \liminf\limits_{t\to+\infty} S(t) \geq v_1 := \dfrac{B}{\mu+\beta B/\mu}, \\ \liminf\limits_{t\to+\infty} I(t) \geq v_2 := qI^* \exp\left(-(\mu+\gamma)(h+\rho h)\right),\ \liminf\limits_{t\to+\infty} R(t) \geq v_3 := \dfrac{\gamma v_2}{\mu+\delta}, \end{cases}$$

where $0 < q < 1$ and $\rho > 0$ satisfy $S^* < S^{\triangle} := \frac{B}{k}(1-\exp(-k\rho h))$, $k = \mu + \beta q I^*$.

3. **Global stability of the endemic equilibrium E_* for $R_0 > 1$.** In this section, we study the global stability of E_*. From Lemma 2.1, system (1.4) has the following limit system:

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu S(t) - \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t) \displaystyle\int_0^h f(\tau) I(t-\tau) d\tau - (\mu + \gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma \left(\dfrac{B}{\mu} - S(t) - R(t) \right) - (\mu + \delta) R(t). \end{cases} \tag{3.1}$$

For system (3.1), we obtain the following result:

Theorem 3.1. *Let $R_0 > 1$. If (1.8) holds, then the endemic equilibrium E_* of system (3.1) is globally asymptotically stable for any $h > 0$. In particular, (1.8) is equivalent to (1.9).*

Proof. We construct the following Lyapunov functional which combines (1.2) by McCluskey [6] for system (1.1) with a quadratic term for $R(t)$:

$$U(t) = \frac{1}{\beta I^*} U_S(t) + \frac{1}{\beta S^*} U_I(t) + U_+(t) + \frac{\delta}{\gamma \beta (S^*)^2 I^*} U_R(t),$$

where $U_S(t)$, $U_I(t)$ and $U_+(t)$ are defined in (1.3) and $U_R(t) = \frac{1}{2}(R(t) - R^*)^2$. We now show that $\frac{dU(t)}{dt} \le 0$. We use the following notations if necessary.

$$x_t = \frac{S(t)}{S^*}, \ y_t = \frac{I(t)}{I^*}, \ y_{t,\tau} = \frac{I(t-\tau)}{I^*}, \ z_t = \frac{R(t)}{R^*}.$$

First, we calculate $\frac{dU_S(t)}{dt}$. Substituting $B = \mu S^* + \beta S^* I^* - \delta R^*$ gives as follows:

$$\begin{aligned} \frac{dU_S(t)}{dt} &= \frac{1}{S^*} \left(1 - \frac{S^*}{S(t)} \right) \left(B - \mu S(t) - \beta S(t) \int_0^h f(\tau) I(t-\tau) d\tau + \delta R(t) \right) \\ &= -\mu \left(1 - \frac{S^*}{S(t)} \right) \left(\frac{S(t)}{S^*} - 1 \right) + \frac{\delta R^*}{S^*} \left(1 - \frac{S^*}{S(t)} \right) \left(\frac{R(t)}{R^*} - 1 \right) \\ &\quad + \beta I^* \int_0^h f(\tau) \left(1 - \frac{S^*}{S(t)} \right) \left(1 - \frac{S(t)}{S^*} \cdot \frac{I(t-\tau)}{I^*} \right) d\tau \\ &= -\mu \left(1 - \frac{1}{x_t} \right) (x_t - 1) + \frac{\delta R^*}{S^*} \left(1 - \frac{1}{x_t} \right) (z_t - 1) \\ &\quad + \beta I^* \int_0^h f(\tau) \left(1 - \frac{1}{x_t} \right) (1 - x_t y_{t,\tau}) \, d\tau. \end{aligned} \tag{3.2}$$

We secondly calculate $\frac{dU_I(t)}{dt}$. Substituting $\mu + \gamma = \beta S^*$, we obtain

$$\begin{aligned} \frac{dU_I(t)}{dt} &= \frac{1}{I^*} \left(1 - \frac{I^*}{I(t)} \right) \left(\beta S(t) \int_0^h f(\tau) I(t-\tau) d\tau - \beta S^* I(t) \right) \\ &= \beta S^* \int_0^h f(\tau) \left(1 - \frac{I^*}{I(t)} \right) \left(\frac{S(t)}{S^*} \cdot \frac{I(t-\tau)}{I^*} - \frac{I(t)}{I^*} \right) d\tau \\ &= \beta S^* \int_0^h f(\tau) \left(1 - \frac{1}{y_t} \right) (x_t y_{t,\tau} - y_t) d\tau. \end{aligned} \tag{3.3}$$

Next, we calculate $\frac{dU_+(t)}{dt}$. Since $U_+(t) = \int_0^h f(\tau) \int_{t-\tau}^t g(\frac{I(s)}{I^*})dsd\tau$ holds, we have that

$$\frac{dU_+(t)}{dt} = \int_0^h f(\tau)(g(y_t) - g(y_{t,\tau}))d\tau. \tag{3.4}$$

By applying techniques on deformation of equation in McCluskey [6], we use the following relation:

$$\begin{aligned} & \left(1 - \frac{1}{x_t}\right)(1 - x_t y_{t,\tau}) + \left(1 - \frac{1}{y_t}\right)(x_t y_{t,\tau} - y_t) + (g(y_t) - g(y_{t,\tau})) \\ = \; & 2 - \frac{1}{x_t} + y_{t,\tau} - \frac{x_t y_{t,\tau}}{y_t} - y_t + (y_t - \ln y_t - y_{t,\tau} + \ln y_{t,\tau}) \\ = \; & -g\left(\frac{1}{x_t}\right) - g\left(\frac{x_t y_{t,\tau}}{y_t}\right) \leq 0. \end{aligned}$$

Then, combining (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned} & \frac{d}{dt}\left(\frac{1}{\beta I^*}U_S(t) + \frac{1}{\beta S^*}U_I(t) + U_+(t)\right) \\ = \; & \frac{\mu}{\beta I^*}\left(1 - \frac{1}{x_t}\right)(x_t - 1) + \frac{\delta R^*}{\beta S^* I^*}\left(1 - \frac{1}{x_t}\right)(z_t - 1) \\ & - \int_0^h f(\tau)\left(g\left(\frac{1}{x_t}\right) + g\left(\frac{x_t y_{t,\tau}}{y_t}\right)\right)d\tau. \end{aligned} \tag{3.5}$$

Since $B = \mu S^* + (\mu+\gamma)I^* - \delta R^*$ and $\gamma I^* = (\mu+\delta)R^*$ imply that $\frac{B}{\mu} = S^* + \frac{\mu+\gamma+\delta}{\gamma}R^*$, calculating $\frac{dU_R(t)}{dt}$ gives

$$\begin{aligned} \frac{dU_R(t)}{dt} &= (R(t) - R^*)\left\{\gamma\left(\frac{B}{\mu} - S(t) - R(t)\right) - (\mu+\delta)R(t)\right\} \\ &= (R(t) - R^*)\left\{-\gamma(S(t) - S^*) - (\mu + \gamma + \delta)(R(t) - R^*)\right\} \\ &= \gamma R^* S^*(z_t - 1)(1 - x_t) - (\mu + \gamma + \delta)(R^*)^2(z_t - 1)^2. \end{aligned} \tag{3.6}$$

Under the condition (1.8), we have

$$\begin{aligned} & \frac{\mu}{\beta I^*}\left(1 - \frac{1}{x_t}\right)(x_t - 1) + \frac{\delta R^*}{\beta S^* I^*}\left(1 - \frac{1}{x_t}\right)(z_t - 1) + \frac{\delta R^*}{\beta S^* I^*}(z_t - 1)(1 - x_t) \\ = \; & \frac{\mu}{\beta I^*}\left(2 - x_t - \frac{1}{x_t}\right) + \frac{\delta R^*}{\beta S^* I^*}(z_t - 1)\left\{\left(1 - \frac{1}{x_t}\right) + (1 - x_t)\right\} \\ = \; & \frac{1}{\beta S^* I^*}\left\{\mu S^* + \delta R^*(z_t - 1)\right\}\left(2 - x_t - \frac{1}{x_t}\right) \\ \leq \; & \frac{1}{\beta S^* I^*}(\mu S^* - \delta R^*)\left(2 - x_t - \frac{1}{x_t}\right) \leq 0. \end{aligned}$$

Therefore, combining (3.5) and (3.6), it follows that

$$\begin{aligned}
&\frac{dU(t)}{dt}\\
=\ & \frac{\mu}{\beta I^*}\left(1-\frac{1}{x_t}\right)(x_t-1)+\frac{\delta R^*}{\beta S^* I^*}\left(1-\frac{1}{x_t}\right)(z_t-1)+\frac{\delta R^*}{\beta S^* I^*}(z_t-1)(1-x_t)\\
& -\frac{\delta(\mu+\gamma+\delta)(R^*)^2}{\gamma\beta(S^*)^2 I^*}(z_t-1)^2-\int_0^h f(\tau)\left(g\left(\frac{1}{x_t}\right)+g\left(\frac{x_t y_{t,\tau}}{y_t}\right)\right)d\tau\\
\leq\ & -\frac{\delta(\mu+\gamma+\delta)(R^*)^2}{\gamma\beta(S^*)^2 I^*}(z_t-1)^2-\int_0^h f(\tau)\left(g\left(\frac{1}{x_t}\right)+g\left(\frac{x_t y_{t,\tau}}{y_t}\right)\right)d\tau\leq 0.
\end{aligned}$$

Let M be the largest invariant set of $\{\frac{dU(t)}{dt}=0\}$ that is invariant with respect to system (3.1). We show that M consists of only the endemic equilibrium E_*. Recalling that $g(\frac{1}{x_t})+g(\frac{x_t y_{t,\tau}}{y_t})=0$, if and only if $x_t=1$ and $x_t y_{t,\tau}/y_t=1$, $\frac{dU(t)}{dt}=0$ follows, if and only if

$$S(t)=S^*, I(t)=I(t-\tau) \text{ for almost all } \tau\in[0,h] \text{ and } R(t)=R^*. \tag{3.7}$$

Since M is invariant, we have $\frac{dS(t)}{dt}=0$ in M. Then, from the first equation of (3.1) and (3.7), it holds that

$$\begin{aligned}
0 &= \frac{dS(t)}{dt}\\
&= B-\mu S^*-\beta S^*\int_0^h f(\tau)I(t-\tau)d\tau+\delta R^*\\
&= B-\mu S^*-\beta S^* I(t)+\delta R^*\\
&= \beta S^*\left(I^*-I(t)\right),
\end{aligned}$$

which implies that $I(t)=I^*$. Therefore, each element of M satisfies $S(t)=S^*$, $I(t)=I^*$ and $R(t)=R^*$ for all t. From Theorem 2.1 and LaSalle's invariance principle (see Kuang [5, Corollary 5.2]), E_* of system (3.1) is globally asymptotically stable.

Finally, we show that the condition (1.8) is equivalent to the condition (1.9). From (1.6) and (1.7), it follows $R_0=\frac{S_0}{S^*}$ and

$$\mu-\frac{\delta R^*}{S^*}=\mu-\frac{\delta\gamma}{\mu+\delta+\gamma}\frac{S_0\left(1-\frac{1}{R_0}\right)}{S^*}=\mu-\frac{\delta\gamma}{\mu+\delta+\gamma}(R_0-1).$$

Thus, we obtain

$$\begin{aligned}
\mu-\frac{\delta R^*}{S^*} &= \frac{\mu(\mu+\delta+\gamma)-\delta\gamma(R_0-1)}{\mu+\delta+\gamma}\\
&= \frac{\mu(\mu+\gamma)-\delta\gamma\left(R_0-1-\frac{\mu}{\gamma}\right)}{\mu+\delta+\gamma}\\
&= \frac{\mu+\gamma}{\mu+\delta+\gamma}\left\{\mu-\delta\left(\frac{R_0}{1+\frac{\mu}{\gamma}}-1\right)\right\}.
\end{aligned}$$

Therefore, one can see that (1.8) is equivalent to the condition (1.9). This completes the proof. □

Proof of Theorem 1.1. By Theorem 3.1, we obtain Theorem 1.1. □

Parameter	Value	Dimension
β	0.0001017-0.00022	Number of individual^{-1} $\cdot$ Year^{-1}
B	1600	Number of individual $\cdot$ Year^{-1}
μ	0.0125	Year^{-1}
γ	13	Year^{-1}
δ	0.35	Year^{-1}
τ	0.065	Year

TABLE 1. Parameters and their values in Section 4.

Finally, we consider the following corresponding discrete delay model of (1.4):

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu S(t) - \beta S(t)I(t-\tau) + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t-\tau) - (\mu+\gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu+\delta)R(t). \end{cases} \tag{3.8}$$

It is easy to see that (3.8) also has the same endemic equilibrium E_* of (1.4) if and only if $R_0 > 1$. The condition (1.8), which is equivalent to (1.9), becomes the sufficient condition for the global stability of the endemic equilibrium of (3.8) similar to the case of SIR epidemic model with distributed delays (cf. McCluskey [6, Section 5]).

4. **Numerical examples.** In this section, using parameter values given in Table 1, we present numerical examples to illustrate our global stability condition.

Example 4.1. *Set the parameter values B, μ and γ fixed in Table 1 but with $\beta = 0.0001017$. Then, we obtain $1 < R_0 = 1.00039\cdots \le 1 + \frac{\mu}{\gamma} = 1.00096\cdots$. From the first case of the condition (1.9) in Theorem 1.1, the endemic equilibrium E_* of system (1.4) is globally asymptotically stable for any $\delta \ge 0$.*

Example 4.2. *Set the parameter values B, μ and γ fixed in Table 1 but with $\beta = 0.00022$. Then, we obtain $R_0 = 2.1640\cdots > 1 + \frac{\mu}{\gamma} = 1.00096\cdots$. From the second case of the condition (1.9) in Theorem 1.1, the endemic equilibrium E_* of system (1.4) is globally asymptotically stable for $0 \le \delta \le \overline{\delta} = 0.0107\cdots$.*

Finally, we consider a case where the condition (1.9) does not hold. We fix the parameter values as in Table 1 but with $\beta = 0.00022$. Then, we have $R_0 = 2.1640\cdots > 1 + \mu/\gamma$ and $\delta = 0.35 > \overline{\delta} = 0.0107\cdots$. Thus the condition (1.9) does not hold. In Figure 2, we present the graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of a discrete delay model (3.8) with three different initial conditions and one can see that the endemic equilibrium seems to be globally stable. We leave as an open problem if the endemic equilibrium is globally stable for $\delta > \overline{\delta}$ and $R_0 > 1 + \mu/\gamma$.

REFERENCES

[1] E. Beretta and Y. Takeuchi, Global stability of an SIR epidemic model with time delays, *J. Math. Biol.* **33** (1995) 250-260.

[2] J. Chen, An SIRS epidemic model, *Appl. Math. J. Chinese Univ.* **19** (2004) 101-108.

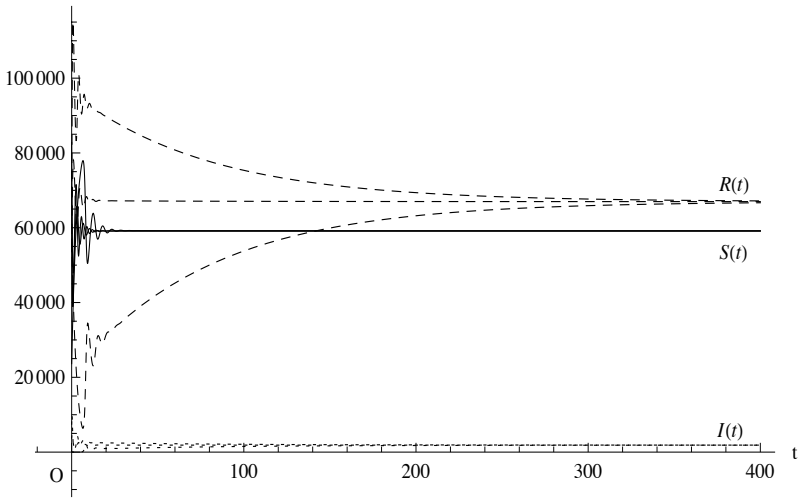

FIGURE 2. Behaviour of $S(t)$, $I(t)$ and $R(t)$ for $R_0 = 2.1640\cdots > 1 + \frac{\mu}{\gamma} = 1.00096\ldots$ and $\delta = 0.35 > \overline{\delta} = 0.0107\ldots$.

[3] Y. Enatsu, Y. Nakata and Y. Muroya, Global stability of SIRS epidemic models with a class of nonlinear incidence rates and distributed delays, *Acta Math. Sci.* To appear.
[4] H.W. Hethcote, The mathematics of infectious diseases, *SIAM review* **42** (2000) 599-653.
[5] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*. Academic Press, San Diego, 1993.
[6] C.C. McCluskey, Complete global stability for an SIR epidemic model with delay-Distributed or discrete, *Nonlinear Anal. RWA*. **11** (2010) 55-59.
[7] J. Mena-Lorcat and H.W. Hethcote, Dynamic models of infectious diseases as regulators of population sizes, *J. Math. Biol.* **30** 7 (1992) 693-716.
[8] Y. Muroya, Y. Enatsu and Y. Nakata, Global stability of a delayed SIRS epidemic model with a non-monotonic incidence rate, *J. Math. Anal. Appl.* **377** (2011) 1-14.
[9] Y. Muroya, Y. Enatsu and Y. Nakata, Monotone iterative techniques to SIRS epidemic models with nonlinear incidence rates and distributed delays, *Nonlinear Anal. RWA*. **12** (2011) 1897-1910.
[10] Y. Takeuchi, W. Ma and E. Beretta, Global asymptotic properties of a delay SIR epidemic model with finite incubation times, *Nonlinear Anal.* **42** (2000) 931-947.
[11] R. Xu and Z. Ma, Stability of a delayed SIRS epidemic model with a nonlinear incidence rate, *Chaos, Solitons and Fractals.* **41** (2009) 2319-2325.
[12] T. Zhang and Z. Teng, Global behavior and permanence of SIRS epidemic model with time delay, *Nonlinear Anal. RWA*. **9** (2008) 1409-1424.
[13] J. Zhen, Z. Ma and M. Han, Global stability of an SIRS epidemic model with delays, *Acta Math. Sci.* **26** (2006) 291-306.

Received July 2010; revised August 2010.

E-mail address: nakata@bcamath.org
E-mail address: yo1.gc-rw.docomo@akane.waseda.jp
E-mail address: ymuroya@waseda.jp