European Congress on Computational Methods
in Applied Sciences and Engineering (ECCOMAS 2012)
J. Eberhardsteiner et.al. (eds.)
Vienna, Austria, September 10-14, 2012

CAUSALITY, 'SUPERLUMINALITY', AND RESHAPING IN UNDERSIZED WAVEGUIDES

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Keywords: wave propagation, apparent 'superluminality', the Hartman effect

Abstract. *We analyse the reshaping mechanism leading to apparently 'superluminal' advancement of a pulse traversing an undersized section of a waveguide. For frequencies below the first inelastic threshold (cut off one), there are only evanescent modes in the narrow region, and the problem becomes similar to quantum mechanical tunnelling across an effective rectangular 'barrier'. In the coordinate representation, the barrier is shown to act as an effective beamsplitter, recombining envelopes of the freely propagating pulse with various spacial shifts. Causality ensures that none of the constituent envelopes are advanced with respect to free propagation, yet the resulting pulse is advanced due to a peculiar interference effect, similar to the one responsible for 'anomalous' values which occur in Aharonov's 'weak measurements'. In the momentum space, the effect is understood as a bandwidth phenomenon, where the incident pulse probes local, rather than global, analytical properties of the transmission amplitude $T(p)$. The advancement is achieved when $T(p)$ mimics locally an exponential behaviour, similar to the one occurring in Berry's 'superoscillations'.*

1 INTRODUCTION

In the early 1930's MacColl [1]noticed that quantum tunnelling appears to take no time or little time, in the sense that a wavepacket, transmitted across a classically forbidden region, may arrive at a detector earlier than that of a transmitted one. If the advanced peak is used to predict the time τ the particle has spent in the barrier region, the result is nearly zero. Dividing the barrier width by τ yields a velocity exceeding the speed of light c, suggesting that the transmission has a 'superluminal' aspect. The effect has been predicted and observed for various systems (for a review see Refs. [2]-[5]), such as potential barriers, semi-transparent mirrors, refraction of light, fast-light materials and microwaves in undersized waveguides [6]-[9].
Superluminal velocities are strictly forbidden by Einstein's causality, yet one might entertain the suspicion that the ban could be violated for extremely rare classically forbidden events. Such radical interpretations can be found, for example, in Ref. [6] which states that 'the effect ... violates relativistic causality', in [7] which suggests that 'evanescent waves do exist in a space free of time', and in Ref.[8] which report 'macroscopic violation of general relativity'. There is, however, general consensus that there is no conflict with relativity since the transmitted wavepacket undergoes in the barrier region a drastic reshaping. The fallacy is in identifying the transmitted peak with the incident one, while the causal connection between the two is broken in reshaping.
Even with reshaping understood to be the main cause of the apparent 'superluminality', one still faces the task of identifying the reshaping its mechanism, and clarifying the precise role played by the causality. The latter is usually related to the analytic properties of the scattering (transmission) amplitude. It was shown in Ref.[10] that the reshaping of the transmitted pulse occurs through an interference effect very similar to the one which causes the appearance of 'unusual' values in the so-called 'weak' quantum measurement introduced by Aharonov and co-workers in [11]-[13]. The main purpose of this paper, which is largely based on the similar approach developed in Refs. [14]-[15], is to analyse the origin of the 'superluminal paradox' observed in in undersized waveguides [6]-[9].
While most authors study the temporal variation of the signal at a given point in space (see e.g., [16]), we choose to analyse the shape of the transmitted pulse at a given moment in time, from which the time of arrival at a given location is then deduced. This approach is mathematically more transparent and, in addition, provides a useful connection with the quantum measurement theory [10]. The rest of the paper is organised as follows: Section 2 explores the similarity between quantum tunnelling and wavepacket propagation in an undersized waveguide. In Section 3 we analyse the causal reshaping mechanism which determines the form of the transmitted pulse. In Sect. 4 we discuss apparently 'superluminal' behaviour of the tunnelled pulse. Section shows how a complex coordinate shift experienced by the transmitted pulse results in its early arrival at a remote detector. In Section 6 the problem is analysed in the momentum space and 'superluminality' is related to 'superoscillatory' behaviour of the transmission amplitude. Section 7 contains our conclusions.

2 UNDERSIZED WAVEGUIDE AS A POTENTIAL BARRIER

Here we briefly review the well known [6]-[9] similarity between propagation of EM waves in waveguides of variable width and transmission of a quantum particle across a rectangular barrier. We start with a 2D waveguide which has a narrow segment of a width d. The scalar wave equation for the electric field component $\mathcal{E}$ ($TE01$ mode was used in experiments of

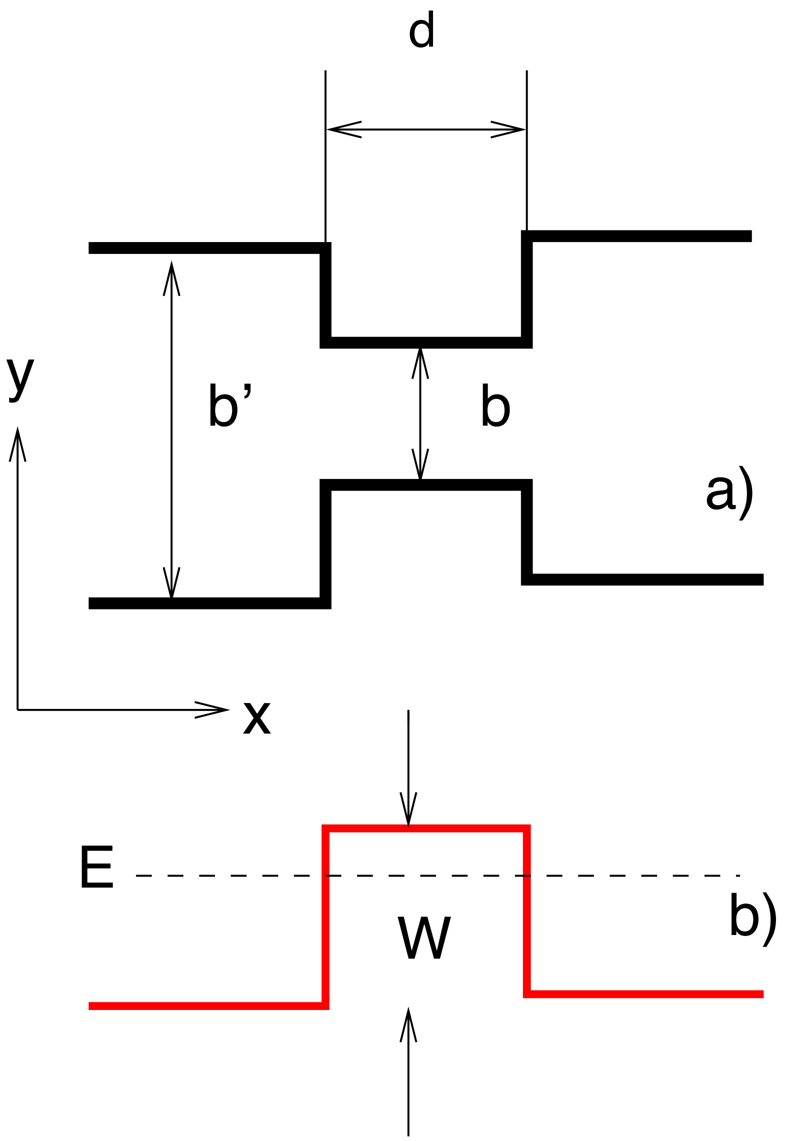

Figure 1: Schematic diagram showing a) an undersized waveguide in two dimensions; b) equivalent one dimensional potential barrier for motion in the x-direction.

Ref.[9]),

$$\partial_t^2 \mathcal{E} = c^2[\partial_x^2 + \partial_y^2]\mathcal{E}, \tag{1}$$

where c is the speed of light. Requiring the filed to vanish on the surface of the waveguide, we look for a solution of the form $\mathcal{E}(x, y, t) = \exp(i\omega t)sin(q_1 y)\Psi(x)$ where $q_1(x) = \pi/b$ for $0 \leq x \leq d$ and $q_1(x) = \pi/b'$ elsewhere. Now $\Psi(x)$ satisfies the equation (we neglect excitations, if any, of the higher transversal modes with $q_n = n\pi/b$, $n = 2, 3, ..$)

$$\partial_x^2 \Psi + [\omega^2/c^2 - q_1(x)^2]\Psi = 0. \tag{2}$$

This can be compared with one-dimensional Schroedinger equation for a particle of a mass μ and energy E , scattered off a rectangular potential barrier, $V(x) = W$ for $0 \leq x \leq d$ and 0 otherwise,

$$\partial_x^2 \Psi + 2\mu[E - V(x)]\Psi = 0. \tag{3}$$

Making the identification

$$\omega^2/c^2 - \pi^2/b'^2 \to 2\mu E, \quad q_1(x)^2 - \pi^2/b'^2 \to 2\mu V(x), \tag{4}$$

we may now discuss waveguide transmission it terms of a simple quantum mechanical scattering problem as shown in Fig.1. In particular, to an incident plane wave $\exp(ipx)$ corresponds the transmitted wave

$$T(p)\exp(ipx), \quad p = (2\mu E)^{1/2} = [\omega^2/c^2 - \pi^2/b'^2], \tag{5}$$

where $T(p, W)$ is the transmission amplitude for a rectangular barrier with the parameters defined by Eq.(4). For the discussion that will follow, it is convenient to write $T(p, W)$ as a geometric progression,

$$T(p, W) = \frac{4pk\exp[-i(p-k)d]}{(p+k)^2} \sum_{n=0}^{\infty} \frac{(p-k)^{2n}}{(p+k)^{2n}} \exp(-i2nkd), \tag{6}$$

where $k(p) \equiv [p^2 - 2\mu W)]^{1/2}$. Ultimately we want to use plane waves in order to build a pulse of a given shape (e.g., Gaussian), with a mean momentum p_0, and then study its temporal evolution. Noting that the dispersion law now reads

$$\omega(p) = \sqrt{c^2p^2 + \epsilon_0}, \quad \epsilon_0 \equiv c^2\pi^2/b'^2, \tag{7}$$

for the transmitted pulse we find

$$\Psi^T(x, t) = \int T(p)A(p - p_0)\exp[ipx - i\omega(p)]dp \tag{8}$$

where $A(p - p_0)$, peaked at $p = p_0$, is the momentum distribution of the initial pulse.

3 EXPLICITLY CAUSAL RESHAPING

Equation (8) can be transformed to the coordinate space by introducing the free propagating pulse (i.e., a pulse propagating in a waveguide of equal width b', without the narrowing shown in Fig.1)

$$\Psi^0(x,t) = \int A(p-p_0)\exp[ipx - i\omega(p)t]dp$$

and two slowly varying 'envelopes',

$$G^{T,0}(x,t,p_0) = exp[-ip_0x + i\omega(p_0)t]\Psi^{T,0}(x,t).$$

Rewriting the Fourier integral (8) as a convolution, we have [14]

$$G^T(x,t,p_0) = T(p_0)\int_{-\infty}^{\infty} G^0(x-x',t,p_0)\eta(x',p_0)dx', \tag{9}$$

where $\eta(x,p_o)$ is, essentially, the Fourier transform of $T(p)$,

$$\eta(x,p_0) = [2\pi T(p_0)]^{-1}\exp(-ip_0x)\int_{-\infty}^{\infty} T(p)\exp(ipx)dp, \tag{10}$$

The poles of the transmission amplitude $T(p)$ in the upper half of the momentum ($p-$) plane can only lie on the imaginary p-axis, where they correspond to bound states of the potential $V(x)$. In the case of a rectangular barrier, there are no bound states and closing the integration contour in Eq.(10) into the upper and lower half-planes yields [14]

$$\eta(x,p_0) = \delta(x) + \tilde{\eta}(x,p_0), \tag{11}$$

where $\int \eta(x,p_0)dx = 1$, and

$$\tilde{\eta}(x,p_0) \equiv 0, \quad for \quad x > 0. \tag{12}$$

Note that the δ term in Eq(11) occurs because transmission at high energies is not affected by the barrier, and $T(p) \to 1$ as $|p| \to \infty$. The alternating distribution $\eta(x, p = 0)$ is shown in Fig.2.

It is easy now to identify the *reshaping mechanism* which determines the form of the transmitted pulse: the barrier acts like a beamsplitter with an infinite (continuum) number of 'arms'. The transmitted pulse builds, therefore, from free envelopes with various spacial shift x', weighted with complex quantities $\eta(x',p_0)$. The transmission is explicitly *causal*: no matter where lies the peak of the transmitted pulse, none of the constituent envelopes emerge from the barrier advanced relative to free propagation. The position of the peak is determined by the interference between $G^0(x-x',t,p_0)$, and ultimately by the properties of the delay amplitude distribution (DAD) $\eta(x,p_0)$ in Eq.(10).

Equivalently, the causality argument can be made in the momentum space by stating that the Fourier spectrum of $T(p)$ does not contain negative frequencies. Indeed, we have

$$T(p) = \int_0^{\infty} \exp(ipx)\xi(-x)dx, \tag{13}$$

where $\xi(x) \equiv (2\pi)^{-1}\int_{-\infty}^{\infty} T(p)\exp(ipx)dp$, with $\xi(x > 0) = 0$ since $T(p)$ has no poles in the upper half of the p-plane. Accordingly, an incident plane wave with a momentum p_0 upon transmission is transformed into a weighted superposition of plane waves with all possible backward shifts,

$$\exp(ip_0x) \to T(p_0)\exp(ip_0x) = \int_{-\infty}^{0} \xi(x')\exp[ip_0(x-x')]dx. \tag{14}$$

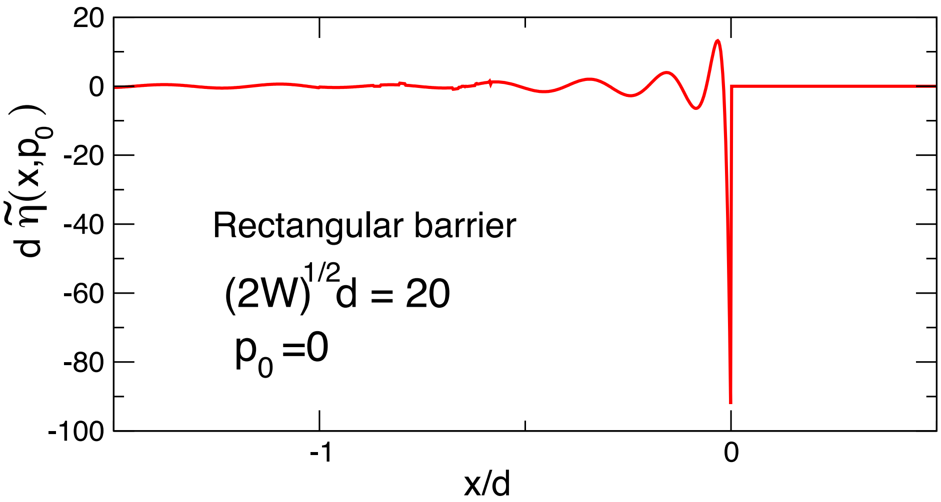

Figure 2: (colour online) Regular part of the delay amplitude distribution, $\tilde{\eta(x, p_0)}$ in Eq.(11), for a rectangular barrier of a width d calculated numerically from Eq.(10) ($Im\tilde{\eta} = 0$ since $T(p, W) = T(-p, W)$). Note the sharp negative peak at $x \approx 0$ which serves to cancel the contribution from the δ-function in Eq.(11) provided the initial pulse is sufficiently broad.

4 APPARENT 'SUPERLUMINALITY'

Like most of the authors [3]-[9], we will analyse the case of an initial pulse of a Gaussian shape prepared at $t = 0$ at some $x_0 < 0$. Depending on the ratio $\beta \equiv \epsilon_0/cp$, the dispersion law (7) can describe massless photons, $\omega(p) \approx cp$, for $\beta << 1$, or imitate non-relativistic particles of a mass $M = \epsilon_0/c^2$, $\omega(p) \approx \epsilon_0 + c^2p^2/2\epsilon_0$ for $\beta >> 1$. For simplicity, we consider here the first case, thus assuming that the free pulse propagates without spreading, and its envelope takes the form

$$G^0(x, t, p_0) = [2\sigma^2/\pi\sigma^4]^{1/4} \exp[-(x - ct + x_0)^2/\sigma^2]. \tag{15}$$

Well above a broad barrier, $p_0 d >> 1$, $p_0^2/2 >> W$, $(p_0 - 2/\sigma)^2/2 >> W$, evaluating integral in Eq.(10) by the stationary phase method shows that $\eta(x, p_0)$ in Eq.(9) is highly oscillatory, with a stationary region near $x' = x_s$, which selects a single shape $G^0(x - x_s, t, p_0)$ from the collection of retarded envelopes. Indeed, retaining only the first term in the expansion (6), we find that the phase of the integrand of Eq.(10), $s(p, x) = px - (p - k)d$, has a stationary point $p_s(x)$ determined by the condition

$$x - d[1 - p_s/k(p_s)] = 0. \tag{16}$$

The stationary point x_s of the phase of $\eta(x, p_0)$, $S(x, p_0) = -ip_0x + s(p_s(x), x)$, is determined by requiring that $\partial_x S'(x, p_0) = p_s(x_s) - p_0 = 0$, which yields

$$x_s = d[1 - p_0/k_0] < 0, \tag{17}$$

where $k_0 \equiv k(p_0)$. This recovers the classical result: a particle passes over a potential hill with a reduced velocity and, therefore, lags behind the free one.

Well below the barrier, $p_0^2/2 < W$, $(p_0 + 2/\sigma)^2/2 < W$, $(p_0 - 2/\sigma)^2/2 > 0$, $k(p_0)$ is imaginary, and $\eta(x, p_0)$ has no stationary region on the real axis., The tunnelling pulse is greatly reduced, and the peak of the transmitted probability lies approximately the distance d *ahead* of the freely propagating one (see Fig.3 d). This seems to suggest that 'there is no appreciable time delay in transmission' [1] and then prompt one to conclude that the barrier region has been traversed infinitely fast.

5 COMPLEX SPACIAL DELAYS AND THE PHASE TIME

To discuss the origin of this apparently 'superluminal' behaviour in more detail, we inspect the moments of $\eta(x, p_0)$,

$$\bar{x^n} \equiv \int_{-\infty}^{0} x^n \eta(x, p_0) dx = i^n \partial_p^n T(p)/T(p)|_{p=p_0}, \tag{18}$$

where the second equality follows from the Fourier relation (10). In the following we will consider the case of a broad barrier, $p_0 d >> 1$ (tunnelling across a high barrier, $W/p_0^2 >> 1$ can be treated in a similar way, details given in Ref. [14]). We obtain

$$\bar{x^n} = \alpha^n + O(d^{n-1}), \quad \alpha \equiv d\left(1 + \frac{ip_0}{\sqrt{2\mu W - p_0^2}}\right). \tag{19}$$

Increasing the width of the barrier, $d \to \infty$, and ignoring for the moment in Eq.(19) the corrections of order $O(d^{n-1})$, we conclude that $[G^n(x) \equiv \partial_x^n G(x)]$.

$$G^T(x,t,p_0) = T(p_0)\sum_n G^{(n)}/n! \int (x-x')^n \eta(x')dx' \tag{20}$$

$$\approx T(p_0)\sum_n G^{(n)}(x-\alpha)^n/n! \approx T(p_0)G^0(x-\alpha,t,p_0),$$

so that the barrier tends to shift the original pulse into the complex coordinate plane by α, and at the same time reduce its magnitude by the factor of $T(p_0)$, This simple mathematical result must be translated into the language of physical observables, such as the time the peak of the transmitted intensity, $P^T(x,t,p_0) \equiv |G^T(x,t,p_0)|^2$, arrives at a given location. For the Gaussian pulse (15) we have

$$P^T(x,t,p_0) \approx [2/\pi\sigma^2]^{1/2}|T(p_0)|^2 \exp(2Im\alpha^2/\sigma^2) \times \exp[-2(x-ct+x_0-Re\alpha)^2/\sigma^2]. \tag{21}$$

Thus, the peak of the tunnelled intensity lies approximately the distance $Re\alpha$ ahead of the freely propagating one. Equivalently, this peak would arrive at a fixed detector a time $\delta\tau$ earlier $[T(p) = |T(p)|\exp(i\Phi(p)]$,

$$\delta\tau = Re\alpha/c = -c^{-1}\partial_p\Phi(p)|_{p=p_0}, \tag{22}$$

than it would do by free propagation.
Naively interpolating this result in order to evaluate the time a photon has spent in the narrow part of the waveguide, one gets the well known *phase time* [5],

$$\tau_{phase} = c^{-1}[d + \partial_p\Phi(p_0)], \tag{23}$$

Such interpretation leads to known inconsistencies, e.g., that τ_{phase} can be shorter than the traversal time allowed by relativity. Thus, the said interpretation of Eq.(23) should best be avoided, whereas Eq.(22) is, of course, legitimate.

6 HARTMAN EFFECT AND SUPEROSCILLATIONS

Combining Eqs. (18), (19) and (23) shows that the phase time τ_{phase} (which, as we have demonstrated above, is not a physical duration) does not increase with the barrier width. This fact, known in the literature as 'the Hartman effect', suggests that by increasing the barrier width one would obtain a (greatly reduced) transmitted pulse lying ever further ahead of the one that moves freely. Hartman [18], who studied quantum mechanical tunnelling, has found that a given wavepacket does exhibit such 'superluminal' advancement for certain barrier widths d, but the effect disappears as $d \to \infty$, when the tunnelling becomes negligible and the transmission is dominated by the momenta passing over the barrier top. The discussion about whether the Harman effect and its variants do persist in the limit $d \to \infty$ continues to date [19]-[22].

Next we show, that it is possible to increase the pulse's width σ along with the barrier width in such a way that, as $d \to \infty$, the ratio of σ to the 'superluminal' advancement d tends to zero, and Eqs.(20) and (21) are satisfied ever more accurately. Under these conditions the transmitted signal is extremely weak, a patient observer might have to wait a long time to see the first photon, yet the first photon is guaranteed to arrive at the detector roughly a time d/c ahead of the peak of the free pulse.

The proof given in [15] is based on a simple estimate. An exponential of the form $\exp[d\sum_{n=0}^{\infty} f^{(n)}\sigma^n]$, with $d \to \infty$, $\sigma \to 0$ and $f^{(n)} \sim 1$ can be approximated by $\exp(df^{(0)})\exp(d\sigma f^{(1)})$ provided $d\sigma \sim 1$ and $d\sigma^n << 1$ for $n > 1$. To obtain the shape of the transmitted pulse we retain only the first term in Eq.(5) and expand the $f \equiv p - i|k|$ in a Taylor series around $p = p_0$. For a Gaussian pulse $A(p)$ in Eq.(7) is proportional to $\exp(-p^2\sigma^2/4)$ and $|T(p)|$ cannot exceed unity on the real p-axis, so that the typical value of $(p - p_0)$ is of order of $1/\sigma$. Accordingly, we estimate the terms in the exponent of $T(p)$ as

$$d\sum_{n=0} f^n(p_0)(p-p_0)^n \sim f(p_0)d + \frac{d}{\sigma}f'(p_0) + \sum_{n=2}\frac{d}{\sigma^n}f^n(p_0)/n!. \tag{24}$$

Thus, provided

$$\sigma = const \times d^{\frac{1+\epsilon}{2}}, \quad 0 < \epsilon \le 1, \tag{25}$$

the last sum in Eq.(24) can be neglected, and the incident pulse would 'see', as $d \to \infty$, the transmission amplitude

$$T(p) \approx T(p_0)\exp[-i\alpha(p-p_0)]. \tag{26}$$

Inserting Eq.(26) into Eq.(8), for such Gaussian pulses, we recover Eqs.(20) and (21), so that the peak of the transmitted density lies approximately the distance d ahead of a freely propagating one, while its width σ can be made much smaller, $\sigma/d \approx const/d^{\frac{1-\epsilon}{2}}$ than the 'superluminal' shift d. The accuracy of the approximation (20) improves as d increases (see Fig.3). Further numerical examples can be found in Ref. [15].

Finally, the explicit form of the transmission amplitude (26) 'seen' by a pulse, narrow in the momentum space, is

$$T(p) \approx \frac{4p_0k_0\exp[-|k_0|d]}{(p_0+k_0)^2}\exp\left[\frac{p_0}{|k_0|}(p-p_0)\right]\exp(-ipd). \tag{27}$$

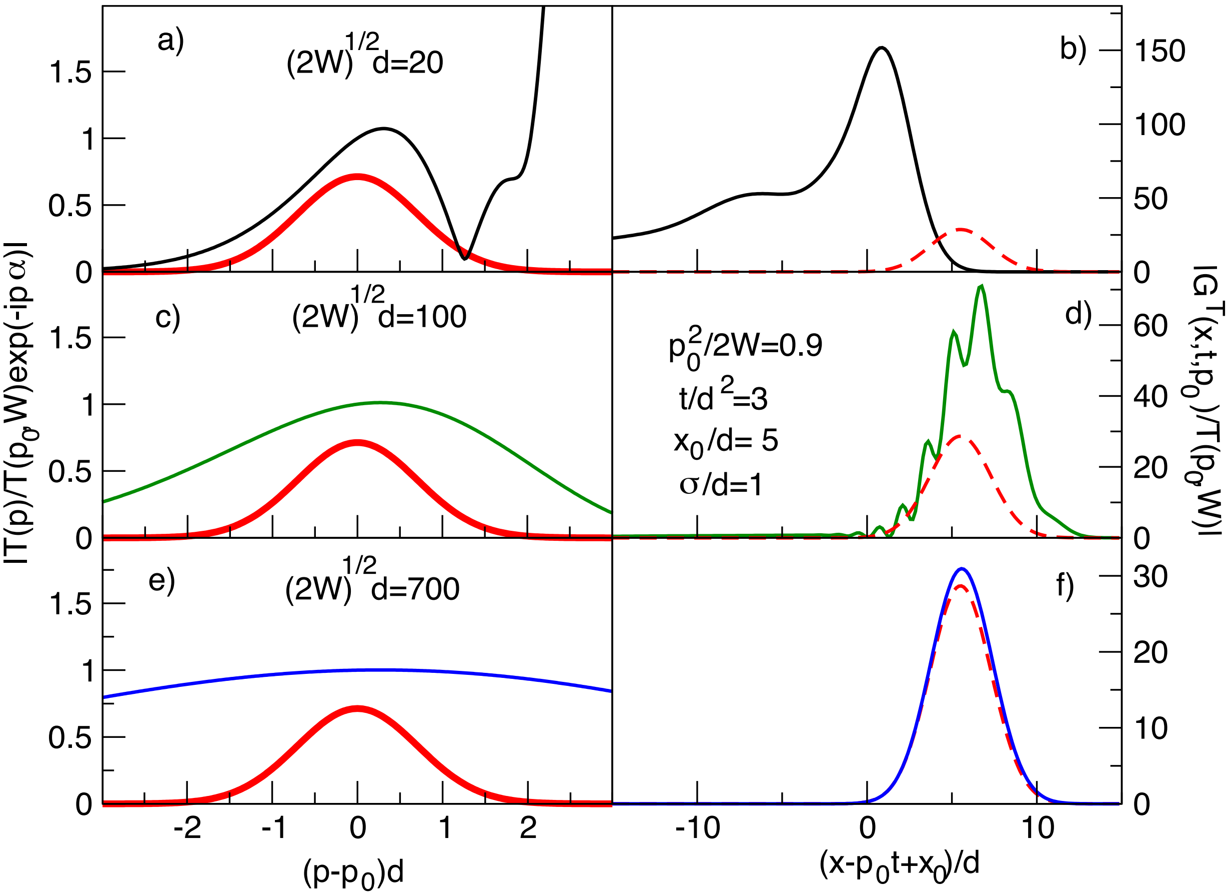

Figure 3: Transmission of a Gaussian wavepacket across rectangular barriers of different widths d. The barrier height W and the mean momentum p_0 are kept constant, but the coordinate width of the pulse increases in proportion to d, $\sigma/d = 1$. The panels (a), (c) and (e) show the ratio between $T(p)$ and the r.h.s. of Eq.(27) (solid) in the region containing the momentum distribution of the incident pulse (thick solid). Panels (b), (d) and (f) show the exact transmitted pulse (solid) and as given by Eq.(20) (dashed). As d increases, the incident pulse probes ever narrower range of momenta, and the transmitted envelope becomes a reduced copy of the original one shifted by α in Eq.(19) into the complex x-plane.

In Eq.(27) the last exponential, responsible for advancing the transmitted pulse along the real x-axis, has the negative frequency $-d$. This seems to contradicts causality, which ensures [cf. Eq.(13)] that there are no negative frequencies in the spectrum of $T(p)$. The resolution is, however, well known: Eq.(27) is valid only locally in p. Exponentials with only positive frequencies can be combined to locally reproduce the behaviour of an exponential with a negative frequency. The relevance of this phenomenon, called *supersocillations* [23]-[24], for the appearance of 'superluminal' tunnelling times was first discussed in [25]. In general, 'superluminal' advancement of a carefully chosen pulse can be related to the 'superoscillatory' behaviour of the transmission amplitude in the region probed by the pulse.

7 CONCLUSIONS AND DISCUSSION

In summary, the phenomenon of apparent 'superluminality' can be analysed in the coordinate or in the momentum space. Each approach has its advantages. In the coordinate representation, a potential barrier (or, indeed, any system with a linear relation between the incident and transmitted amplitudes) acts of an effective beamsplitter, recombining weighted copies of the incident pulse with all spacial shifts x', into the transmitted one. For a potential supporting no bound states, there are no positive shifts, i.e., none of the pulses leaving the beamsplitter are advanced, $x' \leq 0$. (Note the classical analogy: a particle cannot be sped up unless it passes over a potential well where its velocity increases). The causal nature of the propagation is, therefore, stated explicitly.

While the coordinate representation is best suited to expose the causal nature of transmission, the momentum space offers a description in terms of 'superoscillations'. Causality, formulated above as the absence of negative shifts, also requires that the Fourier integral of $T(p)$ contain only exponentials with positive frequencies. In the momentum space the 'paradox' consists in that to advance a pulse by α one needs $T(p)$ to behave as $\exp(-i\alpha p)$, while such exponentials are absent from its Fourier integral. This is a particular case of the 'superoscilltions' phenomenon studied by Berry and others [23]-[25]: the ability of exponentials with non-negative frequencies to mimic, locally, the behaviour of a plane wave with a negative one. If so, a pulse narrow in the momentum space may probe this local superoscillatory behaviour of the transmitted amplitude and is advanced, unconcerned about the global analytical properties of $T(p)$.

Finally we note that the above analysis can also be be applied to transmission of wavepackets of various nature in different types of physical media.

8 ACKHOWLEDGEMENTS

One of us (DS) acknowledges support of the Basque Government Grant No. IT472 and MICINN (Minis- terio de Ciencia e Innovacion) Grant No. FIS2009- 12773-C02-01.

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